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# Handbook of the Shapley Value


Edited by

Encarnación Algaba

Vito Fragnelli

Joaquín Sánchez-Soriano



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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742



Printed on acid-free paper

International Standard Book Number-13: 978-0-8153-7468-8 (Hardback)



**Library of Congress Control Number: 2019954409**

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# Contents

# *Foreword*

**The Shapley Value, a Giant Legacy, and Ongoing Research Agenda**
*Alvin E. Roth*

Game theory, as formulated by von Neumann and Morgenstern (1944), was divided into two parts, which they proposed were appropriate for two different kinds of games. Noncooperative game theory was focused on games in which players made independent individual decisions. Games were modeled and analyzed in terms of the strategies available to the players, and the primary focus of the theory was what sets of decisions would be in equilibrium.

Cooperative game theory was focused on games in which the players could reach binding agreements, and the primary focus of the theory was to discern what kinds of agreements rational players might reach. Games were modeled in terms of the outcomes that could be attained by coalitions of players. A convenient simple model is of games with transferable utility (TU) in characteristic function form. For a given set of players $N$, a TU game is a real valued function on the subsets of $N$ (the possible coalitions of players), $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$, and for each nonempty subset $S$ of $N$, $v(S)$ represents the total amount that the coalition $S$ could effectively distribute among its members regardless of what any players outside of the coalition did. Often an additional assumption is that the function $v$ is super-additive, i.e., that for any disjoint coalitions $S$ and $T$, $v(S \cup T) \geq v(S) + v(T)$, so that the maximum sum that could be distributed equals $v(N)$. A simple interpretation of the function $v$ is that all the players in the game are risk neutral expected utility maximizers and, for each coalition $S$, $v(S)$ is the amount of money that the coalition $S$ can distribute among its members in any way that it wishes. An outcome of the game would be a payoff vector $x \in \mathbb{R}^n$ with $x_i$ representing a payoff for each player $i \in N$ and $\sum_{i \in N} x_i = v(N)$. The goal of the cooperative theory envisioned by von Neumann and Morgenstern was to predict likely outcomes or sets of outcomes of the game.

Shapley (1953) proposed a different approach, or rather a solution to a different problem. Here are the opening sentences of the paper in which he introduced what has become known as the Shapley value:

*At the foundation of the theory of games is the assumption that the players of a game can evaluate, in their utility scales, every 'prospect' that might arise as a result of a play. In attempting to apply the theory to any field, one* *would normally expect to be permitted to include in the class of 'prospects' the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance.*

That is, Shapley proposed not to predict the outcome of the game, but instead to formulate a function that could be interpreted along the lines of the expected utility of playing a game, from each of its positions. However, the formal approach he considered did not involve the kinds of preferences over risky outcomes that von Neumann and Morgenstern had modeled with expected utility. Instead, Shapley considered a set of axioms on the functional form such a value might take.

Shapley considered a value $\phi$ for games $v$ among a universe of $N$ players to be a function such that for any game $v$, $\phi(v) \in \mathbb{R}^n$ satisfies $\sum_{i \in N} \phi_i(v) = v(N)$ (*efficiency*), $\phi_i(v) = 0$ for any $i$ that contributes 0 to every coalition (i.e., such that $v(S \cup \{i\}) - v(S) = 0$, for all subsets $S$ of $N$) (the *null player* axiom), $\phi(v + w) = \phi(v) + \phi(w)$ for any two games $v$ and $w$ (*additivity*), and $\phi_i(v) = \phi_j(v)$ for any $i, j$ which play symmetric roles in the game (i.e., for any permutation $\pi$ of $N$, with $\pi v$ being the permuted game, $\phi_i(v) = \phi_{\pi(i)}(\pi v)$ *symmetry*).

A critical observation is that for any nonempty subset $R$ of $N$, the *unanimity* or *pure bargaining* games $v_R$ defined by $v_R(S) = 1$ if $R \subseteq S$ and $v_R(S) = 0$, otherwise form a basis of the space of games $v$, so that every game $v$ can be written as a weighted sum of the games $v_R$. But since the axioms other than additivity determine that $\phi_i(v_R) = 1/r$ for $i \in R$ and 0, otherwise, this implies that, together with additivity, the axioms determine a unique (Shapley) value for all games,

$$\phi_i(v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \setminus \{i\})],$$

where $s = |S|$ and $n = |N|$ are the number of players in the coalition $S$ and $N$, respectively.

Shapley and Shubik (1954) studied this value for the class of political models called simple games, in which every coalition is either winning or losing, which can be interpreted as TU games such that for each coalition $S$, $v(S)$ equals either zero or one. Dubey (1975) shows how the value can be characterized within this class of games. Aumann and Shapley (1974) extended the Shapley value to study games with a nonatomic continuum of players, and used it to study its relationship to competitive equilibria of markets modeled in this way.

A useful mnemonic for remembering the formula for the Shapley value is that it is the expected marginal contribution of each player $i$ to the coalition it joins if the method of coalition formation is that the players enter one at a time in some possible ordering of the players, with each ordering being equally likely. Thus, there are $n!$ possible orderings, and in $(s-1)!(n-s)!$ of those orderings the coalition $S$ is exactly the coalition that is present after player $i$

enters, since the other members of $S$ can enter before player $i$ in $(s-1)!$ of the permutations and the remaining players can enter in $(n-s)!$ permutations.

Of course, this is not a game-theoretic model of coalition formation. One of the directions in which the Shapley value has been explored is to specify a coalition formation game from which the Shapley value could arise. A pioneering paper in this regard is Aumann and Myerson (1988), who consider what links players would mutually choose to form with other players, if the opportunity to form links occurs in some specified order and if the final payoffs will be the Shapley value for networked coalition structures explored by Myerson 1977 (in which the network where everyone is linked to everyone else yields the Shapley value).

My own work on values of games focused on Shapley's original goal of defining something like an expected utility of playing a game, which should depend on the preferences of the observer whose preferences are being modeled (see Roth 1977a,b; 1988a). An expected utility function of such an individual would model preferences over lotteries whose possible outcomes were that he would play a given position in a particular game. Consider an individual faced with a lottery of the form $[pv_i; (1-p)w_i]$, which with probability $p$ would have him play position $i$ in game $v$ and with probability $(1-p)$ position $i$ in game $w$. If that individual was indifferent between that lottery, and playing position $i$ in the weighted sum of the two games, i.e., indifferent between participating in the lottery or instead playing $(pw + (1-p)v)_i$, then I would call him neutral to probabilistic risk, and so his utility function $u$ would have $u([pv_i; (1-p)w_i]) = pu(v_i) + (1-p)u(w_i) = u(pw + (1-p)v)_i$ with the first equality following from expected utility and the second from neutrality to probabilistic risk. Hence, the utility function of such an individual would have the Shapley value's additivity property.

But ordinary risk is not the only kind of risk faced by a player in a game, and I used the pure bargaining games $v_R$ to assess a player's attitude towards strategic risk. In particular, I called an individual neutral to strategic risk if he was indifferent between playing the game $v_R$ as one of the players $i \in R$ or receiving a payment of $1/r$ for certain. For example, a player who worried that the game $v_R$ had a positive probability of ending in disagreement (with all players receiving zero) would likely prefer to receive $1/r$ instead of playing the game in one of the $r$ symmetric positions. But a player whose utility function equals the Shapley value must be neutral to strategic risk, and indeed the Shapley value is the utility function of an individual who is neutral to both probabilistic and strategic risk (and is indifferent between symmetric positions in a game).

Utility functions of individuals who are neutral to probabilistic risk but not to strategic risk are also weighted sums of marginal contributions, with the weights depending on the strategic risk posture, as captured in the numbers $f(r)$ for $r \in \{2, ..., n\}$ such that the individual is indifferent between receiving $f(r)$ for certain or playing the pure bargaining game $v_R$ among $r$ players. If the individuals are neutral to probabilistic risk, these indices are additive but not efficient. Inefficient indices are studied as semi-values, see, e.g., the important early contribution of Dubey, Neyman and Weber (1981).[1]

As the present volume will make clear, the Shapley value has inspired a continued stream of both theoretical and applied work, not only in economics and mathematics, but also in political science and related fields of application. However, two developments served to divert the attention of much of the game theoretic literature away from the Shapley value.

The first was a general shift in interest from cooperative game theory towards noncooperative theory, which came about in part from the view that the more detailed models of noncooperative games could serve equally well to model cooperative games: If players could reach binding agreements, how they did so could be modeled strategically. The second development was that TU games in characteristic function form came to be viewed as too simple to model many of the things economists were interested in, in which utility could hardly be transferred without restriction.

Although there were a number of attempts to generalize the results for the Shapley value to models with nontransferable utility, called NTU games (see particularly Harsanyi 1963, and Shapley 1969), these attempts came to be regarded as less successful than the original Shapley value for TU games (see, e.g., the discussions by Roth 1980 and Shafer 1980, and the subsequent exchange by Aumann 1985, 1986 and Roth 1986).

Today, particularly in areas of applied economics such as market design, cooperative and noncooperative game theory are viewed more as models at different levels of detail than as models of different kinds of games (Roth and Wilson, 2019). The question that Shapley posed, of how to evaluate the prospect of playing a game, is as important as ever, particularly because market designers have to take into account that participants in a market have the opportunity to choose which marketplaces to participate in, and hence must make comparisons of the kind that the Shapley value is intended to help answer. So one important area of future research will involve how to extend the insights gained, and the tools developed primarily from TU games to the wide class of other kinds of models that economists and game theorists now explore, as well as those explored by other social scientists.

Lloyd Stowell Shapley was one of the founding giants of game theory who helped lay the foundations of both cooperative and noncooperative game theory, and who influenced everything and everyone in the field. He was born in 1923. His paper defining the Shapley value was published in 1953, when he was 30 years old. A previous volume on the Shapley value, Roth (1988b), was published in honor of his 65th birthday. The present volume brings up to date the important stream of research on the Shapley value that has continued, unabated, ever since Shapley first proposed it, and that I expect will continue for the foreseeable future.

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[1] One well-studied value that encodes quite a bit of strategic risk aversion for large pure bargaining games ($f(r) = 1/2^{r-1}$) is the Banzhaf index, for which see Banzhaf (1965) and Dubey and Shapley (1979).

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# *Preface*

The book, *The Shapley Value: Essays in Honor of Lloyd S. Shapley*, edited by Alvin Roth and published in 1988, began by saying, "*This volume is in honor of the 1000001st (binary) birthday of Lloyd Shapley*"; in the case of this book, we could start by saying that it is to commemorate the 65th anniversary of the publication in 1953 of the article, "*A Value for n-Person Games*" by Lloyd Shapley, in which the one known today as the Shapley value was introduced for the first time.

This book contains 24 contributed chapters within which different aspects of the Shapley value are revised. The book is divided into four parts. The first part consists of two chapters which introduce the framework of the book and the Shapley value. The second part consists of eight chapters and is devoted mainly to theoretical aspects of the Shapley value (from Chapter 3 through Chapter 10). The third part consists of five chapters related to theoretical and applied issues of the Shapley value (from Chapter 11 through Chapter 15) and, finally, the fourth part has nine chapters which are devoted to applications of the Shapley value to different problems coming from very distinct fields (from Chapter 16 through Chapter 24).

We hope that this book will help highlight the importance of the Shapley value and its validity and interest more than 65 years after its introduction by Lloyd Stowell Shapley.

Finally, we would like to thank all those who have made this volume dedicated to the value of Shapley possible. Firstly, to all the authors who have contributed to this book, second to all the reviewers of the works included here, as well as Sarfraz Khan for inviting and encouraging us to write or edit a book in July 2016, Callum Fraser for his support during the process, Shashi Kumar for his technical support, Mansi Kabra for her editorial assistance at the last steps of this book and Arun Kumar for managing and overseeing the production phase of this project. Thanks to all of them for being part of this. We hope that this book venture, initiated with caring and enthusiasm more than three years ago, serves to bring gratification and inspiration to those who read it.

**Encarnación Algaba**
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# Chapter 1

## The Shapley Value, a Crown Jewel of Cooperative Game Theory

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### 1.1 Introduction

Game theory is a towering intellectual achievement of the post-World-War-II era and the Shapley value (Shapley, 1953b) a centerpiece of the branch of game theory known as “cooperative game theory”. Introduced in the early days of the subject when mathematicians were its main contributors, it was quickly adopted by economists, political scientists, and operations researchers. Its popularity is reflected in the multiple theoretical analyses of which it has been the object over the years and in the ever expanding scope of its applications. Although the fortunes of some other concepts of game theory have waxed and waned, the Shapley value is as fascinating today as it was when first defined.

Shapley’s 1953b paper is his most cited paper. Together with the Nash bargaining solution (Nash, 1950), it is an obligatory reference in general game theory texts and, of course, it is given detailed attention in all comprehensive treatments of cooperative game theory. It even has an important place in the leading graduate microeconomics textbook (Mas-Colell *et al.*, 1995).

The rich palette of essays gathered in this volume testifies to its remarkable resilience and versatility.

This Introduction first describes how the Shapley value fits in the conceptual apparatus of game theory. It reviews basic definitions of the key axioms that have been invoked in its classic characterizations. The second part briefly comments on how each of the chapters contributes to honoring Shapley and his value.

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## 1.2 Coalitional Games and their Values

A painter on his own can secure a certain income and the same is true of a plumber. When a painter and a plumber get together, they can save on advertising, billing and insurance costs; they can recommend each other to their respective customers; as a result, they can get more in total than the sum of what they would get on their own. By joining forces with a carpenter, an electrician and an architect, they can set up a construction company and build houses. The company can achieve more than the sum of what the components of any partition of its employees could attain. The question it faces then is to distribute among them the revenue it generates, taking into account what each group would attain on its own.

More generally, consider a group of people, or "players", who can get together in "coalitions". Each coalition can engage in activities that create value. By pooling together the resources its members control, exploiting the technology at their disposal, putting to the best use their skills and their knowledge, it can achieve "something", called its "worth". The simplest case is when the "something" is given as a single number, referred to as "utility". The assumption is made that utility can be transferred at a one-to-one rate among any two players. The vector collecting the worths of all the coalitions is a "transferable utility coalitional game", for short a "TU game". Here, the term "game" will simply be used. Utility is an abstract concept, however, and it will be easier to think of what coalitions can achieve and of what is assigned to players as money. This will be the most natural interpretation of the data of a game in almost all of the applications considered in this volume. Sometimes, players get in each other's way—for instance, if they have to use the same facility to produce worth—and together, they achieve less that the sum of what the subgroups into which they could arrange themselves could achieve. In calculating the worth of a coalition, all of the opportunities and organizational constraints the coalition faces should be identified and properly taken into account. The amount that has to be divided between the players is the worth of the grand coalition.

A "dual" interpretation of the model is possible, in which worths are replaced by costs. To each group is associated the cost that the group would incur in order to satisfy some demand it has for a service for example, or to undertake some project. Most of what follows covers this kind of applications, but we use language that is best suited to situations in which what is to be divided is a desirable entity.

A "solution concept" is a mapping that provides, for each game, a payoff vector, or a set of payoff vectors. When the data of a game are interpreted in monetary terms, a player's payoff is an amount of money, a salary, or a share of profits. A vector chosen by a solution concept for a game can also be thought of as a recommendation that an impartial arbitrator could make as to what the various players should get, or as a prediction of the compromise that they would agree on through negotiations. The negotiation process is left unspecified but one can imagine a conversation they would have, arguments they would make, in favor of this or that payoff vector, or to support general principles that could be invoked to select a payoff vector, not only for the game they face at this point, but also for each game they could have faced or could face in the future.

Single-valued solution concepts are often called "values" because the payoff specified for a player involved in some game is interpreted as the value to the player of participating in the game; alternatively, when the variables of the model are thought of in monetary terms, it is the amount that she would be willing to pay to get involved in it (or would have to pay if the cost-sharing interpretation of the model is taken).

The goal of the theory of coalitional games is to identify the most desirable solution concepts. Before going into the reasons why the Shapley value is widely regarded as one of them, it will be useful to show where it belongs in an organized inventory of solution concepts.

Two main categories can be distinguished (Hokari and Thomson, 2015). On the one hand, a "coalition-centric" solution concept attempts to satisfy coalitions. Each coalition has a claim on the worth of the grand coalition based on its own worth. Payoffs are not assigned to coalitions, however, but to players, so a coalition has to assess how well it is treated in terms of the sum of the payoffs assigned to its members. If this sum is too small, the coalition will object. To placate a coalition, the only instrument at one's disposal is the payoffs to its members, a blunt instrument because as a function of the number of players, the number of possible coalitions increases very quickly. Also, raising a player's payoff in an attempt to satisfy a coalition will imply lowering some other player's payoff, and this will impact negatively other coalition(s). Hence, the balancing act that one faces in selecting payoff vectors.

On the other hand, a "player-centric" solution concept rewards each player directly based on an assessment of what she can achieve on her own and how valuable she is to others, that is, based on the worths of the various coalitions to which she belongs. The challenge here is the opposite of what it is for coalition-centric solution concepts; it is to aggregate or summarize this rich information in some fashion, to distill it into a payoff for each player.

Examples of coalition-centric solution concepts are the "core" (Shapley, 1953a)[1] and the "nucleolus" (Schmeidler, 1969). For a payoff vector to be in the core of a game, each coalition should get in total at least its worth.

[1]Zhao (2018) provides compelling support for attributing to Shapley instead of to Gillies (1959), as is common, the formal definition of the core.

Otherwise, the coalition would refuse to participate. Depending upon the context, whether to participate may or may not be an option, but even if not, the scenario according to which a group of players would leave to collect its worth is a meaningful counterfactual on which to anchor the choice of a payoff vector. For the nucleolus, not only the sign, for each coalition, of the difference between what its members have been assigned in total and its worth, but also the magnitude of this difference, are taken into account. A large difference means that the coalition is "treated well" and a small difference the opposite. Expecting the loudest complaints from coalitions that are treated the worse leads to a lexicographic search for payoff vectors at which, dealing with coalitions in the reverse order of how well they are treated, each is treated as well as possible.

A first example of a player-centric solution concept is the "plain egalitarian value", which divides the worth of the grand coalition equally among all players. This solution concept has the disadvantage of disregarding the worths of all other coalitions.[2] The "equal-division-over-individual-worths value" first assigns to each player its own worth, then splits what remains equally among all players. This value is a little more responsive to the data of the game. Next, say that a player's "contribution" to a coalition to which she belongs is the change in the worth of the coalition if she leaves; if she does not belong to a coalition, her "contribution" to it is the change in the worth of the coalition if she joins. (It is with a slight abuse of language that the term is applied to a quantity that may be negative.) Calling a player's contribution to the grand coalition her "principal contribution"—it is this contribution that is most meaningful to an economist—the "equal-division-over-principal-contributions value" first assigns to each player her principal contribution, and then splits what remains equally among all players. This value too ignores most of the coordinates of a game, but instead of using as reference the worths of individual players, it uses their principal contributions.

The Shapley value is also a player-centric solution concept. However, by contrast to the three solution concepts just defined, it has the merit of taking all coordinates of a game into account: It assigns to each player a weighted average of her contributions to all coalitions, the weights being combinatorial expressions that are most easily derived from the following scenario. Imagine players arriving one at a time in some order and assign to each player what she contributes to the coalition consisting of all the players who were there when she arrived; then take the simple average of her contributions over all orders of arrival.

The Shapley value satisfies a number of appealing properties and various axiomatizations that have been given of it are brought up in several of the chapters of this volume. Shapley's (1953b) own characterization is based on

[2] This solution concept is often referred to in the literature as the "equal division value". The equal-division-over-individual-worths value, mentioned below, is known as the "equal surplus division value" and the equal-division-over-principal-contributions value as the "equal allocation of non-separable costs value".

four axioms: "efficiency" says that the sum of everyone's payoffs should be equal to the worth of the grand coalition; "the null player axiom" says that if all of a player's contributions are equal to 0, that is what she should get; "symmetry" says that two players who play symmetric roles in a game should be assigned equal payoffs; "additivity" says that the payoff vector selected for the sum of two games should be the sum of the payoff vectors selected for each of the games.

The first three axioms are punctual requirements that are quite acceptable in most applications. As noted in Algaba *et al.*'s introduction, additivity has often been criticized, however, as being merely a "technical" requirement. We would like to offer a defense of it, as a legitimate member, in the taxonomy developed in Thomson (2018, 2019a), of the category of "robustness axioms", a category with economic significance. Such an axiom expresses the idea that when a situation can be viewed from two different perspectives, neither of which having more legitimacy than the other, a solution concept should deliver the same payoff vector. Otherwise, whoever gets less if one perspective is taken could justifiably challenge the choice. Robustness of the compromise to choices of perspectives underlies many of the invariance axioms that have been considered in game theory and resource allocation theory. Here, when a society faces two separate problems, it could solve them separately or it could consolidate them into one. Additivity says that these two equally valid viewpoints should result in the same payoff vector.

Other characterizations of the Shapley value have been derived (again, see Algaba *et al.*). Each involves a strong relational invariance axiom. One says that a player's payoff should only depend on the profile of her contributions to the various coalitions (Young, 1985). The next two are variable-population axioms. First, if a player leaves a game, the impact that her departure has on any other player's payoff should be equal to the impact that the departure of that other player would have on her payoff (Myerson, 1977). Second is a "consistency" requirement. It asserts that a player's payoff in a game should be equal to her payoff in the related game obtained by imagining that some players leave. In this game, the worth of any coalition of remaining players is set equal to what would be left after these departing players are assigned what the solution would prescribe for them in the subgame involving them and the coalition under consideration (Hart and Mas-Colell, 1989). This type of reduction is an alternative to one proposed by Davis and Maschler (1965) and the consistency notion (called "self-consistency" in Thomson's 2019b survey, as the definition refers to the solution concept that is being operated) is a more demanding one, not in the logical sense, but because by itself, it is close to ruling out most other solution concepts. Another characterization stands out: It identifies the Shapley value as the only efficient mapping to have a "potential" (Hart and Mas-Colell, 1989).

The Shapley value has been generalized in multiple directions, to begin with, by Shapley (1969) himself, to games in which utility (or payoff) may still be transferable between players but at a rate that depends on the point of departure. (These are usually called non-transferable utility games, but it is more natural to think of them as games with variable rates of utility transfers.)

The robustness of the classic characterization of the Shapley value has been successfully tested on a variety of subdomains of the full domain of games (Dubey, 1975; Neyman, 1989).

By contrast to the core, whose non-emptiness requires that restrictions be placed on the game, to the nucleolus and its variants, for which explicit formulas can rarely be obtained, and to various "bargaining sets" as well as the "von Neumann-Morgenstern solution" (the definitions are omitted), which are notoriously difficult to compute, not to mention the existence issues that they raise, the Shapley value is always well defined and it is given by an explicit formula.

Concerning the various monotonicity properties that have been formulated in the literature, the Shapley value is one of the most satisfactory solution concepts. Without any restriction being imposed on games, it is such that an increase in the worth of a coalition brings about an increase in the payoff of each of its members (Young, 1985); also, on the class of convex games, the arrival of new players makes each of the players initially present at least as well off as he was initially (Sprumont, 1990).

The Shapley value is sometimes equivalent to other solution concepts. In spite of the fact that their definitions seem to have nothing to do with each other, for special classes of games, the Shapley value and the nucleolus coincide (Chun and Hokari, 2007). Most notably, when applied in exchange economies with a large number of consumers modeled as a continuum, it coincides with the Walrasian solution (Aumann and Shapley, 1974).

Applications of the Shapley value have been numerous and as already noted, the scope of these applications keeps expanding. First was Shapley's own calculation of how much each of the universities where he had given talks should contribute to the expenses he had incurred on his tour. The present volume presents ample additional evidence of the usefulness of the Shapley value in providing compelling recommendations for a variety of games; games obtained by requiring that the worths of coalitions be related in some fashion, games enriched with additional information, and games derived from some concretely specified allocation problems.

All of this is not to say that the Shapley value is the ideal solution concept, and it would do a disservice to it and to the field not to recognize its limitations and the merits of the competing solution concepts. As usual, there are tradeoffs, and for us, these tradeoffs are between properties.

First, an important drawback of the Shapley value is that it does not always select from the core. Of course doing so may not be an option since the core of a game is not guaranteed to be non-empty, but given the intuitive appeal of the core, one would hope that when it is not empty, a well-behaved solution concept would select from it. The Shapley value does not necessarily do that. Classes of games for which it does have been identified (the class of convex games is the most prominent one), but admittedly, they are quite narrow.

It is also important to note that other values, in particular the various player-centric solution concepts enumerated above, have gained some prominence recently. However, new structural results pertaining to the space of solution concepts have revealed interesting and unexpected ways of linking the Shapley value to these solution concepts (Casajus and Huettner, 2013, 2014; Béal *et al.*, 2015; Yokote *et al.*, 2019a, 2019b).

As for consistency-type properties, the Shapley value satisfies only one of the central ones, self-consistency, mentioned above.

Next is the complexity of calculating the Shapley value of a game: It is known to be NP-complete (Deng and Papadimitriou, 1994; Castro *et al.*, 2009), but here one should note that sampling techniques sometimes help. Also, simple formulas for the Shapley value, when the number of players is large, may be available. Several chapters deliver such formulas.

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## 1.3 A Short Guide to the Chapters

Most of the chapters in this volume are surveys. They range from the very theoretical to the very applied. A fair number of them start with theory and close with one or several applications and some with real-world data. The theoretical chapters state formal results and a few develop complete proofs. References to primary sources are given for any reader eager to get a complete treatment of an issue.

The chapters contribute to the subject in several dimensions. (The enumeration below is not meant to be a partition.)

**Mathematical foundations.** Shapley's and several subsequent characterizations of the Shapley value rely on the "unanimity games" being a vector basis of the linear space of all games. New bases have recently been described whose identifications have allowed new characterizations of the Shapley value. They are central to two chapters.

**Axiomatic foundations.** New axiomatic perspectives have played an important role in recent literature and they are discussed in several chapters.

**Computations.** We already mentioned that calculating Shapley values is out of reach even for relatively small numbers of players. However, in some situations identified in several chapters, simple formulas can be derived.

**Applications to special classes of games; power indices.** A "voting game" is one in which the worth of a coalition can take only two values, 0 and 1, interpreted as losing or winning an election. A "power index" is not meant to distribute some aggregate payoff, the worth of the grand coalition, but to assess the power of each player in allowing a win. The "Shapley-Shubik index", which is simply the Shapley value applied to such games, and the

"Banzhaf index" are primary examples. Several chapters derive power indices from novel considerations.

**Applications to enriched classes of games.** The abstract model of the theory of coalitional games can be enriched in a variety of ways. A coalition structure may be added that has to be respected, or a graph structure on the player set may be included from which the feasibility of coalitions is derived; a hierarchical structure on the player set may also be listed; externalities may be present; there may be uncertainty about the coalitional form. Multiple chapters deal with such enriched models.

**Applications to concretely specified allocation problems: An ever widening range.** Applications of the Shapley value to problems of cost allocation started with the Tennessee Valley Authority pricing problem, the pricing of telephone services, the sharing of the cost of a runway, and taxation. Recent applications to queueing, two-sided matching problems with money, and minimum cost spanning tree problems are the object of several chapters, and other chapters cover new applications to telecommunications, loss allocation in energy transmission networks, terrorism, biology, finance, politics, and molecular genetics.

**Mapping allocation problems into games: How?** To use a solution concept for coalitional form games so as to obtain recommendations for a class of concretely specified allocation problems, one has to map these problems into games. There is rarely a unique way of assessing the worth of a coalition, however, and depending upon the mapping that is adopted, applying the same solution concept will deliver one or the other of several alternative allocation rules. This is illustrated in this volume by applications to queueing problems and minimum cost spanning tree problems. For each of these two classes of problems, "positive" and "negative" views of the situation faced by each coalition can be taken and used to bracket the elusive "true" worth of the coalition.

**Implementation.** A non-cooperative mechanism is said to "implement" a solution concept on some domain of economies if for each economy in the domain, the mechanism has equilibria and the associated equilibrium outcomes are outcomes that are selected by the solution concept for the economy. One chapter exhibits a mechanism that implements the Shapley value.

We proceed with highlights of each of the chapters, indicating in each case in which of the directions just enumerated the chapter principally contributes.

After this Introduction, the editors made an essay about the central axiomatizations that have been given of the Shapley value.

In "An Index of Unfairness" (Chapter 3), Aguiar *et al.* develop a measure of how far a payoff vector for a game is to the payoff vector selected by the Shapley value. They propose an additive decomposition of this measure into three terms, each of which corresponds to a violation of one of the three axioms that enter Young's (1985) characterization of the Shapley value. These axioms are efficiency, symmetry, and the invariance requirement that each player's payoff in a game depend only on the vector of her contributions. [Mathematical foundations]

In "The Shapley Value and Games with Hierarchies" (Chapter 4), Algaba and van den Brink consider a formulation of the model of coalitional games enriched by the addition of a hierarchy among players. For such a model, two approaches have been developed: The "permission structure approach" and the "precedence constraint approach". The authors present a number of characterizations for games with permission structures and games with precedence constraints. [Mathematical foundations; enriching the model]

In "Values, Nullifiers and Dummifiers" (Chapter 5), Alonso-Meijide *et al.* formulate variants of the "nullifier axiom". A "nullifier" is a player such that the worth of any coalition containing her is 0. It is only recently that the significance of the concept, due to Deegan and Packel (1978) has been fully brought out (van den Brink, 2007; Casajus and Huettner, 2013, 2014; Béal *et al.*, 2016). The "nullifier axiom" requires that a nullifying player's payoff be 0. The authors formulate variants of the axiom and establish characterizations of the Shapley value and of an efficient version of the Banzhaf power index that involve them. They also define a "dummifier" as a player such that the worth of any coalition containing her is equal to the sum of the individual worths of its members. They propose the requirement that a dummifier gets her own worth. Here too, they define variants of this axiom and establish parallel characterizations of the Shapley value and the Banzhaf power index by substituting dummifier for nullifiers. This chapter contains extensive proofs. [Mathematical and axiomatic foundations]

In "Games with Identical Shapley Values" (Chapter 6), Béal *et al.* ask how to partition the space of all games into classes of games whose Shapley values are equal, a question first raised by Kleinberg and Weiss (1985). The resolution they propose involves identifying several new bases for the domain of all games. Armed with this knowledge, they develop additional characterizations of the Shapley value. [Mathematical and axiomatic foundations]

In "Several Bases of a Game Space and an Application of the Shapley Value" (Chapter 7), Funaki and Yokote address similar issues. They too identify a family of new bases for the space of games and explore their properties. This allows them to identify the null space of the Shapley value. They also derive conditions on games under which the nucleolus and the Shapley value make the same recommendations. This chapter contains extensive proofs. [Mathematical and axiomatic foundations]

In "Extensions of the Shapley Value for Environments with Externalities" (Chapter 8), Macho-Stadler *et al.* revisit cooperative games with externalities (Thrall and Lucas, 1963). Theirs is a detailed survey of the approaches that have been taken to solve this class of games. They also design a non-cooperative game that implements a variant of the Shapley value that they define. [Extensions; axiomatic foundation; implementation]

In "The Shapley Value and Other Values" (Chapter 9), Bernardi and Lucchetti discuss the concept of a "semi-value", namely a mapping that does not necessarily satisfy the efficiency axiom. They define the concept of a probabilistic semi-value, of which the Shapley value and the Banzhaf power index are two special cases. A novel feature of their paper is an application to molecular genetics. [Mathematical foundations; power indices; applications]

In "Power and the Shapley Value" (Chapter 10), Peters also discusses power indices. His objective is to extend the theory of power indices to situations in which the structure of decision power is described by means of "effectivity functions" or by means of "control structures". How can one derive from such information a measure of the power of each player? He characterizes a family of indices that can be seen as weighted versions of the Shapley value. [Enriching the model; power indices]

In "Cost Allocation with Variable Production and the Shapley Value" (Chapter 11), Albizuri *et al.* review the earliest applications of the Shapley value. When there are many players, they can be modeled as points in a non-atomic continuum, a model whose analysis was the object of a landmark book by Aumann and Shapley (1974). Various characterizations of the cost allocation method they proposed are presented next. When demands are natural numbers, a Shapley value can also be calculated by numbering the units each player demands, providing units in a random order, with each player being held responsible for the sum of the added costs that satisfying each of the units she demanded caused over what was needed until she was served. This is the discrete "Aumann-Shapley cost allocation rule". Several characterizations of this rule have also been proposed, one based on the balanced contributions axiom, another involving an axiom of invariance with respect to merging and splitting of demands, the final one being based on an adaptation of Young's invariance axiom. [Mathematical and axiomatic foundations; applications]

In "Pure Bargaining Problems and the Shapley Rule: A Survey" (Chapter 12) Carreras and Owen explore a connection between bargaining games à la Nash and games with a coalition structure. They first compare what they call a "proportional rule" and the Shapley value when applied to a class of problems with a simple structure (problems that are essentially bargaining problems). They argue that the Shapley value is better behaved than the proportional value. They then enrich the model by adding a coalition structure. They extend the Shapley value to the resulting class of games and characterize their extension by adapting Shapley's own characterization. [Enriching the model; axiomatic foundations]

In "The Shapley Value as a Tool for Evaluating Groups: Axiomatizations and Applications" (Chapter 13), Flores *et al.* associate to each game and each group of players (as opposed to each player) a measure of the group's power. They call such a mapping a "generalized value". Given a standard coalitional game, they define for each coalition a new game by amalgamating its players, as in Lehrer (1988), and applying the Shapley value to it, thereby obtaining the "Shapley group value". They base a characterization of this value on a version of the balanced contributions axiom. They identify desirable properties that this mapping satisfies. They develop two applications, one to measure the power of the political parties in contemporary Spain and the other to "inventory cost games". They conclude by revisiting their definitions in the context of a model enriched with a graph structure. [Axiomatic foundations; power indices; enriching the model]

In "A Value for $j$-Cooperative Games: Some Theoretical Aspects and Applications" (Chapter 14), Freixas considers a class of games in which agents can participate at several levels. These games generalize the standard model of coalitional games. He proposes a value to solve the games and shows that when restricted to standard games, it reduces to the Shapley value. He derives alternative formulas for this new value and offers several axiomatic characterizations of it. He also defines a value inspired by the Banzhaf power index and gives characterizations of it. All characterizations include an axiom of additivity. [Mathematical and axiomatic foundations; applications; enriching the model]

In "The Shapley Value of Corporation Tax Games with Dual Benefactors" (Chapter 15), Meca *et al.* formulate a model of tax collection when evasion is possible and associate with it a coalitional game. They establish the concavity of the game. They propose to solve it by applying the Shapley value and show that the Shapley value can be given an explicit and simple expression. This chapter includes the proofs for most results. [Applications; enriching the model; mapping allocation problems into games; computation]

In "The Shapley Value in Telecommunication Problems" (Chapter 16), Sanchez-Soriano considers various applications of the Shapley value to telecommunications, in particular to wireless networks, Internet pricing, and routing in communication networks. He raises the issue of computational cost of the Shapley value and identifies special cases in which it can be given an explicit and simple expression. [Applications; computation]

In "The Shapley Rule for Loss Allocation in Energy Transmission Networks" (Chapter 17), Bergantiños *et al.* address the issue of distributing among the owners of the components of an energy network the inevitable losses that occur along the network. They propose to associate a game with each situation of this kind and to apply the Shapley value to it. They note that this solution is actually outperformed by other rules they had studied in an earlier paper of theirs. They consider a wide variety of properties having to do with punctual fairness, independence, and incentives. However, when applied to the Spanish electrical network, they note that the Shapley value allocation is actually quite similar to the allocation recommended by another rule, which they call the "proportional tracing rule". They then perform simulations, changing the parameters of the problem, and find that this correlation persists, leaving it as an open question why that might be the case. [Applications; computation and simulation]

In “On Some Applications of the Shapley-Shubik Index for Finance and Politics” (Chapter 18), Bertini *et al.* consider an application of the Shapley value to the construction of power indices. The paper enumerates alternatives to the Shapley-Shubik power index, as well as the properties of this index. Its focus is on financial applications, and it discusses how power is redistributed when a transfer of shares between two shareholders occurs, when some large shareholders face an “ocean” of small shareholders, and when control is indirect (an investor holds shares in a company that holds shares in some second company). [Applications; power indices; enriching the model]

In “The Shapley Value in the Queueing Problem” (Chapter 19), Chun considers the problem of assigning users of a service to a queue where they will receive it. All users require service for the same length of time but each user has her own cost of waiting. The question is to decide the order in which they should be served and how much each of them should pay for the service. To apply the Shapley value to solve an allocation problem, the problem has to be mapped into a coalitional game. A favorable position can be taken in evaluating the worth of a coalition, which consists in assuming that the members of the coalition are given priority over the complementary coalition in the assignment of slots; the unfavorable position consists of course in given priority to the complementary coalition. Whether the Shapley value is applied to the first game or to the second game delivers two distinct rules, each with its own properties. [Applications; mapping allocation problems into games]

In “Sometimes the Computation of the Shapley Value is Simple” (Chapter 20), Dall’Aglio *et al.* address the practical issue of calculating Shapley values. We have already noted that, for certain classes of games, explicit formulas can be obtained. They are reviewed here. They include games associated to the problem of sharing the cost of partially overlapping public goods (“airport games”); the problem of cleaning pollutants from a river when the pollutants are carried downstream; as well as a certain kind of auctions, and four classes of games for which a decomposability property holds. They are queueing games in which players differ in both their unit cost of waiting and the length of the service they need, “maintenance games”, which have to do with providing a service to a set of users who are organized in a hierarchy, “microarray games”, encountered in biology, and “coverage games”, which concern the choice of the best location for an ambulance, taking into account the coverage provided by each location. [Computations; applications]

In “Analysing ISIS Zerkani Network Using the Shapley Value” (Chapter 21), Hamers *et al.* undertake a provocative application of the Shapley value to the ranking of the members of the terrorist networks that carried out the attacks on 2015 and 2016 in France and Belgium, respectively. Weights are assigned to members of the networks based on the resources they control. Also, weights are assigned to the links between any two members, based on the frequency of their information exchanges. Two games are constructed from these data. Because of the complexity of calculations of Shapley values as the number of players increases, they apply a sampling technique to approximate them. They derive from it a ranking of individual members of the terrorist groups and compare the result based on the analysis of the structure of the graph connecting them. Thanks to this work, surveillance resources can be targeted more precisely to the individuals posing the greatest threat. [Applications; computation]

In "A Fuzzy Approach to Some Shapley Value Problems in Group Decision Making" (Chapter 22), Gladysz *et al.* use the tools of fuzzy numbers to model situations in which the parameters of a game are not known with certainty. They apply their approach to several voting situations, the Polish Parliament, the unification of Germany, and Brexit. [Mathematical foundations; enriching the model; computation]

In "Shapley Values for Two-Sided Assignment Markets" (Chapter 23), Núñez and Rafels consider the well-studied class of games in which the player set is partitioned into two "sides" and when a pair is formed, one player on each side, "worth" is generated. What is to be distributed is the sum of the worths of the pairs that are formed. By contrast to other types of pairing problems that have been the object of much literature, a player here does not care about whom she is paired with, only about her share. It is natural to expect a player's payoff to depend not just on the worth of the pair consisting of herself and her assigned partner but also on the worths of the other pairs that she could be part of. The authors provide a number of theorems identifying conditions under which the Shapley value, when applied to the coalitional game associated with a problem of this type in a natural way belongs to its core, and establish an axiomatic characterization of it. [Mathematical and axiomatic foundations; applications; mapping allocation problems into games; computation]

In "The Shapley Value in Minimum Cost Spanning Tree Problems" (Chapter 24), Trudeau and Vidal-Puga consider the problem of connecting a group of players to a "source" where they get supplies, say. Given are the costs of connecting each player to the source and to each other player. If a link between two players is established, it can be used by any number of players at no extra cost. Once a network whose total cost is minimal is identified (several algorithms have been defined that achieve this objective), the question to be addressed is how much each player should contribute to its cost. This chapter offers a detailed survey of the literature on the subject, in which the Shapley value has played a central role. Here too, several perspectives can be taken in associating a coalition game to a spanning tree problem. Indeed, in calculating the worth of a coalition, different assumptions can be made about its access to the links involving the other players. As a result, more than one rule can be obtained by applying the Shapley value. These various Shapley value-based rules have been characterized, on the basis of a variety of relational independence and monotonicity properties. [Applications; mapping allocation problems into games]

## 1.4 Acknowledgments

Thanks to Encarnación Algaba, Bas Dietzenbacher, and Yuki Tamura for their helpful comments.

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# Chapter 2

# *The Shapley Value, a Paradigm of Fairness*

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## 2.1 Introduction

Game theory deals with the analysis of conflict of interest situations in which two or more individuals are involved. Roughly speaking, we may distinguish among two main classes of games depending on the possibility for the agents of subscribing binding agreements or not. In the latter case, we have a non-cooperative game, while in the former case we have a cooperative game. When a non-cooperative game is analyzed, the main aim is to identify a *"good"* strategy for each agent, as is intended when using the Nash equilibrium [15]. Perhaps the Nash equilibrium is the most important solution concept for this class of games. On the other hand, when a cooperative game is analyzed, the main aim is to determine whether there are reasonable ways to divide among the agents involved in a coalition, particularly the grand coalition, the total utility they would obtain by signing a binding agreement of collaboration. Therefore, in this case, agents' main interest is not on a strategy choice for each of them, but on the share that they will obtain of the utility of the coalition.

When the utility can be transferred among the agents in whatever way losslessly, the reasoning above leads to the representation of a transferable utility game (TU game, or simply game, in the sequel) using the characteristic function form. A *TU game* in characteristic function form is a pair $(N, v)$ where $N = \{1, 2, ..., n\}$ is the set of agents, called *players*, and $v : 2^N \to \mathbb{R}$ is the *characteristic function* that assigns to each coalition of players $S \subseteq N$ a real number $v(S)$ with the condition that $v(\varnothing) = 0$. The worth $v(S)$ may be interpreted as the utility that the players in $S$ may obtain by themselves, independently of the behavior of the other players not in $S$. Then, a solution for a TU game is a possible way for dividing the utility of a coalition, in general the grand coalition $N$, among its members.

In their pivotal book of 1944, *Theory of Games and Economic Behavior*, John von Neumann and Oskar Morgenstern [25] proposed an important solution concept, the *imputation set* $I(v)$, whose definition leads to a set of possible ways for sharing the total utility of the grand coalition. Given a game $(N, v)$, the imputation set includes all the $n$-dimensional vectors which are *efficient*, i.e., share the whole utility, and *individually rational*, i.e., assign to each player at least her standing alone worth. Formally:

$$I(v) = \left\{ x \in \mathbb{R}^n \text{ s.t. } \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}), \forall i \in N \right\}. \tag{2.1}$$

In the same line, Donald Gillies proposed in 1953 [5] another set solution, the *core* $C(v)$, that can be viewed as a refinement of the imputation set, considering only the *coalitionally rational* imputations, i.e., those that assign to the members of each coalition at least the worth they can obtain all together. Recently, also the next formal definition of the core is attributed to Shapley (see [27]).

$$C(v) = \left\{ x \in I(v) \text{ s.t. } \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \right\}. \tag{2.2}$$

All these solutions are very important for describing the features that a reasonable allocation should have, but they also have two negative aspects. First, being set solutions, they are not able to provide a definite answer to the question of how to distribute the utility. Second, they may be empty.

In 1953, Lloyd Stowell Shapley [21][1], awarded with the Nobel Memorial Prize in Economic Sciences in 2012 together with Alvin Roth, proposed a

[1] It was first published as a Research Memorandum of the Rand Corporation on the 21st of August, 1951 [20].

point solution that overpasses the two above-mentioned negative aspects. In fact, it is given by a formula that provides the exact amount of utility that each player receives after the division and it can be applied to any cooperative game with transferable utility. Formally, a *point solution* or *value* is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that associates to each TU game $(N, v)$ in the set $\mathcal{G}^N$ of TU games with player set $N$ a $n$-dimensional vector $\psi(v) \in \mathbb{R}^N$, where $\psi_i(v)$, $i \in N$, is the amount of utility assigned to player $i$.

In the rest of this chapter, we do not intend to make an exhaustive review of everything that the Shapley value has meant in the scientific literature, but simply to show some brushstrokes that serve as the initial and introductory chapter for this book dedicated to the Shapley value. We will first present some of the best-known mathematical expressions, starting with those introduced by Lloyd Shapley in 1953. Secondly, we present some of the most important characterizations that show the large number of nice and interesting properties that this value satisfies. Finally, we select a sample of the Shapley value extensions to a large number of contexts and their applications to very different fields and problems. All of this in order to show that the Shapley value is highly regarded by many researchers as a reference to analyze allocation problems in the most general sense.

## 2.2 The Mathematical Expression

Shapley introduced his value referring to a superadditive game $(N, v)$, i.e., whose characteristic function satisfies the condition $v(S \cup T) \geq v(S) + v(T)$, with $S \cap T = \varnothing, S, T \subseteq N$. This is because von Neumann and Morgenstern considered this assumption to define cooperative games. In addition, superadditivity captures the idea that cooperation can be beneficial for all the players involved. Anyhow, the formula by Shapley holds for every game. When the game $(N, v)$ is superadditive, then the Shapley value is an imputation. Moreover, Shapley introduced his value by using three axioms that can be considered fair and crucial for accepting a solution concept. The axiomatic approach of the Shapley value will be discussed below.

From those three axioms, Shapley demonstrated that the idea behind his value, which it will be denoted here by $\phi$, is the *marginal contribution* of a player $i \in N$ to a coalition $S \subseteq N \setminus \{i\}$ that is defined as the variation of the worth of the coalition after player $i$ joins it, i.e., $v(S \cup \{i\}) - v(S)$. He obtained that the value for a player $i$ is her weighted average marginal contribution over all the possible coalitions which player $i$ could be joined to, where the weight associated with each coalition depends on its size, and can be interpreted as the probability that a coalition of that size will be formed. Therefore, the probability that a coalition is formed does not depend on the players that form it but only on its size. The well-known mathematical formula he obtained for his value was:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)], \forall i \in N, \qquad (2.3)$$

where $s = |S|$ and $n = |N|$. Hereinafter, we use the same notation.

John Charles Harsanyi, awarded with the Nobel Memorial Prize in Economic Sciences in 1994 together with John Forbes Nash and Reinhard Selten, proposed in 1959 [6] the following alternative formula:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{\Delta_v(S \cup \{i\})}{s+1}, \forall i \in N, \qquad (2.4)$$

where $\Delta_v(T)$ are the so-called *Harsanyi dividends*, which are defined as follows:

$$\Delta_v(T) = \sum_{R \subseteq T} (-1)^{|T|-|R|} v(R), \forall T \subseteq N. \qquad (2.5)$$

Harsanyi dividends can be obtained by means of a recursive procedure. Hence, they can be interpreted as a measure of the net surpluses of coalitions by discounting the surpluses already created by their subcoalitions. In this sense, the Shapley value is the solution that divides equally these surpluses among the players involved in each coalition.

Shapley also derived his value by a *"bargaining model"* in which the grand coalition would be formed in such a way that the players would be admitted one by one until everyone had been added to the grand coalition. Each player on her admission would demand her marginal contribution to the already added players. The order of entrance in the grand coalition would be randomly selected with all orderings being equally probable. Thus, the expected payoff to each player is precisely the Shapley value. In this sense, as Shapley [21] said, *"the value is best regarded as an a priori assessment of the situation, based on either ignorance or disregard of the social organization of the players."* Formally, every ordering of the players $\pi$ in which the grand coalition may form is considered, i.e., $\pi \in \Pi$ where $\Pi$ is the set of all the permutations of the agents in the set $N$. When considering a permutation $\pi$, it is important to account the worth of the coalition $P(\pi, i)$ of the agents that precede player $i$ in the permutation $\pi$ and the worth of the same coalition after player $i$ joins. Thus, following the bargaining procedure described by Shapley, his value is given by

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))], \forall i \in N. \qquad (2.6)$$

There are other nice mathematical expressions for the Shapley value in the literature, but this chapter does not pretend to be exhaustive in each of the aspects of the Shapley value, but rather to illustrate some of them in a concise way.

Perhaps, the biggest drawback of the formula presented before for the computation of the Shapley value can be observed in its own mathematical expression, since it is necessary to know the worth of each coalition and this may be an intractable problem when the number of players is large. However, in many papers from the literature, there exist classes of games where the characteristic function is not essential to be computed and it is easy to calculate the Shapley value. In this handbook, we can find several examples of this special situation. For the general case, an alternative is to resort to compute it approximately by sampling techniques (see [3, 4] and [10]). Therefore, this drawback can be avoided in different ways.

## 2.3 Some Characterizations

As previously mentioned, Shapley characterized his value using three axioms. This approach is very interesting and has been followed in many articles in the literature because it allows choosing between different solutions considering a set of properties that can be considered fair and relevant so that a solution is accepted, instead of the simple evaluation of the amount assigned to each agent. The used axioms for a solution concept $\psi$ were:

**Axiom 2.1** (Anonymity[2]) *Given a TU game $(N, v)$ and a permutation $\pi$ of the players in $N$, let $\pi v$ be the game defined by $\pi v(S) = v(\pi^{-1}(S)), \forall S \subseteq N$. Then $\psi$ satisfies the axiom of anonymity if $\psi_{\pi(i)}(\pi v) = \psi_i(v)$, for each $i \in N$.*

**Axiom 2.2** (Carrier) *Given a TU game $(N, v)$, a coalition $T \subseteq N$ is a* carrier *if $v(S) = v(S \cap T), \forall S \subseteq N$. Then $\psi$ satisfies the axiom of carrier if $\sum_{i \in T} \psi_i(v) = v(T) = v(N)$, for any carrier $T$.*

**Axiom 2.3** (Additivity) *Given two TU games $(N, v)$ and $(N, u)$ with the same set of players $N$, let $(N, (v + u))$ be the* sum *game where $(v + u)(S) = v(S) + u(S), \forall S \subseteq N$. Then $\psi$ satisfies additivity if $\psi(v + u) = \psi(v) + \psi(u)$.*

In the literature, the axiom of anonimity is often changed for the axiom of *symmetry*, which refers to the following property:

**Axiom 2.4** (Symmetry) *Given a TU game $(N, v)$ and two players $i, j \in N$, then $i, j$ are called* symmetric players *in game $v$, if $v(S \cup \{i\}) = v(S \cup \{j\}), \forall S \subseteq N \setminus \{i, j\}$. Then $\psi$ satisfies the axiom of symmetry if $\psi_i(v) = \psi_j(v)$ when $i$ and $j$ are symmetric players in game $v$.*

[2]Shapley [21] called this property *symmetry*.

The axiom of carrier is usually divided in the literature into two different axioms, *efficiency* and *null player*. The axioms of efficiency and null player are stated as follows:

**Axiom 2.5** (Efficiency) *Given a TU game* $(N, v)$*, then* $\psi$ *satisfies the axiom of efficiency if* $\sum_{i \in N} \psi_i(v) = v(N)$*.*

**Axiom 2.6** (Null player) *Given a TU game* $(N, v)$ *and a player* $i \in N$*, then* $i$ *is called* null player *in game* $v$*, if* $v(S \cup \{i\}) = v(S), \forall S \subseteq N \setminus \{i\}$*. Then* $\psi$ *satisfies the axiom of null player if* $\psi_i(v) = 0$*, when* $i$ *is a null player in game* $v$*.*

Instead of the null player axiom, some authors prefer the *dummy player* axiom, which is written as follows:

**Axiom 2.7** (Dummy player) *Given a TU game* $(N, v)$ *and a player* $i \in N$*, then* $i$ *is called* dummy player *in game* $v$*, if* $v(S \cup \{i\}) = v(S) + v(\{i\}), \forall S \subseteq N \setminus \{i\}$*. Then* $\psi$ *satisfies the axiom of dummy player if* $\psi_i(v) = v(\{i\})$*, when* $i$ *is a dummy player in game* $v$*.*

The additivity axiom 1.2 has been the most controversial because it is too technical, there is no interaction between the two games and the structure of the sum game may induce a behavior that may be unrelated to the behavior induced by the two games separately. Thus, there are many characterizations of the Shapley value in the literature that change this axiom for other axioms which have a *"better"* interpretation or which are at least less controversial. We shall only mention three of these characterizations. One by Peyton Young based on monotonicity properties and another two by Sergiu Hart and Andreu Mas-Colell based on balanced contributions and consistency.

In [26] monotonic solutions for cooperative games were studied. For this, Young [26] introduced several monotonicity properties and studied which solutions satisfied those properties. One of these properties was the *strong monotonicity*. This property says that if all marginal contributions of a player increase, this player cannot be worse off when we apply the same allocation procedure to both situations. This is a principle of justice because it seems reasonable that if a player contributes more to all coalitions in a new situation, she receives at least the same as she received previously. This axiom is written as follows for an arbitrary solution $\psi$.

**Axiom 2.8** (Strong monotonicity) *Given TU games* $(N, v)$ *and* $(N, w)$ *and a player* $i$ *such that* $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S), \forall S \subseteq N \setminus \{i\}$*, then* $\psi$ *satisfies the axiom of strong monotonicity if* $\psi_i(v) \geq \psi_i(w)$*.*

By using the strong monotonicity axiom together with the axioms of efficiency and anonymity, Young [26] characterized the Shapley value. Note that strong monotonicity and additivity are not equivalent, but they are related to each other by using also the other axioms in both characterizations.

In [7] the Shapley value is characterized by using two different concepts. First, the concept of *preservation of differences* that is closely related to the principle of *balanced contributions* introduced by Myerson [14]. Paraphrasing Myerson, this principle of fairness in cooperation says that when players cooperate with each other, any two players should gain or lose the same from their cooperation together, relative to what they would obtain without cooperation. This fairness principle reads as follows for an arbitrary solution $\psi$.

**Axiom 2.9** (Balanced contributions) *Given a TU game $(N, v)$, and a player $k$, let $(N \setminus \{k\}, v_{-k})$ be the TU game obtained by restricting $v$ to the subsets of $N \setminus \{k\}$ only. Then $\psi$ satisfies the axiom of balanced contributions if $\psi_i(v) - \psi_i(v_{-j}) = \psi_j(v) - \psi_j(v_{-i}), \forall i, j \in N$.*

By using the balanced contributions axiom together with the efficiency axiom, Hart and Mas-Colell [7] characterized the Shapley value. In this case, the balanced contributions axiom has a nice interpretation as a fairness condition because it can be regarded as a generalization of the equal division of the surplus idea for two person problems [7]. Likewise, it seems intuitive and common sense that the gains or losses of two agents that cooperate with each other are the same with respect to the situation of non-cooperation, as previously stated. Hence, the Shapley value can be considered as the unique efficient allocation which satisfies the so-called equal-gains principle. Thus, the Shapley value can be regarded as a benchmark for fairness.

The last characterization we will present in this chapter is related to a second concept, that of *consistency*. This establishes what happens to a solution when a subset of players leaves the grand coalition. In this sense, a solution would be *consistent* if, when a subset of players leave the grand coalition, those players would still obtain the same payoffs. The first step is to define the so-called *reduced game*, i.e., the game associated with the leaving players when the remaining players retain their payoffs. There are several ways to define a reduced game but we consider the following introduced in [7].

Given a TU game $(N, v)$, a solution $\psi$ and a subset $T \subset N$, the *reduced game* $(T, v_T^{\psi})$ is defined as follows for an arbitrary solution $\psi$

$$v_T^{\psi}(S) = v(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \psi_i(S \cup (N \setminus T), v), \ \forall S \subseteq T,$$

where $(S \cup (N \setminus T), v)$ is the TU game obtained by restricting $v$ to the subsets of $S \cup (N \setminus T)$ only. Obviously, when $S = T$, we precisely obtain that $v_T^{\psi}(T) = v(N) - \sum_{i \in N \setminus T} \psi_i(N, v)$.

By considering this definition of reduced game, Hart and Mas-Colell [7] introduced the following axiom of consistency for an arbitrary solution $\psi$.

**Axiom 2.10** (Consistency) *Given a TU game $(N, v)$ and a subset $T \subset N$, then $\psi$ satisfies the axiom of consistency if $\psi_j(T, v_T^{\psi}) = \psi_j(N, v), \forall j \in T$.*

Consistency almost characterizes the Shapley value but not completely, because knowing firstly what happens with two-player games is necessary.

**Axiom 2.11** (Standard for two players) *Given a TU game* $(\{i, j\}, v)$ *such that* $i \neq j$*, then* $\psi$ *satisfies the axiom of standard for two-person games, if*

$$\psi_k(v) = v(\{k\}) + \frac{1}{2}\left(v(\{i, j\}) - v(\{i\}) - v(\{j\})\right), \; k = i, j.$$

By using the axioms of consistency and standard for two-person games, Hart and Mas-Colell [7] characterized the Shapley value. Note that in this characterization the axiom of efficiency is not used, but it is implied by the two used axioms. The standard for two person games property reflects the idea of the equal division of the surplus when two players cooperate, which is a simple and intuitive idea that can be easily justified as fair. Consistency property says that it is neither beneficial nor detrimental for any group of players to leave the grand coalition, when they have to compensate those who remain by giving them what they would have obtained from the cooperation of all the players. Thus, the consistency property implicitly hides that the binding agreement of cooperation includes two clauses. The first one refers to the solution that will be applied to distribute the profit or utility generated by the cooperation, and the second clause is that it is possible to abandon the agreement as long as the rest of the players who remain collaborating are conveniently compensated. Therefore, a solution satisfying consistency reinforces the cooperation agreement, in the sense that no group of players will have incentives to leave the grand coalition. Of course, the way to compensate to the remainers is defined by the reduced game, so there will be as many ways of compensation as reduced games can be defined. As Hart and Mas-Colell [7] said *"...which definition is more appropriate will depend on the context being modeled (and the way the characteristic function is defined)."*

Therefore, we observe that the Shapley value satisfies numerous properties that can make it suitable to be applied to many different situations. Some properties are related to equity or fairness concepts such as the axioms of symmetry, null player, dummy player or balanced contributions, others with concepts such as monotonicity or consistency, and many others which have not been included in this chapter for the sake of brevity.

Thus, taking into account the different arguments and reasonings that lead to the Shapley value, some of which have been presented in Section 2.2, from the fact that it always exists and it gives a unique allocation, together with the numerous reasonable properties that it satisfies, it follows that the Shapley value can be considered, without a doubt, an exceptional and remarkable solution for many cooperative games.

## 2.4 Some Extensions and Applications

The importance of the Shapley value in game theoretical applications is witnessed by the volume edited in 1988 by Alvin Eliot Roth [19], awarded with the Nobel Memorial Prize in Economic Sciences in 2012 jointly with Shapley, and by many other outstanding contributions that improve and extend the relevance of the Shapley value. In fact, it is impossible even to simply mention the high number of papers that refer to the Shapley value and/or to its extensions, we just remember that the paper by Shapley in 1953 [21] has been cited more than 7.5 thousand times in Google Scholar until now. In the next lines, we will try to show some of the extensions of the Shapley value and its applications to many different fields. Of course, this book also reflects to what extent the Shapley value is important in game theory and its applications.

In 1954, Lloyd Shapley and Martin Shubik [23] proposed the extension of the Shapley value to simple games, with the so-called *Shapley-Shubik power index*, in order to evaluate the power of voters in voting systems. This value has attracted great attention in political science and it is possible to find a large amount of literature on it, both in the field of game theory and in political science or group decision making.

In 1969, Shapley [22] generalized his value for TU games to the case of *non transferable utility games* (NTU games), i.e., when it is not possible to transfer the utility in any way between players. In this case, the characteristic function defining the game is set-valued. Later, in 1985, Robert John Aumann, awarded with the Nobel Memorial Prize in Economic Sciences in 2005, axiomatically characterized the Shapley value for NTU games [1].

As mentioned above, one of the most important problems of the Shapley value is its calculation, and, for that reason, in 1972, Guillermo Owen [17] proposed a more efficient method for computing the Shapley value, via the *multilinear extension of a game*. Likewise, this multilinear extension of a game is useful to define new values with a probabilistic interpretation of players to join the coalitions.

In 1974, Aumann and Shapley [2] studied non-atomic games in which the set of players is modeled by a non-atomic continuum. For these games, they introduced the so-called *Aumann-Shapley value* that can be seen as an extension of the Shapley value to the non-atomic context. In fact, they utilized three approaches to value theory: The axiomatic, the random order and the asymptotic approaches, which is analogous to that used by Shapley when he introduced his well-known value in 1953 [21].

In 1977, three new extensions of the Shapley value are introduced. First, Roger Bruce Myerson, awarded with the Nobel Memorial Prize in Economic Sciences in 2007, studied games when players may cooperate only if they are connected by links in a graph and proposed a solution, known as *Myerson value*, that is the application of the Shapley value to a modified game that takes into account the restrictions to cooperation defined by a graph [12]. Moreover, Myerson [13] also extended the Shapley value to the case of *games in partition function form* which were introduced in [24]. Finally, again Guillermo Owen considered the extension of the Shapley value to the situation in which the players are partitioned in different groups, called a priori unions, defining the so-called *Owen value* [18].

Barry O'Neill, in 1982, introduced the bankruptcy games derived from a problem of right arbitration from the Talmud. In order to give a solution to that problem, he introduced the so-called *recursive completion rule*, which is the Shapley value of the game [16].

In 1987, Ehud Kalai and Dov Samet [9] extended the possibility of representing other real-world situations in which the assumption of symmetry is not realistic due to existing structural asymmetries among the players or different bargaining abilities, adding to each agent a weight that collects these asymmetries, defining the *weighted Shapley value*.

When in 1993 Chih-Ru Hsiao and T.E.S. Raghavan [8] introduced the multichoice cooperative games, they also extended the Shapley value to this new context. Therefore, we can observe in all cases that, in a certain sense, there is always a need to extend the Shapley value to all the new models that come up in the theory of cooperative games, which shows the great relevance of this solution concept.

Regarding the applications of the Shapley value, we can say that it has been applied to almost as many fields as cooperative game theory itself has been. Thus, Stefano Moretti and Fioravante Patrone [11] reported, without being exhaustive, applications of the Shapley value to the following fields:

- cost allocation,
- social networks,
- water-focused issues,
- biology,
- reliability theory,
- belief formation.

But to these applications, we can add many others, some of which appear in this book and the comments on Moretti and Patrone's article, such as statistics, algorithmics, telecommunications, political science, right arbitration or allocation of scarce resources, among others. This shows, as Moretti and Patrone say, the great transversality of the Shapley value. Moreover, they also say that, *"It is worth mentioning also that the Shapley value is used both as a* normative *tool and as a* descriptive *tool, quite similarly to what happens for the Nash bargaining solution."* Therefore, the Shapley value has a great importance both from a theoretical and an applied point of view.

## 2.5 Conclusions

Along the lines of this chapter, we have only wanted to highlight some of the basic and most relevant aspects of the Shapley value. Based on the appealing interpretations and properties that the Shapley value satisfies, it can be concluded that this solution has many elements to be considered attractive from the point of view of fairness. In addition, this is supported by the large number of applications of the Shapley value that we can find in the literature, which means that it is an excellent reference in many situations for many researchers and practitioners. Therefore, we can conclude without a doubt that the Shapley value can be considered a paradigm of FAIRNESS as the title of this chapter indicated.

## 2.6 Acknowledgments

We thank Francesc Carreras and Stefano Moretti for helpful comments and suggestions that have improved this chapter. Financial support from the Ministerio de Economía y Competitividad (MINECO) of Spain and FEDER funds under project MTM2014-54199-P and from Fundacion Séneca de la Región de Murcia through grant 19320/PI/14 are gratefully acknowledged.

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# Chapter 3

## An Index of Unfairness

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## 3.1 Introduction

Assume that an organization[1] compensates its agents using a pay scheme that possibly violates one or more of the following ideals of justice:

1- Symmetry: Equally productive agents receive the same pay.

2- Efficiency: The entire output of the organization is shared among the agents.

3- Marginality: If the adoption of a new technology increases the marginal productivity of an agent, that agent's pay should not decrease relative to the old technology.

How can we measure violations of these ideals of justice for the compensation rule utilized by the organization? As an answer to this question, Aguiar, Pongou and Tondji ([1]) propose the *Shapley distance*, which, for a given production technology $f$, measures the distance between an arbitrary pay profile and the Shapley pay profile at $f$ given by the Shapley value ([15]). The Shapley value is the only pay scheme that satisfies all of the three aforementioned ideals ([18]). In fact, the axioms characterizing the Shapley value make it a desirable concept of fairness (or distributive justice), as is generally acknowledged in the literature ([11], [14], [17]). Moreover, [1] provides an orthogonal decomposition of the Shapley distance into terms that indicate violations of each of the Shapley axioms. This chapter continues this line of research by analyzing the properties characterizing the Shapley distance.

Our main contribution is to axiomatize the *Shapley distance* as a measure of injustice. We also show that the Shapley distance can be used to determine the outcome of a bargaining procedure. We imagine a situation in which agents have to implement a *fairness prescription* $F$, defined as the set of payoffs induced, under a fixed technology $f$, by a set of compensation rules $\mathcal{F}$ satisfying certain ideals of justice. There is an initial pay profile $\phi$ that works as a reference point. Agents may want to depart from $\phi$, but they should implement an outcome that belongs to the fairness prescription $F$. This defines a bargaining function that maps any pair $(F, \phi)$ to an element of $F$. We show that the Shapley distance is the unique (up to monotone transformations) index defining a bargaining function that satisfies *Anonymity* and *Independence of Irrelevant Alternatives* (IIA), for the set of compensation rules that obey symmetry, efficiency, and marginality.[2]

Using several illustrations that include favoritism, egalitarianism, and tax distortions, we show how the Shapley distance can be applied to determine the extent to which a given income distribution departs from the fair ideal, and how unfairness can be further unbundled to determine its origins.

[1] An organization is defined as a body of agents (including the owner, if any) that operates a production technology by assigning each agent to a specific task.

[2] Some of our ideas are reminiscent of Nash's (1950) pioneering axiomatic characterization of a bargaining solution; see [4], [16], or [13] for surveys of the bargaining literature.

Together with [1], we contribute to the literature that studies economic inequality using game theory (e.g., [5] and [8]). In particular, we provide an axiomatic foundation to a notion of unfairness, namely the Shapley distance. A similar axiomatic approach can be used to characterize the decomposition of this distance as provided in [1].

The rest of this chapter is organized as follows. After dealing with preliminaries in Section 3.1, Section 3.2 introduces the Shapley distance and our notion of unfairness and contains our main results. Section 3.3 presents several applications showing the different ways in which favoritism, egalitarianism, and taxation distort fairness in revenue sharing. Section 3.4 concludes the chapter.

### 3.1.1 Organization and Data Set

In this section, we introduce preliminary definitions. We follow [1]. Let $N$ be a nonempty and finite set of agents, with $|N| = n$. A coalition is a nonempty subset $C$ of agents: $C \subseteq N$, $C \neq \emptyset$.

An organization is a pair $(N, f)$ where $f : 2^N \mapsto \mathbb{R}$ is a technology such that $f(\emptyset) = 0$. In what remains, we fix $N$, so that an organization is completely defined by a technology $f$. We denote by $\Gamma$ the set of all organizations.

A pay scheme is a way to share the output produced by the grand coalition $N$ of agents.[3]

**Definition 3.1** ***(Pay scheme)*** *A pay scheme is a function $\Phi : \Gamma \mapsto \mathbb{R}^n$ that maps any technology $f$ to a vector $\Phi(f) = (\Phi_1(f), \Phi_2(f), \ldots, \Phi_n(f)) = \phi \in \mathbb{R}^n$ such that $\sum_{i \in N} \Phi_i(f) \leq f(N)$. $\phi$ is called a pay profile, and for each agent $i \in N$, $\phi_i \in \mathbb{R}$ is interpreted as the payoff of $i$ out of the output $f(N)$. The set of all pay schemes is denoted $\Theta$.*

Notice that we allow for negative payoffs, interpreted as taxation. We also recall the notions of observation and data generating pay scheme introduced by [1].

An **observation** is a pair $(f, \phi)$ where $f$ is a technology and $\phi \in \mathbb{R}^n$ is a pay profile, defined as a distribution of the output generated by the grand coalition: $\sum\limits_{i \in N} \phi_i \leq f(N)$. In the sequel, any vector $\phi \in \mathbb{R}^n$ such that $\sum_{i \in N} \phi_i \leq f(N)$ is called a pay profile, even if it is not the result of applying a pay scheme.

**Definition 3.2** ***(Data generating pay scheme)*** *We say that $\Phi : \Gamma \to \mathbb{R}^n$ is a data generating pay scheme if it is the unique pay scheme such that $\Phi(f) = \phi$ for any observation $(f, \phi)$.*

In the context of a limited data set, given by a single observation, we do not have the details about how the data generating pay scheme $\Phi$ distributes

[3] Our framework also works if the organization is sharing total cost or total profit. The interpretation of the axioms will have to be done in terms of the context in those cases.

the total output for a technology that is not the observed technology $f$. We only know the realized pay profile $\phi$ for $f$. However, we have full information on $f$, (i.e., we know the exact magnitudes of $f(C)$ for all $C \subseteq N$).

### 3.1.2 The Shapley Value as an Ideal for Fairness

In this subsection, we recall the definition of the Shapley value as well as its fundamental characterization as a fair pay scheme. This characterization provides an axiomatic basis for analyzing the different ways in which an arbitrary pay scheme might violate basic principles of fairness, as departures from the Shapley value prescription. The following definition will be needed for the statement of these characterizations.

**Definition 3.3** *Let $i, j \in N$ be two agents, and $f$ be a technology.*

1. *The marginal contribution at $f$ of agent $i \in N$ to a set $C \subseteq N$ such that $i \notin C$ is $f(C \cup \{i\}) - f(C)$, and it is denoted by $mc(i, f, C)$.*
2. *Agent $i$ is a null-agent at $f$ if for any set $C \subseteq N$ such that $i \notin C$, we have $mc(i, f, C) = 0$.*
3. *Agents $i$ and $j$ are said to be substitutes at $f$ if for any coalition $C \subseteq N$ such that $i, j \notin C$, $mc(i, f, C) = mc(j, f, C)$.*

We now define the axioms that characterize the Shapley value.

**Axiom 3.1** ***(Symmetry)***
*A pay scheme $\Phi$ satisfies symmetry if for any technology $f$, and any agents $i$ and $j$ that are substitutes at $f$, $\Phi_i(f) = \Phi_j(f)$.*

**Axiom 3.2** ***(Efficiency)***
*A pay scheme $\Phi$ is efficient if for any technology $f$, $\sum_{i \in N} \Phi_i(f) = f(N)$.*

**Axiom 3.3** ***(Marginality)***
*A pay scheme $\Phi$ satisfies marginality if for any technologies $f$ and $g$, any agent $i \in N$, $[mc(i, f, C) \geq mc(i, g, C); \forall C \subseteq N \setminus \{i\}] \Rightarrow [\Phi_i(f) \geq \Phi_i(g)]$.*

The symmetry axiom is a no-discrimination condition (horizontal equity), requiring that agents who have identical marginal contributions under a technology $f$ receive the same pay. Efficiency requires that the output of the grand coalition be fully shared among the various contributors, and it can also be justified in terms of Pareto optimality. Marginality means that, if a new technology increases the marginal productivity (or the vector of marginal contributions) of an agent, that agent's pay should not decrease relative to the old technology. This is an old property in neoclassical economic theory, requiring that the payoff of an agent depend only on his marginal productivity given other agents' inputs.

The result set out below establishes necessity and sufficiency to characterize the Shapley payoff function (defined by Equation (3.1) below). The axioms just presented also establish the Shapley value as a fairness ideal.

**Theorem 3.1** *([18]) There exists a unique pay scheme, denoted* $\boldsymbol{Sh}$*, that satisfies the efficiency, symmetry, and marginality axioms, and, for any technology* $f$*, it is given by:*

$$\boldsymbol{Sh}_i(f) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n-|C|-1)!}{n!}[f(C \cup \{i\}) - f(C)], \ \textit{for all } i \in N. \quad (3.1)$$

---

## 3.2 The Shapley Distance as a Measure of Unfairness

In this subsection, we provide an axiomatic characterization of the notion of the *Shapley distance* introduced in [1]. It measures the level of unfairness associated with any pay profile $\phi$ by the distance between that pay profile and the Shapley value. [1] shows that it can be decomposed into terms that indicate violations of the axioms that characterize the Shapley value. We recall this decomposition and illustrate it through several examples.

### 3.2.1 An Axiomatic Characterization of the Shapley Distance

In this section, we provide an axiomatic characterization of the *Shapley distance*. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a distance in $\mathbb{R}^n$. Denote the Euclidean norm defined in $\mathbb{R}^n$ by $|| \cdot ||$. Also, denote the inner product associated with the Euclidean norm by $< \cdot, \cdot >$. We have the following definition of the Shapley distance:

**Definition 3.4** ***(Shapley distance)*** *For any technology* $f$*, the Shapley distance of a pay profile* $\phi \in \mathbb{R}^n$ *for* $f$*, denoted* $d(\phi, \boldsymbol{Sh}(f))$*, is the distance between* $\phi$ *and the Shapley pay profile* $\boldsymbol{Sh}(f) \in \mathbb{R}^n$ *at* $f$*.*

We axiomatize below the Shapley distance. First, we need some definitions.

We consider the set of fairness prescriptions of an arbitrary set of pay schemes.

**Definition 3.5** ***(Fairness prescription)*** *Given a technology* $f$ *and a set of pay schemes* $\mathcal{F} \subseteq \Theta$*, a set of profiles* $F \subseteq \mathbb{R}^n$ *is a fairness prescription at* $f$ *with respect to* $\mathcal{F}$ *if, for each* $\phi \in F$*, there exists* $\Phi \in \mathcal{F}$ *such that* $\phi = \Phi(f)$*.*

Our fairness index will be the result of a bargaining procedure, where an original pay profile $\phi$ works as a reference point. The intuition is that an arbitrator requires all agents to implement a fairness prescription, but the agents are free to choose a new pay profile. They may want to depart from the status-quo $\phi$ altogether. The result of this procedure is a fairness bargaining function.

**Definition 3.6** ***(Fairness bargaining function)*** *A fairness bargaining function is a mapping $C : \{F\} \times \{\phi\} \to F$ for any fairness prescription $F$ and pay profile $\phi$.*

We propose an axiomatic approach to studying the properties that the fairness bargaining function ought to have.

Let $\sigma : N \to N$ be a permutation of agents. We define $\sigma(F)$ as the set of fairness prescriptions such that $\varphi \in \sigma(F)$ is a permutation of an element $\eta \in F$. The first axiom requires that the fairness bargaining function is invariant with respect to permutations of the prescriptions and the reference pay profile $\phi$.

**Axiom 3.4** ***(Anonymity)***
*For all $F \subseteq \mathbb{R}^n$, all $\phi \in \mathbb{R}^n$, and any permutation $\sigma$ on $N$, $(C_{\sigma(i)}(F,\phi))_{i\in N} = C(\sigma(F),(\phi_{\sigma(i)})_{i\in N})$.*

The second condition requires that the solution to the fairness bargaining problem be optimal.

**Axiom 3.5** ***(Independence of Irrelevant Alternatives (IIA))***
*For any set $S \subseteq F \subseteq \mathbb{R}^n$ and any $\phi \in \mathbb{R}^n$, $C(F,\phi) \in S$ implies $C(F,\phi) = C(S,\phi)$.*

Without loss of generality, we also assume that any $F \subseteq \Theta$ is convex and closed.

**Lemma 3.1** *The only fairness bargaining function that satisfies Anonymity and IIA is the minimal distance bargaining function*

$$C(F,\phi) = argmin_{v\in F} d(v,\phi).$$

*Proof.* To check that the minimal distance bargaining function satisfies Anonymity and IIA is trivial. To prove uniqueness, we observe that Anonymity implies the following two axioms: Invariance to Permutations (IP) and Nash Symmetry (NS). The latter axioms are defined below.

(i) A fairness prescription is closed to permutations if, for any $\phi \in F$, $(\phi_{\sigma(i)})_{i\in N} \in F$ for any permutation of the set of agents $\sigma : N \to N$.

Invariance to Permutations (IP): If $F$ is closed to permutations, then $C_i(F,\phi) = C_j(F,\phi)$ for all $i,j \in N$.

(ii) A fairness prescription $F$ is said to be symmetric if the set $F$ is symmetric relative to the 45-degree line.

Nash Symmetry (NS): If $F$ is symmetric and $\phi_i = \phi_j$ for all $i, j \in N$, then $C_i(F, \phi) = C_j(F, \phi)$ for all $i, j \in N$.

To complete the proof, we define below the axiom of symmetry relative to a line introduced by Rubinstein and Zhou ([12]).

A line $< \phi, \alpha >$, where $\phi \in \mathbb{R}^n$ is a reference and $\alpha \in \mathbb{R}^n$ is a direction, is the set of all points of the form $\phi + t\alpha$ for some real number $t$. We say that $F$ is symmetric relative to a line $< \phi, \alpha >$ if for every orthogonal direction $\beta$ ($\beta'\alpha = 0$), $\phi + t\alpha + \beta \in F$ implies that $\phi + t\alpha - \beta \in F$.

Rubinstein and Zhou Symmetry (RZS): If $F$ is symmetric relative to a line $< \phi, \alpha >$, then $(C_i(F, \phi))_{i \in N} \in < \phi, \alpha >$.

If axioms (IP) and (NS) hold, then axiom (RZS) holds. In fact, axiom (IP) implies that if $F$ is symmetric relative to the line $(t, \cdots, t)'$ for any real number $t$, then $(C_i(F, \phi))_{i \in N} \in (t, \cdots, t)'$. Moreover, axiom (NS) implies that, if $F$ is symmetric relative to the 45-degree line and $\phi_i = \phi_j$ for all $i, j \in N$, then $(C_i(F, \phi))_{i \in N} \in (t, \cdots, t)'$ (i.e., $C_i(F, \phi) = C_j(F, \phi)$ for all $i, j \in N$). It follows that axioms (IP) and (NS) imply that if $F$ is symmetric relative to any line going through $\phi$, then the solution will be on that line. In other words, axioms (IP) and (NS) imply axiom (RZS). We conclude that Anonymity implies axiom (RZS), which together with (IIA), implies, thanks to Proposition 2.1 in [12], that

$$C(F, \phi) = argmin_{v \in F} d(v, \phi).$$

■

Next, we define our fairness index.

**Definition 3.7** ***(Fairness index)*** *A fairness index is a mapping ($\rho : \mathbb{R}^n \times \{\phi\} \mapsto \mathbb{R}_+$) such that there exists a fairness bargaining function $C$ defined as follows:*

$$C(F, \phi) = argmin_{v \in F} \rho(v, \phi).$$

We are ready to present our main result.

**Theorem 3.2** ***(Shapley distance)****. Let $C$ be a bargaining function that satisfies Anonymity and IIA. Then the Shapley distance is the unique (up to monotone transformations) value of the fairness index defining $C$ at any point $(F, \phi)$ where $F$ is induced by the set $\mathcal{F}$ of pay schemes that satisfy symmetry, efficiency, and marginality.*

*Proof.* By Lemma 3.1, the bargaining function $C$ is defined by the minimal distance function: $C(F, \phi) = argmin_{v \in F} d(v, \phi)$ for any convex and closed set $F$. By Theorem 3.1 (see also [18]), we know that, for any technology $f$, the fairness prescription $F$ induced by the set of pay schemes that satisfy symmetry, efficiency, and marginality is the singleton $\{\boldsymbol{Sh}(f)\}$, which is a convex and closed set. It follows that $C(F, \phi) = argmin_{v \in \{\boldsymbol{Sh}(f)\}} d(v, \phi)$. But $min_{v \in \{\boldsymbol{Sh}(f)\}} d(v, \phi) = d(\boldsymbol{Sh}(f), \phi)$, which completes the proof. ■

Different choices of the distance function provide different fairness indices. We focus now on a particular choice, the Euclidean distance, which is shown by [1] to have an additive (and orthogonal) comparability property in terms of the different axioms of fairness, hence justifying its use. As recalled below, the square of the Shapley distance has a unique **decomposition** into terms that measure violations of the classical axioms of the Shapley value. This approach is analogous to that of [2] who study departures of a demand function from rationality. Despite the similarities in the two approaches, in this paper we address a different question in a different environment.

Moreover, in finite data sets, these terms can be used to make partial inferences about the violations of the axioms defined for complete data sets, and to make complete inference about the violations of the axioms defined for a fixed technology, for the subset of monotone technologies (see also [3]). This is of interest because the observer usually does not have information about a pay scheme under different technologies, making it practically impossible to check the validity of the axioms that require comparisons between different technologies.

### 3.2.2 A Decomposition of the Shapley Distance with Limited Data Sets

We now present a decomposition of the Euclidean Shapley distance, or Shapley Distance for short. In this section, we follow the set-up in [1]. Let $f$ be a technology and $\phi \in \mathbb{R}^n$ an observed pay profile generated by a pay scheme that may not be known (to the observer). We can always decompose it into a sum of the Shapley value at the observed technology $f$ and an error term $\phi = \boldsymbol{Sh}(f) + e^{sh}$, by defining $e^{sh} = \phi - \boldsymbol{Sh}(f) \in \mathbb{R}^n$. Moreover, we show that the error term $e^{sh}$ can be further decomposed uniquely into three vectors that are orthogonal to each other, with these vectors being respectively connected to the violation of symmetry ($sym$), efficiency ($eff$), and marginality ($mrg$). Formally, this means that we can write $e^{sh} = e^{sym} + e^{eff} + e^{mrg}$ such that the inner product of these axioms errors (roughly their correlation) is zero.

[1] finds this orthogonal decomposition to be the result of the following procedure. First they find the closest pay scheme to $\phi$ that satisfies $sym$; then they find the closest pay scheme to $\phi$ that satisfies $eff$ in addition to $sym$; and finally they find the closest pay scheme to $\phi$ that satisfies $mrg$ in addition to $sym$ and $eff$, which is simply the Shapley value itself. The described order, in which these constraints are imposed, is the only one that produces the orthogonality of the different error vectors. This decomposition is also meaningful as each component measures a quantity of economic interest that completely and effectively "isolates" one of the three conditions $sym$, $eff$ and $mrg$.

Begin by fixing a pair consisting of an observed pay profile and a technology $(f, \phi)$ and consider the Shapley distance of $\phi$ at this point, which is:

$$||e^{sh}|| = ||\phi - \boldsymbol{Sh}(f)||.$$

Let $\boldsymbol{v}^{sym}$ be the closest pay scheme to $\phi$ that satisfies symmetry (pointwise under the chosen norm) (i.e., $\boldsymbol{v}^{sym} \in argmin_{\boldsymbol{v} \in \boldsymbol{\Theta}} ||\phi - \boldsymbol{v}(f)||$ s.t. $\boldsymbol{v}$ satisfies

$sym$).[4] [1] proves that each entry evaluated at $f$ is given by $v_i^{sym}$ that corresponds to the average pay according to $\phi$ among the agents who are substitutes of $i$ under $f$. They then establish that $\phi$ can be written uniquely as the sum of its symmetric part $v^{sym} = \boldsymbol{v}^{sym}(f)$ and a residual $e^{sym}$ that is orthogonal to $v^{sym}$ under the Euclidean inner product:

$$\phi = v^{sym} + e^{sym}.$$

In a similar way, let $\boldsymbol{v}^{sym,eff}$ be the pay scheme that is pointwise closest to the symmetric pay scheme $v^{sym}$ and that satisfies efficiency (i.e., $\boldsymbol{v}^{sym,eff} \in argmin_{\boldsymbol{v}\in\boldsymbol{\Theta}} ||v^{sym} - \boldsymbol{v}(f)||$ s.t. $\boldsymbol{v}$ satisfies $sym$ and $eff$). [1] proves that $v_i^{sym,eff} = \boldsymbol{v}_i^{sym,eff}(f)$ is given by the summation of $v_i^{sym}$ and the output wasted by $\phi$ divided by the number of agents in $N$. It follows that $v^{sym}$ can be uniquely written as:

$$v^{sym} = v^{sym,eff} + e^{eff},$$

where $e^{eff}$ is the negative of the wasted output by $\phi$ divided by the number of agents in $N$.

Finally, remark that the pay scheme satisfying the axiom of marginality that is pointwise closest to the symmetric and efficient pay scheme $v^{sym,eff}$, which we denote by $\boldsymbol{v}^{sym,eff,mrg}$, must be the Shapley value because of the uniqueness established in Theorem 3.1. Thus, $v^{sym,eff,mrg} = \boldsymbol{Sh}(f)$. Thus, we let $e^{mrg} = v^{sym,eff} - \boldsymbol{Sh}(f)$. Notice that we can always decompose $\phi$ (pointwise) as:

$$\phi = \boldsymbol{Sh}(f) + e^{sh},$$

because $\phi$ and $\boldsymbol{Sh}(f)$ belong to the same vector space. With all this in hand, [1] establishes the following main result.

**Theorem 3.3** *([1]) For any given observation $(f, \phi)$, we have the unique pointwise decomposition:*

$$\phi = \boldsymbol{Sh}(f) + e^{sym} + e^{eff} + e^{mrg}.$$

*Moreover, the distance to the Shapley pay scheme can be uniquely decomposed as:*

$$||e^{sh}||^2 = ||e^{sym}||^2 + ||e^{eff}||^2 + ||e^{mrg}||^2,$$

*into its symmetry, efficiency, and marginality departures, such that for any $i, j \in \{sym, eff, mrg\}$, $i \neq j$, $< e^i, e^j >= 0$.*

The proposed decomposition of the Shapley distance that we just stated has economic meaning described hereunder:

**a)** $||e^{sym}||^2 = \sum_{i\in N} [\phi_i - v_i^{sym}]^2$, where for any agent $i$, $v_i^{sym}$ is the average payoff within the class $[i]^f$ of agents who are substitutes of $i$ at $f$. This

[4] Existence is easy to verify noticing that the space of symmetric pay schemes is convex and closed.

means that $||e^{sym}||^2$ is a dispersion measure within equivalence classes of agents. In other words, this quantity measures *horizontal inequity*, which is the inequality among agents who are identical.

**b)** $||e^{eff}||^2 = E^2/n$, where $E = [f(N) - \sum_{i\in N} \phi_i]$ is the total waste produced by the pay profile. This means that $||e^{eff}||^2$ increases solely due to the lack of efficiency.

**c)** $||e^{mrg}||^2 = \sum_{i\in N} [v^{sym,eff} - \boldsymbol{Sh}(f)]^2$, where $v^{sym,eff}$ is the symmetrized and efficient pay profile that is closest to the original pay profile $\phi$. This means that $||e^{mrg}||^2$ is a measure of departures from the marginality principle conditional on fulfilling horizontal equity and efficiency.

To the best of our knowledge, $||e^{sh}||^2$, introduced in ([1]), is the first measure of departures from the Shapley axioms. It has the advantage of providing a unified treatment of the three axioms in the form of a numerical and additive decomposition. Furthermore, in the decomposition analysis, each component of $||e^{sh}||^2$ measures a violation of a Shapley axiom, with the main result providing a formal and unified theoretical foundation for using the three components.

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## 3.3 Some Applications

In this section, we feature several applications of our analysis. They are attempts to enhance our understanding of inequality, and answer the question of when income inequality can be considered unfair. The different applications show how favoritism, egalitarianism, and taxation distort fairness in revenue distribution.

### 3.3.1 Favoritism

Consider the following simple example:

**Example 3.1** *The nephew's problem. Let an organization consist of a set of agents* $N = \{1, 2, 3\}$ *and a technology* $f$ *defined as follows:* $f(N) = 10$, $f(\{1, 2\}) = 4$, $f(\{1, 3\}) = f(\{2, 3\}) = 9$, $f(\{i\}) = 0$ *for* $i = 1, 2, 3$. *The environment describes a firm owned by agent 3, who employs a nephew (agent 1). Agent 2 is also employed in the firm, with no family connections to the other two people. Although from the point of view of productivities, agents 1 and 2 are substitutes, agent 3, exhibiting favoritism toward agent 1, allows him to show up to work only half of the time, leading to output waste. In addition, the uncle has set the pay scheme* $\Phi(f) = (2, 1, 4)$. *Note that the Shapley value* *yields the pay profile* $\boldsymbol{Sh}(f) = (2.5, 2.5, 5)$*. Thus, the overall (squared) Shapley distance is 3.5, decomposed as 0.5 (attributed to the violation of symmetry) and 3 (attributed to the violation of efficiency). No violation of marginality is observed, after one corrects for the other two failures: The moves in* $\mathbb{R}^3$ *describe a first transition from* $(2, 1, 4)$ *to* $(1.5, 1.5, 4)$ *-correcting for symmetry-, and then to* $(2.5, 2.5, 5)$ *-correcting for efficiency-, which is the Shapley value. In this example, favoritism causes an efficiency flaw that, according to our measure, is 6 times as important as the lack of symmetry.*

### 3.3.2 Egalitarianism versus Fairness

Our second illustration relates to the egalitarian pay scheme. Before showing it, we need to present a generalization, due to [6], [9], and [10], of the framework of an organization, to an environment where agents have more than two options (i.e., active or inactive). A **production environment** is modeled as a list $\mathcal{G} = (N, L, G)$ where $N = \{1, 2, \ldots, n\}$ is a nonempty finite set of agents of cardinality $n$; $L = \{0, 1, 2, \ldots, l\}$ is a nonempty finite set of hours of labor or effort levels that an agent can supply, with 0 denoting a situation of inaction; and $G$ is a production function that maps each action profile $x = (x_1, \ldots, x_n) \in L^n$ to a real number –output– $G(x)$. The function $G$ can also be interpreted as the aggregate profit or cost function. Interpreting it as the profit function might be useful in certain settings, in that it could be incorporating both production and cost functions. Regardless of the interpretation, we assume that $G(0, 0, \ldots, 0) = 0$, which means that no output is produced when all the agents are inactive.

We denote by $e_i$ the $i^{th}$ unit vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where all the entries are zero except the $i^{th}$ component which is one. We will also use the symbols $\trianglelefteq$ and $\triangleleft$, which we define as explained hereunder. Let $\overline{x}, x \in L^n$ be two effort profiles. We write $x \trianglelefteq \overline{x}$ to mean that $x_i \neq \overline{x}_i \Rightarrow x_i = 0$, and we write $x \triangleleft \overline{x}$ to mean that $x \trianglelefteq \overline{x}$ and $x \neq \overline{x}$. For example, $(1, 7, 5, 0, \ldots, 0) \triangleleft (1, 7, 5, 1, 5, 0, \ldots, 0)$. We denote by $|x| = |\{i \in N : x_i > 0\}|$ the number of agents who are not inactive at $x$. We maintain the assumption of monotonicity in the production function environment. The analogous *monotonicity property* for the production function says that $G(x) \leq G(y)$ whenever $x \trianglelefteq y$.

For any production environment $\mathcal{G} = (N, L, G)$, a pay scheme for the production maps any effort profile $\overline{x} \in L^n$ to a nonnull payoff profile $\Phi^G(\overline{x}) = (\Phi_1^G(\overline{x}), \Phi_2^G(\overline{x}), \ldots, \Phi_n^G(\overline{x}))$, where for all $i \in N$, $\Phi_i^G(\overline{x}) \in \mathbb{R}$ is interpreted as the payoff earned by $i$ out of the output $G(\overline{x})$. In the production environment, an **observation** is a triple $(\overline{x}, G, \Phi^G(\overline{x}))$ where $\phi = \Phi^G(\overline{x})$ is an observed pay profile for any production function $G$ and for any effort profile $\overline{x}$.

The corresponding Shapley value for the environment $G$, denoted by $\boldsymbol{Sh}^G$, is given by:

$$\boldsymbol{Sh}_i^G(\overline{x}) = \sum_{x \triangleleft \, \overline{x},\ x_i = 0} \frac{(|x|)!(|\overline{x}| - |x| - 1)!}{(|\overline{x}|)!}[G(x + \overline{x}_i e_i) - G(x)], \ \textit{for all } i \in N. \tag{3.2}$$

[1] shows that, for a fixed level of efforts $\overline{x}$, all the information given by the production environment can be equivalently expressed using a technology.

We now show how the egalitarian pay scheme distorts fairness in revenue distribution. This pay scheme is the benchmark that implements perfect equality. It divides the output in equal parts to each agent. So, this pay scheme is clearly efficient. Evidently, given different levels of efforts and productivities, the egalitarian pay scheme may not be fair, failing marginality. Our aim is to measure the divergence of the egalitarian pay scheme from the Shapley value and to identify the sources of this divergence. We do this through the following example in which, for simplicity, we assume two agents, with each choosing his effort level from a set that contains two levels.

**Example 3.2** *Consider a production environment $\mathcal{G} = (N, L, G)$ where $N = \{1, 2\}$ is the set of agents, $L = \{0, 1\}$ is the set of effort levels, and $G$ is the (monotone) production function defined as follows:*

$$G(x) = \begin{cases} 1 & if \quad x \neq (0,0) \\ 0 & if \quad x = (0,0) \end{cases} \tag{3.3}$$

*Consider the egalitarian pay scheme $\boldsymbol{Eq}$ defined as follows:*

$$\boldsymbol{Eq}_1(x) = \frac{1}{2}G(x) \text{ and } \boldsymbol{Eq}_2(x) = \frac{1}{2}G(x), \ for \ each \ x \in L^2.$$

*For each $x \in L^2$, we have $\boldsymbol{Eq}_1(x) + \boldsymbol{Eq}_2(x) = G(x)$, which means that $\boldsymbol{Eq}$ is efficient.*

*In order to quantify the violations of the properties that characterize the Shapley value, let us first derive the Shapley payoff of each agent at each vector $\overline{x}$. The Shapley payoff profile at each $\overline{x}$ is given by the following matrices: $\boldsymbol{Sh}^G(\overline{X}) = \begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (\frac{1}{2}, \frac{1}{2}) \end{pmatrix}$, where $\overline{X} = \begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (1,1) \end{pmatrix}$ is the matrix that contains all of the possible vectors of effort levels, with the first component of each cell denoting the effort level of agent 1, and the second component denoting the effort level of agent 2.*

*The egalitarian payoff profile is given by: $\boldsymbol{Eq}(\overline{X}) = \begin{pmatrix} (0,0) & (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}) \end{pmatrix}$.*

*Using the difference between the two matrices, $\boldsymbol{Sh}^G(\overline{X}) - \boldsymbol{Eq}(\overline{X}) = \begin{pmatrix} (0,0) & (\frac{-1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{-1}{2}) & (0,0) \end{pmatrix}$, we can compute the Shapley distance $\|\boldsymbol{Sh}^G - \boldsymbol{Eq}\|^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.*

*Note that Theorem 3.3 applies for each fixed effort level, equivalently for each entry of the matrix $\overline{X}$.*

*We now determine how the amount by which the violation of each property characterizing the Shapley value contributes to the total violation of fairness by an egalitarian payoff for any production function and any number of agents.*

*We know that:*

$$\boldsymbol{Eq}(\overline{x}) = \boldsymbol{Sh}^G(\overline{x}) + e^{sym} + e^{eff} + e^{mrg}.$$

1. *Let* $e^{sym} = Eq - v^{sym} = 0$*. For all effort levels* $x$*, because* $\boldsymbol{Eq}$ *satisfies symmetry trivially.*

2. *Let* $e^{eff} = v^{sym} - v^{sym,eff} = 0$*. For all effort levels* $x$*, because* $\boldsymbol{Eq}$ *satisfies efficiency trivially.*

3. *Let* $e^{mrg} = v^{sym,eff} - \boldsymbol{Sh}^G = \boldsymbol{Eq} - \boldsymbol{Sh}^G$*. This means that the Shapley distance in general for this case is equal to* $||\boldsymbol{Sh}^G - \boldsymbol{Eq}||^2 = ||e^{mrg}||^2$*. This implies that a perfectly egalitarian pay profile may still be unfair given certain productivity and effort levels.*

### 3.3.3 Taxes

In our third example, we illustrate how a tax levied over a fair wage can alter the fairness in an economy.

**Example 3.3** *Consider a small economy of two agents* 1 *and* 2 *who have to work to produce goods and services. Each agent has two options, either go to work (option* $W$*), or stay at home (option* $H$*). The production function is given by:* $f(H,H) = 0$*,* $f(H,W) = 2$*,* $f(W,H) = 1$*, and* $f(W,W) = 5$*. We observe that both agents work (i.e., we observe the effort profile* $(W,W)$*). This implies that the Shapley wage function allocates a payoff of* 2 *dollars to agent* 1*, and a payoff of* 3 *dollars to agent* 2*.*

*We assume that both agents have to contribute for a public good. For simplicity, we assume that the benefits from the public good are not received immediately and we can ignore them in the payoff profile. The vector* $\Phi = (2(1-\alpha), 3(1-\alpha))$ *represents the revenues of agents net of contributions, given that each agent contributes a positive proportion* $\alpha$ *of his/her revenue. How far is* $\Phi$ *from the Shapley allocation* $\boldsymbol{Sh}^f = (2,3)$*?*

*The Shapley distance is given by* $||e^{sh}||^2 = ||\boldsymbol{Sh}^f - \Phi||^2 = 13\alpha^2$*. We now determine how the amount by which the violation of each fairness property characterizing the Shapley value contributes to the total violation of* $13\alpha^2$*.*

1. $e^{sym} = \Phi - v^{sym}$*. Since agents are not identical, it follows that* $\Phi_i = v_i^{sym}$ *and* $e^{sym} = (0,0)$*.*

2. $e^{eff} = \Phi - v^{sym,eff}$*. For each* $i \in \{1,2\}$*,* $v_i^{sym,eff} = \Phi_i + \frac{5-\sum \Phi_i}{2}$*. After calculations,* $v^{sym,eff} = (\frac{4+\alpha}{2}, \frac{6-\alpha}{2})$*, and* $||e^{eff}||^2 = \frac{25\alpha^2}{2}$*.*

3. $e^{mrg} = \boldsymbol{Sh}^f - v^{sym,eff} = (-\frac{\alpha}{2}, \frac{\alpha}{2})$*. Then,* $||e^{mrg}||^2 = \frac{\alpha^2}{2}$*. A quick verification confirms that* $||e^{mrg}||^2 + ||e^{eff}||^2 = 13\alpha^2$*. In general, we observe that the tax has an increasing and nonlinear distortion of fairness. When* $\alpha \to 0$ *there is no unfairness in the economy, and when* $\alpha \to 1$ *the unfairness level reaches its maximum.*

*Assuming that each agent contributes half of his/her revenue (i.e., $\alpha = \frac{1}{2}$), the departure from the Shapley allocation is $||e^{sh}|| = 1.8$ dollars. In addition,* 96.15 *percent of this value is explained by the violation of efficiency, and* 3.85 *percent by lack of marginality.*

The previous example provides an upper bound to the cost of fairness. However, we made the strong assumption that there is no enjoyment of the public good by the agents. Here we relax that assumption and provide a lower bound of the cost of fairness.

**Example 3.4** *We consider the same economy defined in Example 3.3, but we assume that there is monetary (equivalent) benefit of the public good that can be enjoyed by both agents immediately. The total tax revenue is given by $5\alpha$ dollars. We assume that each agent enjoyment of the public good is $\frac{5}{2}\alpha$ dollars. This implies that the adjusted payoff after taxes and considering the public good utility is $\Phi = (1-\alpha)(2,3) + \alpha(\frac{5}{2}, \frac{5}{2})$. In other words, the government is able to implement a convex combination of the Shapley wage and the egalitarian wage using a fully efficient tax to provide a public good that produces the same enjoyment to both agents.*

*The Shapley distance is given by $||e^{sh}||^2 = ||\boldsymbol{Sh}^f - \Phi||^2 = \frac{\alpha^2}{2}$. We notice that the new pay scheme is both efficient and symmetric, hence $||e^{sh}||^2 = ||e^{mrg}||^2$, which coincides with the marginality error in the previous example. In this example, the government is able to eliminate the efficiency loss and only the marginality loss remains. Note that when $\alpha \to 1$ there is a loss of $||e^{sh}|| = \frac{1}{\sqrt{2}} \approx 0.707$ dollars in terms of unfairness to produce a fully egalitarian income. This is* 14.14% *of the total output. This is of course a lower bound to the cost of fairness (while the previous example represented an upper bound).*

For our final example, we consider a different tax scheme and explore its implications for fairness.

**Example 3.5** *We consider the same economy defined in Example 3.3, but we assume that the investment in the public good is done by using a lump-sum tax scheme, as opposed to the proportional tax scheme. Specifically, each agent contributes the amount $t_i$, $i \in \{1,2\}$, such that $t_1 + t_2 = X$, where $X$ represents the worth of the public good. The vector $\Phi = (2 - t_1, 3 - t_2)$ represents the revenues net of taxes. What could be the values of $t_i$, such that the vector $\Phi$ is close to the Shapley payoff vector $\boldsymbol{Sh}^f = (2,3)$? The distance between the two vectors $\Phi$ and $\boldsymbol{Sh}^f$ is given by the numerical expression $d(t_1,t_2) = t_1^2 + t_2^2$. To answer the question posed, we should solve the following minimization problem:*

$$\begin{aligned} &\underset{x}{\textit{minimize}} \quad t_1^2 + t_2^2 \\ &\textit{subject to} \quad 0 \le t_1 \le 2 \;; 0 \le t_2 \le 3 \;; t_1 + t_2 = X \;; 0 < X \le 5. \end{aligned} \tag{3.4}$$

*Solving problem 3.4 yields* $t_1^* = \min(2, \frac{X}{2})$ *and* $t_2^* = \min(3, X - t_1^*)$. *Assume that the amount of the public good* $X$ *equals* 4.5 *dollars, then agent* 1 *contributes* $t_1^* = 2$ *dollars, agent* 2 *contributes* $t_2^* = 2.5$ *dollars. The payoff vector is* $\Phi = (0, 0.5)$ *net of taxes. The distance between both allocations* $\Phi$ *and* $\boldsymbol{Sh}^f$ *is* $||e^{sh}|| = 3.20$ *dollars. The vector* $\Phi$ *does not violate the symmetry property, since agents are not identical. The violation of efficiency, measured by* $||e^{sh}|| = 3.18$ *dollars, represents* 98.78 *percent of the total measure of unfairness* ($\frac{||e^{eff}||^2}{||e^{sh}||^2} = 98.78$)*, whereas only* ($\frac{||e^{mrg}||^2}{||e^{sh}||^2} = 1.22$) *of unfairness is explained by the lack of marginality. Again, this is an upper bound of the cost of fairness. Due to the decomposition, it is easy to see that the way to reduce the important cost of fairness is to reduce the efficiency error. This can be done by taking into account the benefits of the public good. If the benefits of the public good are fully internalized, only the marginality error will matter, and that is smaller than in the tax schemes of previous examples.*

## 3.4 Conclusions

We have provided an axiomatic characterization of the *Shapley distance*, which is a measure of unfairness in revenue distribution introduced by [1]. It is defined as the distance between an arbitrary pay profile and the Shapley pay profile under a given technology. [1] provides a decomposition of this distance into terms that measure violations of each of the Shapley axioms. In this chapter, we have shown that the Shapley distance is the unique (up to monotone transformations) index defining a bargaining function that satisfies *Anonymity* and *IIA* for the set of pay schemes that obey symmetry, efficiency, and marginality. The analyses are illustrated through examples showing the different ways in which favoritism, egalitarianism, and taxation distort fairness in revenue sharing. We have also identified a tax scheme that minimizes this distortion.

## 3.5 Acknowledgments

We are grateful to the editors of this volume for inviting us to contribute to it, and to an anonymous referee for helpful comments. This work was completed when Pongou was a Fellow in the Takemi Program in International Health at the Harvard T.H. Chan School of Public Health, and Serrano was visiting the Harvard Department of Economics. The hospitality of Harvard University is gratefully acknowledged.

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# Chapter 4

## The Shapley Value and Games with Hierarchies

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## 4.1 Introduction

A situation in which a finite set of players can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game). A TU-game consists of a player set, and for every subset of the player set, called a *coalition*, a real number which is the *worth* that the coalition of players can earn when they agree to cooperate.

In a TU-game there are no restrictions on the cooperation possibilities of the players, i.e., every coalition is feasible and can generate a worth. Various models with restrictions on coalition formation are discussed in the literature. In this chapter, we focus on restrictions arising from the players belonging to some hierarchical structure that is represented by a digraph. Two of these models are the *games with a permission structure* introduced by Gilles et al. (1992) and *games under precedence constraints* introduced by Faigle and Kern (1992). In both cases, the hierarchy can be represented by a directed graph which restricts the possibilities of coalition formation. Whereas solutions for games with a permission structure are based on a restricted game that is defined from a set of feasible coalitions that typically is a proper subset of the power set of the full player set (i.e. it focuses on feasible *combinations*), in games under precedence constraints the coalition formation process is restricted in the sense that not all orders by which players enter a coalition can form (i.e. it focuses on feasible *permutations*). These two approaches led to two different type of solutions in the literature. In this chapter, we focus on acyclic digraphs.

In a *game with a permission structure*, the hierarchy or digraph is referred to as a *permission structure*, and this models the idea that there are players that need permission from other players before they are allowed to cooperate. Various assumptions can be made about how a permission structure affects the cooperation possibilities. In this chapter, we focus on the *conjunctive approach*, as developed in Gilles et al. (1992) and van den Brink and Gilles (1996), where it is assumed that every player needs permission from *all* its predecessors before it is allowed to cooperate.[1]

To take account of the limited cooperation possibilities, for every game with a permission structure a modified game is defined which assigns to every coalition the worth of its largest feasible subcoalition in the original game. A *solution* for games with a permission structure is a function that assigns to every such game a payoff distribution over the individual players. Applying solutions for TU-games to the modified game yields solutions for games with a permission structure. In this chapter, we consider the *Shapley value*

[1] Alternatively, for games with an acyclic quasi-strongly connected permission structure in the *disjunctive approach*, as considered in Gilles and Owen (1994) and van den Brink (1997), it is assumed that every player needs permission from *at least one* of its predecessors (if it has any) before it is allowed to cooperate with other players.

(Shapley (1953)) yielding a solution that is called the *conjunctive (Shapley) permission value*.

On the other hand, in a game under precedence constraints, the order in which players enter to form the 'grand coalition' is restricted by the digraph, in the sense that players can only enter when their 'subordinates' in the hierarchy have already entered. Instead of taking the average of all marginal contribution vectors, as done by the classical Shapley value, the *precedence (Shapley) value* of Faigle and Kern (1992), takes the average of the marginal vectors over these *admissible* permutations. Alternatively, this precedence Shapley value can be written as an allocation of the *Harsanyi dividends*, where the dividend of every feasible coalition is allocated proportional to the so-called *hierarchical strength* being a power measure for digraphs that assigns to every player the number of admissible permutations where it is the last to enter. Algaba et al. (2017) showed that in this solution the payoff allocation is influenced by the presence of irrelevant players. These are players who do not generate worth in the game and, moreover, also all the players who depend on their presence do not generate worth in the game. Requiring that the payoff allocation does not depend on the presence of these irrelevant players, they modified the precedence Shapley value by requiring the allocation of Harsanyi dividends proportional to the hierarchical strength only in case all players are necessary to generate worth, (this means that the game is a multiple of the unanimity game of the 'grand coalition'). Moreover, they showed that instead of the hierarchical strength, any (positive) power measure can be used yielding the so-called *precedence power solutions*. In this way, the game theoretic problem of payoff allocation is linked with the social network literature on power and centrality measures.

After reviewing some known axiomatizations of the conjunctive permission value, the precedence Shapley value and precedence power solutions, we will show that also the conjunctive permission value can be axiomatized with an axiom that applies a network power measure to the permission structure. Moreover, similar to the precedence power solutions, we can apply any (positive) power measure and obtain a class of *permission power solutions*. In this way, we have two classes of solutions for games with a hierarchy, one based on permission structures and another based on precedence constraints, that are characterized by similar axioms. Moreover, the solutions are linked with network power measures.

This chapter is organized as follows. After some preliminaries on cooperative transferable utility games and digraphs, in Section 4.2, we introduce the two models of games with a hierarchy. In Section 4.3, we discuss the conjunctive permission value, the precedence Shapley value and the hierarchical solution for these two models. In Section 4.4, we generalize these solutions by applying network power measures. In Section 4.5, we show logical independence of the axioms in the main theorems. Finally, Section 4.6 contains concluding remarks.

## 4.2 Games with Hierarchies

This section contains preliminaries on TU-games, digraphs, games with a permission structure and games under precedence constraints.

### 4.2.1 TU-Games

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair $(N, v)$ where $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition $S$, i.e., the members of coalition $S$ can obtain a total payoff of $v(S)$ by agreeing to cooperate. We denote the collection of all TU-games $(N, v)$ by $\mathcal{G}$. We denote the collection of all characteristic functions $v$ on player set $N$ by $\mathcal{G}^N$.

A *payoff vector* for game $(N, v)$ is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function that assigns a payoff vector to every TU-game. One of the most widely used solutions for TU-games is the *Shapley value* (Shapley (1953)), given by

$$Sh_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} m_i^\pi(N, v), \text{ for all } i \in N,$$

where $\Pi(N)$ is the collection of all permutations $\pi\colon N \to N$ on $N$, and for every permutation $\pi \in \Pi(N)$,

$$m_i^\pi(N, v) = v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\}), \quad (4.1)$$

is the *marginal contribution* of player $i$ to the players that are ranked before him in the order $\pi$.

For each $T \subseteq N$, $T \neq \emptyset$, the *unanimity* game $(N, u_T)$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well known that the unanimity games form a basis for $\mathcal{G}^N$. For every $v \in \mathcal{G}^N$, it holds that $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$, where $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ are the *Harsanyi dividends*; see Harsanyi (1959).

For $(N, v), (N, w) \in \mathcal{G}$, the sum game $(N, v+w)$ is defined by $(v+w)(S) = v(S) + w(S)$, and for $c \in \mathbb{R}$, the game $(N, cv) \in \mathcal{G}$ by $(cv)(S) = cv(S)$ for $S \subseteq N$. For $(N, v) \in \mathcal{G}$ and $S \subseteq N$, the *subgame* $(S, v_S)$ is given by $v_S(T) = v(T)$ for all $T \subseteq S$.

### 4.2.2 Digraphs

An *irreflexive directed graph* or *irreflexive digraph* is a pair $(N, D)$ where $N$ is the set of nodes and $D \subseteq \{(i, j) \mid i, j \in N, i \neq j\}$ is an (irreflexive) binary relation on $N$ consisting of ordered pairs called directed links or *arcs*. Since we assume irreflexivity throughout the full chapter, we refer to these just as digraphs. Since the nodes will represent players, we often refer to the nodes as players. For $i \in N$, the nodes in $F_D(i) := \{j \in N \mid (i,j) \in D\}$ are called the *followers* or *successors* of $i$ in $D$, and the nodes in $P_D(i) := \{j \in N \mid (j,i) \in D\}$ are called the *predecessors* of $i$ in $D$. Further, by $\widehat{F}_D(i)$ we denote the set of successors of $i$ in the *transitive closure* of $D$, i.e., $j \in \widehat{F}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in F_D(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$. We refer to the players in $\widehat{F}_D(i)$ as the *subordinates* of $i$ in $D$, and to the players in the set $\widehat{P}_D(i) = \{j \in N \mid i \in \widehat{F}_D(j)\}$ consisting of all predecessors of $i$ in the transitive closure of $D$, as $i$'s *superiors*. We denote by $\widehat{P}_D(T) = \bigcup_{i \in T} \widehat{P}_D(i)$ the set of all superiors of players in $T$. The digraph $(N, D)$ is called *acyclic* if $i \notin \widehat{F}_D(i)$ for all $i \in N$. We denote the collection of all acyclic digraphs by $\mathcal{D}$, and the collection of all acyclic binary relations (which we will also often refer to as digraphs) on $N$ by $\mathcal{D}^N$. For $S \subseteq N$ and $(N, D) \in \mathcal{D}$, the digraph $(S, D(S))$ is given by $D(S) = \{(i,j) \in D \mid \{i,j\} \subseteq S\}$. By $TOP(N,D) = \{i \in N \mid P_D(i) = \emptyset\}$ we denote the set of 'top players' in $(N, D)$, i.e., the set of players without predecessors. Note that $TOP(N,D) \neq \emptyset$ if $(N, D)$ is acyclic.

### 4.2.3 Games with a Permission Structure

A game with a permission structure describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate with other players. Formally, a *permission structure* is a directed graph on $N$. In this context, a triple $(N, v, D)$ with $N$ a finite set of players, $v \in \mathcal{G}^N$ a TU-game and $D \in \mathcal{D}^N$ a digraph on $N$ is called a *game with a permission structure*. In the *conjunctive approach* as introduced in Gilles et al. (1992) and van den Brink and Gilles (1996), it is assumed that a player needs permission from *all* its predecessors in order to cooperate with other players. In this sense, a coalition is feasible if and only if for every player in the coalition all its predecessors are also in the coalition. So, for permission structure $D$, the set of *conjunctive feasible coalitions* is given by

$$\Phi^c(N,D) = \{S \subseteq N \,|\, P_D(i) \subseteq S \text{ for all } i \in S\}\,.$$

For every $S \subseteq N$, let $\sigma^c_D(S) = \bigcup_{\{F \in \Phi^c(N,D) | F \subseteq S\}} F = S \setminus \widehat{F}_D(N \setminus S)$ be the largest conjunctive feasible subset[2] of $S$ in the collection $\Phi^c(N,D)$. Then, the induced *conjunctive restricted* game of $(v, D)$ is the game $r^c_{v,D} \colon 2^N \to \mathbb{R}$ that assigns to every coalition $S \subseteq N$ the worth of its largest conjunctive feasible subset[3], i.e.,

$$r^c_{v,D}(S) = v(\sigma^c_D(S)) \text{ for all } S \subseteq N. \tag{4.2}$$

We denote the class of all games with a permission structure by $\mathcal{G}_{PS}$.

[2]Every coalition having a unique conjunctive largest feasible subset follows from the fact that $\Phi^c(N,D)$ is union closed.

[3]Alternatively, for acyclic and quasi-strongly connected permission structures, in the *disjunctive approach* as introduced in Gilles and Owen (1994) and van den Brink (1997) (see also Gilles (2010)), it is assumed that a non-top player needs permission from *at least one* of its predecessors. By a similar approach as described here, one can define the *disjunctive restricted game*.

### 4.2.4 Games under Precedence Constraints

Faigle and Kern (1992) consider situations where a partial order or acyclic directed graph represents a *precedence relation* meaning that the order in which players enter the grand coalition is restricted. Assuming that a player can only enter after all its subordinates have entered, a coalition is feasible if for every player in the coalition all of its successors in the digraph are also present in the coalition. The set $\Phi^p(N,D)$ of feasible coalitions according to digraph $(N,D) \in \mathcal{D}$ is thus given by

$$\Phi^p(N,D) = \{S \subseteq N \mid F_D(i) \subseteq S \text{ for all } i \in S\}.$$

Instead of considering a restricted game on the collection of all coalitions (i.e., subsets of $N$), Faigle and Kern (1992) consider cooperative games, where for acyclic digraph $(N,D) \in \mathcal{D}$ the domain of the characteristic function is given by the set $\Phi^p(N,D)$. In this context, we call a triple $(N,v,D)$, where $N \subseteq \mathbb{N}$ is a finite set of players, $(N,D) \in \mathcal{D}$ is an acyclic digraph, and $v : \Phi^p(N,D) \to \mathbb{R}$, with $v(\emptyset) = 0$, is a characteristic function that is defined only on $\Phi^p(N,D)$, a *game under precedence constraints*.

We denote the class of all games under precedence constraints by $\mathcal{G}_{PC}$, and we denote the class of games under precedence constraints on graph $(N,D) \in \mathcal{D}$ by $\mathcal{G}_{PC}^{(N,D)}$. For $(N,v,D),(N,w,D) \in \mathcal{G}_{PC}$, the sum game $(N,v+w)$ is defined by $(v+w)(S) = v(S)+w(S)$, and for $c \in \mathbb{R}$, the game $(N,cv) \in \mathcal{G}$ by $(cv)(S) = cv(S)$ for $S \in \Phi^p(N,D)$. The game under precedence constraints obtained from $(N,v,D) \in \mathcal{G}_{PC}$ by considering only feasible coalition $S$ and its subsets is denoted by $(S, v_S, D(S))$, where $v_S(T) = v(T)$ for all feasible coalitions $T \subseteq S$. We refer to $(S, v_S, D(S))$ as the subgame on $S$ of $(N,v,D)$.

Because of the difference in interpretation, we refer to a triple $(N,v,D)$ with $v$ a characteristic function on $2^N$ as a game with a permission structure, and to a triple $(N,v,D)$ with $v$ a characteristic function on $\Phi^p(N,D)$ as a game under precedence constraints. Sometimes, we refer to these situations in general as a game with a hierarchy.

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## 4.3 Solutions for Games with Hierarchies

In this section, we discuss several solutions for games with a permission structure and games under precedence constraints.

### 4.3.1 The Conjunctive Permission Value for Games with a Permission Structure

A *solution* for games with a permission structure is a function $f$ that assigns a payoff distribution $f(N,v,D) \in \mathbb{R}^N$ to every game with permission structure $(N,v,D)$. The *conjunctive (Shapley) permission value* $\varphi^c$ is the solution that assigns to every game with a permission structure the Shapley value of the conjunctive restricted game[4], i.e.,

$$\varphi^c(N, v, D) = \mathit{Sh}(N, r^c_{v,D}).$$

Next, we discuss one of the axiomatizations of the conjunctive permission value. Player $i \in N$ is *inessential* in game with permission structure $(N, v, D)$ if $i$ and all its subordinates are *null* players in game $v$, i.e., if $v(S) = v(S \setminus \{j\})$ for all $S \subseteq N$ and $j \in \{i\} \cup \widehat{F}_D(i)$. Player $i \in N$ is *necessary* in game $v$ if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.

Next, we mention some axioms of solutions for games with a permission structure. Efficiency and linearity are straightforward generalizations of TU-game solution axioms. The inessential player property requires that inessential players earn a zero payoff. The necessary player property requires that necessary players earn at least as much as any other player if the game is monotone. A game $(N, v)$ is monotone if $v(S) \leq v(T)$, for all $S \subseteq T \subseteq N$. Notice that a necessary player is a 'strong' player in a monotone game. Structural monotonicity requires that in monotone games, players earn at least as much as their successors. From now on, the class of monotone TU-games on $N$ will be denoted by $\mathcal{G}^N_M$.

**Efficiency** For every $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

**Linearity** For every $v, w \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, it holds that $f(N, v + w, D) = f(N, v, D) + f(N, w, D)$, and for $c \in \mathbb{R}$ it holds that $f(N, cv, D) = cf(N, v, D)$.

**Inessential player property** For every $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, if $i \in N$ is an inessential player in $(N, v, D)$, then $f_i(N, v, D) = 0$.

**Necessary player property** For every $v \in \mathcal{G}^N_M$ and $D \in \mathcal{D}^N$, if $i \in N$ is a necessary player in $(N, v)$, then $f_i(N, v, D) \geq f_j(N, v, D)$ for all $j \in N$.

**Structural monotonicity** For every $v \in \mathcal{G}^N_M$ and $D \in \mathcal{D}^N$, if $j \in F_D(i)$, then $f_i(N, v, D) \geq f_j(N, v, D)$.

The above five axioms characterize the conjunctive permission value.[5]

**Theorem 4.1 (van den Brink and Gilles (1996))** *A solution for games with a permission structure is equal to the conjunctive permission value $\varphi^c$ if and only if it satisfies efficiency, linearity, the inessential player property, the necessary player property and structural monotonicity.*

[4]Alternatively, for acyclic and quasi-strongly connected permission structures, the *disjunctive permission value* is obtained as the Shapley value of the disjunctive restricted game, see Footnote 3.

[5]We remark that, instead of linearity, van den Brink and Gilles (1996) use the weaker additivity axiom.

If $D = \emptyset$, then there are no restrictions in coalition formation (i.e., $\Phi^c(N, D) = 2^N$), and then $\varphi^c(N, v, D) = Sh(N, v)$. In this sense, the conjunctive permission value generalizes the Shapley value for TU-games. Notice that the axiomatization in Theorem 4.1 gives an axiomatization of the Shapley value for TU-games by taking $D = \emptyset$. In that case, efficiency and linearity just boil down to the corresponding axioms for TU-game solutions. Since no player has subordinates, a player is inessential if and only if it is a null player in the game, and thus the inessential player property boils down to the null player property for TU-game solutions. The necessary player property does not depend on the permission structure anyway, and can be stated as well for TU-game solutions by requiring that a necessary player in a monotone game earns at least as much as any other player.[6] Efficiency, linearity, the inessential (null) player property and the necessary player property then give uniqueness as in Shapley (1953). Note that structural monotonicity has no meaning when $D = \emptyset$.

In Gilles at al. (1992) it is shown that the conjunctive permission value can also be obtained by allocating the Harsanyi dividends in the conjunctive restricted game, equally over all players in the corresponding coalition and their superiors, i.e.,

$$\varphi_i^c(N, v, D) = \sum_{\substack{S \subseteq N \\ i \in S \cup \widehat{P}_D(S)}} \frac{\Delta_{r^c_{v,D}}(S)}{|S \cup \widehat{P}_D(S)|} \text{ for all } i \in N.$$

## 4.3.2 The Precedence Shapley Value and the Hierarchical Solution for Games under Precedence Constraints

### 4.3.2.1 The Precedence Shapley Value

Faigle and Kern (1992) introduce the precedence Shapley value as solution for games under precedence constraints. First, a permutation $\pi \in \Pi(N)$ is called *admissible* in acyclic digraph $(N, D)$ if $\pi(i) > \pi(j)$ whenever $(i, j) \in D$, i.e., successors enter before their predecessors in the digraph.[7] The set of admissible permutations $\Pi_D(N)$ in $D$ is denoted by

$$\Pi_D(N) = \{\pi \in \Pi(N) \mid \pi(i) > \pi(j) \text{ if } (i, j) \in D\}. \tag{4.3}$$

Note that the set of admissible permutations in $D$ is the same as that of its transitive closure $tr(D)$: $\Pi_D(N) = \Pi_{tr(D)}(N)$.

[6] Since all players in $T \subseteq N$ are necessary players in the unanimity game $u_T$ on $T$, they should earn the same in that unanimity game, which in the axiomatization of the Shapley value is guaranteed by symmetry.

[7] The terminology looks somewhat counterintuitive, but this is because of the different interpretations of the hierarchy in games with a permission structure and games under precedence constraints, see also the last paragraph of Section 4.6.

The *precedence marginal vector* $m^\pi(N,v,D) \in \mathbb{R}^N$, associated with the game under precedence constraints $(N,v,D)$, permutation $\pi \in \Pi_D(N)$, and player $i \in N$, is given by

$$m_i^\pi(N,v,D) = v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\}). \quad (4.4)$$

Recall from Section 4.2 that the Shapley value assigns to the players the average over all marginal vectors associated with all permutations of the player set $N$. The *precedence Shapley value* $H$ is the solution on $\mathcal{G}_{PC}$ given by

$$H_i(N,v,D) = \frac{1}{|\Pi_D(N)|} \sum_{\pi \in \Pi_D(N)} m_i^\pi(N,v,D), \quad \text{for all } i \in N,$$

and assigns to the players in $N$ the average over all precedence marginal vectors of game under precedence constraints $(N,v,D)$. For $(N,v,D) \in \mathcal{G}_{PC}$, all permutations in $\Pi(N)$ are admissible when $D = \emptyset$. In that case, the domain of characteristic function $v$ is given by $2^N$, and thus is a classical characteristic function of a TU-game. So, also the precedence Shapley value $H$ generalizes the Shapley value for TU-games.

Faigle and Kern (1992) give an axiomatization of the precedence Shapley value using the following axioms. Efficiency and linearity are the same as for the conjunctive permission value, but defined on the domain $\mathcal{G}_{PC}$.

**Efficiency** For each game $(N,v,D) \in \mathcal{G}_{PC}$ it holds that $\sum_{i \in N} f_i(N,v,D) = v(N)$.

**Linearity** For every pair of games $(N,v,D)$ and $(N,w,D) \in \mathcal{G}_{PC}^{(N,D)}$ it holds that $f(N,v+w,D) = f(N,v,D) + f(N,w,D)$, and for $(N,v,D) \in \mathcal{G}_{PC}^{(N,D)}$ and $c \in \mathbb{R}$ it holds that $f(N,cv,D) = cf(N,v,D)$.

A player $i \in N$ is a *null player* in game under precedence constraints $(N,v,D)$, if for every $\pi \in \Pi_D(N)$ it holds that $m_i^\pi(N,v,D) = 0$.

**Null player property** For each $(N,v,D) \in \mathcal{G}_{PC}$, if $i \in N$ is a null player in $(N,v,D)$, then $f_i(N,v,D) = 0$.

Besides these three axioms[8], Faigle and Kern (1992) introduce an axiom that is based on the hierarchical strength of players. First, for all $i \in S$, $S \in \Phi^p(N,D)$, the set of permutations $\Pi_D^i(N,S)$ is defined by

$$\Pi_D^i(N,S) = \{\pi \in \Pi_D(N) \mid \pi(i) > \pi(j) \text{ for all } j \in S \setminus \{i\}\}, \quad (4.5)$$

[8]We remark that, similar to Shapley (1953), Faigle and Kern (1992) combine efficiency and the null player property into a carrier axiom.

being the collection of those admissible permutations in $\Pi_D(N)$ where $i$ enters after the players in $S \setminus \{i\}$. Note that the collection $\{\Pi_D^i(N,S)\}_{i\in S}$ is a partition of $\Pi_D(N)$.

The *absolute hierarchical strength* is the function $h$ that assigns to every $(N,D) \in \mathcal{D}$ and coalition $S \in \Phi^p(N,D)$, the vector $h(N,D,S) \in \mathbb{R}^S$, where $h_i(N,D,S) = |\Pi_D^i(N,S)|$ is the number of permutations in $\Pi_D(N)$ where $i \in S$ enters after the players in $S \setminus \{i\}$.

The *normalized hierarchical strength* is the function $\overline{h}$ that assigns to every $(N,D) \in \mathcal{D}$ and coalition $S \in \Phi^p(N,D)$, the vector $\overline{h}(N,D,S) \in \mathbb{R}^S$, where $\overline{h}_i(N,D,S) = \frac{|\Pi_D^i(N,S)|}{|\Pi_D(N)|}$ is the fraction of permutations in $\Pi_D(N)$ where $i \in S$ enters after the players in $S \setminus \{i\}$. Note that $\sum\limits_{i\in S} \overline{h}_i(N,D,S) = 1$ for all $S \in \Phi^p(N,D)$.

Unanimity games under precedence constraints are defined similar to classical unanimity TU-games. For each $T \in \Phi^p(N,D)$, $T \neq \emptyset$, the *unanimity* game under precedence constraints $(N,u_T,D) \in \mathcal{G}_{PC}$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, $S \in \Phi^p(N,D)$. Note that, different from classical TU-games, the unanimity game (called simple game by Faigle and Kern) $u_T$ is only defined on the set $\Phi^p(N,D)$. Faigle and Kern (1992) also consider the *dividend* of a coalition $S \in \Phi^p(N,D)$ in game under precedence constraints $(N,v,D)$, given by $\Delta_v^D(S) = v(S) - \sum_{T\subset S, T\in\Phi^p(N,D), T\neq\emptyset} \Delta_v^D(T)$.

For every $(N,v,D) \in \mathcal{G}_{PC}$, Faigle and Kern (1992) show that the characteristic function in $(N,v,D)$ can be written as a linear combination of the characteristic functions of unanimity games under precedence constraints $(N,u_T,D)$:

$$v = \sum_{\substack{T\in\Phi^p(N,D)\\ T\neq\emptyset}} \Delta_v^D(T) u_T. \tag{4.6}$$

The axiom of hierarchical strength of a solution for games under precedence constraints states that, in unanimity games under precedence constraints, the earnings are distributed among the players in the unanimity coalition proportional to their normalized hierarchical strength in that coalition. Obviously, this is equivalent to distributing the dividends proportional to the absolute hierarchical strength of the players.

**Hierarchical strength** For every $(N,D) \in \mathcal{D}$, every $S \in \Phi^p(N,D)$ and every $i,j \in S$, it holds that $\overline{h}_i(N,D,S) f_j(N,u_S,D) = \overline{h}_j(N,D,S) f_i(N,u_S,D)$.

**Theorem 4.2 (Faigle and Kern, 1992)** *A solution on $\mathcal{G}_{PC}$ is equal to the precedence Shapley value $H$ if and only if it satisfies efficiency, linearity, the null player property and hierarchical strength.*

Alternatively, the precedence Shapley value can be defined as the solution that allocates the dividend of a coalition $S \in \Phi^p(N,D)$ proportional to the hierarchical strength $h(N, D, S)$ of the players in $S$:

$$H_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(N, D, S)}{\sum_{j \in S} h_j(N, D, S)} \Delta_v^D(S) \text{ for all } i \in N. \tag{4.7}$$

#### 4.3.2.2 The Hierarchical Solution

Algaba et al. (2017) introduce the solution for games under precedence constraints that is obtained by weakening hierarchical strength in Theorem 4.2 to weak hierarchical strength and adding irrelevant player independence.

Weak hierarchical strength is a weaker version of the hierarchical strength axiom in the sense that it only requires the equality for unanimity games of the grand coalition.

**Weak hierarchical strength** For every $(N, D) \in \mathcal{D}$ and every $i, j \in N$, it holds that $\overline{h}_i(N, D, N) f_j(N, u_N, D) = \overline{h}_j(N, D, N) f_i(N, u_N, D)$.

This is a considerable weakening, also in interpretation. If unanimity among all players must be reached before any non-zero worth can be generated, we might consider the players equals with respect to the game. Therefore, worth allocation should depend only on the strength of the players in the digraph. The strength of each player in the digraph is measured by the hierarchical strength.

Player $i \in N$ is called an *irrelevant player* in game under precedence constraints $(N, v, D)$ if $i$ is a null player, and any $j \in \widehat{P}_D(i)$ is also a null player (this implies that any $j \in \widehat{P}_D(i)$ is also irrelevant). So, an irrelevant player is a null player such that all players who depend on its presence are also null players in the game. We call a player $i \in N$ *relevant* if it is not an irrelevant player.

Let $Irr(N, v, D)$ be the set of irrelevant players in game under precedence constraints $(N, v, D)$. Irrelevant player independence states that removal of irrelevant players from the game does not affect the payoff to relevant players.

**Irrelevant player independence** For every $(N, v, D) \in \mathcal{G}_{PC}$, it holds that $f_i(N, v, D) = f_i(N', v_{N'}, D(N'))$ for $i \in N'$, with $N' = N \setminus Irr(N, v, D)$.

For a collection of sets $\mathcal{F} \subseteq 2^N$, let $\mathcal{F}_S = \{T \in \mathcal{F} \mid T \subseteq S\}$ be the collection of subsets of $S$ in $\mathcal{F}$. It can be seen that, for $N' = N \setminus Irr(N, v, D)$, it holds that $\Phi^p_{N'}(N, D) = \Phi^p(N', D(N'))$, i.e., the collection of feasible subsets of coalition $N'$ obtained from graph $(N, D)$ is equal to the collection of feasible sets obtained from subgraph $(N', D(N'))$. (Note that this does not have to be the case for all subsets of $N$.) This means that removing irrelevant players from the game does not have an effect on the ability of relevant players to cooperate with each other.

We consider irrelevant player independence a desirable property for a solution for games under precedence constraints to satisfy. Since irrelevant players are null players, they do not make any contribution to their subordinates in the digraph. Moreover, their superiors are also null players, and thus irrelevant players do not make any contribution through players that need them to be present in any admissible permutation. Therefore, they should not be able to affect the payoffs of those players who do make a contribution in the game. The precedence Shapley value does not satisfy irrelevant player independence, as illustrated by the following example.

**Example 4.1** *Consider the following acyclic digraph used by Faigle and Kern (1992). Let* $N = \{1, 2, 3, 4\}$, *and* $D = \{(3, 1), (3, 2), (4, 2)\}$, *see Figure 4.1.*

*The set of admissible permutations is*

$$\Pi_D(N) = \{(1, 2, 3, 4), (1, 2, 4, 3), (2, 1, 3, 4), (2, 1, 4, 3), (2, 4, 1, 3)\}.$$

*In this case, for* $S = \{1, 2, 4\} \in \Phi^p(N, D)$, *we have*

$$h_1(N, D, S) = 1, \;\; h_2(N, D, S) = 0, \;\; h_4(N, D, S) = 4.$$

*Consider the game* $v = u_{\{1,2,4\}}$. *Then, the precedence Shapley value is given by* $H(N, v, D) = \left(\frac{1}{5}, 0, 0, \frac{4}{5}\right)$.

*Notice that player 3 is an irrelevant player. Deleting player 3 gives the game under precedence constraints* $(N', v', D')$ *with* $N' = \{1, 2, 4\}$, $v' = u_{\{1,2,4\}}$ *(but on a different domain), and* $D' = \{(4, 2)\}$.

*The set of admissible permutations on subgraph* $(S, D(S))$ *is given by*

$$\Pi_{D(S)}(S) = \{(1, 2, 4), (2, 1, 4), (2, 4, 1)\}.$$

*Therefore,* $h_1(S, D(S), S) = 1$, $h_2(S, D(S), S) = 0$, $h_4(S, D(S), S) = 2$, *yielding the precedence Shapley value* $H(N, v, D) = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$.

*The presence of irrelevant player 3 changes the payoffs of players 1 and 4 according to the precedence Shapley value.*

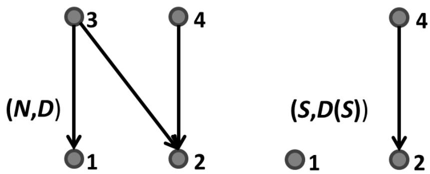


**FIGURE 4.1**: Digraphs $(N, D)$ and $(S, D(S))$ of Example 4.1.

It can be shown that for games $(N_m, u_{\{1,2\}}, D_m)$, where $N_m$ is given by $\{1, \ldots, m\}$ and $D_m$ by $\{(3, 1), (4, 3), \ldots, (m, m-1)\}$, the precedence Shapley value is given by $H_1(N_m, u_{\{1,2\}}, D_m) = \frac{1}{m}, H_2(N_m, u_{\{1,2\}}, D_m) =$

$\frac{m-1}{m}$ and $H_i(N_m, u_{\{1,2\}}, D_m) = 0$ for $i \in N_m \setminus \{1,2\}$ and so $\lim_{m\to\infty} H_1(N_m, u_{\{1,2\}}, D_m) = 0$ and $\lim_{m\to\infty} H_2(N_m, u_{\{1,2\}}, D_m) = 1$. We find that the fact that player 1 has many irrelevant players as superiors in the digraph is detrimental to its payoff, even though, for different values of $m$, player 1 is present in exactly the same feasible coalitions that contain only relevant players.

Algaba et al. (2017) provide a characterization where the null player property is replaced by the following weaker property on irrelevant players.[9]

**Irrelevant player property** For each $(N, v, D) \in \mathcal{G}_{PC}$, if $i \in N$ is an irrelevant player in $(N, v, D)$, then $f_i(N, v, D) = 0$.

The unique solution for games under precedence constraints that satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence, and weak hierarchical strength is the hierarchical solution which allocates the dividend of every feasible coalition over the players in that coalition proportional to the hierarchical strength in the subgraph on that coalition.

The *hierarchical solution* $\widetilde{H}$ is the solution on $\mathcal{G}_{PC}$ given by

$$\widetilde{H}_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(S, D(S), S)}{\sum_{j \in S} h_j(S, D(S), S)} \Delta_v^D(S), \; i \in N.$$

A main difference with the precedence Shapley value is that in that value, the dividends are allocated proportional to the hierarchical strength in the full digraph (see (4.7)), while in the hierarchical solution, when allocating the dividend of a feasible coalition, we consider the hierarchical strength of the *subgraph* on the corresponding coalition.

**Theorem 4.3 (Algaba et al., 2017)** *A solution for games under precedence constraints is equal to the hierarchical solution $\widetilde{H}$ if and only if it satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence, and weak hierarchical strength.*

Next, we provide an example which calculates the hierarchical solution and the precedence Shapley value highlighting that in general both solutions are different, and are also different from the conjunctive permission value.

**Example 4.2** *Consider the game under precedence constraint given in Example 4.1. By taking the appropriate domain for the characteristic function, this can also be seen as a game with a permission structure. In Example 4.1, we already computed the precedence Shapley value $H(N, v, D) = \left(\frac{1}{5}, 0, 0, \frac{4}{5}\right)$.*

*For $S = \{1, 2, 4\} \in \Phi^p(N, D)$, in Example 4.1, we also found that the set of admissible permutations on subgraph $(S, D(S))$ is given by*

$$\Pi_{D(S)}(S) = \{(1, 2, 4), (2, 1, 4), (2, 4, 1)\},$$

[9] It is straightforward to show that the null player property can also be replaced by the irrelevant player property in the axiomatization of the precedence Shapley value.

*and thus* $h_1(S, D(S), S) = 1$, $h_2(S, D(S), S) = 0$, $h_4(S, D(S), S) = 2$. *This yields the hierarchical solution* $\widetilde{H}(N, v, D) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right)$ .

*The conjunctive restricted game is given by* $r^c_{v,D} = u_N$, *and thus the conjunctive permission value gives an equal allocation of the payoff,* $\varphi^c(N, v, D) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

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## 4.4 Power Measures for Digraphs and Solutions for Games with Hierarchies

Power or centrality measures for digraphs are applied to define solutions for games under precedence constraints by Algaba et al. (2017). In this section, we first review their result, and then apply this approach to games with a permission structure.

### 4.4.1 Precedence Power Solutions for Games under Precedence Constraints

Algaba et al. (2017) considered a class of solutions for games under precedence constraints that contain the hierarchical solution. Similar as van den Brink et al. (2011a) generalize the communication ability property of Borm et al. (1992) for communication graph games (see Myerson (1977)), this class is obtained by applying a power measure for digraphs to allocate the dividends, and apply this power measure in a corresponding version of the weak hierarchical strength axiom.

A *power measure* for acyclic digraphs is a function $p$, that to every acyclic digraph $(N, D) \in \mathcal{D}$ assigns a vector $p(N, D) \in \mathbb{R}^N$. For a player $i \in N$, $p_i(N, D)$ is a measure of its relational 'power' or 'influence' in $(N, D)$. We call a power measure $p$ positive if $\sum_{j \in N} p_j(N, D) > 0$ for all $(N, D) \in \mathcal{D}$ with $D \neq \emptyset$. Notice that a power measure is defined for any set of nodes $N \subset \mathbb{N}$, and thus also for any $S \subseteq N \subset \mathbb{N}$, $p(S, D(S))$ is defined. In this chapter, we only consider positive power measures.

For positive power measure $p$, we define the *p-hierarchical solution* as the solution that allocates the dividend of a coalition $S \in \Phi^p(N, D)$ among the players in $S$ proportional to $p(S, D(S))$.

**Definition 4.1** *For positive power measure* $p$, *the* $p$-*hierarchical solution is the solution on* $\mathcal{G}_{PC}$ *given by*

$$H^p_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{p_i(S, D(S))}{\sum_{j \in S} p_j(S, D(S))} \Delta^D_v(S) \text{ for all } i \in N.$$

We refer to the class consisting of all $p$-hierarchical solutions as the class of *precedence power solutions*.

In order to axiomatize the $p$-hierarchical solution, the $p$-strength axiom is introduced. This axiom has an interpretation similar to that of weak hierarchical strength from Theorem 4.3. If unanimity among all players must be reached to generate any non-zero worth, we might consider the players equals with respect to the game. Therefore, worth allocation should only depend on the strength of the players in the digraph. The $p$-hierarchical solution uses the power measure $p$ to measure the strength of each player in the digraph. The $p$-strength axiom requires that in a game where all players are necessary to generate worth, the payoffs are allocated proportional to the power measure $p$.

**$p$-strength** Let $p$ be a positive power measure. For every $(N, D) \in \mathcal{D}$ and every $i, j \in N$, it holds that
$p_i(N, D) f_j(N, u_N, D) = p_j(N, D) f_i(N, u_N, D)$.

The $p$-hierarchical solution is axiomatized by replacing in Theorem 4.3 weak hierarchical strength by $p$-strength.

**Theorem 4.4 (Algaba et al., 2017)** *A solution for games under precedence constraints is equal to the p-hierarchical solution $H^p$ if and only if it satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence and p-strength.*

Note that this gives Theorem 4.3 as a corollary by taking the hierarchical strength as power measure. The axioms of Theorem 4.4 are not logically independent. It can be shown that efficiency and irrelevant player independence together imply the irrelevant player property.

**Proposition 4.1** *Consider a solution $f$ on $\mathcal{G}_{PC}$. If $f$ satisfies efficiency and irrelevant player independence, then $f$ satisfies the irrelevant player property.*

*Proof.* Suppose that solution $f$ satisfies efficiency and irrelevant player independence. We show that $f$ must satisfy the irrelevant player property by induction on the number of irrelevant players. Suppose that $|Irr(N, v, D)| = 1$, and let $j \in N$ be the irrelevant player in $(N, v, D) \in \mathcal{G}_{PC}$. By irrelevant player independence, we have that $f_i(N, v, D) = f_i(N \setminus \{j\}, v_{N\setminus\{j\}}, D(N \setminus \{j\}))$ for all $i \in N \setminus \{j\}$. By efficiency, it then follows that $\sum_{i\in N} f_i(N, v, D) = v(N) = v(N\setminus\{j\}) = \sum_{i\in N\setminus\{j\}} f_i(N\setminus\{j\}, v_{N\setminus\{j\}}, D(N\setminus\{j\})) = \sum_{i\in N\setminus\{j\}} f_i(N, v, D)$, and thus $f_j(N, v, D) = 0$.

By induction, we assume that irrelevant players get a zero payoff for all $(N', v', D') \in \mathcal{G}_{PC}$ with $|Irr(N', v', D')| < |Irr(N, v, D)|$. Take a $j \in Irr(N, v, D)$ such that all successors of $j$ are relevant players. (Existence of such players can be shown as follows. Consider an irrelevant player who has an irrelevant successor. If this successor has at least one irrelevant successor, then consider this successor. Continuing in this way, by acyclicity and finiteness of $D$, eventually we reach an irrelevant player whose successors are all relevant players, possibly being an irrelevant player who has no successors.) Consider the game $w = u_{(N \setminus Irr(N,v,D)) \cup \{j\}}$, i.e., the unanimity game on the set of all relevant players in $v$ with player $j$. Then $|Irr(N,w,D)| < |Irr(N,v,D)|$, specifically $Irr(N,w,D) = Irr(N,v,D) \setminus \{j\}$, and thus by the induction hypothesis

$$\sum_{h \in Irr(N,v,D) \setminus \{j\}} f_h(N,w,D) = 0. \tag{4.8}$$

Since also $|Irr(N,v+w,D)| = |Irr(N,w,D)| = |Irr(N,v,D)| - 1$, we have similar that

$$\sum_{h \in Irr(N,v,D) \setminus \{j\}} f_h(N,v+w,D) = 0. \tag{4.9}$$

Since linearity implies that $f(N,v,D) = f(N,v+w,D) - f(N,w,D)$, with (4.8) and (4.9), it follows that

$$\begin{aligned} \sum_{h \in Irr(N,v,D) \setminus \{j\}} f_h(N,v,D) &= \sum_{h \in Irr(N,v,D) \setminus \{j\}} f_h(N,v+w,D) \\ &- \sum_{h \in Irr(N,v,D) \setminus \{j\}} f_h(N,w,D) = 0. \end{aligned} \tag{4.10}$$

By efficiency and irrelevant player independence it follows, similar as in the initial step, that $\sum_{i \in N} f_i(N,v,D) = v(N) = v(N \setminus Irr(N,v,D)) = \sum_{i \in N \setminus Irr(N,v,D)} f_i(N \setminus Irr(N,v,D), v_{N \setminus Irr(N,v,D)}, D(N \setminus Irr(N,v,D))) = \sum_{i \in N \setminus Irr(N,v,D)} f_i(N,v,D)$, and thus $\sum_{h \in Irr(N,v,D)} f_h(N,v,D) = 0$. With (4.10) it then follows that $f_j(N,v,D) = 0$. This shows that the irrelevant player property is satisfied. ■

Proposition 4.1 and Theorem 4.4 immediately give the following result as a corollary.[10]

**Theorem 4.5** *A solution for games under precedence constraints is equal to the p-hierarchical solution $H^p$ if and only if it satisfies efficiency, linearity, irrelevant player independence and p-strength.*

Logical independence of the axioms is shown in Section 4.5.

[10]Similarly, the irrelevant player property is superfluous in Theorem 4.3.

### 4.4.2 Power Measures, Solutions for Games with a Permission Structure and Permission Values

The conjunctive permission value satisfies efficiency and linearity[11]. Although the conjunctive permission value does not satisfy the weak hierarchical strength axiom, it satisfies a version of the $p$-strength axiom, where in the unanimity game of the 'grand coalition', the payoffs are allocated equally over the players, i.e., proportional to the equal power measure where all players have equal power in any digraph.[12]

**Equal-strength** For every $(N, D) \in \mathcal{D}$ and every $i, j \in N$, it holds that $f_i(N, u_N, D) = f_j(N, u_N, D)$.

The conjunctive permission value does not satisfy the null player property, irrelevant player independence and the irrelevant player property[13]. However, it satisfies similar properties. Instead of irrelevant player independence, the conjunctive permission value satisfies inessential player independence, requiring that payoffs of essential players do not depend on the presence of inessential players (instead of requiring that payoffs of relevant players do not depend on the presence of irrelevant players). Let $Iness(N, v, D)$ be the set of inessential players in $(N, v, D)$.

**Inessential player independence** For every $(N, v, D) \in \mathcal{G}_{PS}$, it holds that $f_i(N, v, D) = f_i(N', v_{N'}, D(N'))$ for $i \in N'$, with $N' = N \setminus Iness(N, v, D)$.

In a similar way, the irrelevant player property can be modified. This gives the inessential player property that requires that a null player whose subordinates are all null players, earns a zero payoff. Similar as in Proposition 4.1, efficiency and inessential player independence imply the inessential player property.

Note that, similar to the irrelevant player property, the inessential player property is weaker than the null player property. The null player property deals with all players who are null players in the game, while the inessential player property only takes care of the null players whose subordinates are also null players. For permission tree games, van den Brink et al. (2015) deal with this by the axiom which requires that the payoff distribution does not change if a predecessor $i$ becomes necessary for its successor $j$ in the sense that the marginal contribution of player $j$ to every coalition that does not

[11] For games with a permission structure, these axioms are defined the same as for games under precedence constraints, by simply replacing the domain $\mathcal{G}_{PC}$ by the domain $\mathcal{G}_{PS}$ in the definitions in the previous sections.

[12] This is a weaker version of *necessary player symmetry* used by van den Brink et al. (2015) for the more specific permission tree games, requiring that all necessary players in the game earn the same payoff irrespective of their position in the digraph.

[13] Also stating these axioms for games with a permission structure, we can simply replace the domain $\mathcal{G}_{PC}$ by the domain $\mathcal{G}_{PS}$, but we also need to redefine what is a null player as a player whose marginal contribution is zero to any coalition in $2^N$.

contain player $i$ becomes zero. For game $(N, v)$ and players $i, j \in N$, the game $(N, v_j^i)$ is defined by $v_j^i(S) = v(S \setminus \{j\})$ for all $S \subseteq N \setminus \{i\}$, and $v_j^i(S) = v(S)$ otherwise.

**Predecessor necessity** For every $(N, v, D) \in \mathcal{G}_{PS}$ and $i, j \in N$ such that $(i, j) \in D$, it holds that $f(N, v, D) = f(N, v_j^i, D)$.

Although predecessor necessity is used for axiomatization on permission tree games, it is also satisfied for all games with an acyclic permission structure.

Next, we give a new characterization of the conjunctive permission value that uses similar axioms as used to characterize the precedence Shapley value and the hierarchical solution for games under precedence constraints.

**Theorem 4.6** *A solution for games with acyclic permission structure is equal to the conjunctive permission value $\varphi^c$ if and only if it satisfies efficiency, linearity, inessential player independence, predecessor necessity and equal-strength.*

*Proof.* It is straightforward to verify that the conjunctive permission value satisfies the five axioms.

To prove uniqueness, suppose that solution $f$ for games with an acyclic permission structure satisfies the five axioms. Consider any $(N, D) \in \mathcal{D}$ and $\emptyset \neq T \subseteq N$.

Efficiency and inessential player independence imply the inessential player property, and thus

$$f_i(N, u_T, D) = 0 \text{ for all } i \in N \setminus (T \cup \widehat{P}_D(T)). \tag{4.11}$$

Repeated application of predecessor necessity implies that $f(N, u_T, D) = f(N, u_{T \cup \widehat{P}_D(T)}, D)$.

Inessential player independence implies that for all $i \in T \cup \widehat{P}_D(T)$

$$f_i(N, u_{T \cup \widehat{P}_D(T)}, D) = f_i(T \cup \widehat{P}_D(T), u_{T \cup \widehat{P}_D(T)}, D(T \cup \widehat{P}_D(T))). \tag{4.12}$$

Equal strength implies that there is an $\alpha \in \mathbb{R}$ such that

$$f_i(T \cup \widehat{P}_D(T), u_{T \cup \widehat{P}_D(T)}, D(T \cup \widehat{P}_D(T))) = \alpha \text{ for all } i \in T \cup \widehat{P}_D(T). \tag{4.13}$$

Efficiency then implies that $\alpha = \frac{1}{|T \cup \widehat{P}_D(T)|}$.

With (4.12) it then follows for all $i \in T \cup \widehat{P}_D(T)$,

$$f_i(N, u_{T \cup \widehat{P}_D(T)}, D) = f_i(T \cup \widehat{P}_D(T), u_{T \cup \widehat{P}_D(T)}, D(T \cup \widehat{P}_D(T))) = \frac{1}{|T \cup \widehat{P}_D(T)|}. \tag{4.14}$$

With (4.11) then $f(N, u_T, D)$ is determined.

Since efficiency and inessential player independence imply the inessential player property, for a null game $v^0(S) = 0$ for all $S \subseteq N$, and thus we have that $f_i(N, v^0, D) = 0$ for all $i \in N$ and $(N, D) \in \mathcal{D}$.

Since $f$ satisfies linearity, the solution $f(N, v, D)$ is uniquely determined and coincides with the conjunctive permission value, for any $(N, v, D) \in \mathcal{G}_{PS}$.
■

Now, an obvious next question is if we can generalize the equal strength to $p$-strength also in this context. This can be done, and it yields the following class of solutions where the Harsanyi dividends in the conjunctive restricted game are allocated proportional to a network power measure.

**Definition 4.2** *For positive power measure $p$, the $p$-permission value is the solution for games with a permission structure given by*

$$\begin{aligned} \widehat{H}_i^p(N, v, D) &= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{p_i(S, D(S))}{\sum_{j \in S} p_j(S, D(S))} \Delta_{r^c_{v,D}}(S) \\ &= \sum_{\substack{S \in \Phi^c(N,D) \\ i \in S}} \frac{p_i(S, D(S))}{\sum_{j \in S} p_j(S, D(S))} \Delta_{r^c_{v,D}}(S) \text{ for all } i \in N. \end{aligned} \tag{4.15}$$

A main difference with the precedence power solutions is that now we consider the Harsanyi dividends in the conjunctive restricted game $r^c_{v,D}$ instead of the original game $v$ on the domain.

**Theorem 4.7** *A solution for games with an acyclic permission structure is equal to the $p$-permission value $\widehat{H}^p$ if and only if it satisfies efficiency, linearity, inessential player independence, predecessor necessity and $p$-strength.*

The proof of uniqueness follows straightforward from the proof of Theorem 4.6 by replacing (4.13), which followed from equal strength, applying $p$-strength:

$$p_i(N, D) f_j(N, u_N, D) = p_j(N, D) f_i(N, u_N, D),$$

when $(N, D) = (T \cup \widehat{P}_D(T), D(T \cup \widehat{P}_D(T)))$. Together with efficiency, this determines the payoffs in $f(T \cup \widehat{P}_D(T), u_{T \cup \widehat{P}_D(T)}, D(T \cup \widehat{P}_D(T)))$. We will refer to the solutions characterized in Theorem 4.7 as *permission power solutions*. Logical independence of the axioms is again shown in Section 4.5.

An interesting question is now also to see which precedence power solution on $\mathcal{G}_{PC}$ satisfies equal strength, i.e., the $p$-strength axiom with the equal power measure that is used to characterize the conjunctive permission value on $\mathcal{G}_{PS}$ in Theorem 4.6. It turns out that this gives essentially the Shapley value, i.e., the solution $\widetilde{Sh}$ on $\mathcal{G}_{PC}$ that to every game under precedence constraints assigns the Shapley value of the unrestricted game (extended to the power set of $N$), i.e.,

$$\widetilde{Sh}_i(N, v, D) = Sh(N, \overline{v}) \text{ for all } (N, v, D) \in \mathcal{G}_{PC},$$

where $\overline{v} \in \mathcal{G}^N$ is given by $\Delta_{\overline{v}}(S) = \Delta_v^D(S)$ if $S \in \Phi^p(N, D)$, and $\Delta_{\overline{v}}(S) = 0$ if $S \notin \Phi^p(N, D)$.

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## 4.5 Logical Independence

In this section, we present alternative solutions for games with a hierarchy that show logical independence of the axioms in the main theorems in Section 4.4.

### 4.5.1 Logical Independence of the Axioms in Theorem 4.5

The following alternative solutions each satisfy all but one of the axioms in Theorem 4.5.

1. Consider the solution $f^{zero}$ on $\mathcal{G}_{PC}$ that always assigns zero payoff to every player in every game under precedence constraint, i.e.,

$$f_i^{zero}(N, v, D) = 0 \text{ for all } i \in N \text{ and } (N, v, D) \in \mathcal{G}_{PC}.$$

   This solution satisfies all axioms of Theorem 4.5 except efficiency.

2. For positive power measure $p$, consider the solution $f$ on $\mathcal{G}_{PC}$ that allocates the dividend of every coalition proportional to the power measure $p$ among *all* relevant players, i.e., for all $i \in N$ and $(N, v, D) \in \mathcal{G}_{PC}$

$$f_i(N, v, D) = H_i^p(N, v(N)u_{\bigcup_{\{T \in \Phi^p(N,D), \Delta_v^D(T) \neq 0\}} T}, D).$$

   This solution satisfies all axioms of Theorem 4.5 except linearity.

3. For positive power measure $p$, consider the solution $f$ on $\mathcal{G}_{PC}$ that allocates the dividends of every coalition in every game under precedence constraints proportional to the power values $p(N, D)$, i.e., for positive power measure $p$,

$$f_i^p(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{p_i(N, D)}{\sum_{j \in S} p_j(N, D)} \Delta_v^D(S) \text{ for all } i \in N.$$

   Compared to the precedence power solutions, these solutions allocate the dividend of coalition $S$ proportional to the power values $p(N, D)$ in the original digraph instead of the power values $p(S, D(S))$ in the subgraphs on $S$. This solution satisfies all axioms of Theorem 4.5 except irrelevant player independence.

4. Let the equal network power measure $\gamma$ be given by $\gamma_i(N,D) = \frac{1}{|N|}$ for all $i \in N$ and $(N,D) \in \mathcal{D}$. For $p \neq \gamma$, consider the solution $\widetilde{Sh} = H^{\gamma}$ on $\mathcal{G}_{PC}$. This solution satisfies all axioms of Theorem 4.5 except $p$-strength. For $p = \gamma$, the hierarchical solution satisfies all axioms of Theorem 4.5 except $\gamma$-strength.

### 4.5.2 Logical Independence of the Axioms in Theorem 4.7

The following alternative solutions each satisfy all but one of the axioms in Theorem 4.7.

1. Solution $f^{zero}$ assigning zero payoff to every player in every game with a permission structure satisfies all axioms of Theorem 4.7 except efficiency.

2. For positive power measure $p$, consider the solution $f$ for games with a permission structure that allocates the dividend of every coalition in the conjunctive restricted game proportional to the power measure $p$ among *all* essential players, i.e., for all $i \in N$ and $(N,v,D) \in \mathcal{G}_{PS}$,

$$f_i(N,v,D) = \widehat{H}_i^p(N, v(N)u_{\bigcup_{\{T\subseteq N, \Delta_{r^c_{v,D}}(T)\neq 0\}} T}, D).$$

   This solution satisfies all axioms of Theorem 4.7 except linearity.

3. For positive power measure $p$, consider the solution $f$ for games with a permission structure that allocates the dividends of every coalition in the conjunctive restricted game proportional to the power values $p(N,D)$, i.e., for positive power measure $p$,

$$f_i^p(N,v,D) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{p_i(N,D)}{\sum_{j\in S} p_j(N,D)} \Delta_{r^c_{v,D}}(S) \text{ for all } i \in N.$$

   This solution satisfies all axioms of Theorem 4.7 except inessential player independence.

4. For positive power measure $p$, consider the solution that allocates the dividend of every coalition $S$ in the original game proportional to the power measure $p(S, D(S))$ in the subgraph on $S$, i.e.,

$$\widetilde{H}_i^p(N,v,D) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{p_i(S,D(S))}{\sum_{j\in S} p_j(S,D(S))} \Delta_v(S) \text{ for all } i \in N.$$

   This solution satisfies all axioms of Theorem 4.7 except predecessor necessity.

5. For $p \neq \gamma$, the conjunctive permission value satisfies all axioms of Theorem 4.7 except $p$-strength. For $p = \gamma$, the $p$-permission power value with $p = h$ being the hierarchical strength satisfies all axioms of Theorem 4.7 except $\gamma$-strength.

## 4.6 Conclusions

The goal of this chapter is to review and compare two well-known approaches to games with a hierarchy in the literature: The permission structure approach and the precedence constraint approach. Moreover, by a new axiomatization of the conjunctive permission value, we could extend this solution and its axiomatization to define the new class of permission power solutions which is characterized by axioms that make it comparable with the class of precedence power solutions.

There are several extensions of the model that can be considered. For example, instead of digraphs, other combinatorial structures might represent some relational structure among the players. A 'natural' extension of games with a permission structure are games on *antimatroids*, see Algaba et al. (2003, 2004b). Antimatroids are combinatorial structures introduced by Dilworth (1940), see also Edelman and Jamison (1985) which, besides permission structures, also generalize other models such as *ordered partition voting* where players are partitioned into levels, and a coalition in a certain level can be active only if a majority of players in higher levels approve. Since antimatroids are union closed (i.e., the union of any two feasible coalitions is also feasible), a similar approach for games with a permission structure can be followed by defining a restricted game that assigns to every coalition the worth of its largest feasible subset in the original game, and applying the Shapley value (or any other TU-game solution) to this restricted game. Different extensions of the Shapley value for games on union closed systems are considered in van den Brink et al. (2011b). An even more general model is games on union stable systems, see Algaba et al. (2000, 2001a), and Algaba et al. (2001b, 2004a), where feasibility of the union of two feasible coalitions is only required if the two coalitions have a non-empty intersection, which reflects the communication feature. In this framework, Algaba et al. (2015) applied power measures to distribute dividends in games on union stable systems extending some results given in Algaba et al. (2012) about the Myerson and position values. Network structures taking into account both hierarchical and communication features have been introduced in Algaba et al. (2018). A 'natural' structure to extend games under precedence constraints is augmenting systems (see Bilbao (2003) and Algaba et al. (2010)) and regular set systems (see Honda and Grabisch (2006) and Lange and Grabisch (2009)).

In this chapter, (i) we recall one axiomatization of the conjunctive permission value, and one from the precedence Shapley value from the literature, (ii) we applied network power measures to define two classes of solutions from them, and (iii) developed these axiomatizations further into two comparable axiomatizations of these two classes. Further work can be done to see if other types of axioms can be part of comparable axiomatizations, such as axiomatizations using some type of fairness axiom (see van den Brink (1997) for games with a permission structure, Algaba et al. (2003) for games on antimatroids, and Algaba et al. (2001a) for games on union stable systems). Further research will include the introduction and analysis of the disjunctive permission approach in the context of games under precedence constraints.

The connection between game theoretic payoff allocation and social network power measures gives insight in different solutions which might be helpful in applications. In this chapter, on one hand, within each of these two classes of solutions, different network power measures yield different solutions from the class. But on the other hand, taking one specific power measure we obtain two solutions, one from each class.

In this chapter, we applied power measures to define solutions for cooperative games with a hierarchical network structure. The other way around, solutions for games on (directed or undirected) networks can be used to measure power or centrality in networks. Taking a (symmetric) game and applying a solution for graph games yield such a power measure. For example, Gómez et al. (2003) apply the Myerson value for communication graph games to certain symmetric games restricted on an undirected graph in the sense of Myerson (1977), measuring different types of centrality in undirected graphs. Similarly, we can apply the precedence power or permission power solutions to games with a hierarchical structure to obtain power measures for acyclic digraphs. Obviously, when we use the unanimity game of the grand coalition, we get the same power measure that we use for the solution. But, as done in Gómez et al. (2003) for undirected graphs, other games can be used.

Without going into power measurement, notice that network centrality has a very different effect in the precedence approach than in the permission approach. Consider a strict hierarchical network that is represented by a linear order $\{(i_k, i_{k+1}) \mid k = 1, \ldots n-1\}$. Both in the permission as well as the precedence approach, the top player $i_1$ is 'powerful' in obtaining a share in the worth of the game. However, there is a difference considering the networks $\Phi^c(N,D)$ and $\Phi^p(N,D)$. Since $\Phi^c(N,D) = \{\{i_1, \ldots, i_l\} \mid l = 1, \ldots, n\}$, the top player $i_1$ can be considered to be the most central since it belongs to every feasible coalition. However, since $\Phi^p(N,D) = \{\{i_l, \ldots, i_n\} \mid l = 1, \ldots, n\}$, in the precedence approach the top player $i_1$ seems to be the least central since the only feasible coalition it belongs to is the grand coalition. On the other hand, the bottom player $i_n$ seems to be the most central and belongs to every feasible coalition. Notice that in the precedence approach, being a bottom player means that you are feasible as a singleton, but then you can be the first player to enter in an admissible permutation, and if you have a predecessor in the hierarchy, you will never be the last to enter which is disadvantageous in, for example, convex games. On the other hand, being in few feasible coalitions means that you more often enter as the last player, which in many games (in particular in convex games) gives you a benefit.

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## 4.7 Acknowledgments

We are grateful to a referee for useful suggestions to improve this chapter. Financial support from the Ministerio de Economía y Competitividad (MINECO) of Spain and FEDER funds under project MTM2014-54199-P is gratefully acknowledged.

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# Chapter 5

## Values, Nullifiers and Dummifiers

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### 5.1 Introduction

In game theory, the axiomatic approach to a solution $\mathcal{S}$ for a particular problem consists of identifying a collection of properties that a sensible solution for the problem should satisfy and, then, proving that $\mathcal{S}$ is the unique solution that satisfies those properties. The axiomatic approach is a useful tool to understand better the available solutions to address a particular problem; the game theorist can use this approach to identify the most appropriate solution, simply identifying that one whose supporting properties are more adequate for the situation at hand.

In general, there can be several axiomatic characterizations of a particular solution. For instance, a number of axiomatic characterizations of the Shapley value of TU-games have been proposed in the literature: The seminal one in [8], one without the additivity property in [10], one based on the potential approach in [6], etc. Probably the most common way to introduce axiomatically the Shapley value is to use the properties of efficiency, symmetry, additivity and null player (see for example [7]).

In [4] the *nullifying players* are defined; a nullifying player is one such that all the coalitions containing him have zero worth (i.e., the characteristic function of the TU-game maps every coalition containing a nullifying player to zero). Recently, some properties in relation with nullifying players have been proposed and their relations with well-known values of TU-games have been explored. In [9] the *nullifying players get nothing* property was introduced: A value satisfies this property if it allocates zero to any nullifying player of any TU-game. Moreover, [9] proves that the equal division value is the unique value for TU-games that satisfies efficiency, symmetry, additivity and nullifying players gets nothing. In [1] two other properties concerning nullifying players are studied. First, the *nullifying players pay for the mean* property states that a value must allocate to any nullifying player of any TU-game minus the average worth nullified by him. Second, the *nullifying players pay for the weighted mean* property states that a value must allocate to any nullifying player of any TU-game minus the weighted average worth nullified by him, the weights being related to the coalition sizes (this property had already been introduced in [2] for the so-called games with optimistic aspirations, a kind of generalization of TU-games). Besides, [1] characterizes the Shapley value using the nullifying player pays for the weighted mean property, and the Banzhaf value using the nullifying player pays for the mean property.

Later on, the concept of dummifying players is introduced in [3] to illustrate the difference between the Shapley value, the equal division value and the equal surplus division value. A dummifying player is one such that each coalition containing him has a worth equal to the sum of the individual payoffs of its members. The authors also introduce the *dummifying players get their individual payoffs* property; a value satisfies this property if it allocates his individual payoff to any dummifying player of any TU-game. Moreover, [3] proves that the equal surplus division value is the unique value for TU-games that satisfies efficiency, symmetry, additivity and dummifying players get their individual payoffs.

In this chapter we first provide new axiomatic characterizations of the Shapley and Banzhaf values using properties concerning nullifying players (in Section 5.2). Then, in Section 5.3 we introduce a novel value for TU-games using an axiomatic approach: The e-Banzhaf value. Specifically, we take the properties of efficiency, symmetry, additivity, and combine them with the property for nullifying players that we consider most natural: The nullifying players pay for the mean property. We prove that these four properties characterize the e-Banzhaf value. This novel value possesses some properties of the Shapley value and some of the Banzhaf value, and can be seen as a kind of efficient variation of the Banzhaf value. However, it is not invariant to $S$-equivalence. In Section 5.4 we first provide new axiomatic characterizations of the Shapley and Banzhaf values using properties concerning dummifying players. We also show that the e-Banzhaf value does not satisfy the property for dummifying players that we consider most natural: The dummifying players pay for the mean property. Finally, in Section 5.5 we introduce an alternative novel value for TU-games: The ie-Banzhaf value. It is the unique value for TU-games that satisfies efficiency, symmetry, additivity and the dummifying players pay for the mean property. We prove that the ie-Banzhaf value is a kind of efficient variation of the Banzhaf value and that it is invariant to $S$-equivalence. To finish this chapter, we include a section of conclusions.

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## 5.2 Axiomatic Characterizations and Nullifying Players

A cooperative game with transferable utility (in short, a TU-game) is given by a finite set of players $N$, with cardinality $n$, and a map $v : 2^N \to \mathbb{R}$ that satisfies $v(\emptyset) = 0$; $v$ is called the characteristic function of the game. Let $(N, v)$ be a TU-game; we say that

- $i, j \in N$ are symmetric players in $(N, v)$ if $v(S \cup i) = v(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$ (notice that we denote singletons $\{i\}$ and $\{j\}$ simply as $i$ and $j$),
- $i \in N$ is a null player in $(N, v)$ if $v(S \cup i) = v(S)$ for every $S \subseteq N \setminus i$,
- $i \in N$ is a nullifying player in $(N, v)$ if $v(S) = 0$ when $i \in S$.

A value for TU-games is a map $f$ that assigns to each TU-game $(N, v)$ a vector $(f_i(N, v))_{i \in N} \in \mathbb{R}^N$. Two important values for TU-games are the Shapley value $\varphi$ and the Banzhaf value $\beta$ defined below:

$$\varphi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n - s - 1)!}{n!} (v(S \cup i) - v(S)),$$

for every TU-game $(N, v)$ and every $i \in N$, where $s$ denotes the cardinality of $S$, and

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} (v(S \cup i) - v(S)),$$

for every TU-game $(N, v)$ and every $i \in N$.

We list now a collection of properties that have been used to provide characterizations of the Shapley and Banzhaf values. Let $f$ be a value for TU-games.

**Efficiency (EFF)**. $f$ satisfies efficiency if for any TU-game $(N,v)$, it holds that

$$\sum_{i\in N} f_i(N,v) = v(N).$$

**Total power (TP)**. $f$ satisfies total power if for any TU-game $(N,v)$, it holds that

$$\sum_{i\in N} f_i(N,v) = \frac{1}{2^{n-1}} \sum_{i\in N} \sum_{S\subseteq N\setminus i} (v(S\cup i) - v(S)).$$

**Symmetry (SYM)**. $f$ satisfies symmetry if for any TU-game $(N,v)$ and for all pair of symmetric players $i,j\in N$ in $(N,v)$ it holds that

$$f_i(N,v) = f_j(N,v).$$

**Null player (NP)**. $f$ satisfies null player if for any TU-game $(N,v)$ and for any null player $i\in N$ in $(N,v)$ it holds that

$$f_i(N,v) = 0.$$

**Additivity (ADD)**. $f$ satisfies additivity if for any pair of TU-games $(N,v),(N,w)$ and for any player $i$, it holds that

$$f_i(N,v+w) = f_i(N,v) + f_i(N,w).$$

The Shapley value is the unique value for TU-games satisfying efficiency, symmetry, null player and additivity (see for example [7]). A similar characterization of the Banzhaf value is obtained changing efficiency by total power; i.e., the Banzhaf value is the unique value for TU-games satisfying total power, symmetry, null player and additivity (see [5]).

Recently, some properties in relation with nullifying players have been proposed and their relations with well-known values of TU-games have been explored. In [9] the following property was introduced.

**Nullifying players get nothing (NPN)**. $f$ satisfies nullifying players get nothing if for any TU-game $(N,v)$ and for any nullifying player $i\in N$ in $(N,v)$ it holds that

$$f_i(N,v) = 0.$$

In words, a value satisfies nullifying players get nothing if it allocates zero to any nullifying player of any TU-game. To show the relevancy of this property, we define below the equal division value $\delta$:

$$\delta_i(N,v) = \frac{v(N)}{n},$$

for every TU-game $(N,v)$ and every $i\in N$. [9] proves that commuting the properties null player and nullifying players get nothing, two similar axiomatic characterizations of the Shapley and equal division values are obtained. More specifically, [9] shows that the equal division value is the unique value for TU-games satisfying efficiency, symmetry, nullifying players get nothing and additivity.

In [1] two other properties concerning nullifying players are studied. First, the *nullifying players pay for the mean* property states that a value must allocate to any nullifying player of any TU-game minus the average worth nullified by him. This property adapts to TU-games a similar property introduced in [2] for the so-called games with optimistic aspirations, a kind of generalization of TU-games. Second, the *nullifying players pay for the weighted mean* property states that a value must allocate to any nullifying player of any TU-game minus the weighted average worth nullified by him, the weights being related to the coalition sizes.

**Nullifying players pay for the mean (NPM)**. $f$ satisfies nullifying players pay for the mean if for any TU-game $(N, v)$ and for any nullifying player $i \in N$ in $(N, v)$ it holds that

$$f_i(N, v) = -\frac{1}{2^{n-1}} \sum_{S \subseteq N} v(S).$$

**Nullifying players pay for the weighted mean (NPWM)**. $f$ satisfies nullifying players pay for the weighted mean if for any TU-game $(N, v)$ and for any nullifying player $i \in N$ in $(N, v)$ it holds that

$$f_i(N, v) = -\frac{1}{n} \sum_{S \subseteq N} \frac{1}{\binom{n-1}{s}} v(S).$$

The two properties above are used in [1] to provide characterizations of the Shapley and Banzhaf values (jointly with additivity and other properties concerning the so-called necessary players). Now we present novel characterizations of the Shapley and Banzhaf values that follow easily from the results in [2] and [1]. For the proofs we use that the set of characteristic functions of TU-games with player set $N$ is a vector space whose canonical basis is $\{e_S\}_{S \in 2^N \setminus \emptyset}$, where $e_S(S) = 1$ and $e_S(T) = 0$ for all $T \subseteq N$ with $T \neq S$.

**Theorem 5.1** *The Shapley value is the unique value for TU-games that satisfies efficiency, symmetry, nullifying players pay for the weighted mean and additivity.*

*Proof.* It is well known that the Shapley value satisfies efficiency, symmetry and additivity. According to [1], it also satisfies nullifying players pay for the weighted mean. It is easy to check that efficiency, symmetry and nullifying players pay for the weighted mean characterize the Shapley value within the class of TU-games corresponding to the elements of the canonical basis of the set of characteristic functions of TU-games with player set $N$. Then, additivity implies the uniqueness. ■

**Theorem 5.2** *The Banzhaf value is the unique value for TU-games that satisfies total power, symmetry, nullifying players pay for the mean and additivity.*

*Proof.* It is well known that the Banzhaf value satisfies total power, symmetry and additivity. According to [1], it also satisfies nullifying players pay for the mean. It is easy to check that total power, symmetry and nullifying players pay for the mean characterize the Banzhaf value within the class of TU-games corresponding to the elements of the canonical basis of the set of characteristic functions of TU-games with player set $N$. Then, additivity implies the uniqueness. ∎

From the point of view of fairness, which of the three properties for nullifying players discussed above is the most natural? It is true that only nullifying players get nothing is respectful with individual rationality for the nullifying players, because if $i \in N$ is a nullifying player in a TU-game $(N, v)$, then $v(i) = 0$ and nullifying players pay for the weighted mean and nullifying players pay for the mean imply that all nullifying players receive negative allocations. However, individual rationality has more to do with stability than with fairness. It is also true that a nullifying player will somehow be forced to leave the game if a value satisfying nullifying players pay for the weighted mean or nullifying players pay for the mean is used. Nevertheless our question is about fair allocations of $v(N)$ among the *fixed* players of a game, and not about allocations of $v(N)$ that will encourage the players to stay or not in the game. From this point of view of fairness, we believe that nullifying players pay for the weighted mean and nullifying players pay for the mean are preferable to nullifying players get nothing. But once we have decided that nullifying players have to return the value they nullify, it seems most natural to treat all the coalitions equally, i.e., we prefer nullifying players pay for the mean to nullifying players pay for the weighted mean. In the next section, we introduce a novel value for TU-games that satisfies nullifying players pay for the mean and, moreover, is efficient.

## 5.3 The e-Banzhaf Value for TU-Games

We start this section proving that efficiency, symmetry, nullifying players pay for the mean and additivity characterize a unique value for TU-games.

**Theorem 5.3** *There exists a unique value $\sigma$ for TU-games that satisfies efficiency, symmetry, nullifying players pay for the mean and additivity. It is given by:*

$$\sigma_i(N, v) = \frac{1}{2^{n-1}} \left( \sum_{S \subset N, i \in S} \frac{n-s}{s} v(S) - \sum_{S \subset N, i \notin S} v(S) \right) + \frac{v(N)}{n} \tag{5.1}$$

*for every TU-game $(N, v)$ and every $i \in N$.*

*Proof.* (Existence). It is clear that $\sigma$ satisfies nullifying players pay for the mean and additivity. To check that it satisfies efficiency notice that, for every TU-game $(N, v)$,

$$\begin{aligned}\sum_{i\in N}\sigma_i(N,v) &= \frac{1}{2^{n-1}}\sum_{i\in N}\left(\sum_{S\subset N, i\in S}\frac{n-s}{s}v(S) - \sum_{S\subset N, i\notin S}v(S)\right) + v(N)\\ &= \frac{1}{2^{n-1}}\left(\sum_{S\subset N}s\frac{n-s}{s}v(S) - \sum_{S\subset N}(n-s)v(S)\right) + v(N)\\ &= v(N).\end{aligned}$$

To check that $\sigma$ satisfies symmetry, take a TU-game $(N, v)$ and a pair of symmetric players in $(N, v)$ $i, j \in N$. Notice that:

$$\sum_{S\subset N, i\in S}\frac{n-s}{s}v(S) - \sum_{S\subset N, i\notin S}v(S) =$$

$$\sum_{S\subseteq N\setminus i}\frac{n-s-1}{s+1}v(S\cup i) - \sum_{S\subseteq N\setminus i}v(S) =$$

$$\sum_{S\subseteq N\setminus\{i,j\}}\left(\frac{n-s-1}{s+1}v(S\cup i) + \frac{n-s-2}{s+2}v(S\cup i\cup j)\right) - \sum_{S\subseteq N\setminus\{i,j\}}\left(v(S) + v(S\cup j)\right).$$

Now, since $i, j$ are symmetric in $(N, v)$, the last expression is equal to:

$$\sum_{S\subseteq N\setminus\{i,j\}}\left(\frac{n-s-1}{s+1}v(S\cup j) + \frac{n-s-2}{s+2}v(S\cup i\cup j)\right) - \sum_{S\subseteq N\setminus\{i,j\}}\left(v(S) + v(S\cup i)\right),$$

and then it is clear that $\sigma_i(N, v) = \sigma_j(N, v)$.

(Uniqueness). Take $f$, a value for TU-games that satisfies efficiency, symmetry, nullifying players pay for the mean and additivity and take a TU-game $(N, v)$. We will prove that $f(N, v) = \sigma(N, v)$. Consider the canonical basis of the vector space of characteristic functions of TU-games with set of players $N$: $\{e_S\}_{S\in 2^N\setminus\emptyset}$. $v$ can be written in a unique way as a linear combination of the elements of the canonical basis: $v = \sum_{S\in 2^N\setminus\emptyset} v(S)e_S$. Since $f$ satisfies additivity,

$$f(N,v) = \sum_{S\in 2^N\setminus\emptyset} f(N, v(S)e_S).$$

Notice that efficiency, symmetry and nullifying players pay for the mean characterize a unique value in the class of games $\{(N, v(S)e_S) \mid S \subseteq N, S \neq \emptyset\}$. Hence, $f(N, v) = \sigma(N, v)$. ∎

**Example 5.1** *Consider the TU-game $(N, v)$ with*

- $N = \{1, 2, 3\}$,
- $v(1) = v(2) = v(3) = v(23) = 0$, $v(12) = v(13) = v(N) = 1$.

*$(N, v)$ is the glove game. It is easy to check that $\sigma(N, v) = (\frac{7}{12}, \frac{5}{24}, \frac{5}{24})$. Notice that the Shapley value for this game is $(\frac{8}{12}, \frac{1}{6}, \frac{1}{6})$, which means that the Shapley value and our novel value behave in a similar way in the glove game, although the Shapley value is slightly more beneficial for the player possessing the scarce glove.*

Comparing Theorem 5.2 and Theorem 5.3, it is clear that $\sigma$ is a kind of efficient version of the Banzhaf value; from now on, we refer to $\sigma$ as the *e-Banzhaf* value. In view of the similarities between the Banzhaf and the e-Banzhaf values, we provide an expression that relates them. Take a TU-game $(N, v)$ and a player $i \in N$. Then,

$$\begin{aligned}
\sigma_i(N, v) &= \frac{1}{2^{n-1}} \left( \sum_{S \subset N, i \in S} \frac{n-s}{s} v(S) - \sum_{S \subset N, i \notin S} v(S) \right) + \frac{v(N)}{n} \\
&= \frac{1}{2^{n-1}} \left( \sum_{S \subseteq N \setminus i} \frac{n-s-1}{s+1} v(S \cup i) - \sum_{S \subseteq N \setminus i} v(S) \right) + \frac{v(N)}{n} \\
&= \frac{1}{2^{n-1}} \left( \sum_{S \subseteq N \setminus i} (v(S \cup i) - v(S)) - \sum_{S \subseteq N \setminus i} (2 - \frac{n}{s+1}) v(S \cup i) \right) \\
&\quad + \frac{v(N)}{n} \\
&= \beta_i(N, v) - \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} (2 - \frac{n}{s+1}) v(S \cup i) + \frac{v(N)}{n}.
\end{aligned}$$

It is interesting to note that the e-Banzhaf value is an ideal value in the sense of [11]. In fact, it is proved in [11] that a value belongs to the family of ideal values if and only if it is linear and it satisfies efficiency, symmetry and the property coalitional monotonicity given below.

**Coalitional monotonicity (CMON).** $f$ satisfies coalitional monotonicity if for any pair of TU-games $(N, v)$ and $(N, w)$ fulfilling that there exists $T \subseteq N$ with $v(T) > w(T)$ and $v(S) = w(S)$ for all $S \subseteq N$, $S \neq T$, it holds that

$$f_i(N, v) \geq f_i(N, w).$$

for all $i \in T$.

In view of (5.1), it is clear that the e-Banzhaf value satisfies coalitional monotonicity. Since the e-Banzhaf value is linear and satisfies efficiency and symmetry, it is an ideal value.

Let us see now a new example concerning the e-Banzhaf value. It is somehow puzzling, although it eventually shows a property of the e-Banzhaf value that might be desirable in some circumstances.

**Example 5.2** *Consider the TU-game* $(N, v)$ *with*

- $N = \{1, 2, 3\}$,
- $v(1) = v(2) = 1$, $v(3) = 10$, $v(12) = 2$, $v(13) = v(23) = 11$, $v(N) = 12$.

*Notice that* $(N, v)$ *is an additive game and then*

$$I(N, v) = C(N, v) = \{(1, 1, 10)\},$$

*where* $I(N, v)$ *and* $C(N, v)$ *denote the set of imputations and the core of* $(N, v)$*, respectively. It is easy to check that*

$$\sigma(N, v) = (0.625, 0.625, 10.75) \notin I(N, v) = C(N, v).$$

*In principle, one might expect that a fair value proposes for this game the allocation* $(1, 1, 10)$*. However, notice that the e-Banzhaf value is rather severe with the nullifying players and this severity is somehow transferred to the players who contribute little. That is the reason why* $(0.625, 0.625, 10.75)$ *is proposed for the game in this example; it is as if the e-Banzhaf value is based on the principle, "those who contribute little can even lose part of what they contribute".*

Let us formulate now a property for the e-Banzhaf value that partially collects the intuition we obtained from Example 5.2. First, we remember the definition of dummy player and the dummy player property. We say that $i \in N$ is a dummy player in a TU-game $(N, v)$ if $v(S \cup i) = v(S) + v(i)$ for every $S \subseteq N \setminus i$.

**Dummy player (DP)**. A value for TU-games $f$ satisfies dummy player if for any TU-game $(N, v)$ and for any dummy player $i \in N$ in $(N, v)$ it holds that

$$f_i(N, v) = v(i).$$

Consider now a TU-game $(N, v)$ and assume that $i, j \in N$ are two dummy players in $(N, v)$ with $v(i) > v(j)$. If $f$ is a value satisfying the dummy player property, then $f_i(N, v) - f_j(N, v) = v(i) - v(j)$. Differently, the next theorem shows that if $n \geq 3$, then $\sigma_i(N, v) - \sigma_j(N, v) > v(i) - v(j)$.

**Theorem 5.4** *Consider a TU-game* $(N, v)$ *and assume that* $n \geq 3$ *and that* $i, j \in N$ *are two dummies in* $(N, v)$ *with* $v(i) > v(j)$*. Then,*

$$\sigma_i(N, v) - \sigma_j(N, v) > v(i) - v(j).$$

*Proof.* In the proof of Theorem 5.3 we showed that for every pair of different players $i, j$ in $(N, v)$ it holds that

$$\begin{aligned}\sigma_i(N,v) &= \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} \left(\frac{n-s-1}{s+1} v(S\cup i) + \frac{n-s-2}{s+2} v(S\cup i \cup j)\right) \\ &\quad - \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} (v(S) + v(S\cup j)) + \frac{v(N)}{n}.\end{aligned}$$

Since $i$ and $j$ are dummies in $(N, v)$, then

$$\begin{aligned}\sigma_i(N,v) &= \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} \frac{n-s-1}{s+1} (v(S) + v(i)) \\ &\quad + \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} \frac{n-s-2}{s+2} (v(S) + v(i) + v(j)) \\ &\quad - \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} (v(S) + v(S) + v(j)) + \frac{v(N)}{n}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sigma_i(N,v) - \sigma_j(N,v) &= \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} \frac{n-s-1}{s+1} (v(i) - v(j)) \\ &\quad + \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i,j\}} (v(i) - v(j)) \\ &= (v(i) - v(j)) \frac{2^{n-2} + \sum_{s=0}^{n-2} \binom{n-2}{s} \frac{n-s-1}{s+1}}{2^{n-1}}.\end{aligned}$$

Notice that

$$\sum_{s=0}^{n-2} \binom{n-2}{s} \frac{n-s-1}{s+1} = \sum_{s=0}^{n-2} \binom{n-2}{s} \frac{n}{s+1} - 2^{n-2} =$$

$$n \sum_{s=0}^{n-2} \frac{(n-1)!}{(s+1)!(n-s-2)!} \frac{1}{n-1} - 2^{n-2} = \frac{n}{n-1} \sum_{s=0}^{n-2} \binom{n-1}{s+1} - 2^{n-2} =$$

$$\frac{n}{n-1}(2^{n-1} - \binom{n-1}{0}) - 2^{n-2}.$$

Finally,

$$\frac{2^{n-2} + \sum_{s=0}^{n-2} \binom{n-2}{s} \frac{n-s-1}{s+1}}{2^{n-1}} = \frac{\frac{n}{n-1}(2^{n-1} - 1)}{2^{n-1}} = \frac{n}{n-1}(1 - \frac{1}{2^{n-1}}),$$

and the last expression is greater than one if $n \geq 3$; therefore, the proof is concluded. ■

We finish this section indicating a negative property of the e-Banzhaf value that is implied by Example 5.2. Remember that two TU-games with the same set of players $(N, v)$ and $(N, w)$ are said to be $S$-equivalent if there exist $a \in \mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}^N$ such that, for every $T \subseteq N$, it holds that

$$w(T) = av(T) + \sum_{j \in T} b_j.$$

Next we remember an important property of a value for TU-games.

**Invariance to $S$-equivalence (INV)**. A value for TU-games $f$ satisfies invariance to S-equivalence if for any pair of S-equivalent TU-games $(N, v)$ and $(N, w)$ such that $w(T) = av(T) + \sum_{j \in T} b_j$ for all $T \subseteq N$ (with $a \in \mathbb{R}$, $a > 0$ and $b \in \mathbb{R}^N$) it holds that, for every $i \in N$,

$$f_i(N, w) = af_i(N, v) + b_i.$$

It is easy to check that if a value $f$ for TU-games satisfies invariance to S-equivalence, efficiency and symmetry and $(N, v)$ is an additive TU-game, then $f(N, v) = (v(j))_{j \in N}$. Hence, in view of Example 5.2 and taking into account that the e-Banzhaf value satisfies efficiency and symmetry, it is clear that it does not satisfy invariance to S-equivalence. In Section 5.5 we introduce a variant of the e-Banzhaf value that satisfies invariance to S-equivalence.

---

## 5.4 Dummifying Players

In [3] the concept of dummifying players is introduced to illustrate the difference between the Shapley value, the equal division value and the equal surplus division value. We start this section by reminding you of this definition. Let $(N, v)$ be a TU-game. We say that $i \in N$ is a dummifying player in $(N, v)$ if $v(S) = \sum_{j \in S} v(j)$ for all $S \subseteq N$ with $i \in S$. We give now a property of a value $f$ for TU-games introduced in [3] concerning the dummifying players.

**Dummifying players get their individual payoffs (DPIP)**. $f$ satisfies dummifying players get their individual payoffs if for any game $(N, v)$ and for any dummifying player $i \in N$ in $(N, v)$ it holds that

$$f_i(N, v) = v(i).$$

The property dummifying players get their individual payoffs is used in [3] to characterize the equal surplus division value. Before introducing this value, we give a preliminary definition. Take a TU-game $(N, v)$; its corresponding zero normalized TU-game is $(N, v^0)$, where $v^0(S) = v(S) - \sum_{j \in S} v(j)$ for all $S \subseteq N$. Now, the equal surplus division value $ES$ for TU-games is given by:

$$ES_i(N, v) = v(i) + \frac{v^0(N)}{n}$$

for every TU-game $(N, v)$ and every $i \in N$. In [3] it is proven that the equal surplus division value is the unique value for TU-games satisfying efficiency, symmetry, dummifying players get their individual payoffs and additivity; remember that [9] shows that the equal division value is the unique value for TU-games satisfying efficiency, symmetry, nullifying players get nothing and additivity.

In the remainder of this section, we present some results relating the dummifying players with the Shapley, Banzhaf and e-Banzhaf values. To start with, we introduce two new properties concerning a value for TU-games that somehow adapt nullifying players pay for the mean and nullifying players pay for the weighted mean to dummifying players.

**Dummifying players pay for the mean (DPM)**. $f$ satisfies dummifying players pay for the mean if for any game $(N, v)$ and for any dummifying player $i \in N$ in $(N, v)$ it holds that

$$f_i\,(N, v) = v(i) - \frac{1}{2^{n-1}} \sum_{S \subseteq N \backslash i} \left( v(S) - \sum_{j \in S} v(j) \right).$$

**Dummifying players pay for the weighted mean (DPWM)**. $f$ satisfies dummifying players pay for the weighted mean if for any game $(N, v)$ and for any dummifying player $i \in N$ in $(N, v)$ it holds that

$$f_i\,(N, v) = v(i) - \frac{1}{n} \sum_{S \subseteq N \backslash i} \frac{1}{\binom{n-1}{s}} \left( v(S) - \sum_{j \in S} v(j) \right).$$

Next we give two new characterizations of the Shapley and Banzhaf values using the properties above.

**Theorem 5.5** *The Shapley value is the unique value for TU-games that satisfies efficiency, symmetry, dummifying players pay for the weighted mean and additivity.*

*Proof.* It is well known that the Shapley value satisfies efficiency, symmetry and additivity. Let us check that it satisfies dummifying players pay for the weighted mean. Take a TU-game $(N, v)$ and a dummifying player $i \in N$ in

$(N, v)$. Then,

$$\begin{aligned}
\varphi_i(N,v) &= \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}(v(S\cup i)-v(S)) \\
&= \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}\left(\sum_{j\in S\cup i} v(j)-v(S)\right) \\
&= v(i)-\frac{1}{n}\sum_{S\subseteq N\setminus i}\frac{1}{\binom{n-1}{s}}\left(v(S)-\sum_{j\in S}v(j)\right).
\end{aligned}$$

To prove the uniqueness, assume that $f$ is a value for TU-games satisfying efficiency, symmetry, dummifying players pay for the weighted mean and additivity. Take now a TU-game $(N, v)$. Notice that $v = v^0 + a^v$, where $a^v$ is the additive game given by $a^v(S) = \sum_{j\in S} v(j)$ for all $S \subseteq N$. It is clear that $v^0$ can be written in a unique way as a linear combination of the canonical basis of the vector space of characteristic functions of TU-games with player set $N$ as:

$$v^0 = \sum_{S\subseteq N, |S|>1} v(S)e(S).$$

Notice that efficiency, symmetry and dummifying players pay for the weighted mean characterize a unique value in the class of games $\{(N, v(S)e_S) \mid S \subseteq N, |S| > 1\}$ and that dummifying players pay for the weighted mean implies that $f(a^v) = \varphi(a^v) = (v(i))_{i\in N}$. Hence, $f(N, v) = \varphi(N, v)$. ■

**Theorem 5.6** *The Banzhaf value is the unique value for TU-games that satisfies total power, symmetry, dummifying players pay for the mean and additivity.*

*Proof.* It is well known that the Banzhaf value satisfies total power, symmetry and additivity. Let us check that it satisfies dummifying players pay for the mean. Take a TU-game $(N, v)$ and a dummifying player $i \in N$ in $(N, v)$. Then,

$$\begin{aligned}
\beta_i(N,v) &= \frac{1}{2^{n-1}}\sum_{S\subseteq N\setminus i}(v(S\cup i)-v(S)) \\
&= \frac{1}{2^{n-1}}\sum_{S\subseteq N\setminus i}\left(\sum_{j\in S\cup i} v(j)-v(S)\right) \\
&= v(i)-\frac{1}{2^{n-1}}\sum_{S\subseteq N\setminus i}\left(v(S)-\sum_{j\in S}v(j)\right).
\end{aligned}$$

The uniqueness is proven analogously to the uniqueness in Theorem 5.5 using total power and dummifying players pay for the mean instead of efficiency and dummifying players pay for the weighted mean. ■

Remember that the e-Banzhaf value is the unique value for TU-games that satisfies efficiency, symmetry, nullifying players pay for the mean and additivity. Nevertheless, the e-Banzhaf value does not satisfy dummifying players pay for the mean. To check it, just consider the game $(N, v)$ in Example 5.2. In $(N, v)$ all the players are dummifying and, thus, a value satisfying dummifying players pay for the mean associates to each player $i$ the amount $v(i)$. However,

$$\sigma(N, v) = (0.625, 0.625, 10.75) \neq (v(1), v(2), v(3)) = (1, 1, 10).$$

This feature suggests that a new value for TU-games can be defined using the properties efficiency, symmetry, dummifying players pay for the mean and additivity. We do it in the next section.

## 5.5 The ie-Banzhaf Value for TU-Games

We start this section proving that efficiency, symmetry, dummifying players pay for the mean and additivity characterize a unique value for the class of TU-games with set of players $N$.

**Theorem 5.7** *There exists a unique value $\rho$ for TU-games that satisfies efficiency, symmetry, dummifying players pay for the mean and additivity. It is given by:*

$$\rho_i(N, v) = \sigma_i(N, v^0) + v(i) \tag{5.2}$$

*for every TU-game $(N, v)$ and every $i \in N$.*

*Proof.* It is clear that $\rho$ satisfies additivity. Since $\sigma$ satisfies efficiency and symmetry, it is immediate to check that $\rho$ also satisfies efficiency and symmetry. To check that $\rho$ satisfies dummifying players pay for the mean, take a TU-game $(N, v)$ and a dummifying player $i \in N$ in $(N, v)$. Since $i$ is dummifying in $(N, v)$, then it is nullifying in $(N, v^0)$. Hence, taking into account that $\sigma$ satisfies nullifying players pay for the mean, it holds that:

$$\begin{aligned} \rho_i(N, v) = \sigma_i(N, v^0) + v(i) &= -\frac{1}{2^{n-1}} \sum_{S \subseteq N} v^0(S) + v(i) \\ &= v(i) - \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} \left( v(S) - \sum_{j \in S} v(j) \right). \end{aligned}$$

The uniqueness is proven analogously to the uniqueness in Theorem 5.5 using dummifying players pay for the mean instead of dummifying players pay for the weighted mean. ■

Comparing Theorems 5.6 and 5.7 it is clear that, like $\sigma$, $\rho$ is a kind of efficient version of the Banzhaf value. However, it is easy to check that, unlike $\sigma$, $\rho$ is invariant to $S$-equivalence. In fact, take a couple of $S$-equivalent TU-games $(N,v)$ and $(N,w)$ such that, for every $T\subseteq N$, $w(T)=av(T)+\sum_{j\in T}b_j$ ($a\in\mathbb{R}$ with $a>0$ and $b\in\mathbb{R}^N$). Notice that, for all $i\in N$, taking into account that $av^0=w^0$ it holds that

$$\begin{aligned} a\rho_i(N,v)+b_i &= a\left(\sigma_i(N,v^0)+v(i)\right)+b_i \\ &= \sigma_i(N,av^0)+av(i)+b_i \\ &= \sigma_i(N,w^0)+w(i)=\rho_i(N,w), \end{aligned}$$

and thus $\rho$ is invariant to $S$-equivalence. From now on, we refer to $\rho$ as the *ie-Banzhaf* value.

It is a well-known feature that the Shapley value is the unique value for TU-games satisfying efficiency, symmetry, dummy player and additivity (see for instance [7]). Then, since the ie-Banzhaf value is efficient, symmetric and additive, it is clear that it cannot satisfy dummy player. The following example illustrates this feature; it also shows a supperadditive TU-game for which the ie-Banzhaf value does not provide an imputation (i.e., it is not individually rational).

**Example 5.3** *Consider the TU-game* $(N,v)$ *with* $N=\{1,2,3,4\}$ $W^m=\{\{2,3\},\{2,4\},\{3,4\}\}$*, and*

$$v(S)=\begin{cases} 1 & \textit{if there exists } T\in W^m \textit{ such that } T\subseteq S, \\ 0 & \textit{otherwise.}\end{cases}$$

*It is easy to check that* $\rho(N,v)=(-\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8})$ *and, thus,* $\rho_1(N,v)<v(1)$ *(notice that* $1$ *is a dummy player in this game).*

One may wonder if the ie-Banzhaf value is an ideal value in the sense of [11]. The answer is negative because the ie-Banzhaf value does not satisfy coalitional monotonicity, as we check below. Notice that for every TU-game $(N,u)$ and for every $i\in N$, in view of (5.1) and (5.2), $\rho_i(N,u)$ can be written as:

$$\begin{aligned} \rho_i(N,u) &= \frac{1}{2^{n-1}}\sum_{S\subset N, i\in S}\frac{n-s}{s}\left(u(S)-\sum_{j\in S}u(j)\right) \\ &\quad -\frac{1}{2^{n-1}}\sum_{S\subset N, i\notin S}\left(u(S)-\sum_{j\in S}u(j)\right) \qquad (5.3) \\ &\quad +\frac{1}{n}\left(u(N)-\sum_{j\in N}u(j)\right)+u(i). \end{aligned}$$

Take now $(N, v)$, $(N, w)$ and $T \subseteq N$ as in the statement of coalitional monotonicity. Using (5.3), it can be checked that

1. If $T$ has two or more elements, then $\rho_i(N, v) \geq \rho_i(N, w)$ for all $i \in T$.
2. For every $i \in N$, if $T = \{i\}$ then
$$\rho_i(N, v) \geq \rho_i(N, w) \Longleftrightarrow n \leq 6.$$

Observe that item 2 implies that the ie-Banzhaf value does not satisfy coalitional monotonicity. Item 1 is clearly true. In order to check that item 2 is true, notice that, according to (5.3), the coefficients of $v(i)$ and $w(i)$ are identical and given by:

$$\begin{aligned}
& -\frac{1}{2^{n-1}} \sum_{S \subset N, i \in S, S \neq i} \frac{n-s}{s} - \frac{1}{n} + 1 \\
= & -\frac{1}{2^{n-1}} \sum_{s=2}^{n-1} \frac{n-s}{s} \binom{n-1}{s-1} - \frac{1}{n} + 1 \\
= & -\frac{1}{2^{n-1}} \sum_{s=2}^{n-1} \binom{n-1}{s} - \frac{1}{n} + 1 \\
= & -\frac{1}{2^{n-1}} (2^{n-1} - 1 - (n-1)) - \frac{1}{n} + 1 = \frac{n}{2^{n-1}} - \frac{1}{n}.
\end{aligned}$$

It is clear that $\frac{n}{2^{n-1}} - \frac{1}{n} \geq 0$ if and only if $n \leq 6$, which implies that item 2 is true.

---

## 5.6 Conclusions

In this chapter, we have provided new axiomatic characterizations for the Shapley and Banzhaf values using properties involving nullifying players (introduced in [4] and recently studied in [9] for the equal division value) or involving dummifying players (introduced and studied in [3] for the equal surplus division value). These new characterizations have prompted the introduction of two new values for TU-games: The e-Banzhaf value and the ie-Banzhaf value. Both are efficient variations of the Banzhaf value; the ie-Banzhaf value is also invariant to $S$-equivalence.

To conclude this chapter, we show in Table 5.1 a matrix summarizing its main results. The matrix rows display the main properties we have dealt with, and the matrix columns display the main values we have dealt with. A blank cell indicates that the corresponding property is not satisfied by the corresponding value. An asterisk in a cell indicates that the corresponding

| Properties | Shapley | Banzhaf | e-Banzhaf | ie-Banzhaf |
|---|---|---|---|---|
| EFF | 1,2 | | 1 | 1 |
| TP | | 1,2 | | |
| SYM | 1,2 | 1,2 | 1 | 1 |
| ADD | 1,2 | 1,2 | 1 | 1 |
| NPM | | 1 | 1 | |
| NPWM | 1 | | | |
| DPM | | 2 | | 1 |
| DPWM | 2 | | | |
| INV | * | * | | * |
| CMON | * | * | * | |

**TABLE 5.1**: Properties and Values

property is satisfied by the corresponding value. A number $i$ in a cell indicates that the corresponding property is satisfied by the corresponding value and that is one of the properties used in the i-th new axiomatic characterization of the value. In this chapter, we have provided two new axiomatic characterizations of the Shapley and Banzhaf values and one of the e-Banzhaf and ie-Banzhaf values.

## 5.7 Acknowledgments

This work has been supported by MINECO grants MTM2014-53395-C3-1-P, MTM2014-53395-C3-2-P, MTM2017-87197-C3-1-P and MTM2017-87197-C3-3-P, and by the Xunta de Galicia through the ERDF (Grupos de Referencia Competitiva ED431C-2016-015 and ED431C 2017/38, and Centro Singular de Investigación de Galicia ED431G/01). The comments of an anonymous referee have been very useful to improve this chapter.

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# Chapter 6

## Games with Identical Shapley Values

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## 6.1 Introduction

In this chapter, we survey the research studying cooperative games with transferable utility that induce the same Shapley values. The problem of identifying all games that generate a given vector of Shapley values has been first considered by Kleinberg and Weiss (1985) and became known as the "inverse problem" in the literature. Since the Shapley value is a linear operator on the space of games, the inverse problem is equivalent to characterizing its kernel—the space of games in which the Shapley value assigns zero payoffs to all players. We discuss several sets of games that reflect a clear balance of power among players and coalitions and constitute bases for the kernel of the Shapley value. We show how these games can be used to develop new axiomatizations of the Shapley value.

Chapter 7 written by Yukihiko Funaki and Koji Yokote also derives bases for the space of games by considering the inverse problem. That chapter focuses on a special class of the factious oligarchic games we define in this chapter. We prove that the special power structure Yokote and Funaki impose on factious oligarchic games is not necessary for the conclusion that each family of such games forms a basis for the kernel of the Shapley value.

The chapter is organized as follows. Section 6.2 provides basic definitions related to the Shapley value. In Section 6.3, we investigate the kernel of the Shapley value. We present three bases for this kernel as well as an intuitive characterization of games in the kernel. These classes of games lead to natural axiomatizations of the Shapley value, which we present in Section 6.4. In Section 6.5, we discuss how the bases for the kernel of the Shapley value can be completed to construct bases for the space of all games. Section 6.6 surveys alternative bases for the kernel of the Shapley value from the literature. Section 6.7 explores other interesting games that belong to the kernel of the Shapley value. Finally, Section 6.8 provides proofs of the new results, and Section 6.9 concludes.

---

## 6.2 The Shapley Value

Fix a set $N$ of $n \geq 2$ *players*. A *coalition* is any subset of players $S \subseteq N$. A game $v$ with transferable payoffs, simply called a *game* henceforth, associates a real number $v(S)$ to any coalition $S$, which represents the *value* coalition $S$ can create and share among its members ($v(\emptyset) = 0$). A *solution* $\psi$ assigns a *payoff* $\psi_i(v)$ to each player $i \in N$ for every game $v$. The *kernel* $\mathcal{K}(\psi)$ of a solution $\psi$ is the space of games in which $\psi$ assigns 0 payoffs to all players: $\mathcal{K}(\psi) = \{v|\psi_i(v) = 0, \forall i \in N\}$.

Shapley (1953) proposed the following solution $\phi$:

$$\phi_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)), \forall i \in N. \tag{6.1}$$

This solution, now known as the *Shapley value*, has the following interpretation. If players are ordered randomly (all orderings being equally likely), then $\phi_i(v)$ represents the expected marginal contribution of player $i$ to the coalition formed by his predecessors. The Shapley value has many elegant properties. For a comprehensive treatment, the reader may consult the monograph edited by Roth (1988) and the textbooks of Moulin (1988) and Osborne and Rubinstein (1994). Here we discuss only some of its properties—most of which Shapley introduced in his original paper—necessary for our analysis. Since these properties have been used in the context of axiomatic characterizations of the Shapley value, we refer to them as *axioms*.

Some preliminary definitions are necessary for stating the classic axioms. Player $i$ is *null* in game $v$ if $v(S \cup \{i\}) = v(S)$ for all coalitions $S$. Players $i$ and $j$ are *interchangeable* in $v$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions $S$ disjoint from $\{i, j\}$. A game $v$ is *inessential* if $v(S) = \sum_{i \in S} v(\{i\})$ for all coalitions $S$.

Given the assumption that the empty coalition has value 0, we view games as column vectors in the linear (vector) space $\mathbb{R}^{2^N \setminus \{\emptyset\}}$, which has dimension $2^n - 1$. Likewise, we represent solutions $(\psi_i(v))_{i \in N}$ for specific games $v$ as column vectors in $\mathbb{R}^N$. Hence, for any pair of games $v$ and $w$ and real number $\alpha$, $v+\alpha w$ is the game in which the value of coalition $S$ is given by $v(S)+\alpha w(S)$; similarly, $\psi(v) + \alpha\psi(w)$ denotes the vector $(\psi_i(v) + \alpha\psi_i(w))_{i \in N}$. We use the notation $\mathbf{0}$ for the zero vector in either $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ or $\mathbb{R}^N$ (the dimension will be clear from the context).

It is well known that the Shapley value $\phi$ satisfies the following axioms:

**Axiom** (Null). Solution $\psi$ satisfies the *null axiom* if $\psi_i(v) = 0$ whenever player $i$ is null in game $v$.

**Axiom** (Linearity). Solution $\psi$ satisfies the *linearity axiom* (or is *linear*) if $\psi(v+\alpha w) = \psi(v)+\alpha\psi(w)$ for every pair of games $v$ and $w$ and real number $\alpha$.

**Axiom** (Symmetry). Solution $\psi$ satisfies the *symmetry axiom* if $\psi_i(v) = \psi_j(v)$ whenever players $i$ and $j$ are interchangeable in game $v$.

**Axiom** (Inessential). Solution $\psi$ satisfies the *inessential axiom* if $\psi_i(v) = v(\{i\})$ for all $i \in N$ in every inessential game $v$.

In his original paper, Shapley identified a salient basis for the linear space of all games—unanimity games—which also plays an important role in our analysis. For every non-empty coalition $T$, the *unanimity game $u^T$ with ruling coalition $T$* is specified as follows:

$$u^T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Shapley proved that the $2^n - 1$ games $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ are linearly independent and thus $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ constitutes a basis for the $(2^n - 1)$-dimensional space of games $\mathbb{R}^{2^N \setminus \{\emptyset\}}$.

## 6.3 The Kernel of the Shapley Value

Given the natural embedding of games and solutions in the corresponding linear spaces, the Shapley value can be expressed as $\phi(v) = Av$, where $A$ is an $n \times (2^n - 1)$ matrix that reflects the coefficients from formula (6.1). For inessential games $v$, we have $Av = \phi(v) = (v(\{i\}))_{i \in N}$ because the Shapley value satisfies the inessential axiom. Since the space of vectors $(v(\{i\}))_{i \in N}$ derived from inessential games $v$ has dimension $n$, the matrix $A$ must have full row rank equal to $n$. It follows that, as Kleinberg and Weiss (1985) noted, the set of games in which all players have Shapley value 0—the kernel $\mathcal{K}(\phi) = \{v | Av = \mathbf{0}\}$—is a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ of dimension $2^n - n - 1$.

In what follows, we construct several sets of games, each spanning a space of dimension $2^n - n - 1$, in which all players have Shapley value 0. Since $\mathcal{K}(\phi)$ has dimension $2^n - n - 1$ and contains each set of games, we conclude that every set spans the full space $\mathcal{K}(\phi)$.

An *oligarchy* is any coalition that consists of at least two players. The members of an oligarchy are called *oligarchs*. Let $\mathcal{O}$ denote the set of oligarchies, $\mathcal{O} = \{O \subseteq N || O| \geq 2\}$. We define multiple games for every oligarchy $O$.

The *dog eat dog game* $\underline{w}^O$ for oligarchy $O$ is specified by

$$\underline{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This game has been introduced by Yokote (2015) and is called the *commander game* in the follow-up paper of Yokote et al. (2016).

The *scapegoat game* $\bar{w}^O$ for oligarchy $O$ is specified by

$$\bar{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = |O| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This game first appears in the study of Béal et al. (2016).

In the games constructed above, oligarchs have some power and are instrumental for value creation but the oligarchy is factious and cannot cooperate effectively to realize any value. In dog eat dog games, a coalition creates value only if it includes a single oligarch—the fierce "dog." In scapegoat games, a coalition generates value only if it contains all but one oligarch—the "scapegoat."

Funaki and Yokote (2019) construct a more general set of games with disharmonious oligarchies as follows. The *factious oligarchic game* for oligarchy $O$ with *parameter* $k$ $(1 \leq k \leq |O| - 1)$ is given by

$$w_k^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = k \\ 0 & \text{otherwise.} \end{cases}$$

In order to generate a basis for the kernel of the Shapley value, we allow for any variation in the parameter $k$ as a function of the oligarchy $O$. Let $f : \mathcal{O} \to \{1, 2, \ldots, n-1\}$ be a function such that $1 \leq f(O) \leq |O| - 1$ for all $O \in \mathcal{O}$. The *family of factious oligarchic games* $(w_f^O)_{O \in \mathcal{O}}$ with *power structure* $f$ is specified by $w_f^O := w_{f(O)}^O$ (with a slight abuse of notation). Note that dog eat dog games and scapegoat games are families of factious oligarchic games with special power functions $f$—the former specified by $f(O) = 1$ for all $O \in \mathcal{O}$, and the latter by $f(O) = |O| - 1$ for all $O \in \mathcal{O}$.

As Yokote (2015), Funaki and Yokote (2019), Yokote et al. (2016), and Béal et al. (2016) show, every player has Shapley value 0 in all types of oligarchic games defined above. To see this, consider the factious oligarchic game $w_k^O$ for oligarchy $O$ with parameter $k \leq |O| - 1$. In $w_k^O$, all players in $N \setminus O$ are null and must obtain Shapley value 0 since $\phi$ satisfies the null axiom. All oligarchs are interchangeable in $w_k^O$ and should obtain the same Shapley value because $\phi$ satisfies the symmetry axiom. The common Shapley value of the oligarchs must be 0 because $w_k^O(N) = 0$. Since $\phi$ is linear and $\phi(w) = \mathbf{0}$ for all games $w$ defined above, the Shapley value satisfies the following axioms.

**Axiom** (Dog Eat Dog).[1] Solution $\psi$ satisfies the *dog eat dog axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game $v$, any dog eat dog game $w$, and all real numbers $\alpha$.

**Axiom** (Scapegoat). Solution $\psi$ satisfies the *scapegoat axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game $v$, any scapegoat game $w$, and all real numbers $\alpha$.

**Axiom** (Factious Oligarchy). Solution $\psi$ satisfies the *factious oligarchy axiom* if there exists a power structure $f$ such that $\psi(v) = \psi(v + \alpha w)$ for every game $v$, any factious oligarchic game $w$ with power structure $f$, and all real numbers $\alpha$.

The intuition for each of the three axioms is that changing the cooperation structure by adding disharmonious oligarchies should not affect the division of payoffs. Note that a solution $\psi$ satisfies the dog eat dog, scapegoat, or factious oligarchy axiom if and only if $\psi(v) = \psi(v + w)$ for every game $v$ and all games $w$ that are linear combinations of dog eat dog, scapegoat, or factious oligarchic games, respectively.

We next introduce a set of games inspired by Hamiache (2001) and Béal et al. (2016). A *synergy function* is a game $\pi$ with the property that $\pi(\{i\}) = 0$ for all $i \in N$. The *paper tiger game* with synergy $\pi$ is defined by

$$w^\pi(S) = \sum_{i \in N} (\pi(S \cup \{i\}) - \pi(S)) \ \left(= \sum_{i \in N \setminus S} (\pi(S \cup \{i\}) - \pi(S))\right).$$

[1]This axiom is a special case of the axiom of $\omega$-weak addition invariance introduced in Yokote (2015) when $\omega$ is a vector of 1.

The interpretation of this game is that every player $i$ is by nature a solitary "tiger", which can add synergies to any group $S$ that excludes him. However, the synergy of the expanded group $S \cup \{i\}$ supersedes the original synergy of $S$, rendering $i$ a "paper tiger." Since only outsiders add value to coalitions, all synergies "wash out" for the grand coalition, $w^{\pi}(N) = 0$.

The set of paper tiger games constitutes a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ that has dimension at most $2^n - n - 1$ because each component of any element $(w^{\pi}(S))_{S \in 2^N \setminus \{\emptyset\}}$ is a linear function of the $2^n - n - 1$ variables $(\pi(S))_{S \in \mathcal{O}}$. Béal et al. (2016) remark that for any oligarchy $O$, the paper tiger game $w^{\pi}$ derived from the synergy function

$$\pi(S) = \begin{cases} 1 & \text{if } O \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

is identical to the scapegoat game $\bar{w}^O$. Thus, the space of paper tiger games contains the linear space spanned by scapegoat games. Béal et al. argue that the space of scapegoat games has dimension $2^n - n - 1$, which implies that the space of paper tiger games has dimension $2^n - n - 1$ and coincides with the space spanned by scapegoat games. Hence, every paper tiger game is a linear combination of scapegoat games. The linearity of the Shapley value, along with the fact that $\phi(\bar{w}^O) = \mathbf{0}$ for all scapegoat games $\bar{w}^O$, implies that $\phi(w^{\pi}) = \mathbf{0}$ for every paper tiger game $w^{\pi}$. Therefore, the Shapley value satisfies the following axiom, which captures the "paper tiger" metaphor.

**Axiom** (Paper Tiger). Solution $\psi$ satisfies the *paper tiger axiom* if $\psi(v) = \psi(v + w)$ for every game $v$ and any paper tiger game $w$.

Yokote (2015) established that the set of dog eat dog games forms a basis for the kernel of the Shapley value, and Béal et al. (2016) proved that the set of scapegoat games has the same property. Funaki and Yokote (2019) generalized these two results to families of factious oligarchic games $(w_f^O)_{O \in \mathcal{O}}$ with special power structures $f$. In the analysis of Funaki and Yokote (2019), $f(O)$ depends only on the size of $O$, i.e., $f(O) = g(|O|)$ where $g$ is a function from $\{2, \ldots, n\}$ to $\{1, \ldots, n-1\}$. Moreover, their main result imposes the following "continuity" restriction on $g$:

$$g(k-1) - 1 \leq g(k) \leq g(k-1) + 1 \text{ for } k \in \{3, \ldots, n\}.$$

We show that neither of these restrictions is necessary for the result: The family of factious oligarchic games $(w_f^O)_{O \in \mathcal{O}}$ is linearly independent and spans the kernel of the Shapley value for every power structure $f$. The proof of this result relies on a new basis of the set of all games consisting of games with oligarchic structures we develop in Section 6.5 (see Theorem 6.3).

**Theorem 6.1** *The set of dog eat dog games constitutes a basis for the linear space $\mathcal{K}(\phi)$, and the same is true about the set of scapegoat games. More* *generally, the family of factious oligarchic games with any power structure forms a basis for $\mathcal{K}(\phi)$. Furthermore, $\mathcal{K}(\phi)$ is given by the set of paper tiger games.*

Section 6.8 provides the proof of Theorem 6.1. We next present two corollaries that invoke paper tiger games. In light of Theorem 6.1, we can restate either corollary using a linear combination of each type of oligarchic game in lieu of the paper tiger game. The first corollary follows from the linearity of the Shapley value.

**Corollary 6.1** *Games $v$ and $w$ yield identical Shapley values if and only if their difference $v - w$ is a paper tiger game.*

Fix a game $v$. The *Shapley inessential game $w$ of $v$* is defined by $w(S) = \sum_{i \in S} \phi_i(v)$ for all coalitions $S$. Since the Shapley value satisfies the inessential axiom, we have that $\phi_i(w) = w(\{i\}) = \phi_i(v)$ for all $i \in N$. Then the linearity of the Shapley value implies that $\phi(v - w) = \mathbf{0}$. Thus, as Kleinberg and Weiss (1985) observed, the game $v$ can be decomposed into its Shapley inessential game $w$ and the game $v - w$, which is an element of $\mathcal{K}(\phi)$. This conclusion leads to another corollary of Theorem 6.1.

**Corollary 6.2** *Every game is the sum of its Shapley inessential game and a paper tiger game.*

## 6.4 Axiomatizations of the Shapley Value Based on its Kernel

If a solution $\psi$ is pinned down for inessential games by the inessential axiom, and the addition of games in $\mathcal{K}(\phi)$ does not affect the solution as implied by any of the dog eat dog, scapegoat, factious oligarchy, or paper tiger axioms, then $\psi$ must coincide with the Shapley value $\phi$. This observation, along with Theorem 6.1 and Corollary 6.2, leads to four axiomatizations of the Shapley value.

**Theorem 6.2** *A solution is the Shapley value if and only if it satisfies the inessential axiom and any one of the dog eat dog, scapegoat, factious oligarchy, and paper tiger axioms.*

We finally comment on a connection between our paper tiger axiom and an axiom due to Hamiache (2001). Derive a synergy function $\pi_v$ from a game $v$ as follows:

$$\pi_v(S) = v(S) - \sum_{i \in S} v(\{i\}). \tag{6.2}$$

Let $w^{\pi_v}$ denote the paper tiger game with synergy $\pi$, and define the game $v_\lambda = v + \lambda w^{\pi_v}$, where $\lambda$ is a positive real number. Algebra leads to

$$v_\lambda(S) = v(S) + \lambda \sum_{i \in N \setminus S} \left( v(S \cup \{i\}) - v(S) - v(\{i\}) \right), \forall S \subseteq N.$$

Since $w^{\pi_v}$ is a paper tiger game, Theorem 6.2 implies that the games $v$ and $v_\lambda$ have the same Shapley value for every $\lambda$. Hamiache (2001) uses this property, coined *associated consistency*, to develop a characterization of the Shapley value. In addition to the inessential axiom, his characterization requires a continuity axiom because associated consistency is a weaker version of our paper tiger axiom that applies only to pairs of games $(v, \lambda w^{\pi_v})$ for which the synergy function $\pi_v$ has the special relation to $v$ described by formula (6.2).

To obtain an alternative proof of Hamiache's result, Béal et al. (2016) remark that the matrix associated with the linear transformation $v \to w^{\pi_v}$ is upper triangular when expressed in the basis of unanimity games. Its kernel is formed by the set of inessential games, and all its non-zero eigenvalues are negative. This ensures that, for sufficiently small $\lambda$, the sequence generated by the iteration of the transformation $v \to v_\lambda$ converges to an inessential game $v^\infty$ for every first term $v$. The associated consistency and continuity of the solution $\psi$ are used to conclude that $\psi(v) = \psi(v^\infty)$. If $\psi$ satisfies the inessential axiom, then $\psi_i(v) = \psi_i(v^\infty) = v^\infty(\{i\})$ for all players $i$, which proves that $\psi(v)$ is uniquely determined.

Kleinberg (2018) extends the work of Hamiache (2001) by exploring linear and anonymous solutions (called *membership solutions*) other than the Shapley value that satisfy associated consistency. A solution is *anonymous* if a change in the label of the players has no effect on the solution. Note that the equal division solution, which divides the value of the grand coalition evenly among all players, is linear and anonymous and satisfies associated consistency. Kleinberg proves that a solution is linear and anonymous and satisfies associated consistency if and only if it is a linear combination of the Shapley value and the equal division solution. An equivalent statement of this result is that a linear and anonymous solution satisfies associated consistency if and only if its kernel contains the kernel of the Shapley value.

## 6.5 Bases for the Space of Games

Recall that Shapley (1953) showed that the set of unanimity games $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ constitutes a basis for the space of all games. We construct a rich class of new bases for the space of games by expanding the set of oligarchic games from Section 6.3. Specifically, we allow for "singleton oligarchies" $O = \{i\}$ and consider the possibility that oligarchies are functional, so parameter $k$ in the specification of the corresponding game $w_k^O$ can take the value $|O|$ (which is necessary for singleton oligarchies to generate a game different from $\mathbf{0}$). Therefore, we redefine an *oligarchy* to be any nonempty coalition $O \subseteq N$ and specify the *oligarchic game* for oligarchy $O$ with *parameter* $k$ as in Section 6.3,

$$w_k^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = k \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq k \leq |O|$, with the novelty that $k = |O|$ is an admissible parameter. Power functions need to be adjusted accordingly—$f : 2^N \setminus \{\emptyset\} \to \{1, 2, \ldots, n\}$ is a *power function* if $1 \leq f(O) \leq |O|$ for all $O \in 2^N \setminus \{\emptyset\}$. The *family of oligarchic games* $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ with *power structure* $f$ is specified as before, $w_f^O := w_{f(O)}^O$.

By definition, for singleton coalitions $O = \{i\}$, every power function $f$ satisfies $f(\{i\}) = 1$ and $w_f^{\{i\}} = u^{\{i\}}$. In general, $w_{|O|}^O := u^O$ for all oligarchies $O$. Thus, the new oligarchic games added to the set of factious ones are exactly the unanimity games. Note that the Shapley value for the unanimity game $u^O$ is given by

$$\phi_i(u^O) = \begin{cases} 1/|O| & \text{if } i \in O \\ 0 & \text{if } i \in N \setminus O. \end{cases}$$

Hence, the newly added games do not belong to the kernel of the Shapley value. We establish that the family of oligarchic games with any power structure constitutes a basis of the space of games, which generalizes the main result of Funaki and Yokote (2019) as discussed in Section 6.3.

**Theorem 6.3** *For any power structure $f$, the set of oligarchic games $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ forms a basis for the space of all games.*

The proof of the theorem can be found in Section 6.8. The key ingredient of the proof is a representation of the oligarchic game for oligarchy $O$ with parameter $k$ in the basis of unanimity games,

$$w_k^O = \sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S.$$

The coefficient of the game $u_S$ in the unique linear decomposition of any game $v$ in the basis $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ is known as the *Harsanyi (1959) dividend* of coalition $S$ in game $v$. Hence, the identity above shows that the Harsanyi dividend of coalition $S$ in the oligarchic game $w_k^O$ is $(-1)^{|S|-k}\binom{|S|}{k}$ for $S \subseteq O, |S| \geq k$ and 0 otherwise. We then reach the desired conclusion by noting that the linear transformation $(u^T)_{T \in 2^N \setminus \{\emptyset\}} \to (w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ derived from the identity above is captured by a lower-triangular matrix with non-zero diagonal elements.

We can build an alternative basis for the space of games by augmenting any basis of the $(2^n - n - 1)$-dimensional kernel $\mathcal{K}(\phi)$ of the Shapley value identified in Theorem 6.1 with any collection of $n$ linearly independent games that span a space whose only intersection with $\mathcal{K}(\phi)$ is game $\mathbf{0}$. One obvious selection for the $n$ games is the set of degenerate unanimity games with singleton ruling coalitions, $(u^{\{i\}})_{i \in N}$, which we call *trivial* games. By Theorem 6.3, trivial games are linearly independent and span the space of inessential games. Since $\phi(v) = (v(\{i\}))_{i \in N}$ for every inessential game $v$, the intersection of the set of inessential games and the kernel of the Shapley value is $\{\mathbf{0}\}$. It follows that any basis of $\mathcal{K}(\phi)$ described in Theorem 6.1 along with the collection of trivial games forms a basis for the set of all games. In a result related to Corollary 6.2, Yokote et al. (2016) show that the coefficient of the trivial game $u^{\{i\}}$ in the decomposition of any game in each of these bases coincides with the Shapley value of player $i$. To see this, note that the discussion above implies that every game $v$ can be uniquely decomposed as a linear combination of games in any basis of $\mathcal{K}(\phi)$ and trivial games $u^{\{j\}}$ for $j \in N$. Let $\alpha_j$ denote the coordinate of $u^{\{j\}}$ in the decomposition of $v$. Then, the linearity of the Shapley value $\phi$ leads to

$$\phi_i(v) = \sum_{j \in N} \alpha_j \phi_i(u^{\{j\}}), \forall i \in N.$$

For any $j \in N$, since $u^{\{j\}}$ is an inessential game, we have $\phi_j(u^{\{j\}}) = u^{\{j\}}(\{j\}) = 1$ and $\phi_i(u^{\{j\}}) = u^{\{j\}}(\{i\}) = 0$ for $i \in N \setminus \{j\}$. Therefore, $\phi_i(v) = \alpha_i$ for all $i \in N$, as asserted. The following theorem collects results from Yokote et al. (2016) and Funaki and Yokote (2019).

**Theorem 6.4** *The collection of trivial games and each family of factious oligarchic games with any power structure constitutes a basis for the space of all games. In every such basis, the coefficient of each trivial game in the decomposition of any given game coincides with the Shapley value of the corresponding player in that game.*

## 6.6 Other Bases

Kleinberg and Weiss (1985) provided the first characterization of the kernel of the Shapley value as a direct sum decomposition of linear spaces. Each game in their decomposition assigns non-zero values only to singletons or coalitions of a fixed size. Their decomposition consists of three types of games:

$$\{v|v(S) = v(S') \text{ if } |S| = |S'| = k; v(S) = 0 \text{ if } |S| \neq k\} \text{ for } 1 \leq k \leq n-1,$$

$$\left\{v|\sum_{i\in N} v(\{i\}) = 0; v(S) = -\sum_{i\in S} v(\{i\}) \text{ if } |S| = k; v(S) = 0 \text{ if } |S| \neq 1, k\right\}$$
$$\text{for } 2 \leq k \leq n-1,$$

$$\left\{v|v(S) = 0 \text{ if } |S| \neq k; \sum_{i\in S} v(S) = 0, \forall i \in N\right\} \text{ for } 2 \leq k \leq n-1.$$

Dragan et al. (1989) develop a different basis for the space of games building on the potential value of Hart and Mas-Colell (1989). Recall that the potential $P(S,v)_{S\subseteq N}$ of a game $v$ is defined recursively by

$$P(S,v) = \frac{1}{|S|}\left(v(S) + P(S\setminus\{i\}, v)\right)$$

with the initial condition $P(\emptyset, v) = 0$. Hart and Mas-Colell showed that the Shapley value can be computed as

$$\phi_i(v) = P(N,v) - P(N\setminus\{i\}, v), \forall i \in N.$$

Dragan et al. pointed out that the potential function $P$ can be interpreted as a linear endomorphism on the space of games, and hence one can derive a basis for this space by identifying a game $w^T$ for every nonempty coalition $T$ with the property that $P(T, w^T) = 1$ and $P(S, w^T) = 0$ if $S \neq T$. They found that

$$w^T(S) = \begin{cases} |S| & \text{if } S = T \\ -1 & \text{if } S = T \cup \{j\} \text{ with } j \notin T \\ 0 & \text{otherwise.} \end{cases}$$

It can then be checked that the set of games $(w^T)_{1\leq|T|\leq n-2}$ together with the game $w^N + \sum_{i\in N} w^{N\setminus\{i\}}$ forms a basis for the kernel of the Shapley value.

Another basis for the kernel of the Shapley value can be obtained by considering a generalization of the Shapley value. Recall that a solution $\psi$ is *efficient* if the total payoffs it allocates equal to the value of the grand coalition, i.e., $\sum_{i\in N} \psi_i(v) = v(N)$ for all games $v$. The Shapley value is a prominent solution which is linear, anonymous, and efficient. Ruiz et al. (1998) show that any linear, anonymous, and efficient solution takes the form $\phi^b$, where

$$\phi_i^b(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(b_{|S|+1}v(S\cup\{i\}) - b_{|S|}v(S)), \forall i \in N$$

for a collection of constants $b = (b_k)_{0\leq k\leq n}$ with $b_n = 1$.

Rojas and Sanchez (2016) analyze the subset of linear, anonymous, and efficient solutions $\phi^b$ with $b_k \neq 0$ for all $k$, which they call *regular* solutions.

They provide a basis for the kernel of each regular solution $\phi^b$ consisting of the games $(v_T^b)_{T\subseteq N,|T|\neq 1}$ defined as follows:

$$v_N^b(S) = \begin{cases} 1 & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and for $T$ such that $2 \leq |T| \leq |N| - 1$,

$$v_T^b(S) = \begin{cases} 1 & \text{if } |S| = 1 \text{ and } S \cap T = \emptyset \\ \dfrac{b_1}{b_{|T|}} \dbinom{|N|-2}{|T|-1} & \text{if } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value is obtained by setting $b_k = 1$ for all $k$, the collection of games $(v_T^{(1,\ldots,1)})_{T\subseteq N,|T|\neq 1}$ is a new basis for the kernel of the Shapley value. As in Section 6.5, the authors further provide a basis of the space of all games by augmenting their basis of the kernel of any regular solution $\psi^b$. Rojas and Sanchez (2016) first prove that the kernel $\mathcal{K}(\psi^b)$ of any such solution $\psi^b$ has dimension $2^n - n - 1$. Then they need to add the following collection of $n$ games $(v_{\{i\}}^b)_{i\in N}$ such that:

$$v_{\{i\}}^b(S) = \begin{cases} \dfrac{1}{b_{|S|}} & \text{if } i \in S \text{ and } S \neq N \\ |N| & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

In another recent study, Faigle and Grabisch (2016) employed the change of basis underlying isomorphic linear operators to construct new bases for the space of games and for the kernel of linear values from existing linear representations of games. Starting from the Shapley interaction transform of Grabisch (1997), Faigle and Grabisch obtain the basis $(b^T)_{T\subseteq N,|T|\neq 1}$ for the kernel of the Shapley value specified by

$$b^T(S) = \sum_{j=0}^{|S\cap T|} \binom{|S\cap T|}{j} B_{|T|-j},$$

where $B_0, B_1, \ldots$ are the cumbersome Bernoulli numbers.

While conceptually interesting, the approaches discussed in this section provide less immediate game theoretic intuitions for the kernel of the Shapley value.

## 6.7 Other Games in the Kernel of the Shapley Value

For any oligarchy $O \in \mathcal{O}$ and every nonempty set $K \subseteq \{1, 2, \ldots, |O| - 1\}$, the game $w_K^O$ defined by

$$w_K^O(S) = \begin{cases} 1 & \text{if } |S \cap O| \in K \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

delivers Shapley value 0 to all players. This follows from the linearity of the Shapley value and the observation that each such game can be decomposed into factious oligarchic games with parameters in $K$,

$$w_K^O = \sum_{k \in K} w_k^O.$$

In particular, note that dog eat dog, scapegoat, and fictitious oligarchic games are all special instances of this set of games in which $K$ is a singleton.

One interesting subset of the games $w_K^O$ is obtained by setting $K = \{1, 2, \ldots, |O| - 1\}$ for every $O \in \mathcal{O}$. Specifically, define the (dysfunctional) *wolf pack game* $\tilde{w}^O$ for oligarchy $O$ as follows:

$$\tilde{w}^O(S) = \begin{cases} 1 & \text{if } 1 \leq |S \cap O| \leq |O| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In dysfunctional wolf pack games, a coalition is productive only if it involves some but not all oligarchs—the "wolf pack" cannot coordinate as a whole. In light of the rich set of bases identified by Theorem 6.1, it is worth pointing out that the $2^n - n - 1$ wolf pack games obtained by varying the composition of the oligarchy are not always linearly independent and hence do not span the kernel of the Shapley value. For instance, for $n = 4$, one can check that the sum of all wolf pack games with oligarchies of size two is identical to the sum evaluated for oligarchies of size three.

Wolf pack games lie at the opposite end on the spectrum of dissent among oligarchs from dog eat dog games: Every subset of oligarchs except for the entire oligarchy operates effectively in wolf pack games, while no two oligarchs can cooperate successfully in dog eat dog games. Funaki and Yokote (2019) consider an intermediate level of power struggle among oligarchs whereby only coalitions formed by half of the oligarchs are effective. This corresponds to setting $K = \{|O|/2\}$ for $|O|$ even and $K = \{(|O| + 1)/2\}$ for $|O|$ odd in (6.3). Theorem 6.4 implies that this set of games augmented with the set of trivial games constitutes a basis for the kernel of the Shapley value. Funaki and Yokote (2019) employ the decomposition of games in this basis to identify games for which the Shapley value coincides with the prenucleolus.

## 6.8 Proofs

**Proof of Theorem 6.1** Since dog eat dog games and scapegoat games are families of factious oligarchic games with two special power functions $f$—the former specified by $f(O) = 1$ for all $O \in \mathcal{O}$, and the latter by $f(O) = |O| - 1$ for all $O \in \mathcal{O}$—the statements about dog eat dog games and scapegoat games are implied by the one about general factious oligarchic games.

To prove the statement regarding factious oligarchic games, fix a power structure $f$ and consider the family of factious oligarchic games $(w_f^O)_{O\in\mathcal{O}}$ it generates. By Theorem 6.3, the elements of the family $(w_f^O)_{O\in\mathcal{O}}$ are linearly independent. Since this family contains exactly $2^n - n - 1$ games, it spans a linear space of dimension $2^n - n - 1$. In Section 6.3, we have argued that $\mathcal{K}(\phi) = \{v|\phi(v) = \mathbf{0}\}$ is a linear subspace of $\mathbb{R}^{2^N\setminus\{\emptyset\}}$ of dimension $2^n - n - 1$ that contains all factious oligarchic games, including the ones in $(w_f^O)_{O\in\mathcal{O}}$. Since the space spanned by $(w_f^O)_{O\in\mathcal{O}}$ has dimension $2^n - n - 1$, it must coincide with $\mathcal{K}(\phi)$. Therefore, $(w_f^O)_{O\in\mathcal{O}}$ constitutes a basis for $\mathcal{K}(\phi)$.

Finally, the conclusion that the space of paper tiger games is identical to $\mathcal{K}(\phi)$ follows from the finding that $\mathcal{K}(\phi)$ spans the set of scapegoat games and the arguments provided after the definition of paper tiger games.

**Proof of Theorem 6.3** Fix a power structure $f$ and consider the family of $2^n-1$ oligarchic games $(w_f^O)_{O\in 2^N\setminus\{\emptyset\}}$ it generates. To establish that the family $(w_f^O)_{O\in 2^N\setminus\{\emptyset\}}$ forms a basis for the $(2^n - 1)$-dimension space of all games, it is sufficient to show that the games in the family are linearly independent.

We first argue that the oligarchic game for oligarchy $O$ with parameter $k$ can be decomposed in the basis of unanimity games as follows:

$$w_k^O = \sum_{S\subseteq O, |S|\geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S.$$

We need to show that for every coalition $T \subseteq N$,

$$w_k^O(T) = \sum_{S\subseteq O, |S|\geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T). \tag{6.4}$$

Fix a coalition $T$, and let $T' = T \cap O$ and $t = |T'|$.

Clearly, if $t < k$, then $w_k^O(T) = w_k^O(T') = 0$ and $u^S(T) = u^S(T') = 0$ for $S \subseteq O$ such that $|S| \geq k$. Hence, for $t < k$, both sides of Equation (6.4) equal zero.

Suppose now that $t \geq k$. We can rewrite the right-hand side term in Equation (6.4) as follows:

$$\begin{aligned}
\sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T) &= \sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T') \\
&= \sum_{S \subseteq T', |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} \\
&= \sum_{s=k}^{t} (-1)^{s-k} \binom{s}{k} \binom{t}{s} \\
&= \sum_{s=k}^{t} (-1)^{s-k} \binom{t}{k} \binom{t-k}{s-k} \\
&= \binom{t}{k} \sum_{s=0}^{t-k} (-1)^{s} \binom{t-k}{s} \\
&= \binom{t}{k} (1-1)^{t-k}.
\end{aligned}$$

The first equality follows from $u^S(T) = u^S(T \cap S) = u^S(T \cap O) = u^S(T')$ for $S \subseteq O$, the second relies on the fact that $u^S(T') = 1$ if $S \subseteq T'$ and $u^S(T') = 0$ otherwise (along with $T' \subseteq O$), and the third accounts for the number $\binom{t}{s}$ of sets $S \subseteq T'$ with $|S| = s \geq k$ given that $|T'| = t$. The fourth equality uses the formulae

$$\begin{aligned}
\binom{s}{k}\binom{t}{s} = \frac{s!}{k!(s-k)!} \frac{t!}{s!(t-s)!} &= \frac{t!}{k!(s-k)!(t-s)!} \\
&= \frac{t!}{k!(t-k)!} \frac{(t-k)!}{(s-k)!(t-s)!} = \binom{t}{k}\binom{t-k}{s-k},
\end{aligned}$$

while the fifth one simply changes the variable $s - k$ to $s$. The final equality follows from the binomial formula.

For $t \geq k$, claim (6.4) then follows from noting that

- if $t = k$, then $\binom{t}{k}(1-1)^{t-k} = 1 = w_k^O(T)$;
- if $t > k$, then $\binom{t}{k}(1-1)^{t-k} = 0 = w_k^O(T)$.

We are now prepared to show that the games $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ are linearly independent. Consider any linear order $\succeq$ on $2^N \setminus \{\emptyset\}$ extending the partial order $(2^N \setminus \{\emptyset\}, \supseteq)$ and construct the $(2^n - 1) \times (2^n - 1)$ matrix of coordinates of the games of $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ in the basis of unanimity games $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$. This matrix is lower-triangular since the coordinates of $w_f^O$ associated with $u^T$ are zero whenever $T \succ O$. Moreover, each diagonal element in the matrix takes the form

$$(-1)^{|O|-f(O)} \binom{|O|}{f(O)} \neq 0.$$

Consequently, the matrix has full rank, which delivers the result.

## 6.9 Conclusions

We introduced several classes of cooperative games in which the Shapley value yields zero payoffs to all players. These games deliver a rich set of bases for the kernel of the Shapley value and lead to multiple characterizations of games with identical Shapley values. Building on these games, we were able to provide new intuitive axiomatizations of the Shapley value. We explained how each basis of the kernel of the Shapley value can be enlarged to create a basis for the space of all games. Many of the games we presented admit straightforward game theoretic interpretations. However, some of the games require a deeper understanding of the power structure they induce among coalitions. It would be useful to develop more connections between the various bases of the kernel of the Shapley value.

## 6.10 Acknowledgments

The first, third and fourth authors benefited from the financial support of IDEXLYON from Université de Lyon (project INDEPTH) within the Programme Investissements d'Avenir (ANR-16-IDEX-0005). The authors thank Aubrey Clark for outstanding research assistance and acknowledges financial support from the NSF (grant 1260744).

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# Chapter 7

## Several Bases of a Game Space and an Application to the Shapley Value

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### 7.1 Introduction

The basis for the linear space of TU games consisting of the unanimity games (Shapley (1953)) has been recognized as a very useful tool to analyze TU cooperative games. The basis is often used in the proof of axiomatic characterization of the (weighted) Shapley value; see Young (1985), Chun (1989), Kalai and Samet (1987) or van den Brink (2002). Since the set of TU games is Euclidean space, to consider a basis is very important. The purpose of this chapter is to introduce new bases and explore their properties.

First, we recall that in the unanimity game, cooperation of **all** players in a given coalition $T$ included the grand coalition $N$ yields the total payoff. That is, for each coalition $T \subseteq N$, the unanimity game for $T$ assigns 1 to a coalition including all players in $T$ and 0 otherwise. In the *commander game* introduced by Yokote et al. (2016), **only one** player in a given coalition $T$ yields the total payoff. That is, for each coalition $T \subseteq N$, the commander game for $T$ assigns 1 to coalitions including only one member in $T$, and assigns 0 otherwise. The set of the commander games forms a basis and has the following properties. When we express a game by a linear combination of this basis, the coefficients related to singletons coincide with the Shapley value. Second, the basis induces the null space of the Shapley value: The set of games such that the Shapley value of the games assigns the 0 vector. In Yokote et al. (2016), it is shown that the payoff vector of each commander game can be uniquely determined by using three axioms: Efficiency, equal treatment property and null player property. Thus, we can use the basis in the proof of axiomatization of the Shapley value. In addition, by using the two properties of the basis, we can solve the inverse problem, that is, characterizing the class of games such that the Shapley value of the games is equal to any given efficient payoff vector. The basis enables us to give a new axiomatization of the weighted Shapley value; see Yokote (2014). Moreover, the basis can be used to investigate the relationship between the Shapley value and other solutions; see Yokote et al. (2017).

Moreover linear algebraic method has made significant contributions to the development of solution theory in TU cooperative games. In recent studies, Xu et al. (2015) characterized the CIS and ENSC values by using infinite multiplications of matrices.

This chapter is located at the intersection of linear algebra and TU cooperative games. We extend the basis consisting of the commander games and provide new mathematical tools for analyzing the Shapley value. The unanimity games and commander games describe two extreme requirements for obtaining payoff: All players in $T$ or only one player in $T$. We consider the intermediates between them. Our new game, which we call the $(T, k)$-intermediate game, assigns 1 to a coalition including $k$ players in $T$ and 0 otherwise, where $1 \leq k \leq |T|$. We show that, if some relationship between the size of coalition $T$ and $k$ holds, then we can construct a basis.

All the new bases obtained in this paper preserve two desirable properties of the commander games. Namely, (i) when we express a game by a linear combination of the basis, the coefficients related to singletons coincide with the Shapley value, and (ii) the basis induces the null space of the Shapley value.

We apply our basis to the analysis of coincidence conditions between the Shapley value and the prenucleolus (Schmeidler (1969)). Our basis enables us to take a linear algebraic approach to a coincidence condition known as the PS property (Kar et al. (2009)) and clarify the mathematical structure behind the coincidence region.

The reminder of this chapter is organized as follows. Section 7.2 provides notations and definitions. Section 7.3 reviews the commander games basis and shows that the set of the games is a basis. Section 7.4 introduces some interesting properties of the commander games basis. Both Sections 7.3 and 7.4 are based on Yokote et al. (2016). Section 7.5 gives definitions of our new bases. Section 7.6 applies one of our new bases to the PS property, a sufficient condition that the Shapley value coincides with the nucleolus. Section 7.7 gives concluding remarks.

## 7.2 Notations and Definitions

For two sets $A$ and $B$, $A \subseteq B$ means that $A$ is a subset of $B$. $A \subset B$ means that $A \subseteq B$ and $A \neq B$. Let $|A|$ denote the cardinality of $A$.

Let $N \subset \mathbb{N}$ denote a finite set of players, here $\mathbb{N}$ is the set of natural numbers. We call $S \subseteq N$ a coalition of $N$. We define $|N| = n$. A characteristic function $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition of $N$. We assume $v(\emptyset) = 0$. We call $v(S)$ the worth of coalition $S$. A pair $(N, v)$ is called a TU cooperative game, or simply a game. In the remaining part, we fix player set $N$ and write $v$ instead of $(N, v)$. Let $\Gamma^N$ denote the set of all games with player set $N$. We say that $v \in \Gamma^N$ is a simple game if $v(S) = 0$ or 1 for all $S \subseteq N$. We regard $\Gamma^N$ as a linear space $\mathbb{R}^{2^n-1}$ by defining addition and scalar multiplication as follows: For any $v$, $w \in \Gamma^N$ and $\alpha \in \mathbb{R}$, we define $v + w$ and $\alpha v$ by $(v + w)(S) = v(S) + w(S)$ and $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$.

A value function is a function from $\Gamma^N$ to $\mathbb{R}^n$. We define the Shapley value, introduced by Shapley (1953), as follows: For any $v \in \Gamma^N$,

$$Sh_i(v) = \sum_{T \subseteq N : i \in T} \frac{(n - |T|)!(|T| - 1)!}{n!} \big(v(T) - v(T \backslash i)\big) \text{ for all } i \in N.$$[1]

We can also calculate the Shapley value by using the dividend introduced by Harsanyi (1959). For any $v \in \Gamma^N$ and $T \subseteq N$, $T \neq \emptyset$, we define the dividend as follows:

$$D(T, v) = \sum_{S \subseteq T} (-1)^{|T \backslash S|} v(S).$$

The following equation holds: For any $v \in \Gamma^N$,

$$Sh_i(v) = \sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T, v) \text{ for all } i \in N.$$

For any $T \subseteq N$, $T \neq \emptyset$, we define the unanimity game $u_T$ for $T$, introduced by Shapley (1953), as follows:

$$u_T(S) = \begin{cases} 1 \text{ if } T \subseteq S, \\ 0 \text{ otherwise.} \end{cases}$$

When we express a game $v \in \Gamma^N$ by a linear combination of the set of unanimity games $\{u_T : \emptyset \neq T \subseteq N\}$, the coefficient of $u_T$ is equal to the dividend $D(T, v)$.

Mathematically, the Shapley value $Sh$ is a surjective[2] linear mapping from $\mathbb{R}^{2^n-1}$ to $\mathbb{R}^n$. Here for the value function $Sh$, the null space of $Sh$

[1] For simplicity, we write $i$ for $\{i\}$.

[2] Surjective is due to the following inessential game property: Let $x \in \mathbb{R}^n$ and consider the game $v$ such that $v(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$, $S \neq \emptyset$. Then, $Sh(v) = x$.

is defined by

$$\{v \in \Gamma^N : Sh(v) = \mathbf{0}\}.$$

The null space is the set of all games to which the Shapley value assigns the 0 vector. The dimension of the space is equal to $2^n - 1 - n$.

We say that a finite set of games $\{v_k\}_{k=1}^{\ell} \subseteq \Gamma^N$ spans $X \subseteq \Gamma^N$ if

$$X = \Big\{\sum_{k=1}^{\ell} \alpha_k v_k : \alpha_k \in \mathbb{R} \text{ for all } k = 1, \ldots, \ell\Big\}.$$

Let $\mathrm{Sp}(\{v_k\}_{k=1}^{\ell})$ denote the set of games spanned by $\{v_k\}_{k=1}^{\ell}$.

Let $v \in \Gamma^N$, $i \in N$ and $S \subseteq N \backslash i$. We define the marginal contribution of player $i$ to coalition $S$ as follows:

$$\Delta_i v(S) = v(S \cup i) - v(S).$$

Let $v \in \Gamma^N$ and $i \in N$. We say that $i$ is a null player in $v$ if $v(S \cup i) - v(S) = 0$ for all $S \subseteq N \backslash i$. The Shapley value satisfies the following axioms:

**Efficiency** For any $v \in \Gamma^N$, $\sum_{i \in N} Sh_i(v) = v(N)$.

**Null Player Property** Let $v \in \Gamma^N$. If $i \in N$ is a null player in $v$, then $Sh_i(v) = 0$.

**Symmetry** Let $v \in \Gamma^N$ and $i, j \in N$. If $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$, then $Sh_i(v) = Sh_j(v)$.

**Linearity** Let $v, w \in \Gamma^N$ and $\alpha, \beta \in \mathbb{R}$. Then, $Sh(\alpha v + \beta w) = \alpha Sh(v) + \beta Sh(w)$.

## 7.3 Commander Games

We recall a game, which is studied in Yokote et al. (2016). Let $T \subseteq N$, $T \neq \emptyset$. We define the game $\bar{u}_T$ as follows:

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |S \cap T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition, $\bar{u}_T$ is a simple game. We call $\bar{u}_T$ the *commander game for* $T$. Note that $\bar{u}_{\{i\}} = u_{\{i\}}$ for any $i \in N$. We consider the following situation behind the game. Each member in $T$ is a commander and has authority to control other players. If there is no member of $T$ in a coalition, the coalition does not have power. If a coalition that includes only one member in $T$ forms, then the member behaves as a commander. The coalition obtains power, which results in the payoff of 1. On the other hand, if a coalition that includes two or more members in $T$ forms, then they compete with each other and the coalition obtains nothing.

**Example 7.1** *For 3 person games $(N, v)$ with $N = \{1, 2, 3\}$, the set of the commander games $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$ is given as follows:*

$$\begin{aligned}
\bar{u}_{\{1\}} &= (\bar{u}_{\{1\}}(\{1\}), \bar{u}_{\{1\}}(\{2\}), \bar{u}_{\{1\}}(\{3\}), \bar{u}_{\{1\}}(\{1,2\}), \bar{u}_{\{1\}}(\{1,3\}), \bar{u}_{\{1\}}(\{2,3\}), \\
&\quad \bar{u}_{\{1\}}(N)) \\
&= (1, 0, 0, 1, 1, 0, 1) \\
\bar{u}_{\{2\}} &= (0, 1, 0, 1, 0, 1, 1) \\
\bar{u}_{\{3\}} &= (0, 0, 1, 0, 1, 1, 1) \\
\bar{u}_{\{1,2\}} &= (1, 1, 0, 0, 1, 1, 0) \\
\bar{u}_{\{1,3\}} &= (1, 0, 1, 1, 0, 1, 0) \\
\bar{u}_{\{2,3\}} &= (0, 1, 1, 1, 1, 0, 0) \\
\bar{u}_N &= (1, 1, 1, 0, 0, 0, 0).
\end{aligned}$$

We prove that the set of the commander games is a basis in the following theorem.

**Theorem 7.1** *The set of games $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$ is a basis of $\Gamma^N$.*

*Proof.* Let $v \in \Gamma^N$. From the fact that the dividend is the coefficient in the linear combination of the unanimity games, we have

$$\begin{aligned}
v &= \sum_{R \subseteq N : R \neq \emptyset} \frac{D(R, v)}{|R|} \cdot |R| u_R \\
&= \sum_{R \subseteq N : R \neq \emptyset} \frac{D(R, v)}{|R|} \cdot \sum_{T \subseteq R : T \neq \emptyset} (-1)^{|T|-1} \bar{u}_T \\
&= \sum_{T \subseteq N : T \neq \emptyset} (-1)^{|T|-1} \sum_{R \subseteq N : T \subseteq R} \frac{D(R, v)}{|R|} \bar{u}_T,
\end{aligned} \tag{7.1}$$

where the second equality holds because

$$\sum_{T \subseteq R : T \neq \emptyset} (-1)^{|T|-1} \bar{u}_T(S) = |R \cap S| \cdot \sum_{k=0}^{|R \setminus S|} \binom{|R \setminus S|}{k} (-1)^k = |R| u_R(S)$$

for $R \subseteq N$, $R \neq \emptyset$, and any $S \subseteq N$.

As a result, any game $v \in \Gamma^N$ can be expressed by a linear combination of the games $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$. In other words, the set $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$

spans $\Gamma^N$. If the set $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$ is linearly dependent, then there exist a coalition $T \subseteq N$, $T \neq \emptyset$, and a vector $(\alpha_S)_{\emptyset \neq S \subseteq N, S \neq T}$ such that

$$\bar{u}_T = \sum_{S \subseteq N : S \neq \emptyset, S \neq T} \alpha_S \bar{u}_S.$$

Together with Equation (7.1), the set $\Gamma^N$ can be spanned by vectors with less than $2^n - 1$ vectors, which is a contradiction.[3] ■

For any $v \in \Gamma^N$, let $d(T, v)$ denote the coefficient in the linear combination of $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$, namely, $v = \sum_{T \subseteq N : T \neq \emptyset} d(T, v) \bar{u}_T$. From Equation (7.1), we obtain the following proposition:

**Proposition 7.1** *For any $v \in \Gamma^N$,*

$$d(\{i\}, v) = \sum_{R \subseteq N : i \in R} \frac{D(R, v)}{|R|} = Sh_i(v) \text{ for all } i \in N.$$

Proposition 7.1 states that the coefficients related to singletons coincide with the Shapley value. The proof is given in Yokote et al. (2016). This is a very unique property of the commander games basis.

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## 7.4 Properties of Commander Games Basis

We refer to the properties of commander games basis stated in Yokote et al. (2016). Since the set $\{\bar{u}_T : T \subseteq N, |T| \geq 2\}$ consists of $2^n - 1 - n$ linearly independent vectors, we obtain the following proposition:

**Proposition 7.2** *The set $\{\bar{u}_T : T \subseteq N, |T| \geq 2\}$ spans the null space of $Sh$.*

Proposition 7.2 states that the Shapley value does not depend on the coefficient $d(T, v)$, $T \subseteq N$, $|T| \geq 2$. Recall that from Proposition 7.1, the coefficients $d(\{i\}, v)$, $i \in N$, coincide with the Shapley value. As a consequence, we obtain the following interesting corollary:

**Corollary 7.1** *Let $x \in \mathbf{R}^n$. Then, $Sh(v) = x$ if and only if there exists a vector $(\alpha_T)_{T \subseteq N : |T| \geq 2} \in \mathbb{R}^{2^n - 1 - n}$ such that*

$$v = \sum_{i \in N} x_i \bar{u}_{\{i\}} + \sum_{T \subseteq N : |T| \geq 2} \alpha_T \bar{u}_T.$$

[3] Recall the following result in linear algebra: If the vectors $x_1, \cdots, x_n$ span a linear space $X$ and the vectors $y_1, \cdots, y_j$ in $X$ are linearly independent, then $j \leq n$.

In Corollary 7.1, we characterize the set of all games to which the Shapley value assigns a fixed vector. This approach is known as the inverse problem. By solving the problem, we can characterize the equivalence relation that relates two different games with the same Shapley value. If two games are equivalent by the relation, we can know that the Shapley value is silent about the difference between the two situations described by the games.

**Example 7.2** *By using Corollary 7.1, we obtain the following result for* 3-*person games: Let* $v \in \Gamma^N$*,* $N = \{1,2,3\}$*,* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$*. Then,* $Sh(v) = x$ *if and only if there exists* $(y_{12}, y_{13}, y_{23}, y_N) \in \mathbb{R}^4$ *such that* $v(N) = x_1 + x_2 + x_3$ *and*

$$v(\{1,2\}) = x_1 + x_2 + y_{13} + y_{23}, v(\{3\}) = x_3 + y_{13} + y_{23} + y_N,$$
$$v(\{1,3\}) = x_1 + x_3 + y_{12} + y_{23}, v(\{2\}) = x_2 + y_{12} + y_{23} + y_N,$$
$$v(\{2,3\}) = x_2 + x_3 + y_{12} + y_{13}, v(\{1\}) = x_1 + y_{12} + y_{13} + y_N.$$

*The above equations imply*

$$v(\{1,2\}) = x_1 + x_2 + v(\{3\}) - x_3 - y_N,$$
$$v(\{1,3\}) = x_1 + x_3 + v(\{2\}) - x_2 - y_N,$$
$$v(\{2,3\}) = x_2 + x_3 + v(\{1\}) - x_1 - y_N.$$

*As a result, we obtain the following: Let* $N = \{1,2,3\}$ *and* $v \in \Gamma^N$ *be a game such that* $v(\{k\}) = 0$ *for all* $k \in N$*. Then,* $Sh(v) = x$ *if and only if there exists* $y \in \mathbb{R}$ *such that* $v(N) = x_1 + x_2 + x_3$ *and*

$$v(\{1,2\}) = x_1 + x_2 - x_3 + y,$$
$$v(\{1,3\}) = x_1 + x_3 - x_2 + y,$$
$$v(\{2,3\}) = x_2 + x_3 - x_1 + y.$$

*The "only if" part says that, given an arbitrary vector* $x$*, we can always find an identical amount* $y$ *for all coalitions with 2 players.*

The null space or the inverse problem have been investigated in previous works. As for the null space, Kleinberg and Weiss (1985) gave a direct sum decomposition of the space. Dragan et al. (1989) characterized the space by using a basis for TU games related to the potential function by Hart and Mas-Colell (1989). As for the inverse problem, see Dragan (2005) or Dragan (2012). The merit of using the commander games basis is that we can solve the above problems by using only simple games with clear meanings.

---

## 7.5 New Bases

Both collections of the unanimity games $\{u_T : \emptyset \neq T \subseteq N\}$ and the commander games $\{\bar{u}_T : \emptyset \neq T \subseteq N\}$ form a basis for the linear space $\Gamma^N$. Note that the two games $u_T$ and $\bar{u}_T$ capture two extreme cases: The cooperation of all players in $T$ yields payoff, or only one player in $T$ yields payoff. We consider the intermediate between the two extreme cases.

Let $T \subseteq N$ and $k \in \mathbb{N}$, $1 \le k \le |T|$. We define the $(T,k)$-*intermediate game* $\bar{u}_T^k$ by

$$\bar{u}_T^k(S) = \begin{cases} 1 & \text{if } |S \cap T| = k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\bar{u}_T^{|T|} = u_T$: The unanimity game, and $\bar{u}_T^1 = \bar{u}_T$: The commander game.

We prove that if there is some relationship between the size of coalition $T$ and $k$, then we can construct a basis. Consider an index function $l : \{1, \cdots, n\} \to \{1, \cdots, n\}$ satisfying the following conditions:

C1: $l(1) = 1$.

C2: $l(t) = l(t-1)$ or $l(t-1)+1$ or $l(t-1)-1$ for all $t = 2, \cdots, n$.

**Example 7.3** *For 3 person games* $(N, v)$ *with* $N = \{1,2,3\}$*, which satisfy C1 and C2, we only have three sets of games,* $\{\bar{u}_T^{(1)} : \emptyset \neq T \subseteq N\}$ *with* $l(2) = 1, l(3) = 2$*,* $\{\bar{u}_T^{(2)} : \emptyset \neq T \subseteq N\}$ *with* $l(2) = 2, l(3) = 2$*,* $\{\bar{u}_T^{(3)} : \emptyset \neq T \subseteq N\}$ *with* $l(2) = 2, l(3) = 1$*, except for the set of the unanimity games and the set of the commander games. These three sets are given as follows:*

$$\begin{aligned}
\bar{u}_{\{1\}}^{(1)} &= (1,0,0,1,1,0,1) \\
\bar{u}_{\{2\}}^{(1)} &= (0,1,0,1,0,1,1) \\
\bar{u}_{\{3\}}^{(1)} &= (0,0,1,0,1,1,1) \\
\bar{u}_{\{1,2\}}^{(1)} &= (1,1,0,0,1,1,0) \\
\bar{u}_{\{1,3\}}^{(1)} &= (1,0,1,1,0,1,0) \\
\bar{u}_{\{2,3\}}^{(1)} &= (0,1,1,1,1,0,0) \\
\bar{u}_{N}^{(1)} &= (0,0,0,1,1,1,0).
\end{aligned}$$

$$
\begin{aligned}
\bar{u}^{(2)}_{\{1\}} &= (1,0,0,1,1,0,1)\\
\bar{u}^{(2)}_{\{2\}} &= (0,1,0,1,0,1,1)\\
\bar{u}^{(2)}_{\{3\}} &= (0,0,1,0,1,1,1)\\
\bar{u}^{(2)}_{\{1,2\}} &= (0,0,0,1,0,0,1)\\
\bar{u}^{(2)}_{\{1,3\}} &= (0,0,0,0,1,0,1)\\
\bar{u}^{(2)}_{\{2,3\}} &= (0,0,0,0,0,1,1)\\
\bar{u}^{(2)}_{N} &= (0,0,0,1,1,1,0).
\end{aligned}
$$

$$
\begin{aligned}
\bar{u}^{(3)}_{\{1\}} &= (1,0,0,1,1,0,1)\\
\bar{u}^{(3)}_{\{2\}} &= (0,1,0,1,0,1,1)\\
\bar{u}^{(3)}_{\{3\}} &= (0,0,1,0,1,1,1)\\
\bar{u}^{(3)}_{\{1,2\}} &= (0,0,0,1,0,0,1)\\
\bar{u}^{(3)}_{\{1,3\}} &= (0,0,0,0,1,0,1)\\
\bar{u}^{(3)}_{\{2,3\}} &= (0,0,0,0,0,1,1)\\
\bar{u}^{(3)}_{N} &= (1,1,1,0,0,0,0).
\end{aligned}
$$

**Theorem 7.2** *Let $l$ be a function satisfying C1 and C2. Then, the set of games $\{\bar{u}_T^{l(|T|)} : \emptyset \neq T \subseteq N\}$ is a basis for $\Gamma^N$.*

Special cases of interest are when $l(k) = k$ for all $k = 1, \cdots, n$ and $l(k) = 1$ for all $k = 1, \cdots, n$. The former coincides with the basis consisting of the unanimity games, while the latter coincides with the basis consisting of the commander games. Thus, Theorem 7.2 generalizes the results by Shapley (1953) and Yokote et al. (2016).

Chapter 6: Games with Identical Shapley Values, in this book by S. Béal, M. Manea, E. Rémila, and P. Solal considers a class of bases for the space of all games, which contains our new basis (See Beál et, al. (2019)). Indeed they consider a set of games called oligarchic games and proved that the set of the games forms a basis for the space of all games in Theorem 6.3. Our results in this chapter are also restated in Theorem 6.4. However, since our proof is independent, we still give the proof of the theorem,

Before providing a formal proof of Theorem 7.2, we prove a lemma.

**Lemma 7.1** *Let $T \subseteq N$, $|T| \geq 2$, $k \in \mathbb{N}$, $2 \leq k \leq |T|$. Then, we have*

$$\bar{u}_T^k = \frac{1}{k}\left(\sum_{i\in T} \bar{u}_{T\setminus i}^{(k-1)} - (|T| - k + 1)\bar{u}_T^{(k-1)}\right).$$

*Proof.* Let $S \subseteq N$, $S \neq \emptyset$. We calculate the worth of $S$ in both sides.

**Case 1** $0 \leq |T \cap S| \leq k-2$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = \bar{u}_T^{(k-1)}(S) = 0$. Consider the game $\bar{u}_{T\setminus i}^{(k-1)}$, $i \in T$.

If $i \in S$, $|(T\setminus i) \cap S| \leq k-3$.
If $i \notin S$, $|(T\setminus i) \cap S| \leq k-2$.

It follows that $\bar{u}_{T\setminus i}^{(k-1)}(S) = 0$ for all $i \in T$.

**Case 2** $k+1 \leq |T \cap S| \leq |T|$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = \bar{u}_T^{(k-1)}(S) = 0$. Consider the game $\bar{u}_{T\setminus i}^{(k-1)}$, $i \in T$.

If $i \in S$, $|(T\setminus i) \cap S| \geq k$.
If $i \notin S$, $|(T\setminus i) \cap S| \geq k+1$.

It follows that $\bar{u}_{T\setminus i}^{(k-1)}(S) = 0$ for all $i \in T$.

**Case 3** $|T \cap S| = k-1$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^k(S) = 0$. Let $i \in T$.

If $i \in S$, $|(T\setminus i) \cap S| = k-2$.
If $i \notin S$, $|(T\setminus i) \cap S| = k-1$.

That is, if $i \in S \cap T$, then $\bar{u}_{T\setminus i}^{(k-1)}(S) = 0$. As a result,

$$\begin{aligned}\sum_{i\in T} \bar{u}_{T\setminus i}^{(k-1)}(S) &= \sum_{i\in T\setminus S} \bar{u}_{T\setminus i}^{(k-1)}(S) \\ &= \sum_{i\in T\setminus S} \bar{u}_T^{(k-1)}(S) \\ &= |T| - (k-1),\end{aligned}$$

where the second equality follows from $(T\setminus i) \cap S = T \cap S$ for $i \in T\setminus S$. Together with $-(|T|-k+1)\bar{u}_T^{(k-1)}(S) = -(|T|-k+1)$, the right-hand side is equal to 0, which is equal to the left-hand side.

**Case 4** $|T \cap S| = k$.

By definition of $\bar{u}_T^k$, we have $\bar{u}_T^{(k-1)}(S) = 0$. Let $i \in T$.

If $i \in S$, $|(T\setminus i) \cap S| = k-1$,
If $i \notin S$, $|(T\setminus i) \cap S| = k$.

Hence, $\sum_{i \in T} \bar{u}_{T \setminus i}^{(k-1)}(S) = k$, which implies that the right-hand side is equal to 1. Since the left-hand side is also equal to 1, the proof completes.

■

The proof of Theorem 7.2 proceeds by induction. First, it is already proved that the set $\{\bar{u}_T^1 : \emptyset \neq T \subseteq N\}$ is a basis. Given this induction base, we increase the sum of the numbers of superscripts. The induction step is completed by using Lemma 7.1.

*Proof. (Proof of Theorem 7.2)* Throughout the proof, we refer to functions $l$ satisfying C1 and C2. For a function $l$, we define

$$
\begin{aligned}
K(l) &= \sum_{k=1}^{n} l(k), \\
M(l) &= \max\{l(k) : k = 1, \cdots, n\}, \\
Q(l) &= \{k : l(k) = M(l)\}.
\end{aligned}
$$

**Induction base:** Suppose $K(l) = n$, namely, $l(k) = 1$ for all $k = 1, \cdots, n$. In this case, by Theorem 7.1, the proof completes.

**Induction step:** Suppose the result holds for all $l$ with $n \leq K(l) \leq p$, and we prove the result for $l$ with $K(l) = p + 1$, where $n \leq p \leq \frac{n(n+1)}{2} - 1$.

Assume, by way of contradiction, that the set of games $\bar{u}_T^{l(|T|)}$ is not a basis. Then, there exists $(\lambda_T)_{\emptyset \neq T \subseteq N} \neq \mathbf{0}$ such that

$$
\sum_{T \subseteq N : T \neq \emptyset} \lambda_T \bar{u}_T^{l(|T|)} = \mathbf{0}. \tag{7.2}
$$

Let $q \geq 2$ denote the natural number such that $l(q) = M(l)$ and $q \leq k$ for all $k \in Q(l)$. Then, the following equation holds:

$$
l(q-1) = l(q) - 1 \geq 1.
$$

By (6.2),

$$
\sum_{T \subseteq N : T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{l(|T|)} + \sum_{S \subseteq N : |S| = q} \lambda_S \bar{u}_S^{l(q)} = \mathbf{0}.
$$

By Lemma 7.1,

$$
\begin{aligned}
&\sum_{T \subseteq N : T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{l(|T|)} \\
&+ \sum_{S \subseteq N : |S| = q} \frac{\lambda_S}{l(q)} \left( \textstyle\sum_{i \in S} \bar{u}_{S \setminus i}^{(l(q)-1)} - (q - l(q) + 1) \bar{u}_S^{(l(q)-1)} \right) = \mathbf{0}.
\end{aligned} \tag{7.3}
$$

We define $l'$ by

$$l'(|T|) = \begin{cases} l(|T|) & \text{if } |T| \neq q, \\ l(q) - 1 & \text{if } |T| = q. \end{cases}$$

We show that $l'$ satisfies C1 and C2. Since $q \geq 2$, $l'(1) = l(1) = 1$, which proves C1. Since $l(q) = M(l)$, we have

$$l(q+1) = l(q) \text{ or } l(q) - 1.$$

If $l(q+1) = l(q)$, then $l'(q+1) = l(q+1) = l'(q)+1$. If $l(q+1) = l(q)-1$, then $l'(q+1) = l(q+1) = l'(q)$. Namely, $l'(q+1) = l'(q) + 1$ or $l'(q)$. In addition, $l'(q) = M(l) - 1 = l(q-1) = l'(q-1)$, which proves C2.

Using the function $l'$, (7.3) can be written as follows:

$$\begin{aligned} &\sum_{T \subseteq N: T \neq \emptyset, |T| \neq q} \lambda_T \bar{u}_T^{l'(|T|)} + \sum_{T \subseteq N: |T| = q} \frac{\lambda_T}{l(q)} \left( \textstyle\sum_{i \in T} \bar{u}_{T \setminus i}^{l'(|T \setminus i|)} - (q - l'(q)) \bar{u}_T^{l'(q)} \right) \\ = &\sum_{T \subseteq N: T \neq \emptyset, |T| \leq q-2, |T| \geq q+1} \lambda_T \bar{u}_T^{l'(|T|)} \\ &+ \sum_{T \subseteq N: |T| = q-1} \left( \lambda_T + \sum_{j \in N \setminus T} \frac{\lambda_{T \cup j}}{l(q)} \right) \bar{u}_T^{l'(|T|)} \qquad (7.4) \\ &- \sum_{T \subseteq N: |T| = q} \frac{\lambda_T (q - l'(q))}{l(q)} \bar{u}_T^{l'(q)} \\ = &\mathbf{0}. \end{aligned}$$

Since $K(l') \leq p$, by the induction hypothesis, all the coefficients in the above equation are 0. We obtain

$$\lambda_T = 0 \text{ for all } T \subseteq N, T \neq \emptyset, |T| \leq q-2, |T| \geq q+1,$$

and, together with $l'(q) = l(q) - 1 \leq q - 1$,

$$\lambda_T = 0 \text{ for all } T \subseteq N, |T| = q.$$

Substituting this equation into the coefficients in (7.4),

$$\lambda_T = 0 \text{ for all } T \subseteq N, |T| = q - 1.$$

We obtain a contradiction to $(\lambda_T)_{\emptyset \neq T \subseteq N} \neq \mathbf{0}$. ■

We prove that, by making an additional assumption on $l$, the basis consisting of the intermediate games preserves the desirable properties as commander games.

**Theorem 7.3** *Let $l$ be a function satisfying $C1$, $C2$ and $l(2) = 1$. Then,* *(1): The set of games* $\{\bar{u}_T^{l(|T|)} : \emptyset \neq T \subseteq N\}$ *is a basis of* $\Gamma^N$.

*(2): When we express a game* $v \in \Gamma^N$ *by a linear combination of this basis, the coefficient of* $\bar{u}_i^1$ *is equal to* $Sh_i(v)$ *for all* $i \in N$.

*(3): The set* $\{\bar{u}_T^{l(|T|)} : T \subseteq N, |T| \geq 2\}$ *spans the null space of the Shapley value.*

*Proof.* The first statement (1) follows from Theorem 7.2. Let $T \subseteq N$, $|T| \geq 2$, and $j \in N \backslash T$. Then, for any $S \subseteq N \backslash j$, we have $|S \cap T| = |(S \cup j) \cap T|$. It follows that $j$ is a null player. By the null player property, we obtain $Sh_j(\bar{u}_T^{l(|T|)}) = 0$. By symmetry and the fact that $\bar{u}_T^{l(|T|)}(N) = 0$, we have $Sh_i(\bar{u}_T^{l(|T|)}) = 0$ for all $i \in N$. As a result, (3) holds. It remains to prove (2). Let $v \in \Gamma^N$ be given. Let $(\alpha_T)_{T \in 2^N \backslash \emptyset}$ denote the coefficients in the linear combination of $v$ by $\bar{u}_T^{l(|T|)}$. Then, for any $i \in N$,

$$\begin{aligned} Sh_i(v) &= Sh_i\Big(\sum_{T \in 2^N \backslash \emptyset} \alpha_T \bar{u}_T^{l(|T|)}\Big) \\ &= \sum_{T \in 2^N \backslash \emptyset} \alpha_T Sh_i(\bar{u}_T^{f(|T|)}) \\ &= \sum_{j \in N} \alpha_j Sh_i(\bar{u}_j^1) \\ &= \alpha_i, \end{aligned}$$

where the fourth equality follows from $Sh_i(\bar{u}_i^1) = 1$ and $Sh_i(\bar{u}_j^1) = 0$ for all $j \in N \backslash i$.[4] ■

## 7.6 Basis and Coincidence Condition

We apply our new basis to the analysis of sufficient conditions for the Shapley value to coincide with the prenucleolus (Schmeidler (1969)).

Consider the following function $f : 2^N \to \mathbb{N}$:

$$f(T) = \begin{cases} 1 & \text{if } |T| = 1, \\ \frac{|T|+1}{2} & \text{if } |T| \text{ is an odd number, } |T| \geq 2, \\ \frac{|T|}{2} & \text{if } |T| \text{ is an even number, } |T| \geq 2. \end{cases}$$

For a coalition $T$ with even number of players, the game $\bar{u}_T^{f(T)}$ captures the situation in which half of the players in $T$ yields payoff. By Theorem 7.2, the

[4] These equations immediately follow from the null player property and efficiency.

set of games $\{\bar{u}_T^{f(T)} : \emptyset \neq T \subseteq N\}$ is linearly independent since $l(|T|) = f(T)$ satisfies conditions C1 and C2.

We define the set of games $\hat{\Gamma}$ by

$$\hat{\Gamma} = \mathrm{Sp}\big(\{\bar{u}_T^{f(T)} : T \in 2^N, |T| \geq 2, |T| \text{ is an even number}\} \cup \{u_{\{i\}} : i \in N\}\big).$$

We prove that $\hat{\Gamma}$ coincides with the set of games satisfying the PS property due to Kar et al. (2009). A game $v \in \Gamma^N$ satisfies the PS property if the following condition holds: For any $i \in N$, there exists $c_i \in \mathbb{R}$ such that

$$\Delta_i v(S) + \Delta_i v\big(N \backslash (S \cup i)\big) = c_i \text{ for all } S \subseteq N \backslash i. \tag{7.5}$$

This means that, for any player $i$, the sum of marginal contributions to a coalition and its complement is always constant. Under the PS property, the Shapley value coincides with the prenucleolus. Let $\Gamma^{PS}$ denote the set of games satisfying the PS property.

**Theorem 7.4** $\hat{\Gamma} = \Gamma^{PS}$.

Before providing a formal proof of Theorem 7.14, we prove a lemma. For any $v \in \Gamma^N$, we define $v^{Sh}$ by

$$v^{Sh} = v - \sum_{i \in N} Sh_i(v) u_{\{i\}}.$$

**Lemma 7.2** $v \in \Gamma^{PS}$ *if and only if* $v^{Sh}(T) = v^{Sh}(N \backslash T)$ *for all* $T \subseteq N$.

*Proof.* If $v$ satisfies the PS property, there exist $c_i \in \mathbb{R}$ for all $i \in N$ such that (7.5) holds. As proven by Kar et al. (2009), we have

$$Sh(v) = \Big(\frac{c_1}{2}, \cdots, \frac{c_n}{2}\Big).$$

By efficiency of the Shapley value, $v^{Sh}(N) = v^{Sh}(\emptyset) = 0$. We proceed by induction. Suppose that $v^{Sh}(R) = v^{Sh}(N \backslash R)$ for all $R \subseteq N$ with $|R| = r$, and we prove the result for $T \subseteq N$ with $|T| = r + 1$, where $r \geq 0$.

Let $i \in T$. Then,

$$v(T) - v(T \backslash i) + v\big((N \backslash T) \cup i\big) - v(N \backslash T) = c_i,$$
$$v^{Sh}(T) - v^{Sh}(T \backslash i) + v^{Sh}\big((N \backslash T) \cup i\big) - v^{Sh}(N \backslash T) = c_i - \frac{c_i}{2} - \frac{c_i}{2} = 0.$$

By the induction hypothesis, $v^{Sh}(T \backslash i) = v^{Sh}\big((N \backslash T) \cup i\big)$. It follows that

$$v^{Sh}(T) = v^{Sh}(N \backslash T).$$

Conversely, suppose that $v^{Sh}(T) = v^{Sh}(N \backslash T)$ for all $T \in 2^N$. Then, for any $i \in N$ and $T \in 2^N$ with $T \ni i$,

$$v^{Sh}(T) - v^{Sh}(T \backslash i) + v^{Sh}\big((N \backslash T) \cup i\big) - v^{Sh}(N \backslash T) = 0,$$
$$v(T) - v(T \backslash i) + v\big((N \backslash T) \cup i\big) - v(N \backslash T) = 2Sh_i(v),$$

which means that the PS property holds. ■

*Proof.* *(Proof of Theorem 7.4)* We first introduce additional notations. For a coalition $T \in 2^N \backslash \emptyset$, we define $e_{T,N\backslash T} \in \Gamma^N$ by

$$e_{T,N\backslash T}(S) = \begin{cases} 1 & \text{if } S = T \text{ or } N\backslash T, \\ 0 & \text{otherwise.} \end{cases}$$

One easily verifies that the following equation holds: For any $T \in 2^N \backslash \emptyset$,

$$Sh(e_{T,N\backslash T}) = \mathbf{0}. \tag{7.6}$$

By Lemma 7.2,

$$\Gamma^{PS} = \left\{ v \in \Gamma^N : v^{Sh}(T) = v^{Sh}(N\backslash T) \text{ for all } T \in 2^N \right\}.$$

Let $k \in N$ be fixed. We define

$$\Gamma^* = \mathrm{Sp}\big(\{e_{T,N\backslash T} : T \in 2^N \backslash \emptyset, k \notin T\} \cup \{u_{\{i\}} : i \in N\}\big).$$

We prove two claims.

**Claim 7.1** *The set of games* $\{e_{T,N\backslash T} : T \in 2^N \backslash \emptyset, k \notin T\} \cup \{u_{\{i\}} : i \in N\}$ *is linearly independent.*

*Proof.* Let $(\beta_T)_{T \in 2^N \backslash \emptyset, k \notin T}$ and $(\alpha_{\{i\}})_{i \in N}$ be such that

$$\sum_{T \in 2^N \backslash \emptyset, k \notin T} \beta_T e_{T,N\backslash T} + \sum_{i \in N} \alpha_{\{i\}} u_{\{i\}} = \mathbf{0}. \tag{7.7}$$

By linearity of the Shapley value,

$$\sum_{T \in 2^N \backslash \emptyset, k \notin T} \beta_T Sh(e_{T,N\backslash T}) = -\sum_{i \in N} \alpha_{\{i\}} Sh(u_{\{i\}}),$$

$$\mathbf{0} = -\alpha. \tag{7.8}$$

By substituting (7.8) to (7.7), we have

$$\sum_{T \in 2^N \backslash \emptyset, k \notin T} \beta_T e_{T,N\backslash T} = \mathbf{0},$$

which implies that $(\beta_T)_{T \in 2^N \backslash \emptyset, k \notin T} = \mathbf{0}$. ■

**Claim 7.2** $\Gamma^{PS} = \Gamma^*$.

*Proof.* **Proof of $\Gamma^{PS} \subseteq \Gamma^*$:** Let $v \in \Gamma^{PS}$. Then, $v^{Sh}(T) = v^{Sh}(N\backslash T)$, which implies $v^{Sh} \in \mathrm{Sp}(\{e_{T,N\backslash T} : T \in 2^N \backslash \emptyset, k \notin T\})$. Since

$$v = \sum_{i \in N} Sh_i(v) u_{\{i\}} + v^{Sh},$$

we obtain $v \in \Gamma^*$.

**Proof of $\Gamma^* \subseteq \Gamma^{PS}$:** Let $v \in \Gamma^*$. Then, by Claim 7.1, there exists a unique vector $(\beta_T)_{T\in 2^N\setminus\emptyset, k\notin T}$, $(\alpha_{\{i\}})_{i\in N}$, such that

$$v = \sum_{T\in 2^N\setminus\emptyset, k\notin T} \beta_T e_{T,N\setminus T} + \sum_{i\in N} \alpha_{\{i\}} u_{\{i\}}. \tag{7.9}$$

By linearity of the Shapley value and (7.6), we have

$$Sh\Big(\sum_{T\in 2^N\setminus\emptyset, k\notin T} \beta_T e_{T,N\setminus T}\Big) = \sum_{T\in 2^N\setminus\emptyset, k\notin T} \beta_T Sh(e_{T,N\setminus T}) = \mathbf{0}.$$

By Theorem 7.3, the set of games $\{\bar{u}_T^{f(T)} : T \subseteq N, |T| \geq 2\}$ spans the null space of the Shapley value. Hence, there exists a unique vector $(\gamma_T)_{T\in 2^N, |T|\geq 2}$ such that

$$\sum_{T\in 2^N\setminus\emptyset, k\notin T} \beta_T e_{T,N\setminus T} = \sum_{T\in 2^N: |T|\geq 2} \gamma_T \bar{u}_T^{f(T)}. \tag{7.10}$$

By substituting (7.10) to (7.9), we obtain

$$v = \sum_{T\in 2^N: |T|\geq 2} \gamma_T \bar{u}_T^{f(T)} + \sum_{i\in N} \alpha_{\{i\}} u_{\{i\}}.$$

By Theorem 7.3,

$$\alpha_{\{i\}} = Sh_i(v). \tag{7.11}$$

By substituting (7.11) to (7.9), we have

$$\begin{aligned} v &= \sum_{T\in 2^N\setminus\emptyset, i\notin T} \beta_T e_{T,N\setminus T} + \sum_{i\in N} Sh_i(v) u_{\{i\}}, \\ v^{Sh} &= \sum_{T\in 2^N\setminus\emptyset, i\notin T} \beta_T e_{T,N\setminus T}. \end{aligned}$$

Hence, $v^{Sh}(T) = v^{Sh}(N\setminus T)$ for all $T \in 2^N$. By Lemma 7.2, $v \in \Gamma^{PS}$. ■

By Claim 7.2, our final goal is to prove that $\Gamma^* = \hat{\Gamma}$. By Theorem 7.3, $\hat{\Gamma} \subseteq \Gamma^*$ holds. To prove the converse set-inclusion, we prove that $\dim \Gamma^* = \dim \hat{\Gamma}$, where $\dim X$ represents the dimension of $X$.

$$\begin{aligned} \dim \Gamma^{PS} &= |\{T \in 2^N\setminus\emptyset : k \notin T\}| + n \\ &= |\{T \in 2^{N\setminus k}\setminus\emptyset\}| + n \\ &= |\{T \in 2^{N\setminus k}\setminus\emptyset : |T| \text{ is an even number}\}| \\ &+ |\{T \in 2^{N\setminus k}\setminus\emptyset : |T| \text{ is an odd number}\}| + n \\ &= |\{T \in 2^N\setminus\emptyset : |T| \text{ is an even number}, k \notin T\}| \\ &+ |\{T \in 2^N\setminus\emptyset : |T \cup k| \text{ is an even number}, k \notin T\}| + n \\ &= \dim \hat{\Gamma}. \end{aligned}$$

■

## 7.7 Conclusions

The key advantage of linear algebraic approach is to clarify the mathematical structure behind solutions in cooperative games. Theorem 7.3 fully describes the game situations to which the Shapley value assigns the 0 vector. Theorem 7.4 tells us the dimension of the linear space $\Gamma^{PS}$, thereby clarifying how "large" the coincidence region is.

As bases are an essential mathematical tool in TU games, new bases might open up new analyses of solutions. For example, Yokote and Funaki (2017) provided a new axiomatization of solutions by using the basis

$$\{\bar{\bar{u}}_T : \emptyset \neq T \subseteq N\} = \{u_{\{i\}} : i \in N\} \cup \{\bar{u}_T^2 : |T| \geq 2\},$$

or equivalently,

$$\bar{\bar{u}}_T(S) = \begin{cases} 1 & \text{if } |T| = 1 \text{ and } T \subseteq S, \\ 1 & \text{if } |T| \geq 2 \text{ and } |S \cap T| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We give several candidates of interesting bases which satisfy conditions C1 and C2.

$$\{\hat{u}_T : \emptyset \neq T \subseteq N\} = \{u_{\{i\}} : i \in N\} \cup \{\bar{u}_T^{|T|-1} : |T| \geq 2\},$$

or equivalently,

$$\hat{u}_T(S) = \begin{cases} 1 & \text{if } |T| = 1 \text{ and } T \subseteq S, \\ 1 & \text{if } |T| \geq 2 \text{ and } |S \cap T| = |T| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\{\hat{\hat{u}}_T : \emptyset \neq T \subseteq N\} = \{u_T : \emptyset \neq T \subset N\} \cup \{\bar{u}_N^{n-1}\},$$

or equivalently,

$$\hat{\hat{u}}_T(S) = \begin{cases} 1 & \text{if } 1 \leq |T| \leq n-1 \text{ and } T \subseteq S, \\ 1 & \text{if } |T| = n \text{ and } |S| = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\{\hat{\hat{\hat{u}}}_T : \emptyset \neq T \subseteq N\} = \{\bar{u}_T : 1 \leq |T| \leq n-1, |S \cap T| = 1\} \cup \{\bar{u}_N^2\},$$

or equivalently,

$$\hat{\hat{\hat{u}}}_T(S) = \begin{cases} 1 & \text{if } 1 \leq |T| \leq n-1 \text{ and } |S \cap T| = 1, \\ 1 & \text{if } |T| = n \text{ and } |S| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For these bases, Theorem 7.3 induces that the following two sets of games span the null space of the Shapley value since $l(2) = 1$.

$$\{\hat{u}_T : T \subseteq N, |T| \geq 2\} = \{\bar{u}_T^{|T|-1} : T \subseteq N, |T| \geq 2\}.$$

$$\{\hat{\hat{\hat{u}}}_T : T \subseteq N, |T| \geq 2\} = \{\bar{u}_T : 2 \leq |T| \leq n-1, |S \cap T| = 1\} \cup \{\bar{u}_N^2\}.$$

It remains as a topic for future study to apply other bases including these bases to the analysis of the Shapley value and other linear values in TU games.

It is known that if $\{w_T : \emptyset \neq T \subseteq N\}$ is a basis of a game space, the set of the dual games $\{w_T^* : \emptyset \neq T \subseteq N\}$ is also a basis of the game space, where the dual game $w_T^*$ of a game $w_T$ is given by $w_T^*(S) = w_T(N) - w_T(N \setminus S)$ for all $S \subseteq N$. It might be interesting and important to find properties of bases given by dual games of $(T, k)$-intermediate games.

In this chapter we have restricted attention to the class of games satisfying the PS property, but there are other classes of games where the coincidence holds. A notable example is the class of clique games (Trudeau and Vidal-Puga, 2017), which is neither a superset nor a subset of the class of games satisfying the PS property. It remains as a topic for future work to discuss the relationship between clique games and our new bases.

## 7.8 Acknowledgments

The authors are grateful to the editors Encarnación Algaba, Vito Fragnelli and Joaquín Sánchez-Soriano of this volume for inviting us to contribute to it, and to an anonymous referee for the valuable comments which helped to improve the manuscript.

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# Chapter 8

## Extensions of the Shapley Value for Environments with Externalities

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### 8.1 Introduction

As is emphasized in the other chapters of this book, the Shapley value, a central concept in cooperative game theory, addresses the question of how players should share the gains from cooperation. Shapley (1953a) formulates his proposal for cooperative games with transferable utility in characteristic function form, that is, for games where the worth of the resources every group of players has available to distribute among its members depends exclusively on the actions of the group members. His proposal has important applications in economics, such as the study of markets with given sets of potential buyers and potential sellers.[1]

However, describing environments through characteristic function form games may imply an important shortcoming, since the worth of a coalition of players often depends on the actions of players outside the group. In fact, the existence of such external effects is one of the key ingredients in most economic, social, or political environments. To mention just a few examples, in treaty agreements the gain of the participant countries depends on the way the non-member countries act, that is, on whether they form a union or they partition into singletons. In economic or political mergers, the gain of the participants in the integration depends on the arrangements reached by the non-included firms or political parties. For cartels and research joint ventures, there are important cross effects, since what a group of players obtains depends on the groups formed by the other players.

The abundance of situations where externalities among coalitions are present calls for extending the class of cooperative games to allow for the presence of such cross effects. The first formal description of settings with externalities is provided by Thrall and Lucas (1963), who introduce games in partition function form. Since then, several cooperative solution concepts, and most notably the Shapley value, have been extended to games with externalities.

In this chapter, we present the extensions of the Shapley value to games in partition function form. One possible avenue to address the task of extending the value is to take the Shapley value axioms for games in characteristic function form and adapt them to that larger class of games. The extension of the Shapley value axioms has to take a stand on the treatment (importance) of the various externalities. Different approaches to these issues lead to distinct systems of axioms, in particular distinct dummy player axioms, all of which reduce to the original Shapley axioms in the absence of externalities. As a consequence, several plausible extensions of the Shapley value are obtained.

Myerson (1977) is the first attempt to extend the Shapley value for games in partition function form. As we will see later, his set of axioms identifies a unique value. However, in environments where externalities are present, natural extensions of the Shapley axioms do not necessarily imply a unique value. That is why most authors have imposed additional and/or different axioms to identify a unique solution (Bolger, 1989; Albizuri, Arin, Rubio, 2005; Macho-Stadler, Pérez-Castrillo, Wettstein, 2007; Pham Do and Norde, 2007; McQuillin, 2009; Hu and Yang, 2010; Grabisch and Funaki, 2012).

Other possible avenues to extend the Shapley value to games in partition function form are based on alternative ways to characterize the Shapley value, such as the marginalistic approach (De Clippel and Serrano, 2008a), the potential avenue (Dutta, Ehlers and Kar, 2010) and the algorithmic route.

[1] In addition, as several authors have underlined, the fact that the Shapley value can be interpreted in terms of "marginal contributions" makes it perhaps the game theoretic concept most closely related to traditional economic ideas (see, e.g., Aumann, 1994).

The chapter is organized as follows. In Section 8.2 we present the environment, and in Section 8.3 the proposals for extending the Shapley value for games with externalities using the axiomatic approach. Section 8.4 presents the extensions of the value based on the agents' marginal contributions. Section 8.5 describes extensions that follow the approaches of the potential, the Harsanyi dividends, and the algorithmic view. Section 8.6 provides non-cooperative foundations to several values for partition function form games. A concluding section offers some examples of applications and avenues for future research.

## 8.2 The Environment

Cooperative games with externalities were first introduced by Thrall and Lucas (1963) as transferable utility (TU) $n$-person games in *partition function form* (PFF) as follows. Given a set of players, $N = \{1, ..., n\}$, a coalition $S$ is a group of $s$ players, that is, a non-empty subset of $N$, $S \subseteq N$. An embedded coalition specifies the coalition, $S$, as well as the structure of coalitions formed by the other players, that is, an embedded coalition is a pair $(S, P)$, where $S$ is a coalition and $P \ni S$ is a partition of $N$. We adopt the convention that the empty set $\varnothing$ is in $P$ for every partition $P$ although we refrain from explicitly inserting it in the partitions. A particular partition is $[N] = \left\{\{i\}_{i\in N}\right\}$, where all the coalitions are singleton coalitions. More generally, we denote by $[S]$ the partition of $S$ consisting of all the singleton players in $S$, that is, $[S] = \left\{\{i\}_{i\in S}\right\}$.

Let $\mathcal{P}(N)$ denote the set of all partitions of $N$ and $\mathcal{P}_S = \{P \in \mathcal{P}(N) \mid S \in P\}$ the set of partitions including $S$. The set of embedded coalitions of $N$ is denoted by $ECL$:

$$ECL = \{(S, P) \mid P \in \mathcal{P}_S \text{ and } S \subseteq N\}.$$

A PFF game is given by a set of players, $N$, and a function, $v : ECL \to \mathbb{R}$, that associates a real number with each embedded coalition. Thus, $v(S, P)$ is the worth of coalition $S$ when the players are organized according to the partition $P$. We assume that $v(\varnothing, P) = 0$. Let $\mathcal{G}^N$ be the set of games in PFF with players in $N$. We will sometimes refer to some particularly simple games which we will denote by $(N, w_{(S,P)})$. The function $w_{(S,P)}$ is defined as $w_{(S,P)}(S, P) = w_{(S,P)}(N, \{N\}) = 1$ and $w_{(S,P)}(S', P') = 0$ for any $(S', P')$ different from $(S, P)$ and $(N, \{N\})$.[2]

Some games in $\mathcal{G}^N$ do not have externalities. A game has no externalities if the worth of any coalition $S$ is independent of the way the other players are organized. A game with no externalities satisfies $v(S, P) = v(S, P')$ for any

[2] The set of games $\left\{w_{(S,P)}\right\}_{(S,P)\in ECL}$ constitutes a basis for $\mathcal{G}^N$.

$P, P' \in \mathcal{P}_S$ and any coalition $S \subseteq N$. We denote a game with no externalities by $\hat{v}$. Since in this case the worth of a coalition $S$ can be written without reference to the organization of the remaining players, we can write $\hat{v}(S) \equiv \hat{v}(S, P)$ for all $P \in \mathcal{P}_S$ and all $S \subseteq N$ for such games. We denote by $G^N$ the set of games without externalities with players in $N$, which corresponds to the set of TU games in *characteristic function form* (CFF). For convenience, we will denote a value for games in characteristic form by $\psi$, that is, $\psi : G^N \to \mathbb{R}^n$. We denote the Shapley value for a CFF game $\hat{v}$ by $\psi^{Sh}(\hat{v})$.

After the introduction of PFF games by Thrall and Lucas (1963), the subsequent literature dealt with both the structure of multi-valued solutions and the construction of single-valued solutions for PFF games. A single-valued solution is given by a function $\varphi : \mathcal{G}^N \to \mathbb{R}^n$, where $\varphi_i(v)$ is the payoff assigned by the solution $\varphi$ to player $i \in N$ in the PFF game $v$. As mentioned in the Introduction, in this chapter we are interested in the extensions of the Shapley value $\psi^{Sh}(\hat{v})$ for PFF games.

## 8.3 Axiomatic Extensions of the Shapley Value for Games with Externalities

One branch of the literature takes as a starting point the axioms underlying the Shapley value for CFF games. These axioms can be extended to PFF games in several ways and give rise to several distinct "Shapley-like" values. New axioms can also be proposed to deal with the externalities.

First of all, let us note that all the values we present in this section assume that the grand coalition will form and thus the value will share the worth of the grand coalition, that is, the value is efficient.[3]

**Efficiency axiom.** A value $\varphi$ is efficient if $\sum_{i \in N} \varphi_i(N, v) = v(N, \{N\})$ for any $v \in \mathcal{G}^N$.

Myerson (1977) was the first to extend the Shapley axioms to PFF games and obtain a value for this class of games. The symmetry and additivity axioms were extended in the following, natural way. Let us define the $\sigma-$ permutation of the game $v \in \mathcal{G}^N$, denoted by $\sigma v$, as $(\sigma v)(S, P) \equiv v(\sigma S, \sigma P)$ for all $(S, P) \in ECL$.

**Symmetry axiom.** A value $\varphi$ is symmetric if $\varphi(\sigma v) = \sigma \varphi(v)$ for any $v \in \mathcal{G}^N$ and for any permutation $\sigma$ of $v$.

[3] This may be the most adequate assumption for games where the grand coalition maximizes joint surplus. Hafalir (2007) shows that a natural extension of superadditivity for PFF games is not sufficient to imply that the grand coalition is efficient, and provides a condition, analogous to convexity, that is sufficient for a game to have this feature.

This symmetry axiom is interpreted as an anonymity axiom.

If we define the addition of two games $v$ and $v'$ in $\mathcal{G}^N$ as the game $v + v'$, where $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$ for all $(S, P) \in ECL$, then the additivity axiom can be written as follows:

**Additivity axiom.** A value $\varphi$ is additive if $\varphi(v + v') = \varphi(v) + \varphi(v')$ for any $v, v' \in \mathcal{G}^N$.

In Myerson (1977), the dummy and efficiency axioms are extended by providing a carrier definition for PFF games. We say that $S \subseteq N$ is a *carrier* for $v$ if and only if

$$v(\tilde{S}, \tilde{Q}) = v(\tilde{S} \cap S, \tilde{Q} \wedge \{S, N \backslash S\}) \text{ for every } (\tilde{S}, \tilde{Q}) \in ECL.$$

That is, $S$ is a carrier for $v$ if the payoff of any embedded coalition $(\tilde{S}, \tilde{Q})$ is determined by the set of players in $\tilde{S}$ that are in $S$ and the meet $\tilde{Q} \wedge \{S, N \backslash S\}$ of the partitions $\tilde{Q}$ and $\{S, N \backslash S\}$ (the largest partition that refines both). The carrier axiom for CFF games is then extended as follows:

**Carrier axiom.** A value $\varphi$ satisfies the carrier axiom if $\sum_{i \in S} \varphi_i(N, v) = v(N, \{N\})$ for any $v \in \mathcal{G}^N$ for which $S$ is a carrier.

The three axioms of symmetry, additivity, and carrier yield a unique value, allowing Myerson (1977) to propose the extension $\varphi^M(v)$ given by

$$\varphi_i^M(v) = \sum_{(S,P) \in ECL} (-1)^{|P|-1} (|P| - 1)! \left[ \frac{1}{n} - \sum_{\substack{T \in P \backslash \{S\} \\ i \notin T}} \frac{1}{(|P| - 1)(n - |T|)} \right] v(S, P)$$

for any $i \in N$, where $|T|$ is the number of agents in $T$ and $|P|$ is the number of non-empty coalitions in $P$.[4]

While the extension of the efficiency axiom through the carrier axiom is natural, the extension of the dummy player axiom may be more problematic. A player $i \in N$ is a dummy player, in the sense of Myerson (1977), if there exists a carrier $S$ with $i \notin S$. The carrier axiom implies that such a dummy player will receive zero according to $\varphi^M$. This is problematic since a dummy player, thus defined, might have an effect on the worth of coalitions. Take, for example, the game with three players $(\{1, 2, 3\}, w_{(\{1\},\{\{1\},\{2,3\}\})})$. In this game, player 1 is a carrier and hence players 2 and 3 are dummy players. Therefore, $\varphi_1^M(w_{(\{1\},\{\{1\},\{2,3\}\})}) = 1$ and $\varphi_2^M(w_{(\{1\},\{\{1\},\{2,3\}\})}) = \varphi_3^M(w_{(\{1\},\{\{1\},\{2,3\}\})}) = 0$. On the other hand, in the possibly similar game $(\{1, 2, 3\}, w_{(\{1\},\{\{1\},\{2\},\{3\}\})})$, player 1 is not a carrier and, in fact, $\varphi_1^M(w_{(\{1\},\{\{1\},\{2\},\{3\}\})}) = 0$.

[4] Albizuri (2010) adapts the axioms in Myerson (1977) to extend $\varphi^M$ to a new class of games, where players can take part in more than one coalition, named "games in coalition configuration function form."

Bolger (1989) is the second author to obtain a value for PFF games by suggesting a different extension of the Shapley axioms. The efficiency and symmetry axioms are extended as above and the additivity axiom is strengthened to a natural linearity axiom, regarding both addition and multiplication by a scalar.[5]

Formally, given the game $v \in \mathcal{G}^N$ and the scalar $\lambda \in \mathbb{R}$, the game $\lambda v$ is defined by $(\lambda v)(S, P) \equiv \lambda v(S, P)$ for all $(S, P) \in ECL$.

**Linearity axiom.** A value $\varphi$ is linear if it is additive and $\varphi(\lambda v) = \lambda\varphi(v)$ for any $v \in \mathcal{G}^N$ and for any scalar $\lambda \in \mathbb{R}$.

Bolger (1989) also introduces a dummy player axiom, which is a natural generalization of the dummy player axiom for CFF games. We will say that player $i$ is a *dummy* player in $v \in \mathcal{G}^N$ if he alone receives zero for any partition of the other players and, furthermore, he has no effect on the worth of any coalition $S$ (i.e., the worth of $S$ in partition $P$ is constant for all possible assignments of player $i$ to some coalition in $P$). That is, player $i$ is a dummy player in $v \in \mathcal{G}^N$ if for every $(S, P) \in ECL$ with $i \in S$ and each $R \in P\backslash\{S\}$, $v(S, P) = v(S\backslash\{i\}, P\backslash \{S, R\} \cup \{S\backslash\{i\}, R \cup \{i\}\})$.[6]

**Dummy player axiom.** A value $\varphi$ satisfies the dummy player axiom if $\varphi_i(v) = 0$ for any game $v \in \mathcal{G}^N$ and any dummy player $i$ in the game $v$.

The final axiom considered by Bolger (1989) is inspired by the desired behavior of the value over simple games, where $v(S, P)$ equals either zero or one. It states that if the sum of marginal contributions of player $i$ to any coalition in $v \in \mathcal{G}^N$ is the same as in $v' \in \mathcal{G}^N$, then player $i$ should receive the same payoff in both games. This axiom is well suited to simple games but it may be less intuitive for general games in PFF.

While Bolger (1989) shows that efficiency, symmetry, linearity, dummy player,[7] plus the additional axiom related to the sum of marginal contributions imply that there is a unique value $\varphi^B$, there is no closed-form expression for $\varphi^B$.

Macho-Stadler, Pérez-Castrillo, and Wettstein (2007 and 2017) introduce a new axiom, strong symmetry, in addition to the efficiency and symmetry axioms (appearing in both Myerson, 1977, and Bolger, 1989). The strong symmetry axiom strengthens the symmetry axiom by requiring that a player's payoff should not change after permutations in the set of players in $N\backslash S$, for any embedded coalition structure $(S, P)$. To illustrate its meaning, consider

[5]In Myerson's (1977) extension, there is no need to introduce the linearity axiom. As is the case for CFF games, additivity together with the carrier axiom imply linearity. However, this is not true for the definitions of dummy player used in most papers (see Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007, for a formal proof).

[6]For $R = \varnothing$, we slightly abuse notation by assuming that the partition $P\backslash \{S, \varnothing\} \cup \{S\backslash\{i\}, \varnothing \cup \{i\}\}$ also includes the empty set.

[7]Sánchez-Pérez (2015) provides a representation of all the values that satisfy efficiency, symmetry, linearity, and dummy player.

the following two games with four players: $(\{1,2,3,4\}, w_{(\{1\},\{\{1\},\{2\},\{3,4\}\})})$ and $(\{1,2,3,4\}, w_{(\{1\},\{\{1\},\{3\},\{2,4\}\})})$. Strong symmetry requires that player 2 should receive the same payoff in both games. Another way to view it is that 2 should receive the same payoff as 3 and 4 in $w_{(\{1\},\{\{1\},\{2\},\{3,4\}\})}$. Since the roles of players 2 and 3 (or 4) are similar (because they only generate the externality if they are organized in a particular way), this axiom can be viewed as a symmetric treatment of the externalities generated by players. Put differently, exchanging the names of the players inducing externalities does not affect the payoff of any player.

Formally, given an embedded coalition $(S, P)$, denote by $\sigma_{(S,P)}P$ a new partition such that $S \in \sigma_{(S,P)}P$, and the other coalitions result from a permutation of the set $N\backslash S$ applied to $P\backslash\{S\}$. That is, in the partition $\sigma_{(S,P)}P$, the players in $N\backslash S$ are reorganized in sets whose size distribution is the same as in $P\backslash\{S\}$. Given the permutation $\sigma_{(S,P)}$, the permutation of the game $v$ denoted by $\sigma_{(S,P)}v$ is defined by $(\sigma_{(S,P)}v)(S,P) = v(S, \sigma_{(S,P)}P)$, $(\sigma_{(S,P)}v)(S, \sigma_{(S,P)}P) = v(S,P)$, and $(\sigma_{(S,P)}v)(R,Q) = v(R,Q)$ for all $(R,Q) \in ECL\backslash \left\{(S,P), (S, \sigma_{(S,P)}P)\right\}$.

**Strong symmetry axiom.** A value $\varphi$ satisfies the strong symmetry axiom if for any game $v \in \mathcal{G}^N$ it is the case that

**1.** $\varphi(\sigma v) = \sigma\varphi(v)$ for any permutation $\sigma$ of $N$, and

**2.** $\varphi(\sigma_{(S,P)}v) = \varphi(v)$ for any $(S,P) \in ECL$ and for any permutation $\sigma_{(S,P)}$.

Note that strong symmetry is implied by symmetry when there are just three players, but it is a more demanding property for games with more players.

The symmetry axioms above are associated with the idea of anonymity. One could instead require a different axiom, often considered in CFF games, usually called equal treatment of equals. This property requires that interchangeable players (that is, players that can be interchanged without affecting the value of any coalition) should receive the same payoff. For games in CFF, the symmetry and equal treatment axioms are equivalent for efficient and additive values.

Macho-Stadler, Pérez-Castrillo, and Wettstein (2017) introduce a strong equal treatment axiom for PFF games by defining a weak version of interchangeability: Players $i$ and $j$ are *weakly interchangeable* in $v \in \mathcal{G}^N$ if for all $(S,P)$ with $i \in S$ and $j \in R \in P\backslash\{S\}$, $v(S,P) = v\left((S\backslash\{i\}) \cup \{j\}, P\backslash\{S,R\} \cup \{(S\backslash\{i\}) \cup \{j\}, (R\backslash\{j\}) \cup \{i\}\}\right)$. That is, players $i$ and $j$ are weakly interchangeable in the game $v$ if for any coalition $S$ including one of them, switching them does not affect the value of any embedded coalition $(S,P)$. For example, in the game $(\{1,2,3,4\}, w_{(\{1\},\{\{1\},\{2\},\{3,4\}\})})$, players $2,3$, and 4 are weakly interchangeable.

**Strong Equal Treatment axiom.** A value $\varphi$ satisfies the strong equal treatment axiom if $\varphi_i(N,v) = \varphi_j(N,v)$ for any pair of weakly interchangeable players $i$ and $j$ in $v$.

Strong equal treatment and strong symmetry axioms are equivalent for linear and efficient values (Macho-Stadler, Pérez-Castrillo, and Wettstein, 2017).

An additional motivation for the strong symmetry axiom is that combined with efficiency and linearity, it provides an axiomatic foundation for the use of an intuitive approach to construct values for PFF games, namely the *average approach*, introduced in Macho-Stadler, Pérez-Castrillo, and Wettstein (2007 and 2017).[8] This approach assigns to each coalition an average of the surpluses it obtains in all the partitions it might belong to. In this way, it first transforms a PFF game to a CFF game. It then uses a value for CFF games to determine the payoffs of the players in the PFF game.

Formally, the average approach constructs a value $\varphi$ for PFF games using a value for CFF games $\psi$ as follows. First, for any $v \in \mathcal{G}^N$, it constructs an average game $\tilde{v} \in G^N$ by assigning to each $S \subseteq N$ the average worth $\tilde{v}(S) \equiv \sum_{P \in \mathcal{P}_S} \alpha(S,P)v(S,P)$, with $\sum_{P \in \mathcal{P}_S} \alpha(S,P) = 1$. We refer to $\alpha(S,P)$ as the "weight" of partition $P$ in the computation of the value of coalition $S \in P$. Second, the value is defined as $\varphi(v) = \psi\left(\tilde{v}\right)$.[9]

Macho-Stadler, Pérez-Castrillo, and Wettstein (2017) show that a value $\varphi$ can be constructed through the average approach using a value for CFF games $\psi$ that satisfies efficiency, linearity, and symmetry if and only if $\varphi$ satisfies efficiency, linearity, and strong symmetry.[10] Adding the dummy player axiom to the desirable requirements for a value implies a constraint on the weights $\alpha(S,P)$, but still, many weighting systems are compatible with the four axioms. If we define the average game $\tilde{v}$ using any of these weights, then $\varphi(v) = \psi^{Sh}\left(\tilde{v}\right)$ is an extension of the Shapley value.

To select a single value, Macho-Stadler, Pérez-Castrillo, and Wettstein (2007) propose a similar influence axiom. This axiom guarantees that similar environments lead to similar payoffs for the players. Consider, for example, the following two games with three players: $(\{1,2,3\}, w_{(\{1\},\{\{1\},\{2,3\}\})})$ and $(\{1,2,3\}, w_{(\{1\},\{\{1\},\{2\},\{3\}\})})$. These two games are very similar: In both, only player 1 can produce some benefits alone. The only difference is that in $w_{(\{1\},\{\{1\},\{2,3\}\})}$, players 2 and 3 need to be together for the benefits to player 1 to be realized, while in $w_{(\{1\},\{\{1\},\{2\},\{3\}\})}$, players 2 and 3 should be separated. The similar influence axiom requires players 2 and 3 to receive the same payoff in both games.

Formally, we say that a pair of players $\{i,j\} \subseteq N, i \neq j$, has *similar influence* in games $v$ and $v'$ if $v(T,Q) = v'(T,Q)$ for all $(T,Q) \in ECL \backslash \{(S,P),(S,P')\}$,

[8] In the previous paper, the average approach was restricted to values satisfying the efficiency, linearity, and dummy player axioms whereas in the latter it was applied to all efficient and linear values.

[9] As will be clear, all the axiomatically based values described in the remainder of this section satisfy this approach.

[10] Hence, the average approach can be used to extend both the Shapley value and several other values, such as the equal division value (van den Brink, 2007), the equal surplus value (Driessen and Funaki, 1991), the $\lambda-$egalitarian Shapley value (Joosten, 1996), the consensus value (Ju, Borm, and Ruys, 2007), and the family of least-square values (Ruiz, Valenciano, and Zarzuelo, 1998).

$v(S,P) = v'(S,P')$, and $v(S,P') = v'(S,P)$, where the only difference between partitions $P$ and $P'$ is that $\{i\},\{j\} \in P\backslash\{S\}$ while $\{i,j\} \in P'\backslash\{S\}$.

**Similar influence axiom.** A value $\varphi$ satisfies the similar influence axiom if for any two games $v, v' \in \mathcal{G}^N$ and for any pair of players $\{i,j\} \subseteq N$ that has similar influence in those games, we have $\varphi_i(v) = \varphi_i(v')$ and $\varphi_j(v) = \varphi_j(v')$.

The axioms of efficiency, linearity, dummy player, strong symmetry, and similar influence characterize a unique solution which can be constructed through the average approach by using the following weights:

$$\alpha^{MPW}(S,P) = \frac{\prod_{T \in P\backslash\{S\}} (|T|-1)!}{(n-|S|)!}.$$

Note that $\alpha^{MPW}(S,P)$ can be interpreted as the probability that partition $P$ is formed, given that coalition $S$ forms.[11] For any $v \in \mathcal{G}^N$, once we have computed the average game $\tilde{v}^{MDW}$ using these weights, we obtain:

$$\varphi^{MPW}(v) = \psi^{Sh}(\tilde{v}^{MDW}).$$

This same value was proposed, but not axiomatized, by Feldman (1996). Let us finally note that $\varphi^{MPW}$ satisfies the strong dummy axiom:

**Strong dummy player axiom.** A value $\varphi$ satisfies the strong dummy player axiom if for any dummy player $i$ in the game $v$, $\varphi_j(N,v) = \varphi_j(N\backslash\{i\},v)$ for all $j$ in $N\backslash\{i\}$.

The strong dummy property requires that adding or subtracting a dummy player from a game leaves the outcomes of the remaining players unchanged.[12]

Albizuri, Arin, and Rubio (2005) provide another extension of the Shapley value for PFF games, using the efficiency, symmetry, and additivity axioms, to which they add two additional properties. First, they introduce the oligarchy axiom (which can be viewed as a type of carrier axiom) for PFF games.

**Oligarchy axiom.** A value $\varphi$ satisfies the oligarchy axiom if for any $v \in \mathcal{G}^N$ for which there exists $R \subseteq N$ such that $v(S,P) = v(N,\{N\})$ if $R \subseteq S$ and $v(S,P) = 0$ if $R \nsubseteq S$, then $\sum_{i \in R} \varphi_i(v) = v(N,\{N\})$.

This axiom states that if there is a (oligarchic) coalition $R$ in a game $v$ such that any coalition that contains $R$ generates the worth of the grand

[11] According to this interpretation, the denominator in the expression that defines $\alpha^{MPW}(S,P)$ is the number of permutations of the players in $N\backslash S$. The numerator counts the number of those permutations of $N\backslash S$ that "generate" the partition $P$, when we write a permutation as a cycle.

[12] This property is satisfied by the Shapley value in CFF games. Note also that for any efficient value, the strong dummy player axiom implies the dummy player axiom.

coalition, whereas any other embedded coalition has zero worth, then all the worth must be shared among the members of the oligarchic coalition. Thus, in some sense, this axiom implies a form of null player axiom, different from the dummy player axiom as defined above.

Finally, to introduce the last axiom, Albizuri, Arin, and Rubio (2005) consider a coalition $S \subseteq N$ and a bijection $\xi_S$ on $\{(S,P) \mid P \in \mathcal{P}_S\}$. For each $v \in \mathcal{G}^N$, denote by $\xi_S v$ the game in $\mathcal{G}^N$ such that $(\xi_S v)(S,P) = v(\xi_S(S,P))$ for any $P \in \mathcal{P}_S$ and $(\xi_S v)(T,P) = v(T,P)$ for any $T \in N \backslash S$ and any $P \in \mathcal{P}_T$.

**Embedded coalition anonymity axiom.** A value $\varphi$ satisfies the embedded coalition anonymity axiom if for any bijection $\xi_S$ on $\{(S,P) \mid P \in \mathcal{P}_S\}$, and for any $v \in \mathcal{G}^N$, it is the case that $\varphi(\xi_S v) = \varphi(v)$.

The embedded coalition anonymity axiom states that the determinant of the players' payoffs is the worth of the embedded coalitions, irrespective of the partitions that generate the worth.

Albizuri, Arin, and Rubio (2005) show that the axioms of efficiency, symmetry, additivity, oligarchy, and embedded coalition anonymity characterize a unique solution. It is given by the Shapley value of the CFF game derived from the PFF game by assigning to each coalition the arithmetic average of its worth for all the possible partitions it may belong to. That is, defining the game $\tilde{v}^{AAR} \in G^N$ as $\tilde{v}^{AAR}(S) = \sum_{Q \in \mathcal{P}_S} \frac{1}{|\mathcal{P}_S|} v(S,Q)$, the value is:

$$\varphi^{AAR}(v) = \psi^{Sh}(\tilde{v}^{AAR}).$$

In another axiomatic proposal, Pham Do and Norde (2007) use the efficiency, additivity, and strong equal treatment axioms. In addition, they introduce an extension of the dummy player axiom that is stronger than the one we previously defined, as they propose a weaker definition of a null player. They call player $i \in N$ a null player if player $i$'s worth as a singleton is zero for any partition in $\mathcal{P}_{\{i\}}$ and his marginal contribution to any other coalition is zero when he joins the coalition from being a singleton. Formally, player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $v(\{i\}, P) = 0$ for every $(\{i\}, P) \in ECL$ and $v(S \cup \{i\}, P \backslash \{S, \{i\}\} \cup \{S \cup \{i\}\}) = v(S,P)$ for each $(S,P)$ with $S \neq \{i\}$ and $\{i\} \in P$. Note that a so-defined null player can affect the worth of coalition $S$ when he moves among coalitions other than $S$. Therefore, a player can be a null player and not dummy.

**Null player axiom.** A value $\varphi$ satisfies the null player axiom if $\varphi_i(v) = 0$ for any $v \in \mathcal{G}^N$ and any null player $i$ in $v$.

Pham Do and Norde (2007) show that there is a unique solution satisfying efficiency, additivity, symmetry, and null player. It is given by the Shapley value of the CFF game defined by $\hat{v}^{PN}(S) \equiv v(S, [P \backslash \{S\}] \cup \{S\})$:

$$\varphi^{PN}(v) = \psi^{Sh}(\hat{v}^{PN}).$$

Note that the value $\varphi^{PN}$ ignores most of the information provided by the whole PFF game.

A simultaneous extension of both the Shapley value and the Owen value (Owen, 1977) for CFF games with an *a priori* coalition structure is provided by McQuillin (2009). He introduces the idea of an extended generalized value (EGV), which is a mapping $\chi : \mathcal{G}^N \to \mathcal{G}^N$. For $v \in \mathcal{G}^N$, $\chi(v)(S,P)$ is the value of coalition $S$ in game $v$ with an initial coalition structure given by $P$. When $P = [N]$, the corresponding function $\chi(v)(\{i\},[N])$ constitutes a standard value extension to PFF games. For partitions different from $[N]$, the values obtained extend values for CFF games with an initial coalition structure.

To obtain an EGV, McQuillin (2009) uses efficiency, symmetry, linearity, and dummy player (which he constructs via an appropriate extension of the carrier axiom). In addition, he introduces a weak monotonicity condition. Let $w^o_{(S,P)}$ denote the function given by $w^o_{(S,P)}(S,P) = 1$ and $w^o_{(S,P)}(R,Q) = 0$ when $(R,Q) \neq (S,P)$; then:

**Weak monotonicity axiom.** A value $\chi$ satisfies weak monotonicity if $\chi(w^o_{(S,P)})(\{i\},[N]) \geq 0$ for any $i \in S$ and any game $w^o_{(S,P)}$.

Three further axioms are related to the behavior of the value in the presence of an *a priori* coalition structure. The first is the *rule of generalization*, implying that given an *a priori* coalition structure, each member of the partition is viewed as a single player. The second is the *cohesion axiom*, which requires that the payoff to any embedded coalition $(S,P)$ depends only on the payoffs to those embedded coalitions with partitions that are coarser than $(S,P)$. The third strengthens the dummy axiom, through a *generalized null player axiom*, by requiring that a dummy player in $v$ is also a dummy player in $\chi(v)$. The final axiom is the *recursion axiom* stating that $\chi(\chi(v)) = \chi(v)$; that is, the solution is the right way to assign payoffs: Once payoffs are assigned according to the solution, the solution will "agree" that these are the appropriate payoffs.

This set of axioms leads to a unique value called the extended Shapley value. It is given by the Shapley value of each player in the CFF game derived from the PFF game by $\hat{v}^{MQ}(S) \equiv v(S,\{N\backslash S, S\})$:

$$\varphi_i^{MQ}(v) = \chi(v)(\{i\},[N]) = \psi_i^{Sh}(\hat{v}^{MQ}) \text{ for all } i \in N.$$

The value $\varphi^{MQ}$ again abstracts from most of the information provided by the whole PFF game; it only takes into account the worth of a coalition $S$ when other players form the complementary coalition $N\backslash S$. McQuillin (2009) interprets it in two ways. From a normative point of view, most information should indeed be ignored based on the properties the extension should satisfy. From a positive point of view, it implies an impossibility result: If all the information in the PFF game is taken into account, it is impossible to satisfy the axioms and the recursion property.

Finally, Hu and Yang (2010) extend the Shapley value using efficiency, symmetry, additivity, and introducing a demanding extension of the dummy player axiom. In their proposal, a player $i \in N$ is an "average dummy player" if his average contribution to every coalition is zero, where the average is taken over all the possible partitions including the coalition. Then, Hu and Yang (2010) require the value to satisfy the axiom that the average dummy players must obtain zero. They show that this set of axioms characterizes a unique extension of the Shapley value for PFF games, which can be written as follows:

$$\varphi_i^{HY}(v) = \sum_{\substack{P \in \mathcal{P}(N) \\ S \subseteq N, S \ni i}} \frac{(|S|-1)!\,(n-|S|)!}{n!\,|\mathcal{P}(N)|} \left(v\left(S, P_{-S}\right) - v\left(S \backslash \{i\}, P_{-(S \backslash \{i\})}\right)\right),$$

where, for $P \in \mathcal{P}(N)$, we denote $P_{-S} = \{T \backslash S \mid T \in P\} \cup \{S\}$, and similarly for $P_{-(S \backslash \{i\})}$.

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## 8.4 Marginal Contributions

The *marginal contribution* of a player to a coalition in a CFF game is the difference between the value of this coalition with and without the player. It can also be understood as a loss incurred by the remaining agents when the player leaves the coalition. For CFF games, the concept of marginal contribution of players plays an important role in the analysis of values both axiomatically and operationally (when calculating the values). In particular, Young (1985) proposes to replace the additivity and dummy player axioms in the characterization of the Shapley value for CFF games with a marginality axiom requiring a player's payoff to depend only on his own productivity measured by marginal contributions. He proves that the Shapley value can be formulated as the average of players' marginal contributions to all coalitions. In other words, the axioms of marginality, efficiency, and symmetry provide a characterization of the Shapley value.

The concept of marginal contribution is easily defined and computed for CFF games. However, defining marginal contributions is not straightforward for games with externalities because the change of worth of a coalition caused by an agent leaving this coalition depends on the partition in which it is embedded and on the identity of the coalition the agent joins.

De Clippel and Serrano (2008a) thoroughly analyze the use of marginal contributions to determine possible sharings of the surplus generated in PFF games. Once they adopt the efficiency and symmetry axioms as above, they focus on properties related to marginal contributions. First, they consider the case where a player may join any other coalition after leaving a coalition $S$. When player $i$ leaves coalition $S$ in partition $P$ to join another coalition $T$ in $P$, the total effect on coalition $S$ is:

$$v(S, P) - v(S \backslash \{i\}, P \backslash \{S, T\} \cup \{S \backslash \{i\}, T \cup \{i\}\}).$$

Therefore, a natural extension of Young's (1985) axiom is:

**Weak marginality axiom.** A value $\varphi$ satisfies the weak marginality axiom if for any two games $v, v' \in \mathcal{G}^N$ for which

$$v(S,P) - v(S\backslash\{i\}, P\backslash\{S,T\} \cup \{S\backslash\{i\}, T \cup \{i\}\}) = v'(S,P) - v'(S\backslash\{i\}, P\backslash\{S,T\} \cup \{S\backslash\{i\}, T \cup \{i\}\}),$$

for any $(S,P) \in ECL$ with $i \in S, T \neq S$ and $T \in P$, then it is the case that $\varphi_i(v) = \varphi_i(v')$.

The three axioms of efficiency, symmetry, and weak marginality impose very few restrictions on values satisfying them. It is possible to strengthen the weak marginality axiom to a "monotonicity axiom" which states that if a player's marginal contributions in game $v$ are greater than or equal to (with at least one strict inequality) the corresponding marginal contributions in game $v'$, then the player's payoff in $v$ must be greater than the payoff in $v'$. This new axiom, together with efficiency and symmetry, imposes upper and lower bounds on the payoffs prescribed by values satisfying them. Still, there is a large family of values satisfying the three axioms.

One way to single out a unique value is by strengthening the weak marginality axiom. To do this, De Clippel and Serrano (2008a) decompose the total effect on coalition $S$ when player $i$ leaves $S$ in $P$ to join another $T \in P$ in the "intrinsic marginal contribution," given by $v(S,P) - v(S\backslash\{i\}, P\backslash\{S\} \cup \{S\backslash\{i\}, \{i\}\})$, and the "externality effect," given by $v(S\backslash\{i\}, P\backslash\{S\} \cup \{S\backslash\{i\}, \{i\}\}) - v(S\backslash\{i\}, P\backslash\{S,T\} \cup \{S\backslash\{i\}, T \cup \{i\}\})$. That is, the intrinsic marginal contribution is the loss incurred due to the player leaving $S$ and becoming a singleton. The externality effect is the additional loss incurred when the player joins coalition $T$.

Then, De Clippel and Serrano (2008a) introduce a "marginality axiom" stating that the value assigned to player $i$ depends only on the intrinsic marginal contributions of the player. The value characterized by the marginality axiom together with efficiency and symmetry coincides with the one proposed by Pham Do and Norde (2007) (we have denoted it $\varphi^{PN}$) and is called, in De Clippel and Serrano (2008a), the *externality-free value*. It is viewed as a reference point from which transfers can be made compatible with the identified bounds rather than as an actual final recommendation of the payoffs for the players.

Skibski, Michalak, and Wooldridge (2013)[13] take a more direct approach and provide a direct link between marginal contributions and values for PFF games. The marginal contribution of a player $i$ to coalition $S$ in a partition $P$ in a game $v$, denoted by $mc_i^{\alpha}(v)(S,P)$, is taken to be a weighted average of

[13]Skibski, Michalak, and Wooldridge (2017) provide a more condensed presentation of this value in Section 4.2.

$i$'s total effects on coalition $S$ over $\mathcal{P}_S$. More formally,

$$mc_i^\alpha(v)(S,P) = \sum_{T \in P\setminus\{S\}} \alpha_i(S\setminus\{i\},T,P)(v(S,P) - v(S\setminus\{i\}, P\setminus\{S,T\} \cup \{S\setminus\{i\}, T\cup\{i\}\})$$

$\alpha_i(S\setminus\{i\},T,P)$ is the weight attached to the effect on the value of $S$, of having player $i$ leave $S$ and join another partition $T \in P$.

A player $i$ is an $\alpha$-null player in a game $v$ if $mc_i^\alpha(v)(S,P) = 0$ for all $(S,P) \in ECL$ with $i \in S$. Then Skibski, Michalak, and Wooldridge (2013) show there is a unique value $\varphi^{SMW}$ on $\mathcal{G}^N$ that satisfies the standard axioms of efficiency, symmetry, additivity, together with the following axiom:

**Null player axiom.** $^\alpha$**.** A value $\varphi$ satisfies the null player axiom$^\alpha$ if $\varphi_i(v) = 0$ for any game $v \in \mathcal{G}^N$ and any $\alpha$-null player $i \in N$.

The closed-form expression for the value $\varphi^{SMW}$, similar to the Shapley value for CFF games, is

$$\varphi_i^{SMW}(v) = \frac{1}{n!} \sum_{\sigma \in \Omega(N)} \sum_{P \in \mathcal{P}(N)} pr_\sigma^\alpha(P) \left( v\left(C_i^\sigma \cup \{i\}, P_{-(C_i^\sigma \cup \{i\})}\right) - v\left(C_i^\sigma, P_{-C_i^\sigma}\right)\right),$$

where $\Omega(N)$ denotes the set of permutations of $N$, $C_i^\sigma$ denotes the set of players before player $i$ in the permutation $\sigma$, $pr_\sigma^\alpha(P)$ is $\Pi_{i\in N}\alpha_i(C_i^\sigma, P_{-C_i^\sigma})$ and $P_{-S}$ is defined as in the expression for $\varphi^{HY}$.

It is worth noting that, in the same way as $\varphi^{SMW}$, several values derived axiomatically $(\varphi^B, \varphi^{AAR}, \varphi^{MPW}, \varphi^{PN}, \varphi^{MQ}, \varphi^{HY})$ also have an interpretation as an average of suitably defined marginal contributions.

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## 8.5 Other Approaches

The Shapley value lent itself to several characterizations besides the classical axiomatic and marginalistic ones. In what follows, we describe extensions of the Shapley value via three of these approaches.

### 8.5.1 The Potential Approach

Hart and Mas-Colell (1989) introduce the concept of a *potential* function, $p$. This function associates with each CFF game $(N,\hat{v})$ a single number, $p(N,\hat{v})$, the potential of the game. The marginal contribution of player $i \in N$ to the game $(N,\hat{v})$, denoted by $D^i(N,\hat{v})$, is then defined as $p(N,\hat{v}) - p(N\setminus\{i\},\hat{v})$, where the game $(N\setminus\{i\},\hat{v})$ is the CFF game given by the restriction of $\hat{v}$

to $N\backslash\{i\}$. Furthermore, for any CFF game $(N,\hat{v})$ the sum of the marginal contributions of the players equals $\hat{v}(N)$. That is, $\sum_{i\in N} D^i(N,\hat{v}) = \hat{v}(N)$ for any CFF game $(N,v)$. Hart and Mas-Colell (1989) show that such a function exists and the marginal contribution of each player is precisely its Shapley value.

In addition to providing a new and exciting way to look at the Shapley value as a marginal contribution, the potential concept leads to a consistency property characterization of the Shapley value. Given a game $(N,\hat{v})$ and a value for CFF games $\psi$, let us define the "reduced" CFF game $(T,\hat{v}_T^{\psi})$ by:

$$\hat{v}_T^{\psi}(S) = \hat{v}(S\cup(N\backslash T)) - \sum_{i\in N\backslash T} \psi^i(S\cup(N\backslash T),\hat{v}) \text{ for all } S\subset T$$

$(S\cup(N\backslash T),\hat{v})$ is the game $(N,\hat{v})$ restricted to $S\cup(N\backslash T)$. A value $\psi$ is consistent if $\psi^j(T,\hat{v}_T^{\psi}) = \psi^j(N,\hat{v})$ for any CFF game $(N,\hat{v})$, any $T\subset N$, and any $j\in T$. Hart and Mas-Colell (1989) show that a value $\psi$ is consistent and "equally splits the surplus" for two-person games if and only if it is the Shapley value.

Dutta, Ehlers, and Kar (2010) extend the potential notion to PFF games by defining restriction operators that quantify the marginal contribution of a player $i\in N$ to a game $v\in\mathcal{G}^N$. A restriction operator $r$ associates with each game $(N,v)$ and each player $i\in N$ a subgame $(N\backslash\{i\},v^{-i,r})$. The worth $v^{-i,r}(S,P)$ of an embedded coalition $(S,P)\in ECL(N\backslash\{i\})$ is a function, implicit in the definition of the mapping $r$, of the values $v(S,P')$, where $P'$ is any partition that can arise from partition $P$ by adding player $i$ (player $i$ may enter as a singleton or join one of the existing coalitions in $P$). This definition imposes very little structure on the subgames. Dutta, Ehlers, and Kar (2010) start by requiring that the restriction operators satisfy path independence. To introduce the assumption, let $v^{-ij,r} = v^{-i,r}(v^{-j,r})$.

**Path independence axiom.** A restriction operator satisfies the path independence axiom if $v^{-ij,r} = v^{-ji,r}$.

That is, the order by which players are removed does not affect the game taking place after their departure.

Given a restriction operator $r$ satisfying path independence, an $r$-potential function, $p^r:\mathcal{G}^N\rightarrow\mathcal{R}$, is similarly defined to the potential definition in CFF games. Marginal contributions of players are given by $D^ip^r(N,v) = p^r\left(N,v\right) - p^r(N\backslash\{i\},v^{-i,r})$ for all $i\in N$ and they sum up to $v(N,\{N\})$. Each potential function $p^r$ gives rise to what Dutta, Ehlers, and Kar (2010) call an $r$-Shapley value.

Still, there are several $r$-Shapley values. For example, $\varphi^{PN}$ (the externality-free value of De Clippel and Serrano, 2008a) is obtained by letting $v^{-i,r}(S,P) = v(S,(P\cup\{i\}))$.[14] The value $\varphi^{AAR}$ is obtained when $v^{-i,r}(S,P)$

[14] Dutta, Ehlers, and Kar (2010) call it the *sing restriction operator*.

is a weighted average of the $v(S, P')$'s ($P'$ is again any partition that can arise from partition $P$ by adding player $i$).[15]

Imposing further axioms on the restriction operators singles out particular families of values for PFF games. Furthermore, Dutta, Ehlers, and Kar (2010) study the relationship between the axioms on the restriction operators and the extension of the standard Shapley axioms to PFF games. The restriction operators also enable the authors, similar to Hart and Mas-Colell (1989), to define a consistency property for PFF games. They show under some further assumptions that the unique value satisfying consistency for a given restriction operator $r$ is the $r$-Shapley value.

### 8.5.2 The Harsanyi Dividends Approach

Another approach that leads to the Shapley value involves the use of "dividends." For any CFF game $(N, \hat{v}) \in G^N$, the dividends that a coalition $S$ generates are recursively defined as follows:

$$\Delta_{\hat{v}}(S) = \begin{cases} 0 & \text{if } S = \varnothing \\ \hat{v}(S) - \sum_{T \subset S, T \neq S} \Delta_{\hat{v}}(T) & \text{if } S \neq \varnothing \end{cases}.$$

Harsanyi (1959) proves that the Shapley value evenly distributes the dividends of each coalition to the players comprising it. That is, $\psi_i^{Sh}(N, \hat{v}) = \sum_{S \subset N, i \in S} \frac{1}{|S|} \Delta_{\hat{v}}(S)$.

Macho-Stadler, Pérez-Castrillo, and Wettstein (2010) show that a similar construction leads to any value $\varphi$ for PFF games that is constructed through the average approach with weights $\alpha(S, P)$. The dividends for any embedded coalition $(S, P)$ are defined recursively as follows:

$$\Delta_v^{\alpha}(S, P) = \begin{cases} 0 & \text{if } S = \varnothing \\ v(S, P) - \sum_{(T,Q) \in ECL, T \subset S, T \neq S} \alpha(T, Q) \Delta_v^{\alpha}(T, Q) & \text{if } S \neq \varnothing. \end{cases}$$

That is, dividends received by subsets of $S$ are all taken into account scaled down in accordance to the weights associated with each partition. As in the CFF case, the value for player $i$, $\varphi_i^{\alpha}(N, v)$ can be expressed as:

$$\varphi_i^{\alpha}(N, v) = \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \frac{1}{|S|} \alpha(S, P) \Delta_v^{\alpha}(S, P).$$

That is, dividends, taking into account the embedded coalition generating them, are equally shared among the players comprising the embedded coalition.[16]

### 8.5.3 Algorithms

One of the most popular interpretations of the Shapley value in CFF games, already present in Shapley's thesis (Shapley, 1953b), is that the value of a

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[15] See Dutta, Ehlers, and Kar (2010) for the full description of the weighting system used.

[16] Modifying the summation of dividends by introducing a vector of player weights, Macho-Stadler, Pérez-Castrillo, and Wettstein (2010) obtain a weighted Shapley value for games with externalities.

player can be computed using the $n!$ orders in which the players can arrive to the game: The Shapley value of the player is his average marginal contribution in a sequential process where each order has the same probability of happening.

Skibski, Michalak, and Wooldridge (2017) extend this interpretation to PFF games. They envision a situation where the partition that a player encounters and the coalition that he joins when he leaves a coalition is the result of the "Chinese restaurant process," where players are sequentially assigned to coalitions; the $k$-th player (except for the first one) is assigned to a coalition in proportion to the size of that coalition, and he remains single with probability $1/k$. Thus, the marginal contribution of a player to a coalition (or, equivalently, the contribution that the coalition loses when the player leaves) is computed as the average of the contributions for all the possible coalitions and partitions that can emerge from the Chinese restaurant process. Skibski, Michalak, and Wooldridge (2017) define the stochastic Shapley value $\varphi^{SMW}$ as the average of the average (according to the previous process) marginal contributions of each player when each permutation has the same probability of happening.

The stochastic Shapley value can be characterized as the unique value that satisfies efficiency, symmetry, additivity, and the CRP-null player axiom. A player is a CRP-null player if his marginal average contribution (where the average is again computed using the Chinese restaurant process) is zero. The CRP-null player axiom requires that his payoff is zero. The stochastic Shapley value coincides with the value proposed by Feldman (1996) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), that is, $\varphi^{SMW} = \varphi^{MPW}$.

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## 8.6 Non-Cooperative Approaches to Value Extensions

The Shapley Value, while a cooperative solution concept, has been extensively analyzed from a non-cooperative point of view as well. In what follows we provide several findings related to the two main noncooperative approaches: implementation and bargaining.

### 8.6.1 Implementation

Values for cooperative games are often viewed as a recommendation of how to share jointly earned profits. A natural question regarding cooperative solutions is whether they can be implemented. In other words: Can a designer, who does not know the CFF or PFF game the agents are facing, design a game-form (a mechanism) leading in equilibrium to the payoffs recommended by the solution?

This question was answered in the affirmative for the Shapley value for CFF games. Winter (1994) and Dasgupta and Chiu (1998) propose demand commitment games in which, for some uniformly chosen random order of the players, each player can either make a demand to the following player or form a coalition satisfying the demands of some of the preceding players. For strictly convex CFF games, these mechanisms implement the Shapley value in expectation, that is, the expected payoff of every player (over all possible orderings) coincides with his Shapley value. Pérez-Castrillo and Wettstein (2001 and 2002) construct bidding mechanisms, where players compete for the right to make a proposal to other players, that implement the Shapley value directly, and not just in expectation, for zero-monotonic CFF games. A CFF game $(N, \hat{v})$ is zero-monotonic if $v(S) + v(i) \leq v(S \cup \{i\})$ for any subset $S \subseteq N$ and any $i \notin S$.

Macho-Stadler, Pérez-Castrillo, and Wettstein (2006) generalize these mechanisms to games with externalities by adding a coalition(partition)-forming stage. They construct two mechanisms implementing solution concepts derived through the average approach. One mechanism is designed for environments with positive externalities and the other for environments with negative externalities. A PFF game $(N, v)$ has negative externalities if $v(S, P) \geq v(S, P')$ for every $P, P'$, when each element in $P'$ is given by a union of elements in $P$, that is, $P$ is a refinement of $P'$. A PFF game $(N, v)$ has positive externalities if $v(S, P) \leq v(S, P')$ for every $P, P'$, where $P$ is a refinement of $P'$.

Similarly, Ju and Wettstein (2008) construct a mechanism implementing $\varphi^{PN}$ through a different generalization of the bidding mechanisms introduced in Pérez-Castrillo and Wettstein (2001) (see also Ju and Wettstein, 2009).

### 8.6.2 A Bargaining Approach

Another common support for values is given by providing reasonable or attractive bargaining procedures realizing them. Note that unlike the implementation approach, it is assumed that promises in utility terms can be enforced or, alternatively, are truthfully carried out. Gul (1989 and 1999) and Hart and Levy (1999) provide bargaining protocols with pairwise meetings that under some conditions on the underlying CFF game (strict convexity or strict super-additivity) lead to expected payoffs coinciding with Shapley value payoffs. Hart and Mas-Colell (1996) construct a bargaining protocol with multilateral meetings leading in expectation to the Shapley value payoffs for CFF games and the Nash bargaining solution for pure bargaining problems. We note that just like Hart and Levy (1999) reveals that efficiency in the Gul procedure does not lead to immediate agreement, nonstationary equilibria in the Hart and Mas-Colell procedure do not lead to the Shapley value payoffs if the probability of breakdown is high enough (see Krishna and Serrano, 1995; for a full characterization of equilibrium payoffs).

McQuillin (2009) shows that a simple adaptation of Gul's (1989) protocol leads to $\varphi^{MQ}$. Also, McQuillin and Sugden (2016) construct another finite bargaining process, the deadline bargaining game, which for PFF games with negative externalities leads again to $\varphi^{MQ}$. The deadline bargaining game assumes the same form as Gul's (1989) bargaining in each period, except for the final period where each active player receives the value of the coalition he represents.

Grabisch and Funaki (2012) propose three values for PFF games, each corresponding to a distinct procedure of coalition formation. The values are different from the values suggested thus far in this chapter, as they do not match the Shapley value for PFF games that are CFF games. Grabisch and Funaki (2012) do suggest modified values that reduce to the Shapley value. However, they argue that "pure" coalition formation values should not reduce to the Shapley value, since in the coalition formation scenarios all players are always "present in the game", whereas in the Shapley value there is a distinction based on the order in which players arrive.

Maskin (2003), in his Presidential Address to the Econometric Society, studies cooperation in the presence of externalities using a set of bargaining procedures, where all orderings of the players are possible at the offset. He draws a clear distinction between environments with negative and positive externalities. He then stresses that in the presence of positive externalities the assumption that the grand coalition forms, even if it is efficient, is problematic and may not be supported by any reasonable bargaining procedure. Several axioms are formulated regarding the bargaining procedures and the payoffs they generate at the various stages. These axioms are satisfied by several sharing schemes, which form a family of generalized Shapley values. These values determine both which coalitions form and how the surplus is shared among their members. As pointed out in De Clippel and Serrano (2008b), the main results are limited to 3-player games.

Borm, Ju, and Wettstein (2015) also take a bargaining perspective to analyze PFF games. They use a sequential approach to calculate the "reasonable" worth of any coalition (when in reality the worth depends on the whole partition) so that the Shapley value can be used to identify the value of each player in the game. To calculate the worth of a coalition $S \subseteq N$, Borm, Ju, and Wettstein (2015) envision a process where coalition $S$ "moves first" by forming a coalition structure within itself, taking into account that the members of $N \backslash S$ would choose a partition that maximizes the value of $N \backslash S$ (and if there is more than one such partition, the one chosen is the most detrimental to $S$). Bearing that in mind, the members of $S$ choose the coalition structure that maximizes their terminal payoff. Once the worth $\hat{v}(S)$ of a coalition is constructed in this way, they define the rational belief Shapley value as the Shapley value of the game $(N, \hat{v})$, that is, $\varphi^{BJW}(N, v) = \psi^{Sh}(N, \hat{v})$. Borm, Ju, and Wettstein (2015) also propose variations of the sequential approach, leading to two further values, and provide mechanisms that share a common bargaining structure and implement the three values.

## 8.7 Conclusions

In this chapter, we have reviewed several extensions of the Shapley value for environments where externalities among coalitions are present. The various approaches that lead to the Shapley value in characteristic function form games (axiomatic, marginalistic, potential, dividends, algorithmic, and non-cooperative) have provided alternative routes to address the question of the most suitable extension of this value for the larger class of games in partition function form. It is worth noting that some of the proposed values emerge from, and can thus be supported through, all or most of the previous approaches.

The main reason to study cooperative solution concepts for games with externalities is that the existence of externalities is the rule rather than the exception in most interesting environments. Therefore, the extensions that we have reviewed should be of interest to researchers looking for solution concepts in such environments.

Interestingly, some of these values have already been applied for studying competitive markets and environmental agreements, both natural fields for applying extensions of the Shapley value for games with externalities. For example, Jelnov and Tauman (2009) consider a game in coalitional form played by the firms in a Cournot industry and an outside innovator who owns a cost-reducing innovation. The firms can form at most two coalitions: The coalition including the innovator and some firms (that will use the new technology in their production processes), and the complementary coalition of firms. Using the Feldman's (1996) and Macho-Stadler, Pérez-Castrillo, and Wettstein's (2007) extension of the Shapley value, Jelnov and Tauman (2009) show that when the industry size goes to infinity, the Shapley value of the innovator approximates the payoff he obtains in a standard non-cooperative setup where he has the entire bargaining power. Another example is provided by Liu, Lindroos, and Sandal (2016) who study both cooperative and competitive solutions for managing a fish stock. In a three-country environment, taking the Norwegian spring-spawning herring as a case study, they analyze the stability of the grand coalition in a rich harvest model where the catch function is density-dependent. In their model, players (Norway, Russia, and the remaining countries fishing there) are asymmetric and, when they cooperate, share the benefits according to the "externality-free" Shapley value introduced by Pham Do and Norde (2007). Their conclusion is that the likelihood of a stable grand coalition increases with the degree of asymmetry in the players' efficiency levels.

The values analyzed in this chapter aim at providing a solution concept that can be applied in any environment where externalities among coalitions exist, independently of the type of externality. Still, we know that the externalities present in some environments are positive (think of the environmental coalitions) whereas they are negative in other situations (as is the case for trading agreements). Some of the solution concepts studied in this chapter may be better suited to some types of externalities than to others. Moreover, it might be advisable to consider extensions of the Shapley value that are suitable just for a subset (for example, the subset of games with positive externalities, or yet a smaller subset where all positive externalities have the same worth) of PFF games. Depending on the features of the subset, it may be possible to propose new axioms, reflecting properties that are desirable for the type of externalities considered, that characterize extensions of the Shapley value well-suited to these environments.

From the opposite point of view, it may be worthwhile extending some of the ideas developed in this chapter to sets larger than the set of partition function form games. Indeed, some environments are characterized by the presence of externalities not only across coalitions, but also across issues that are linked in the sense that the worth of a coalition in one issue depends on the organization of the players on all the issues. Consider countries negotiating both a trade agreement and an environmental agreement. On this occasion, these two issues, trade and environment, are linked. In particular, the accelerated growth triggered by a trade liberalization if countries form a large coalition is likely to raise $CO_2$ emissions, making it more difficult for the participants in an environmental agreement to comply with their obligations. Therefore, the worth of a coalition on trade depends on the partition of the countries following an environmental negotiation. A first attempt in this direction is Diamantoudi, Macho-Stadler, Pérez-Castrillo, and Xue (2015) who extend values for partition function form games (that also satisfy the strong dummy property) to environments where externalities across issues are present.

## 8.8 Acknowledgments

We gratefully acknowledge financial support from the Ministerio de Economía y Competitividad and Feder (ECO2015-63679-P), ICREA Academia, and Severo Ochoa Programme (SEV-2015-0563). The first two authors are fellows of CESifo and MOVE.

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# Chapter 9

## The Shapley Value and other Values

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### 9.1 Introduction

As this book testifies, the Shapley value is probably the most versatile, known and used solution concept for Transferable Utility (TU) games. One of its crucial properties is that of being *efficient*, since it distributes the value of the grand coalition among all players. This is a mandatory requirement for a general solution concept in cooperative setting. However, there is a class of TU games for which the value of the coalition does not usually represent utility of loss, but rather power. It is the case of simple games, where coalitions are partitioned into two subsets, the winning coalitions and the losing coalitions. In this restricted setting, efficiency of a solution is not mandatory, since it represents the power of players, and in this case what really matters is the *relative* power among them, rather than the total sum of their powers. For this reason, other values, also called *power indices*, have been considered in the TU games literature. Actually, another very important and widely used value was defined some years after the Shapley value by Banzhaf [4]. With the further development of the applications of game theory to other disciplines, also outside the setting of the social sciences, the two values became more and more important. Thus, it is natural that the idea behind the definitions of the Shapley and Banzhaf values was further extended to provide more indices.

Among them, the class of the *probabilistic values* [13, 40] plays an important role. The Banzhaf and Shapley values are probabilistic values: Actually, they belong to the special subclass of the probabilistic indices, constituted by the so-called *semivalues*, which can be characterized by interesting properties. In this chapter we shall revise the family of semivalues; in particular, we want to see how they can be defined via unanimity games, and we want to describe in detail two rather unusual applications of them: The first one is connected to molecular genetics, while the second one deals with a classical social choice problem. We now briefly describe the setting of this chapter in some more detail.

Semivalues were introduced and discussed as a subfamily of probabilistic values in [13, 14, 40] and are considered also in the papers [11, 12, 24, 26, 30]. The first application to molecular genetics can be found in [25, 31], where the Shapley and Banzhaf values are considered. It is within this model that in [27] new specific values were defined, with the aim to have other different and comparable tools to rank genes. The leading idea in the paper was to find some values with an intermediate behavior between the Shapley value and the Banzhaf value, in the restricted class under consideration, called the class of the microarray games. Given the finite set $N$ of players, values can be defined on the class of unanimity games, and extended to all games by linearity. On the unanimity game $u_S$, where the unique minimal winning coalition is the coalition $S$, the Shapley value assigns the same power (symmetry property) to the members of the winning coalition, which is $\frac{1}{s}$, where $s$ represents the size of the coalition $S$; moreover, it assigns zero to all other players (null player property). On the other hand, the Banzhaf value fulfills the same symmetry and null player properties, but it assigns a positive power to the players in $S$, which is $\frac{1}{2^{s-1}}$. Thus, the difference of the two indices is in this case of the unanimity games a linear vs. an exponential rate (with respect to the size of the minimal winning coalition). In the paper [27] new indices were considered, considering powers of the size of the winning coalition, and it was proved that these values are actually regular semivalues. Beyond the specific application of the paper, this approach raises the following natural question: define a value by assigning the positive value $a_s$ to the players in $S$ in the unanimity game $u_S$, where $s = |S|$, zero to all other players, and extending it by linearity on the whole space of games; then, under which conditions does the assignment of the positive number $a_s$ actually define a (regular) semivalue? Though the answer can be found by inspecting the results in [13], we report here briefly the characterization we gave in [7], which is more transparent. This will be developed in the first part of the chapter. In the second one, we shall briefly describe the use of the semivalues in two different contexts: The first one is relative to the already mentioned problem in molecular genetics, the other one instead is within the Social Sciences, and it addresses the following question: Given an ordering between the objects of a set $N$, how is it possible to construct an order on the subsets of $N$ "respecting" in some sense the initial one on the objects? Apparently, game theory does not directly relate to this issue, but we used it and, in particular, we exploited semivalues, exactly with the idea of allowing some "interaction" between the objects not so strong however to destroy the initial ranking.

To conclude, we only mention that another field in which semivalues are extensively used is in the theory of centrality of nodes (or edges) in a network. The problem has been extensively studied in biology, social network analysis, and computer science, where it has led to different applications: To quote some of them, we mention the identification of proteins that are critical for the survival of cells (see [18]), the maximisation of influence in social networks [21], and ranking websites in order to improve web search result accuracy [34]. The most famous centrality indices (degree centrality, closeness centrality, betweenness centrality, and eigenvector centrality) focus on individual nodes, without taking into account the various synergies that may occur when nodes are considered in groups. But recently it has been argued that the centrality of a single node should not only depend on individual traits like degree, closeness, or betweenness, but also on the centrality of the groups of nodes it belongs to. This naturally leads to a game theoretical approach, since the evaluation of all synergies between nodes leads to a TU game. On the other hand, a (one point) solution concept in cooperative theory gives back an evaluation of the single elements of the game (i.e., the nodes or the edges of the network), thus providing new and more sophisticated centrality measures. This has been proposed, e.g. in [3, 17, 28, 37]. Moreover, since usually networks have a great number of nodes (i.e., players in the associated game), it is quite important to have conditions under which these measures can be calculated in polynomial time, this has been proposed for regular semivalues in [38].

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## 9.2 Preliminaries and Notation

Let $N = \{1, 2, \ldots, n\}$ be the finite set of $n$ players. A TU-game on $N$ is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. We denote with $\mathcal{G}^N$ the set of games on the finite set $N$, and by $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}^N$ the universal set of all finite TU games.

**Definition 9.1 (Probabilistic values)** *[14, 39] Given a player $a$, let $\{p_S^a, S \subseteq N \setminus \{a\}\}$ be a probability distribution over the set of coalitions not containing $a$, that is a family of real numbers such that $S \subseteq N \setminus \{a\}$, $p_S^a \geq 0$ and $\sum_{S\subseteq N\setminus\{a\}} p_S^a = 1$. Then a* probabilistic value *$\psi$ on $v \in \mathcal{G}^N$ is defined as*

$$\psi_a(v) = \sum_{S\subseteq N\setminus\{a\}} p_S^a[v(S \cup \{a\}) - v(S)]. \tag{9.1}$$

Probabilistic values can be seen as the expected payoff of player $a$ if we see his participation to a game as consisting of joining a coalition $S$ with probability $p^a_S$ and then receiving his marginal contribution $v(S \cup \{a\}) - v(S)$ as a reward.

These solutions can be characterized by simple properties. Let us see how.

**Null player** Define a player $a$ a *null player* in the game $v$ if $v(S \cup \{a\}) = v(S)$ for every $S \subseteq N \setminus \{a\}$. A value $\varphi$ satisfies the *null player property* if given a null player $a$ for the game $v$, then

$$\varphi_a(v) = 0.$$

**Positivity** Given a monotonic game $v$, i.e., a game such that $v(S) \leq v(T)$ for any coalitions $S, T$ such that $S \subseteq T$, a value $\varphi$ satisfies *positivity* if

$$\varphi_a(v) \geq 0$$

for any $a \in N$.

**Linearity** A value $\varphi$ satisfies *linearity* if it is a linear operator on $\mathcal{G}^N$, i.e., $\forall v, w \in \mathcal{G}^N, \lambda \in \mathbb{R}$ it holds

$$\varphi(v + w) = \varphi(v) + \varphi(w) \text{ and } \varphi(\lambda v) = \lambda \varphi(v).$$

Then the following theorem holds [14].

**Theorem 9.1** *Let $\varphi : \mathcal{G}^N \to \mathbb{R}^n$ be a value on $\mathcal{G}^N$ that satisfies linearity, positivity and the null player properties. Then $\varphi$ is a probabilistic value. Moreover, every probabilistic value $\psi$ as in the formula* (9.1) *above satisfies these properties.*

Let us now consider a subfamily of the probabilistic values: The semivalues. Suppose a family $p_s$, $s = 0, \ldots, n-1$ of non-negative real numbers fulfilling the relation

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$$

is given.

**Definition 9.2 (Semivalue)** *A* semivalue *on $\mathcal{G}^N$ is a value $\pi^N$ defined as*

$$\pi^N_a(v) = \sum_{S \subset N \setminus \{a\}} p_s [v(S \cup \{a\}) - v(S)]$$

*for any $a \in N, v \in \mathcal{G}^N$, where $s = |S|$. A semivalue with weighting coefficients $p_s$ is* regular *if $p_s > 0 \, \forall k = 0, \ldots, n-1$.*

Thus, a semivalue is a probabilistic value with the property that $p_s = p_S^a$ for every player $a$ and every coalition $S$ of cardinality $s$.
A semivalue $\pi^N$ can be identified by an element of the simplex

$$\Sigma := \{x \in \mathbb{R}^{n-1} : x_i \geq 0 \quad \wedge \quad \sum_{s=0}^{n-1} \binom{n-1}{s} x_s = 1\}.$$

In the sequel, we shall identify the semivalue $\pi^N$ by means of the $n$-dimensional vector $(p_0, \ldots, p_{n-1})$.

It is clear that a semivalue is a particular probabilistic value with the additional feature of fulfilling a form of anonymity as specified in the following property:

**Anonymity** A value $\varphi$ satisfies *anonymity* if for every permutation $\vartheta$ on $N$, every game $v$ and every player $a$, it holds

$$\varphi_a(\vartheta v) = \varphi_{\vartheta(a)}(v)$$

where $(\vartheta v)(S) = v(\vartheta S)$ for any $S \in 2^N$.

Not surprisingly, the following theorem holds [14].

**Theorem 9.2** *A semivalue is a probabilistic index that satisfies anonymity and, vice versa, any probabilistic index that satisfies anonymity is a semivalue.*

Let us now see some examples of semivalues, taken from the literature. First of all, both the Shapley value and the Banzhaf value are regular semivalues. In particular, they have the following features: The Banzhaf value is the only one for which $p_s = p_t$ for all $s, t \in \{0, 1, \ldots, n-1\}$, or, in other words, the unique semivalue attaching equal probability to all coalitions; the Shapley value, on the other hand, is the only one fulfilling efficiency. Other regular semivalues on $\mathcal{G}^N$ are introduced in [11]; they define the family of the *q-binomial semivalues*, for which the coefficient $p_s$ is defined as $p_s = q^s(1-q)^{n-s-1}$, for some $q$ such that $0 < q < 1$. Furthermore, in [27] the family of *c-values* $\sigma^{N,c}$ has been defined, in the following way: Let $c$ be a positive real number, and define $\sigma^{N,c}$ on unanimity games as

$$\sigma_a^{N,c}(u_S) = \begin{cases} \frac{1}{s^c} & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

where $s = |S|$. Then extend $\sigma^{N,c}$ by linearity. In [27], it is proved that $\sigma^{N,c}$ is a regular semivalue for every $c > 0$. To conclude, observe that the Banzhaf value is the $q$-binomial semivalue corresponding to $q = \frac{1}{2}$, while the Shapley value is the $c$-value corresponding to $c = 1$.

## 9.3 Semivalues and Unanimity Games

The approach used in [27] to define new semivalues raises the following question: Under which condition it is possible to find a new family of values, defining a value on the family of the unanimity games, and extending it by linearity? Let us be more specific, given an $n$-dimensional vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, define the value $\pi^{N,\alpha}$ on $\mathcal{G}^N$, acting in the following way on the class of the unanimity games:

$$\pi_a^{N,\alpha}(u_S) = \begin{cases} \alpha_s & \text{if } a \in S \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

and extending it to the whole space by linearity. Then the question is: Under which conditions on the coefficients $\alpha_s$ is the value $\pi^{N,\alpha}$ a semivalue?

This is what we discuss in this section. The content of this part is taken from [7] (see also [12] and [13]). As we shall see, it is possible to provide a sufficient condition that allows creating new specific families of semivalues. Moreover, it is possible to extend to $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}^N$ the concept of semivalue, by requiring that an operator $\pi$ on $\mathcal{G}$ is a semivalue provided $\pi^N = \pi|_{\mathcal{G}^N}$ is a semivalue for all $n$. In this more general case, it is possible to characterize the conditions under which the sequence $\{\alpha_s\}_{s\in\mathbb{N}}$ generates a regular semivalue.

Let us start with the following preliminary result, proved in [12, 27].

**Proposition 9.3** *Suppose that, for each $t = 1, \ldots, n$, positive numbers $\alpha_t$ are given and let $\pi^{N,\alpha} : \mathcal{G}^N \to \mathbb{R}^n$ be a value that satisfies linearity and that is defined as*

$$\pi_a^{N,\alpha}(u_T) = \begin{cases} \alpha_t & \textit{if } a \in T \\ 0 & \textit{otherwise} \end{cases}$$

*for all unanimity games $u_T$, generated by the minimal winning coalition $T$ of size $t$.*

*Then $\pi^{N,\alpha}$ verifies the following formula:*

$$\pi_a^{N,\alpha}(v) = \sum_{S\subseteq N\setminus\{a\}} \left( \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} \right) [v(S\cup\{a\}) - v(S)]$$

*for every $v \in \mathcal{G}^N$.*

Thus, $\pi^{N,\alpha}$ is characterized by the fact that it is of the form

$$\pi_a^{N,\alpha}(v) = \sum_{S\subseteq N\setminus\{a\}} p_s^n [v(S\cup\{a\}) - v(S)],$$

where

$$p_s^n = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1}.$$

It follows that $\pi^{N,\alpha}$ is a semivalue provided the coefficients $\alpha_s$ fulfill, for any $s = 0, \ldots, n-1$:

$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} \geq 0 \tag{9.3}$$

and

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = 1. \tag{9.4}$$

Condition in Equation (9.4) is easily checked, since the following result holds (see [27]).

**Proposition 9.4** *Take, for every $n \in \mathbb{N}$, real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then*

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = \alpha_1.$$

Thus, Equation (9.4) is fulfilled by requiring $\alpha_1 = 1$; now the problem becomes to provide sufficient conditions under which Equation (9.3) is verified. To analyze this, we need some more preliminaries. We start with the following definition that can be found in [41], (p. 108).

**Definition 9.3** *Given a sequence $\mu_n$ of real numbers, the* backward difference operator $\Delta^k$ *is defined as*

$$\Delta^0 \mu_n = \mu_n \qquad \Delta^k \mu_n = \Delta^{k-1} \mu_{n+1} - \Delta^{k-1} \mu_n$$

*for $n = 0, 1, 2, \ldots,$ and $k = 1, 2, \ldots$*

**Definition 9.4** *The sequence $\{\mu_n\}_{n=0}^{\infty}$ is called* completely monotonic *if its elements are non-negative and*

$$(-1)^k \Delta^k \mu_n \geq 0$$

*for every $k, n = 0, 1, 2, \ldots$*

The backward difference operator can be written also as

$$\Delta^k \mu_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu_{n+k-j}.$$

The following result, whose proof can be found for instance in [23], (p. 171), will be used in detecting the regularity of semivalues.

**Proposition 9.5** *Let $\{\mu_k\}_{k=0}^{+\infty}$ be a completely monotonic sequence. Then*

$$(-1)^k \Delta^k \mu_n > 0$$

*for every $n, k = 0, 1, \dots$ unless $\mu_1 = \mu_2 = \cdots = \mu_n = \dots$, that is the sequence is constant except at most for the first term.*

The following lemma, see [7], is the key ingredient for the main result.

**Lemma 9.1** *Given the value $\pi^{N,\alpha}$, with associated vector $(p_0^n, \dots, p_{n-1}^n)$ the following formula holds, for all $s = 0, \dots, n-1$:*

$$(-1)^{n-s-1} \Delta^{n-s-1} \alpha_{s+1} = p_s^n.$$

From all the above results, we finally have the theorem that shows how to generate new semivalues on $\mathcal{G}^N$ using completely monotonic sequences.

**Theorem 9.6** *Let $\{\alpha_s\}_{s\in\mathbb{N}}$ be a completely monotonic sequence such that $\alpha_1 = 1$ and let $\pi^{N,\alpha}$ be the value defined on unanimity games as in Equation* (9.2) *and extended by linearity on $\mathcal{G}^N$. Then $\pi^{N,\alpha}$ is a semivalue.*

Let us now consider the space $\mathcal{G}$ of all finite games. An element of $\mathcal{G}$ is given by the pair $(N, v)$ where $N$ is the set of players and $v$ is a cooperative game on the set $N$.

**Definition 9.5** *A semivalue on $\mathcal{G}$ is an operator $\pi$ on $\mathcal{G}$ such that its restriction to $\mathcal{G}^N$ is a semivalue for every set $N$ of $n$ players, $n \in \mathbb{N}$.*

Let moreover $\mathcal{S}$ be the space of the real valued sequences:

$$\mathcal{S} = \{\alpha := (\alpha_1, \alpha_2, \dots, \alpha_s, \dots) : \alpha_s \in \mathbb{R}\ \forall s \geq 1, \alpha_1 = 1, \alpha_s \geq 0\ \forall s\}.$$

Finally, given a sequence $\alpha \in \mathcal{S}$ define $\pi^\alpha$ on $\mathcal{G}$ in the following way:

$$\pi^\alpha(v) = \pi^{N,\alpha}(v)$$

for every $v$ such that $v \in \mathcal{G}^N$.

Putting together the above results, we get the following.

**Theorem 9.7** *$\pi^\alpha$ is a semivalue on $\mathcal{G}$ if and only if $\alpha \in \mathcal{S}$ is completely monotonic.*

We explicitly notice that if $\pi^\alpha$ is a semivalue on $\mathcal{G}$, then it is defined for every $n \in \mathbb{N}$ $\mathbf{p^n} = \{p_k^n\}_{k=0}^{n-1}$ such that $\mathbf{p^n}$ is the vector of weighting coefficients associated to $\pi|_{\mathcal{G}^N}$. The coefficients verifiy the formula

$$p_s^n = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1}.$$

In particular, with the choice of $s = n-1$, the formula above shows that $\alpha_n = p^n_{n-1}$ for all $n = 1, 2, \dots$

The next result we want to illustrate deals with semivalues which are not regular. To prove it, we need a preliminary result.

**Proposition 9.8** *For all $s, n$ such that $s \le n-1$, the following formula holds:*

$$p^n_s = p^{n+1}_s + p^{n+1}_{s+1}. \tag{9.5}$$

We now provide the formal definition of regular and irregular semivalues on the space $\mathcal{G}$. Remember that on $\mathcal{G}^N$ regularity of $\pi^\alpha$ means that $p^n_s > 0$ for all $s$.

**Definition 9.6** *A semivalue on $\mathcal{G}$ is called regular if its restriction to $\mathcal{G}^N$ is a regular semivalue for all $n$. A semivalue which is not regular is called irregular.*

In the next definition, we introduce two irregular semivalues, extending the definitions of marginal and dictatorial values given in [12].

**Definition 9.7** *The marginal value $\mu$ on $\mathcal{G}$ is the value such that its restriction to $\mathcal{G}^N$ is described by the vector $(0, 0, \dots, 1)$. The dictatorial value $\delta$ on $\mathcal{G}$ is the semivalue such that its restriction to $\mathcal{G}^N$ is described by the vector $(1, 0, \dots, 0)$.*

**Theorem 9.9** *The values $\mu$ and $\delta$ are semivalues on $\mathcal{G}$. Moreover, let $\pi^\alpha$ be an irregular semivalue on $\mathcal{G}$. Then, there exists $q \in [0, 1]$ such that $\mathbf{p^n}$ is of the form $(1-q, 0, \dots, 0, q)$, for every $n$.*

Thus, an irregular semivalue assigns, for all $n$ a fixed probability to the fact that the players act alone and the complement to the fact that they act all together.

The following is a natural question to address: Since a semivalue on $\mathcal{G}$ automatically (by definition) generates a semivalue on $\mathcal{G}^N$ for all $N$, does something in the opposite direction hold, in the sense that, given a semivalue on some fixed set $N$ of players, can it be extended to a semivalue on $\mathcal{G}$? First of all, as it is easy to see, irregular semivalues cannot be extended, with the exception of those described in Theorem 9.9. Instead given a regular semivalue on $N$, it is possible to extend it on all $T$ such that $|T| < |N|$; but it is not always possible to extend it on bigger sets. For instance it is not possible to extend the semivalue given by the vector $\mathbf{p^3} = (\varepsilon, \frac{1}{2} - \varepsilon, \varepsilon)$ if $\varepsilon < \frac{1}{6}$. Moreover when a semivalue associated to the vector $\mathbf{p^n}$ can be extended, the extension is not unique and there is a family of candidates for $\mathbf{p^{n+1}}$, depending on a parameter. For instance, consider the Shapley value for $n = 4$, that is $\mathbf{p^4} = (\frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{4})$. Then we can choose $p^5_1 = p^5_2 = p^5_3 = \frac{1}{24}$ and get $p^5_0 = p^5_4 = \frac{5}{24}$. In general we can choose $p^n_j = \frac{1}{3 \cdot 2^{n-2}}$ if $j \ne 0, n-1$ and obtain a regular semivalue, with weighting coefficients different from the ones of the Shapley value.

We conclude this section briefly by illustrating how it is possible to generate other family of semivalues using the properties of completely monotonic sequences. First of all, we have the following definition

**Definition 9.8** *A function* $f : [0, \infty[ \to [0, \infty[$ *is said to be* completely monotonic *if it is infinitely times differentiable and* $(-1)^n f^{(n)}(x) \geq 0$ *for all* $x \geq 0$.

The next theorem is Theorem 11d in [41].

**Theorem 9.10** *If* $f(x)$ *is completely monotonic in* $a \leq x < \infty$ *and if* $\delta$ *is any positive number, then the sequence* $\{f(a + n\delta)\}_{n=0}^{\infty}$ *is completely monotonic.*

It follows that in order to have completely monotonic sequences, it is enough to consider completely monotonic functions. In particular, it is immediately seen from Theorems 9.7 and 9.10 that the indices $\sigma^a$ defined in [27] and the binomial indices, generated by the sequence $\alpha_s = q^{s-1}$, are regular semivalues. But a lot of other semivalues can be generated, for instance by using the following simple fact.

**Proposition 9.11** *If* $f, g$ *are completely monotonic functions, then*

- *for any integer* $m$, *also* $f^{(2m)}$ *and* $-f^{(2m+1)}$ *are completely monotonic;*
- *if* $a, b \geq 0$, *then* $af + bg$ *is completely monotonic;*
- $f \cdot g$ *is completely monotonic;*
- *if* $h(x) \geq 0$ *for every* $x$, *and* $h'(x)$ *is completely monotonic, then* $f[h(x)]$ *is completely monotonic.*

We can use Theorem 9.10, with $a = 0$ and $\delta = 1$, to obtain a completely monotonic sequence from a completely monotonic function. Thus, the following are some examples of completely monotonic sequences that can be used to generate semivalues:

$$e^{1-n} \qquad \tfrac{1}{2}(1+n)^{1/n} \qquad e^{1/n-1}$$

$$\frac{1}{\log 2}\frac{\log(1+n)}{n} \qquad \frac{\log 2}{\log(1+n)} \qquad \frac{\log\left(1+\frac{1}{n}\right)}{\log 2}$$

$$\frac{\log(1+n)}{\log 2\; n^2} - \frac{2}{n(1+n)} \qquad \left(\frac{1}{2} + \frac{1}{2n}\right)^k, k \geq 0.$$

After this overview about how to generate new family of semivalues, we change scenario and we propose, as mentioned in the introduction, a pair of examples where general semivalues, not only the Shapley and Banzhaf values, have been used. The leading idea for these applications is to use semivalues to measure the power (strength, cruciality ...) of specific objects. In the first example, developed in the next section, the aim is to single out genes potentially crucial for the development of some specific polygenetic disease. In the second example, developed in the next section, we use semivalues in a problem of Social Choice.

## 9.4 Semivalues and Genetics

Let us now concentrate on the use of semivalues in genetics. The recent availability of the genome sequence information has promoted the development of a number of new technologies, including microarrays. Very briefly, a microarray provides the gene expression of an individual. Thus, a genetic analysis to study a genetic disease is performed in the following way: There are two reference groups, and for each element of both groups the gene expression is considered. The data of each patient are translated in digits, forming matrices, say $A$, a $n \times l$ matrix, and $B$, a $n \times m$ matrix, with real entries. Each row corresponds to a gene, each column to an individual, and the individuals are partitioned in two sets: Call the reference set the group of individual whose data are collected in the matrix $A$. The entries $a_{ij}$, $b_{ij}$ represent the expression of gene $i$ in the individual $j$ (note that individual $j$ relative to the coefficient $a_{ij}$ is different from the individual $j$ relative to the coefficient $b_{ij}$). These matrices usually have a huge number of rows and a relatively small number of columns. The data of the row $a_i$ are used to construct an interval, say $[m_i, M_i]$, called the *normality interval* for the gene $i$. For instance, a possible choice could be taking $m_i$ as the smallest entry of the row $i$ in the matrix $A$, as $M_i$ as the greatest entry of the row in the matrix $A$. Usually, the data from the matrix $A$ are relative to sane individuals, while those from the matrix $B$ are taken from people affected by a specific disease. But this is not the only possible application: For instance we could take two groups of people affected by a similar but not identical form of tumor, and the mutual comparison of the data (taking one time one group as a reference group, and the other one under experiment and repeating the experiment the other way around) can provide another example of microarray game. Finally, a matrix $M$ is built up in the following way: $m_{ij} = 1$ if and only if $m_{ij} \notin [m_i, M_i]$. By means of the matrix $M$, we built the so-called microarray game.

**Definition 9.9** *A microarray game is* $(M, v)$, *where*

- $M = (m_{ij})$ *is an* $(n \times m)$ *matrix, such that* $m_{ij}$ *is either zero or one, and such that for every* $j$ *there is at least an* $i$ *with* $m_{ij} \neq 0$*;*
- *The TU game* $v$ *is built in the following way: given the column* $m_{\cdot j}$, $j = 1, \dots, m$, *define its* support *as the set* $\operatorname{supp} m_{\cdot j} = \{i : m_{ij} = 1\}$ *and consider the associated unanimity game* $v^j$ *generated by* $\operatorname{supp} m_{\cdot j}$, *i.e.,*

$$v^j(T) = u_{\operatorname{supp} m_{\cdot j}}(T) = \begin{cases} 1 & \text{if } T \supset \operatorname{supp} m_{\cdot j} \\ 0 & \text{otherwise} \end{cases}. \tag{9.6}$$

*Then*

$$v = \frac{1}{m}\sum_{j=1}^{m} v^j. \tag{9.7}$$

Let us remark that the choice of considering average of unanimity games is, at least to a first step, strongly motivated by computational needs. As we have already noticed, here the players are genes, and often in the microarrays the detected genes are thousands. This means that calculating a solution is practically impossible because of the intractable number of players, unless one considers very particular games. This explains the use of unanimity games, where the Shapley value assigns, as every semivalue, 0 to players not in the minimal winning coalition, and the reciprocal of the number of the elements in the minimal winning coalition, as it is obvious from the fact that the Shapley value is efficient. By using linearity, the Shapley value can be calculated in a microarray game with the following very simple formula:

$$\sigma_i(v) = \frac{1}{m} \sum_{j=1}^{m} \frac{m_{ij}}{\sum_{l=1}^{n} m_{lj}}. \tag{9.8}$$

with $m_{ij} \in \{0, 1\}$. The formula is easily understood since if player (gene) $i$ is not abnormally expressed in patient $j$ (i.e., $m_{ij} = 0$), its value in null, as it must be, while if $m_{ij} = 1$ the formula provides exactly the reciprocal of the number of the genes abnormally expressed.

With this definition in mind, the Shapley value, and subsequently the Banzhaf values, were axiomatized, within the class of the microarray games, in the papers [25, 31].

Microarray games were used to analyze real data in [1, 25, 27]. However, it turned out that often the Shapley and Banzhaf indices (especially the second one) propose a great number of ties among genes, making it difficult to distinguish them. This happens for the following two reasons. On one side, having few patients versus thousands of genes means that several of them could be grouped in families of symmetric players (for a single column $j$ the genes are divided in just two groups). On the other hand, it is quite possible that roundoff errors do not allow evaluating very small differences. By elaborating some real data it turned out that some patients presented around 200 genes that are abnormally expressed. In such a case, the Banzhaf assign zero power to all players-genes (due to practically negligible roundoff errors made by the computer)[1] thus not distinguishing the abnormally expressed genes from the other ones. This means that actually the unanimity game associated to that patient does not provide useful data, since it considers all genes as null players. This in principle cannot be considered totally useless: In some sense, it indicates that the patient could be considered not meaningful for the analysis, since its abnormally expressed genes are too many. But on the other hand, especially when treating data with few patients, it is of interest to avoid the risk of having a partition of the set of genes made by few elements (i.e., few subsets with a large number of genes).

[1] If the minimal winning coalition has $s$ elements, the Banzhaf value assigns value $\frac{1}{2^{s-1}}$ to the non-null player, which is practically zero already for rather small $s$.

For this reason, in the paper [26] we propose a method to better differentiate the contribution that each gene could give to the disease. In order to do this, we used the idea of *weighted indices* (see [19]). This is done by attaching a weight to any gene for each patient, according to how far the gene is from the normality interval, as explained below. This actually allowed us to better differentiate the genes, since already for a single patient the genes are divided in several (and not only two) classes. We proceed further by using this first ranking to build a new game. The above procedure provides a ranking of thousand of genes, with not so many ties. Then we take the first ones (say e.g. one hundred) genes and to each patient we attach a weighted majority game (weights to attach to the genes are explained later). Finally, we rank the genes by making an average of their ranking over all patients.

Again, the choice of using (average of) weighted majority games is originated by the fact that for this class of games it is possible to evaluate power indices even when the game has a great number of players. In fact, by exploiting the tool of the formal series, algorithms were developed which easily compute the indices for games with many players (see e.g. [2], [8], [29]).

The idea underlying the new version of the microarray game is to allow the matrix $M$ defining the classical microarray game to contain not only zeroes and ones, since we want to take into account how much the genes are abnormally expressed, by giving them a weight gradually increasing depending on how much the gene expression is far from the reference interval. We consider, for each gene $i$, a reference interval, let us call it $N_i^0 = [m_i, M_i]$, to evaluate the standard deviation $s_i$ relative to the data of the gene, to set $N_i^k = [m_i - ks_i, M_i + ks_i]$, $k = 1, \ldots, j$, and to assign the value $k$ to the gene falling in the set $N_i^k \setminus N_i^{k-1}$ ($j$ if it falls outside all these sets). In this way, we can rank the genes according to the weighted index. Now we use the same formula as in Equation (9.8). This is, for every fixed game $j$, nothing else than the expression of the weighted Shapley index, with associated weight $m_{ij}$ for player $i = 1, \ldots, n$ (see [19]).

The subsequent weighted majority game is built by assigning to player $i$ in the game $j$ the coefficient $m_{ij}$. Extending to semivalues, in a straightforward way, the algorithms provided in [8], this idea was applied in the paper [26] to different data sets, i.e., Stroma Rich and Stroma Poor Neuroblastic tumors, Ductal and Lobular breast tumor, two different types of Colon tumor. It must be observed that in one case (Stroma Rich and Stroma Poor Neuroblastic tumors), we did a different experiment: We compared tissues from two groups of cells affected by two different, though similar, forms of tumor. We analyzed both situations arising from taking one of the groups as reference group. In this case, the ranking of the genes assumes the idea of singling out those characterizing one form of tumor with respect to the other one. We refer the interested reader to the paper for the specific results and related comments. Here we add some general comment. It is clear that this is a kind of model for which it is not easy to find data confirming its validity. However, some facts seem to be interesting. The first is that, as it is easy to see, even if in a single weighted majority game, different semivalues do provide the same ranking between players, this is no longer true when considering a game which is the average of several weighted majority games. Thus, in principle one can expect different rankings when using different semivalues. However, our applications to real data gave a stable ranking, in the sense that the differences of positions of the genes were not relevant with respect to the large number of genes. This was a form of stability that shows, in our opinion, that the results are confirmed by the use of different indices. Furthermore, in one case we tried to make a blind research in the medical literature, and the genes that were quoted as potentially relevant for the onset of the disease were ranked in the first places according to our method.

## 9.5 Semivalues and Social Choice

In this section we want to describe how the semivalues have been used in a Social Choice context. The results are taken from [24] to which we refer for further details and for the proofs of the theorems.

In real-life situations, many problems deal with preferences over collections of objects (e.g., alternatives, opportunities, candidates, etc.). Consider, for instance, the selection of the candidate members for the formation of evaluation committees, the assessments of equity of sets of rights inside a society, the comparison of assets in portfolio analysis, the comparison of the stability of groups in coalition formation theory, and so on. As a toy example to better explain the idea, let us consider a woman preparing her luggage before a long trip. We can guess she has a preference relation over the objects she will need, but not over the groups. For instance, her first choice could be to take a specific pair of shoes, then some dresses for the social dinners, and so on. But then the combination of these objects matters, since she should select the group of them that should be taken away, and this explains why it could take a lot of time to prepare a luggage.

The important question in a social choice context, behind these examples, is: given a ranking over the single elements of a set $N$, how to derive a *compatible* ranking over the set of all subsets of $N$? Of course, one of the first points is to clarify what *compatible* means in this context. Mathematically, the question is how to extend a ranking over a set $N$ to a ranking over its *power set* (denoted by $2^N$). This was mainly done by axiomatically characterizing families of ordinal preferences over subsets (see, for instance, [5, 6, 9, 10, 15, 16, 20, 22]). In this context, an order $\succcurlyeq$ on the power set $2^N$ is required to be an *extension* of a primitive order $R$ on $N$. This means that the relative ranking of any two singleton sets according to $\succcurlyeq$ must be the same as the relative ranking of the corresponding alternatives according to $R$ (i.e., for each $a, b \in N$, $\{a\} \succcurlyeq \{b\} \Leftrightarrow aRb$). We shall refer to a total preorder $\succcurlyeq$ on

$2^N$ as an *extension* on $2^N$, with the implicit assumption that such a relation $\succcurlyeq$ ranks singletons in $2^N$ in the same way of some equivalent primitive total preorder $R$ on $N$.

The different axiomatic approaches in literature are, very naturally, related to the interpretation of the properties used to characterize extensions, which is connected to the meaning attributed to sets. According to the survey of [6], the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: (1) *complete uncertainty*, where a decision maker is asked to rank sets which are considered as formed by mutually exclusive objects, taking into account that he cannot influence the selection of an object from a set (see, for instance, [5, 20, 33]); (2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, the decision maker compares sets taking into account that he will be able to select a single element from a set (see, for example, [10, 22, 35]); (3) *sets as final outcomes*, where each set contains objects that are assumed to materialize simultaneously (if that set is selected; for instance, see [9, 15, 36]).

In [24] we focused on the problem of the third class, where sets of elements materialize simultaneously. In this framework, most of the approaches present in literature do not take into account possible interactions between objects. For instance, a usual assumption is the property of *responsiveness* (RESP) ([36]): An extension $\succcurlyeq$ on $2^N$ satisfies RESP if for all $i, j \in N$ and all $S \in 2^N$, $i, j \notin S$

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow S \cup \{i\} \succcurlyeq S \cup \{j\}. \tag{9.9}$$

Thus, the RESP property does not take into account the fact that some objects together can present some form of incompatibility or, on the contrary, of mutual enforcement. In our toy example, it is reasonable to think that the woman, having the possibility to add only one object to the set $S$ of the objects already put in the luggage, prefers to buy a purse, needed for the main social dinner, even if without constraint she would prefer to buy a pair of shoes. In other words, some form of mutual interaction between objects should be allowed.

The first paper to propose this idea is [32]. The model introduced there relies on the simple idea that any utility function attached to a total preorder on $2^N$ represents a coalitional game in characteristic function form (normalized by setting the utility of the empty set to be equal to zero). Thus, a first simple idea can be the following. Given a ranking $\succcurlyeq$ over the set $2^N$ of the set $N$ of the objects, it is possible to associate to it a utility function (normalized in such a way that the utility of the empty set be zero). But this utility function is nothing else than a TU game. Thus, one can apply the Shapley value to this game, and this provides a natural ranking between the objects. Then one could consider $\succcurlyeq$ as an acceptable extension of the primitive order over $N$ if the Shapley value provides the same ranking over the singletons. This approach, however, is too naive for a very precise reason: It is well known and easy to see that different utility functions arising from the same preorder on $2^N$ can produce different ranking, according to the Shapley value. This is a

serious objection to the procedure, since the object we want to be built is a ranking $\succcurlyeq$, and there is no natural utility function associated to it. However, a further step in this direction is possible. There are some rankings on $2^N$ with the property that, no matter which utility function is attached to it, the Shapley value provides always the same ranking between the singletons. This allows, as it is proved, some form of interaction, in the sense that there are preorders which fulfill this property but do not fulfill the RESP property. And of course we can do the same using semivalues different from Shapley's. Actually, in [24] we prove necessary and sufficient conditions for every fixed semivalue, and also characterize those rankings keeping invariant the ranking of the singletons, no matter which utility function is used on the preorder on the subsets, and no matter which semivalue is used to find the ranking.
We now describe the main results.

We start with a bit of necessary notation. In the sequel we deal with some finite set $N$ of cardinality $n$, and the set $2^N$ of its subsets.
A total preorder on a finite set $X$ is a reflexive, transitive and total binary relation $\succcurlyeq \subseteq X \times X$. We shall consider total preorders on both sets $N$ and $2^N$. Now, suppose we have a total preorder $\succcurlyeq$ on $2^N$. This relation naturally induces a TU game for each utility function $v$ *representing* $\succcurlyeq$ (such that $v(\emptyset) = 0$), i.e., $v(S) \geq v(T) \Leftrightarrow S \succcurlyeq T$ for each $S, T \in 2^N$. We shall denote by $V(\succcurlyeq)$ the set of all $v$ representing the total preorder $\succcurlyeq$.
Given $N$ and $i, j \in N$, we use the following notation:

- $\Sigma_i$ ($\Sigma_{ij}$) for the set of all subsets of $N$ which do not contain $i$ (neither $i$ nor $j$).
- $\Sigma_{ij}^s$ for the set of the subsets of $\Sigma_{ij}$ of cardinality $s$.

We now see an important definition.

**Definition 9.10 ($\pi^{\mathbf{p}}$-alignment)** *Given a set $N$, a total preorder $\succcurlyeq$ on $2^N$ and a probabilistic value $\pi^{\mathbf{p}} \in \mathcal{P}$, we shall say that $\succcurlyeq$ is $\pi^{\mathbf{p}}$-aligned if*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \pi_i^{\mathbf{p}}(v) \geq \pi_j^{\mathbf{p}}(v)$$

*for each $v \in V(\succcurlyeq)$.*

The main point of the paper is to use the notion of $\pi^{\mathbf{p}}$-alignment as a compatibility test of an extension to $2^N$ of a total order on $N$. Using different semivalues and/or classes of semivalues provide different types of compatibility, all extending the notion of RESP.

Given the definition of $\pi$-alignment, and fixed a semivalue $\pi^{\mathbf{p}}$, we have to constantly consider the difference $\pi_i^{\mathbf{p}}(v) - \pi_j^{\mathbf{p}}(v)$, so that the following formula plays an important role in our analysis:

$$\pi_i^{\mathbf{p}}(v) - \pi_j^{\mathbf{p}}(v) = \sum_{s=0}^{n-2} (p_s + p_{s+1}) \left[ \sum_{S \in \Sigma_{ij}^s} d_{ij}^S(v) \right]. \tag{9.10}$$

where $d_{ij}^S(v) = v(S \cup \{i\}) - v(S \cup \{j\})$. For easy notation, we shall define $x_{s+1} = p_s + p_{s+1}$. Thus, in order to study the sign of the difference $\pi_i^{\mathbf{p}}(v) - \pi_j^{\mathbf{p}}(v)$, it is clear that it is possible to substitute the quantity $x_{s+1}$, for $s = 0, \dots, n-2$, with $ax_{s+1}$, for any $a > 0$. We are interested in the case when for some $a > 0$ the quantity $ax_{s+1}$ is a natural number for every $s = 0, \dots, n-2$, and thus we shall consider, form now on, only semivalues such that the associated vector $(p_0, \dots, p_{n-1})$ has rational components, and we consider associated quantities $x_{s+1}$ to be *natural numbers.*

Let us then fix a preference relation $\succcurlyeq$ on $2^N$, suppose $\pi^{\mathbf{p}}$ is a given semivalue and fix the associated natural numbers $x_1, \dots, x_{n-1}$. For a given $i \in N$ and a subfamily $\mathcal{F}$ of $2^N$, we write $\theta^{\mathbf{p}}(\mathcal{F}, i)$ for the vector constructed in the following way. Order in decreasing order of preference the sets $S \cup \{i\}$, where $S \in \mathcal{F}$:

$$S_1 \cup \{i\} \succcurlyeq S_2 \cup \{i\} \succcurlyeq \cdots \succcurlyeq S_l \cup \{i\} \succcurlyeq \dots,$$

then replicate each coalition $S_k$ precisely $x_{s_k+1}$ times, if $|S_k| = s_k$, and form the vector

$$\theta^{\mathbf{p}}(\mathcal{F}, i) = (\underbrace{S_1 \cup \{i\}, \dots, S_1 \cup \{i\}}_{x_{s_1+1}\text{times}}, \underbrace{S_2 \cup \{i\}, \dots, S_2 \cup \{i\}}_{x_{s_2+1}\text{times}}, \dots).$$

We provide just a simple example to familiarize with this notation.

**Example 9.1** *Let $N = \{1, 2, 3\}$, consider the preference relation*

$$N \succ \{1\} \succ \{2,3\} \succ \{1,3\} \succ \{2\} \succ \{1,2\} \succ \{3\} \succ \emptyset.$$

*Let $\mathcal{F} = \Sigma_{12} = \{\emptyset, \{3\}\}$. Let $\mathbf{p} = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ and consider the corresponding vector of natural numbers $(1, 1, 2)$; so, $x_1 = 2$ and $x_2 = 3$. Then*

$$\theta^{\mathbf{p}}(\Sigma_{12}, 1) = (\{1\}, \{1\}, \{1,3\}, \{1,3\}, \{1,3\})$$

*and*

$$\theta^{\mathbf{p}}(\Sigma_{12}, 2) = (\{2,3\}, \{2,3\}, \{2,3\}, \{2\}, \{2\}).$$

Now, we write

$$\theta^{\mathbf{p}}(\Sigma_{ij}, i) \succcurlyeq \theta^{\mathbf{p}}(\Sigma_{ij}, j)$$

if

$$(\theta^{\mathbf{p}}(\Sigma_{ij}, i))_k \succcurlyeq (\theta^{\mathbf{p}}(\Sigma_{ij}, j))_k, \quad \forall k \in \{1, \dots, \sum_{s=0}^{n-2} \binom{n-2}{s} x_{s+1}\}$$

and we define a relation $\trianglerighteq_p$ over $N$ such that $\{i\} \trianglerighteq_p \{j\} \Leftrightarrow \theta^{\mathbf{p}}(\Sigma_{ij}, i) \succcurlyeq \theta^{\mathbf{p}}(\Sigma_{ij}, j)$. $\trianglerighteq_p$ induces a partial order on the set of singletons in $N$.

The next definition can be seen as our first definition of compatible extension. Let $\mathbf{p}$ be a semivalue.

**Definition 9.11 (p-WPR)** *We say that the total preorder* $\succcurlyeq$ *on* $2^N$ *satisfies the* $\mathbf{p}$*-weighted permutational responsiveness (*$\mathbf{p}$*-WPR) property if for each* $i, j \in N$ *we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow i \trianglerighteq_p j. \tag{9.11}$$

Here is the first theorem, showing when a preorder is aligned with respect to a fixed semivalue.

**Theorem 9.12** *Let* $\succcurlyeq$ *be a total preorder on* $2^N$ *and let* $\pi^{\mathbf{p}}$ *be a semivalue with rational probabilities. The following two statements are equivalent:*

*1)* $\succcurlyeq$ *is* $\pi^{\mathbf{p}}$*-aligned;*

*2)* $\succcurlyeq$ *satisfies the* $\mathbf{p}$*-WPR property.*

Let us see an example.

**Example 9.2** *Let* $N = \{1,2,3,4\}$ *and let* $\succcurlyeq$ *be a total preorder such that* $\{1,2,3,4\} \succ \{2,3,4\} \succ \{3,4\} \succ \{4\} \succ \{3\} \succ \{2\} \succ \{2,4\} \succ \{1,4\} \succ \{1,3\} \succ \{2,3\} \succ \{1,3,4\} \succ \{1,2,4\} \succ \{1,2,3\} \succ \{1,2\} \succ \{1\} \succ \emptyset$. *It can be seen that this is aligned for the Banzhaf value. All verifications are lengthy, so we just want to see that the condition is verified when considering objects* 1 *and* 2*. Since* $\{2\} \succ \{1\}$*, we need to prove that* $2 \trianglerighteq_p 1$*. The following displays the vectors* $\theta$*.*

| $\theta(\Sigma_{21}, \{2\}$ | $\theta(\Sigma_{21}, \{1\}$ |
|---|---|
| $\{2,3,4\}$ | $\{1,4\}$ |
| $\{2\}$ | $\{1,3\}$ |
| $\{2,4\}$ | $\{1,3,4\}$ |
| $\{2,3\}$ | $\{1,4\}$ |

*It is easy to check that in every line the coalition written on the left is ranked before the corresponding coalition in the right. A similar analysis can be done by comparing* 3 *with* 4 *and* 4 *with* 2 *and this allows to conclude that the given ranking is aligned for the Banzhaf value.*

*Consider now the set* $N = \{1,2,3\}$ *and the ranking*

$$N \succ \{1\} \succ \{2,3\} \succ \{1,3\} \succ \{2\} \succ \{1,2\} \succ \{3\} \succ \emptyset.$$

*Let* $\mathbf{p} = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ *and consider the corresponding vector of natural numbers* $(1,1,2)$*; so,* $x_1 = 2$ *and* $x_2 = 3$*. By comparing objects* 1 *and* 2*, we get the following table:*

*Looking at the third line, since* $\{2,3\} \succ \{1,3\}$*, we can conclude that the given ranking is* not *aligned with respect to the semivalue* $\mathbf{p} = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5})$*.*

The final result instead characterizes the preorders aligned with every regular semivalue. For every $i, j \in N$, call $\mathcal{D}_{ij}^s$ the set $\mathcal{D}_{ij}^s = \Sigma_{ij}^s \cup \Sigma_{ij}^{s+1}$ for $s = 0, \ldots, n-3$. Set also $\mathcal{D}_{ij}^{n-2} = \Sigma_{ij}^{n-1}$.

| | |
|---|---|
| $\theta(\Sigma_{12}, \{1\}$ | $\theta(\Sigma_{12}, \{2\}$ |
| $\{1\}$ | $\{2,3\}$ |
| $\{1\}$ | $\{2,3\}$ |
| $\{1,3\}$ | $\{2,3\}$ |
| $\{1,3\}$ | $\{2\}$ |
| $\{1,3\}$ | $\{2\}$ |

**Definition 9.12 (DPR)** *We say that a total preorder on $2^N$ satisfies the double permutational responsiveness (DPR) property if for each $i, j \in N$ we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \theta(\mathcal{D}^s_{ij}, i)_k \succcurlyeq \theta(\mathcal{D}^s_{ij}, j)_k \tag{9.12}$$

*for every $k = 1, \ldots, |\mathcal{D}^s_{ij}|$ and every $s = 0, \ldots, n-2$.*

**Theorem 9.13** *Let $\succcurlyeq$ be a total preorder on $2^N$. The following statements are equivalent:*

*1) $\succcurlyeq$ fulfills the DPR property;*

*2) $\succcurlyeq$ is $\pi^{\mathbf{p}}$-aligned for all semivalues.*

To conclude, we revisit one of the previous examples.

**Example 9.3** *Let $N = \{1,2,3,4\}$ and let $\succcurlyeq$ be a total preorder such that* $\{1,2,3,4\} \succ \{2,3,4\} \succ \{3,4\} \succ \{4\} \succ \{3\} \succ \{2\} \succ \{2,4\} \succ \{1,4\} \succ \{1,3\} \succ \{2,3\} \succ \{1,3,4\} \succ \{1,2,4\} \succ \{1,2,3\} \succ \{1,2\} \succ \{1\} \succ \emptyset$.

*We have seen that this ranking is aligned with the Banzhaf value. Actually, it can be shown that it fulfills DPR, and thus it is aligned with every (rational) regular semivalue. Observe also that the ranking $\succcurlyeq$ does not satisfy RESP because $\{2\} \succ \{1\}$, $\{2,4\}$ is strictly preferred to $\{1,4\}$ and $\{1,3\}$ is strictly preferred to $\{2,3\}$.*

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## 9.6 Conclusions

In this chapter we have considered symmetric values defined on the space of all TU coalitional games with a fixed set of players. A natural way to generate these values is to define them on the class of unanimity games and extend them on the whole space by linearity. So the question we addressed here was: When does such a definition generate a regular semivalue, i.e., a particularly interesting subclass of the family of the probabilistic values? We were able to provide a characterization in this sense, involving completely monotonic sequences. Since creating completely monotonic sequences is a simple task, this allows to create large families of semivalues. Finally, we have also considered and characterized the case of irregular semivalues. Next, we presented two examples, in different fields, where the extensive use of semivalues, and not only the classical Shapley and Banzhaf values, was proposed. The first one is in molecular genetics, and it is aimed at providing ranking of genes responsible for genetic diseases, starting from microarray data analysis. The second instead is in Social Choice, and it deals with the problem of finding extensions of a ranking between objects to a ranking between the subsets of the objects, respecting in a precise sense the ranking between the singletons. This problem presents many contributions in the related literature, but usually extensions do not take into account possible interactions between objects, in the sense that usually extensions fulfill the property that one object $x$ is better than another object $y$ if and only if $S \cup \{x\}$ is better than $S \cup \{y\}$ for every possible subset $S$ containing neither $x$ nor $y$. In our approach instead, this is allowed under certain conditions.

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# Chapter 10

## Power and the Shapley Value

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### 10.1 Introduction

The Shapley value [19] can be used not only as a rule to divide the gains from cooperation in a game with transferable utility, but also as a measure of power in simple games, that is, games in which the worth of a coalition is zero if it is losing and one if it is winning. Think, for instance, of a parliament where the winning coalitions are those that have a majority. In this case, the Shapley value is also called the Shapley-Shubik power index [20], and it measures, in a specific way, the number of times that a player, for instance a political party, is pivotal – turns a losing into a winning coalition by joining it. A closely related power index is the Banzhaf index or Banzhaf-Coleman index [1], but there are many other indices as well (see [2] for a recent overview).

A drawback of this use of the Shapley value and, for that matter, also of other power indices, is that it takes into account neither the issues at hand nor special relations and structures that may exist among the players. For instance, the political position of a political party – left, right – nor the content of issues on which parliamentary voting takes place, are taken into account when computing the power of a party according to the Shapley-Shubik index.

As a remedy, the political science literature considers spatial models, where political parties are positioned with respect to a number, say $k$, of criteria, and a power index, besides the simple game, takes this constellation in $\mathbb{R}^k$ into account. An example of this is the Owen-Shapley spatial power index [15] which has been axiomatically characterized in [18]. See also [21] for a partial overview of this literature.

The first objective of this chapter is to present a model that generalizes both simple games and spatial models by specifying exactly which issues (alternatives) can be controlled by which players and coalitions. See Section 10.3, which is based on [12]. In particular, we develop a class of power indices that extend the Shapley value.

The second objective is to review and link together a few models in which relations that may exist between the players and that influence their power are taken into account. In Section 10.4 we consider the case where players and coalitions may be controlled by other coalitions, a typical example being provided by firms and investors in a network determined by share holdings. This work was preceded by [5] and [9, 10]; the power indices that will be discussed were developed in [11]. A refinement of this model to directed graphs, based on [17], is discussed in Section 10.5. Section 10.6 concludes.

## 10.2 Preliminaries

We start with some notations. For a set $D$ we denote by $P(D)$ the set of all subsets of $D$, and by $P_0(D)$ the set of all nonempty subsets of $D$. By $|D|$ we denote the number of elements of $D$.

Throughout, $N = \{1, \ldots, n\}$ ($n \in \mathbb{N}$) is the set of *players*. Subsets of $N$ are also called *coalitions*. A game with transferable utility or *TU-game* is a pair $(N, v)$, where $v : P(N) \to \mathbb{R}$ with $v(\emptyset) = 0$. The number $v(S)$ is the *worth* of coalition $S$. The TU-game $(N, v)$ is *simple* if $v(S) \in \{0, 1\}$ for all $S \in P(N)$, $v(N) = 1$, and $S \subseteq T \Rightarrow v(S) \leq v(T)$ for all $S, T \in P(N)$. If $v(S) = 1$ coalition $S$ is *winning*, otherwise it is *losing*. For $T \subseteq N$ the *unanimity game* $(N, u_T)$ is defined by $u_T(S) = 1$ whenever $T \subseteq S$, and $u_T(S) = 0$ otherwise. Instead of $(N, v)$ we also often write $v$.

The Shapley value of a game $(N, v)$ for player $i$ is given by the expression

$$Sh_i(N, v) = \sum_{S \subseteq N : i \notin S} \frac{|S|!(n - |S| - 1)!}{n!}[v(S \cup \{i\}) - v(S)].$$

An alternative expression using so-called dividends will be introduced later in the chapter.

## 10.3 Effectivity and Power

In order to illustrate the aim of this section, which is based on [12], we start with a few examples.

**Example 10.1** *Two men $m_1$ and $m_2$ and a woman $f$ have the following options (alternatives): Each of them stays single, denoted by $s$; $f$ marries $m_1$, denoted by $w_1$; or $f$ marries $m_2$, denoted by $w_2$. Each person has the right to stay single, and for a marriage the consent of both involved persons is required. In this situation, can we say anything about the power of each person? (The example is adapted from [7]; it appears as Example 2.2.3 in [16].)*

**Example 10.2** *Consider the following 'game form':*

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} a & d & c \\ c & b & d \end{pmatrix} \end{array}$$

*Here, $N = \{1, 2\}$, player 1 chooses rows, player 2 chooses columns, and $\{a, b, c, d\}$ is a set of alternatives. (We obtain a bimatrix game if we add utilities of the players over the set of alternatives.) Again the question is: What can we say about the power of the players?*

We will answer the questions raised in these examples by developing a class of power measures for 'effectivity functions'. Let $A$ denote the set of *alternatives*. We fix a set $\mathcal{T} \subseteq P_0(A)$, where $\mathcal{T} = P_0(A)$ if $A$ is a finite set; if $A$ is infinite, endowed with a topology, then $\mathcal{T}$ will be the collection of nonempty closed subsets of $A$.

**Definition 10.1** An *effectivity function* (for $\mathcal{T}$) is a map $E : P(N) \to P(\mathcal{T})$ such that (i) $P(\emptyset) = \emptyset$, (ii) $A \in E(S)$ for every $S \in P_0(N)$, (iii) $E(N) = \mathcal{T}$, and (iv) $B \in E(S)$ implies $B' \in E(T)$ for all $B, B' \in \mathcal{T}$ and $S, T \in P_0(N)$ such that $B \subseteq B'$ and $S \subseteq T$. The set of all effectivity functions is denoted by $\mathcal{E}$.[1]

If $B \in E(S)$, then we say that $S$ is effective for $B$, and this is interpreted as coalition $S$ being able to guarantee that the 'final' alternative is in $B$, or is entitled to this alternative being in $B$. Condition (i) in Definition 10.1 means that the empty coalition is not effective for anything. Condition (ii) means that every coalition is effective for the set of all alternatives, which is a trivial condition reflecting the assumption that there has to be some 'final' outcome. Condition (iii) means that the grand coalition of all players is almighty: it is effective for every nonempty set of alternatives. Condition

[1] The term 'effectivity function' was coined by [14]. For an earlier use of the concept, see for instance [6].

(iv) means that if $S$ is effective for $B$, then every (weakly) larger coalition is effective for every (weakly) larger set of alternatives. The last condition is usually called 'monotonicity'. An additional condition that is usually satisfied by an effectivity function is *superadditivity*, which means that if a coalition $S$ is effective for a set $B$ and $T$ is effective for $C$ and if $S$ and $T$ are disjoint, then $S \cup T$ is effective for $B \cap C$. Here, however, we do not impose this condition on an effectivity function.

**Example 10.3** *(i) In Example 10.1 the set of alternatives is* $A = \{s, w_1, w_2\}$ *and for the associated effectivity function* $E$ *we have* $E(\{m_1\}) = E(\{m_2\}) = \{A\}$, $E(\{f\}) = \{B \in P_0(A) \mid s \in B\}$, $E(\{m_i, f\}) = \{B \in P_0(A) \mid s \in B \text{ or } w_i \in B\}$ *for* $i = 1, 2$, *and* $E(N) = P_0(A)$.

*(ii) For Example 10.2 we have* $E(\{1\}) = \{B \in P_0(A) \mid \{a, d, c\} \subseteq B \text{ or } \{c, b, d\} \subseteq B\}$, $E(\{2\}) = \{B \in P_0(A) \mid \{a, c\} \subseteq B \text{ or } \{b, d\} \subseteq B \text{ or } \{c, d\} \subseteq B\}$, *and* $E(N) = P_0(A)$.

*(iii) As a third example, let* $(N, v)$ *be a simple game, and let* $A$ *be some set of alternatives. Then with* $(N, v)$ *we can associate an effectivity function* $E$ *by letting* $E(S) = \mathcal{T}$ *if* $S$ *is winning and* $E(S) = \{A\}$ *if* $S \neq \emptyset$ *is losing.*

Our aim is to find reasonable measures of power for effectivity functions:

**Definition 10.2** A *power index* on $\mathcal{E}$ is a map $\varphi : \mathcal{E} \to \mathbb{R}^N$ with $\sum_{i \in N} \varphi_i(E) = 1$.

We will impose three basic axioms on a power index for effectivity functions. First a few pieces of notation: For $E, F \in \mathcal{E}$, $E \cup F$ and $E \cap F$ are defined by $E \cup F(S) = E(S) \cup F(S)$ and $E \cap F(S) = E(S) \cap F(S)$ for all $S \in P(N)$. It is straightforward to verify that $E \cup F, E \cap F \in \mathcal{E}$.

The main axiom that we will impose on a power index $\varphi$ is the Transfer Property, which was first formulated for a value on simple games by [4].

**Transfer Property** For all $E, F \in \mathcal{E}$,

$$\varphi(E \cup F) + \varphi(E \cap F) = \varphi(E) + \varphi(F).$$

Throughout, we will also impose anonymity. For a permutation $\pi$ of $N$ and an effectivity function $E \in \mathcal{E}$, let $\pi E \in \mathcal{E}$ be defined by $(\pi E)(\pi(S)) = E(S)$ for all $S \in P(N)$.

**Anonymity** $\varphi_i(E) = \varphi_{\pi(i)}(\pi E)$ for every $E \in \mathcal{E}$, every permutation $\pi$ of $N$, and every $i \in N$.

The third axiom is a monotonicity condition.

**Monotonicity** $\varphi_i(E) \leq \varphi_i(F)$ for all $E, F \in \mathcal{E}$ and every $i \in N$ such that $E(S) \setminus E(S \setminus \{i\}) \subseteq F(S) \setminus F(S \setminus \{i\})$ for all $S \in P(N)$.

The Transfer Property replaces the usual additivity or linearity condition for values of TU-games. Anonymity is clear, and Monotonicity requires that a player whose contributions in an effectivity function $F$ are larger than in $E$, should also be assigned more power in $F$ than in $E$. Monotonicity is somewhat similar to the monotonicity condition of [25] used to characterize the Shapley value and replacing the additivity condition. In our present richer context, both conditions – that is, the Transfer Property and Monotonicity – are complementary and will be used in one and the same characterization.

In order to formulate the results below, for an effectivity function $E$ and a set of alternatives $B \in \mathcal{T}$ we define the simple game $v_B^E$ by $v_B^E(S) = 1$ if $B \in E(S)$ and $v_B^E(S) = 0$ otherwise. In other words, the winning coalitions in $v_B^E$ are exactly those that are effective for $B$.

### 10.3.1 Finitely Many Alternatives

The first result is for a finite set of alternatives. A *weight system* is a collection $\omega = \left(\omega^B\right)_{B \in P_0(A)}$ of nonnegative real numbers such that $\sum_{B \in P_0(A)} \omega^B = 1$. For a weight system $\omega$ we define the power index $\Phi^\omega$ by

$$\Phi^\omega(E) = \sum_{B \in P_0(A)} \omega^B Sh\left(v_B^E\right)$$

for every $E \in \mathcal{E}$.

The following theorem is Theorem 4.10 in [12], to which we refer the reader for a proof.[2]

**Theorem 10.1** *A power index $\varphi$ satisfies the Transfer Property, Anonymity, and Monotonicity if and only if there is a weight system $\omega$ such that $\varphi = \Phi^\omega$.*

**Example 10.4** *We consider the effectivity functions in Example 10.3.*
*(i) In this case, we have, for $(m_1, m_2, f)$:*

$$Sh(N, v_B^E) = \begin{cases} (\frac{1}{2}, 0, \frac{1}{2}) & \textit{if } B = \{w_1\} \\ (0, \frac{1}{2}, \frac{1}{2}) & \textit{if } B = \{w_2\} \\ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & \textit{if } B = \{w_1, w_2, s\} \\ (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}) & \textit{in all other cases.} \end{cases}$$

*Hence,*

$$\begin{aligned} \Phi^\omega(E) &= \omega^{\{w_1\}}(\frac{1}{2}, 0, \frac{1}{2}) + \omega^{\{w_2\}}(0, \frac{1}{2}, \frac{1}{2}) + \omega^A(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \\ &\quad + (1 - \omega^{\{w_1\}} - \omega^{\{w_2\}} - \omega^A)(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}). \end{aligned}$$

[2] The proof in [12] holds for superadditive effectivity functions, but it can be checked that it still holds and even simplifies without the superadditivity condition.

*For instance, a plausible choice of weights could be* $\omega^{\{w_1\}} = \omega^{\{w_2\}} = \omega^{\{s\}} = \frac{1}{3}$ *and* $\omega^B = 0$ *otherwise, and then* $\Phi^\omega(E) = (\frac{2}{9}, \frac{2}{9}, \frac{5}{9})$.
*(ii) In this case,*

$$Sh(N, v_B^E) = \begin{cases} (0,1) & \textit{if } B \in \{\{a,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\}\} \\ (\frac{1}{2},\frac{1}{2}) & \textit{in all other cases.} \end{cases}$$

*Hence,* $\Phi^\omega(E) = \alpha(0,1) + (1-\alpha)(\frac{1}{2},\frac{1}{2})$, *where* $\alpha = \omega^{\{a,c\}} + \omega^{\{b,d\}} + \omega^{\{c,d\}} + \omega^{\{a,b,c\}} + \omega^{\{a,b,d\}}$.
*(iii) We assume that* $A$ *is finite. Now* $(N, v_B^E) = (N, v)$ *for every* $B \in P_0(A) \setminus \{A\}$, *and* $v_A^E(S) = 1$ *for all* $S \in P_0(N)$. *Consequently,* $\Phi^\omega(E) = (1-\omega^A)Sh(N,v) + \omega^A(\frac{1}{n},\ldots,\frac{1}{n})$. *For the (plausible) case where* $\omega^A = 0$ *we therefore have that* $\Phi^\omega(E) = Sh(N,v)$, *i.e.,* $\Phi^\omega$ *is just the Shapley value.*

In [12] further axioms are added, which refine the (large) class of power indices characterized in Theorem 10.1. For instance, in most applications one would expect the weight of a subset of alternatives to decrease with the size of the subset, since being effective for a set implies being effective for every superset.

In the next subsection, we consider the case of an infinite set of alternatives, and at the same time impose further axiomatic restrictions.

### 10.3.2 Infinitely Many Alternatives

We now assume that $A$ is a possibly infinite set, endowed with a topology. More precisely, we assume that for every $a \in A$ the set $\{a\}$ is closed.[3]

Call a player $i \in N$ a *null player* in $E \in \mathcal{E}$ if $E(S) \setminus E(S \setminus \{i\}) = \emptyset$ for all $S \in P(N)$.

We consider the following further axioms for a power index $\varphi$.

**Strong Monotonicity** $\varphi_i(E) \leq \varphi_i(F)$ for all $E, F \in \mathcal{E}$ and every $i \in N$ such that $\{a\} \in E(S) \setminus E(S \setminus \{i\})$ implies $\{a\} \in F(S) \setminus F(S \setminus \{i\})$ for all $S \in P(N)$ and all $a \in A$.

**Null Player** $\varphi_i(E) = 0$ for every $E \in \mathcal{E}$ and every null player $i$ in $E$.

**Continuity** For every sequence $(E_k)_{k\in\mathbb{N}}$ of effectivity functions with $E_1 \subseteq E_2 \subseteq \ldots$ it holds that

$$\varphi\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \varphi\left(E_k\right).$$

Clearly the premiss in the definition of Strong Monotonicity is weaker than the one in the definition of Monotonicity. Jointly with other conditions, Strong

[3] That is, $A$ is a $T_1$-*space*.

Monotonicity will imply that only single alternatives matter for a power index. In the definition of Continuity, the union at the left-hand side of the equality is a well-defined effectivity function: See Lemma 5.1 in [12].

For a probability measure $\mu$ on the $\sigma$-field of Borel sets generated by the topology on $A$ we define the map $\Phi^\mu : \mathcal{E} \to \mathbb{R}^N$ by

$$\Phi_i^\mu(E) = \int_A Sh_i(v_{\{a\}}^E)d\mu(a)$$

for every $E \in \mathcal{E}$ and $i \in N$. Clearly, $\sum_{i \in N} \Phi_i^\mu(E) = 1$ for every $E \in \mathcal{E}$, so that $\Phi^\mu$ is a power index. See [12] for a proof of the following result.

**Theorem 10.2** *Let $\varphi$ be a power index. Then $\varphi$ satisfies the Transfer Property, Anonymity, Strong Monotonicity, Continuity, and the Null Player Property if and only if there is a probability measure $\mu$ such that $\varphi = \Phi^\mu$.*

Due to in particular the strong monotonicity requirement, compared to Theorem 10.1 now only Shapley values of simple games associated with single alternatives occur, and the weight system is replaced by the probability measure $\mu$.

#### 10.3.2.1 An Application: The Owen-Shapley Spatial Power Index

A simple game $(N, v)$ is *proper* if $v(S) = 1$ implies that $v(N \setminus S) = 0$ for each $S \in P(N)$. Let $k \in \mathbb{N}$, $k \geq 2$. A *spatial game* is a pair $g = (v, p)$ where $v$ is a proper simple game and $p = (p^1, \ldots, p^n) \in (\mathbb{R}^k)^N$ with $p^i \neq p^j$ for all $i, j \in N$ with $i \neq j$. Here, $p^i \in \mathbb{R}^k$ is the *position* of player $i$. For instance, $k = 2$, $i$ is a political party, $p_1^i$ reflects $i$'s position with respect to public spending on defense, and $p_2^i$ reflects $i$'s position with respect to public spending on education.

Following [15] we let the set of *issues* $A$ be represented by the unit sphere in $\mathbb{R}^k$, i.e.,

$$A = \{a \in \mathbb{R}^k : ||a|| = 1\} ,$$

where $||\cdot||$ is the Euclidean distance, and we interpret the inner product $p^i \cdot a$ as a measure of the attractiveness of issue $a \in A$ for a player with position $p^i$. More precisely, we interpret the inequality $p^i \cdot a \leq p^j \cdot a$ as player $i$ being more in favor of issue $a \in A$ than player $j$.[4] For a spatial game $g = (v, p)$ and an issue $a \in A$, we say that player $i$ is *pivotal for $a$* if $\{j \in N \mid p^j \cdot a \leq p^i \cdot a\}$ is a winning coalition but $\{j \in N \mid p^j \cdot a \leq p^i \cdot a\} \setminus \{i\}$ is losing. Here, one should think of a coalition being formed in favor of an issue $a$: The players join the coalition in order of their enthusiasm for $a$, and the pivotal player is the player who upon joining the coalition turns this from a losing into a winning coalition.

[4] Of course, this is just a matter of choice: Without loss of generality one could also take the reverse inequality. Further, one can interpret $p^i \cdot a$ as representing the 'utility' of an issue $a$ for player $i$ with position $p^i$ – thus, implicitly linear 'utility' is assumed.

We assume that $A$ is endowed with the relative topology induced by the Euclidean topology on $\mathbb{R}^k$. It is not difficult to see that for almost all $a \in A$ there is a unique pivotal player. Let $\lambda$ be the Lebesgue measure on $A$, and define the probability measure $\nu$ on $A$ by $\nu(B) = \lambda(B)/\lambda(A)$ for every Borel set $B \subseteq A$. Then the *Owen-Shapley spatial power index OS* assigns to each player $i$ the number $OS_i(v,p) = \nu(B)$ if $B$ is the set of issues for which player $i$ is pivotal.

Figure 10.1 illustrates the Owen-Shapley spatial power index for a spatial game in which the simple game is a three-person unanimity game, that is, a spatial game $(\{1,2,3\}, u_{\{1,2,3\}}, (p_1,p_2,p_3))$: In the simple game $u_{\{1,2,3\}}$ only the grand coalition $\{1,2,3\}$ is winning. The issues on the arc of the circle containing the point $p^i$ are those for which player $i$ is pivotal, for each $i = 1,2,3$.

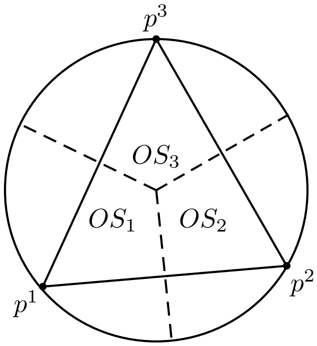


**Figure 10.1** The Owen-Shapley spatial power index in a three-person spatial unanimity game. The dashed lines are perpendicular to the edges of the triangle, and the powers of the players are proportional to the three pieces of the disk.

Now for a spatial game $(v,p)$ we construct an effectivity function $F$ by letting player $i \in N$ be effective for the singleton $\{a\}$ if $i$ is pivotal for $a$. Formally, for every $S \in P_0(N)$,

$$F(S) = \{B \subseteq A \mid B = A \text{ or } i \text{ is pivotal for } b \text{ for some } i \in S \text{ and } b \in B\}.$$

If $i$ alone is pivotal for $a$, then $Sh_i(v^F_{\{a\}}) = 1$ and $Sh_j(v^F_{\{a\}}) = 0$ for all $j \in N \setminus \{i\}$. Therefore, for every player $i$ we have

$$OS_i(v,p) = \Phi^\nu_i(F) = \int_A Sh_i(v^F_{\{a\}})d\nu(a).$$

Thus, the Owen-Shapley spatial power index is a special case of the power indices characterized in Theorem 10.2. See [18] for another axiomatic characterization of OS, and [13] for a generalization.

## 10.4 Control and Power

The results in this section are based on [11]. The leading example in [11] is depicted in Figure 10.2. The diagram describes the Porsche and VW voting rights by the end of 2012, based on the annual reports 2012 of Volkswagen AG and Porsche Automobil Holding SE GmbH. The players are Porsche Families (1), Qatar (2), Lower Saxony (3), Porsche SE (4), Volkswagen AG (5), Porsche AG (6), and other (small) stockholders (7).

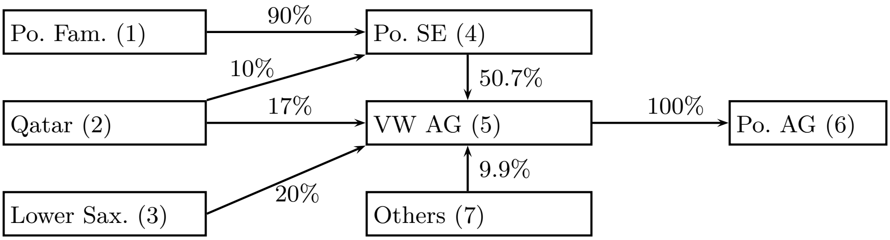


**Figure 10.2** The Porsche-VW case.

Based on this diagram and some further restrictions and laws, for which we refer to [11], one can describe the 'control structure': For each coalition of players, which players are controlled by this coalition? One way[5] in which this can be done is by simple games or the zero game: For each player $i$, the winning coalitions in a simple game $w_i$ are those that control player $i$, which means they have the required percentage of votes over that player. If a player is not controlled by any coalition, then we take for $w_i$ the zero game $(N, z)$, i.e., $z(S) = 0$ for every $S \in P(N)$. For the example in Figure 10.2, we arrive at the following games: For $S \in P(N)$,

$$\begin{aligned} w_1 = w_2 = w_3 = w_7 &\quad = \quad z, \\ w_4(S) = 1 &\quad \Leftrightarrow \quad \{1\} \subseteq S, \\ w_5(S) = 1 &\quad \Leftrightarrow \quad \{2,3,4\} \subseteq S, \\ w_6(S) = 1 &\quad \Leftrightarrow \quad \{5\} \subseteq S. \end{aligned}$$

These simple games express *direct* control. For instance, any coalition containing players 2, 3, and 4, controls player 5. Note, however, that player 4 is controlled by any coalition containing player 1, and therefore player 5 is also *indirectly* controlled by any coalition containing players 1, 2, and 3. By incorporating all such indirect control relations as well we obtain the simple

[5] See [11] for a different but equivalent way.

games $w_i^*$ given by, for each $S \in P(N)$:

$$\begin{aligned} w_1^* = w_2^* = w_3^* = w_7^* &= z, \\ w_4^*(S) = 1 &\Leftrightarrow \{1\} \subseteq S, \\ w_5^*(S) = 1 &\Leftrightarrow \{2,3,4\} \subseteq S \text{ or } \{1,2,3\} \subseteq S, \\ w_6^*(S) = 1 &\Leftrightarrow \{5\} \subseteq S \text{ or } \{1,2,3\} \subseteq S \text{ or } \{2,3,4\} \subseteq S. \end{aligned}$$

We will develop a class of power indices for situations like this. Formally, a *control structure* is an $n$-tuple $\bar{w} = (w_1, \ldots, w_n)$ where, for each $i \in N$, $w_i$ is either the zero game $z$ or a simple game satisfying: For all $j \in N$ and all $S, T \subseteq N$, if $w_i(S) = w_j(T) = 1$, then $w_i((S \setminus \{j\}) \cup T) = 1$. The last condition means that $\bar{w}$ also captures indirect control: Coalition $S$ controls $i$, and if $j$ is a member of $S$ but at the same time controlled by $T$, then $j$'s position in $S$ can be replaced by the coalition $T$. Let $\mathcal{W}$ denote the set of all control structures.[6]

**Definition 10.3** A *power index* on $\mathcal{W}$ is a map $\varphi : \mathcal{W} \to \mathbb{R}^N$.

Call player $i$ a *null player* in $\bar{w} \in \mathcal{W}$ if (i) $w_i = z$, and (ii) for all $j \in N \setminus \{i\}$ and $S \in P(N)$, $w_j(S) = w_j(S \setminus \{i\})$. Hence, a null player is a player who is neither controlled nor adds anything to controlling other players. An example is player 7 in the Porsche-VW case. The first axiom we impose on a power index $\varphi$ is as follows.

**Null Player** $\varphi_i(\bar{w}) = 0$ for every $\bar{w} \in \mathcal{W}$ and every null player $i$ in $\bar{w}$.

A player who does not add anything to control but is controlled by some coalition of players, has in some sense even less power than a null player. For this reason it is natural to assume that a power index can assign negative numbers in this framework. Actually, we impose the following axiom.

**Zerosum** $\sum_{i \in N} \varphi_i(\bar{w}) = 0$ for every $\bar{w} \in \mathcal{W}$.

This axiom implies that, in general, there will be players with positive power as well as players with negative power.

For a permutation $\pi$ of $N$ and a control structure $\bar{w}$ we denote by $\pi\bar{w}$ the control structure with $(\pi\bar{w})_{\pi(i)}(\pi S) = w_i(S)$ for every $i \in N$ and $S \in P(N)$.

**Anonymity** $\varphi_i(\bar{w}) = \varphi_{\pi(i)}(\pi\bar{w})$ for every $\bar{w} \in \mathcal{W}$, every permutation $\pi$ of $N$, and every $i \in N$.

Finally, the transfer property takes the following form.

**Transfer Property** $\varphi_i(\bar{w}) - \varphi_i(\bar{w}') = \varphi_i(\bar{v}) - \varphi_i(\bar{v}')$ for every $i \in N$ and all $\bar{w}, \bar{w}', \bar{v}, \bar{v}' \in \mathcal{W}$ such that, for all $S \in P(N)$, (i) $w_i'(S) = 1 \Rightarrow w_i(S) = 1$ and $v_i'(S) = 1 \Rightarrow v_i(S) = 1$, (ii) $[w_i'(S) = 0$ and $w_i(S) = 1 \Leftrightarrow v_i'(S) = 0$ and $v_i(S) = 1]$.

[6] Control structures are equivalent to so-called command games in [9, 10]. For an earlier approach, see [5].

In words, if $\bar{w}$ arises from $\bar{w}'$ and $\bar{v}$ arises from $\bar{v}'$ by adding the same winning coalitions, then for each player the change in power when going from $\bar{w}'$ to $\bar{w}$ should be equal to the change in power when going from $\bar{v}'$ to $\bar{v}$.

In fact, it can be shown that the Transfer Property is equivalent to the following: For all $\bar{w}, \bar{v} \in \mathcal{W}$,

$$\varphi(\bar{w}) + \varphi(\bar{v}) = \varphi(\bar{w} \vee \bar{v}) + \varphi(\bar{w} \wedge \bar{v})$$

where $\bar{w} \vee \bar{v} = (\max(w_1, v_1), \ldots, \max(w_n, v_n))$ and $\bar{w} \wedge \bar{v} = (\min(w_1, v_1), \ldots, \min(w_n, v_n))$, with the maxima and minima defined coalition-wise. This is similar to the original formula in [4], but the formulation of the Transfer Property above has a more intuitive interpretation.

These four conditions determine a family of power indices. In order to formulate this result, recall that the *dividends* $d(S)$ [8] of a TU-game $v$ with player set $N$ are defined, recursively, by

$$d(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(S) - \sum_{T \subsetneq S} d(T) & \text{otherwise} \end{cases}$$

for all $S \subseteq N$. For a control structure $\bar{w} = (w_1, \ldots, w_n)$ and $i \in N$, we write $d_i^{\bar{w}}$ for the dividends of $w_i$. Also recall that the Shapley value of a TU-game $v$ is alternatively given by

$$Sh_i(v) = \sum_{S: i \in S} \frac{d(S)}{|S|}$$

for every $i \in N$.

For every *weight vector* $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$, we now define the power index $\Phi^\omega$ by

$$\Phi_i^\omega(\bar{w}) = \sum_{k \in N \setminus \{i\}} \left( \sum_{S: i \in S, k \notin S} \frac{d_k^{\bar{w}}(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_k^{\bar{w}}(S)}{|S|} \beta_{|S|} \right)$$
$$- \sum_{k \in N \setminus \{i\}} \left( \sum_{S: i \notin S, k \in S} \frac{d_i^{\bar{w}}(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_i^{\bar{w}}(S)}{|S|} \beta_{|S|} \right) \quad (10.1)$$

for all $\bar{w} \in \mathcal{W}$ and $i \in N$. This formula looks quite complicated, but it nevertheless has a clear interpretation, as follows. The expression in brackets in the first line of (10.1) says that player $i$ receives a weighted sum of dividends in the game $w_k$; this expresses the power player $i$ derives from his role in controlling player $k$. The weights depend both on the size of the coalition of whose dividend player $i$ receives a share, and on whether or not the controlled player $k$ is a member of that coalition. Thus, the first line in (10.1) represents the total power player $i$ acquires from his role in controlling the other players. In the second line, the total (similarly weighted) power that all other players acquire from controlling player $i$, is subtracted.

See [11] for a proof of the following theorem.

**Theorem 10.3** *Let $\varphi : \mathcal{W} \to \mathbb{R}^N$ be a power index. Then $\varphi$ satisfies Null Player, Zerosum, Anonymity, and the Transfer Property if and only if there is a weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ such that $\varphi = \Phi^\omega$.*

In Theorem 10.3 the weights $\omega$ are completely free and can be any real numbers. In [11], several conditions are considered that result in a refinement of this class of power indices. Here, we restrict our attention to the following 'scaling' condition.

**Controlled Player** For all $\bar{w} \in \mathcal{W}$, $j \in N$ with $w_j \neq z$, and $i \in N$ with $w_i = z$,

$$\varphi_j(\bar{w}) = \begin{cases} -1 & \text{if } w_k(S) = w_k(S \setminus \{j\}) \text{ for all } S \subseteq N \text{ and } k \in N \\ \varphi_i(\bar{w}) - 1 & \text{if } w_k(S \setminus \{i\}) = w_k(S \setminus \{j\}) \text{ for all } S \subseteq N \\ & \text{ such that } i, j \in S \text{ and all } k \in N. \end{cases}$$

The first line in the Controlled Player condition says that if $j$ is a 'controlled player', i.e., controlled by at least one coalition and, thus, by $N$, but does not exercise any control himself, then the power of $j$ is fixed at $-1$. Hence, the power of a least powerful player is fixed at $-1$. Further, if $i$ is an uncontrolled player, i.e., controlled by no coalition at all, but $i$ and $j$ exercise the same marginal control with respect to any coalition and player, then their difference in power is fixed at 1, that is, $i$ gets assigned 1 more than $j$. We now have the following corollary (see [11]).

**Corollary 10.1** *There is a unique power index satisfying Null Player, Zerosum, Anonymity, the Transfer Property, and Controlled Player, namely the power index $\Phi^\omega$ with $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$.*

We apply this unique power index to the Porsche-Volkswagen case.

**Example 10.5** *For the Porsche-VW case and $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$ we obtain $\Phi_1^\omega(\bar{w}) = \frac{67}{60}$, $\Phi_2^\omega(\bar{w}) = \Phi_3^\omega(\bar{w}) = \frac{32}{60}$, $\Phi_4^\omega(\bar{w}) = -\frac{53}{60}$, $\Phi_5^\omega(\bar{w}) = -\frac{18}{60}$, $\Phi_6^\omega(\bar{w}) = -1$, and $\Phi_7^\omega(\bar{w}) = 0$. It is interesting to compare the power of Porsche Families with its power at the end of 2007. Figure 10.3 depicts the control structure between the same companies at the end of 2007. At that time, Volkswagen was not controlled by any group of main investors. Although Porsche SE has veto power in the game on Volkswagen AG, we ignore this fact, as it is not clear how this power can be exercised. This situation results in a control structure $\bar{v}$ with coalition $S$ winning in $v_4$ if and only if $1 \in S$, and coalition $S$ winning in $v_6$ if and only if $1 \in S$. Thus, even while ignoring the power of Porsche Families on Volkswagen, we still have $\Phi_1^\omega(\bar{v}) = 2 > \frac{67}{60} = \Phi_1^\omega(\bar{w})$. Hence, according to this power index it had more power in 2007 than it had in the situation described by Figure 10.2 (end 2012).*

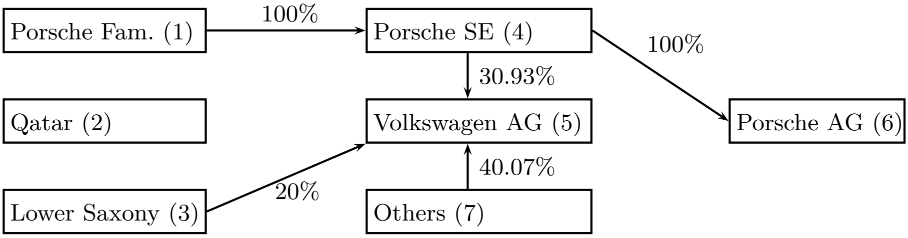


**Figure 10.3** Porsche and VW voting rights by the end of 2007, based on the 2007 annual report of Volkswagen AG and the 2007/2008 annual report of Porsche Automobil Holding SE GmbH.

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## 10.5 Power on Digraphs

In this section we closely follow [17]. The model in [17] is a special case of a control structure as defined in the previous section. More precisely, [17] considers control structures $\bar{w} = (w_1, \ldots, w_n)$ such that each $w_i$ is uniquely determined by the winning singleton coalitions, i.e., for each coalition $S$ we have $w_i(S) = 1$ if and only if there is a $k \in S$ with $w_i(\{k\}) = 1$. Such a control structure can be identified with a directed graph or *digraph* with $N$ as the set of *nodes* and a *link* (edge) from $i$ to $j$ if and only if $w_j(\{i\}) = 1$, i.e., player $j$ is controlled by player $i$. Let $\mathcal{D} \subseteq \mathcal{W}$ denote the set of all such control structures or digraphs.

On a power index $\varphi : \mathcal{D} \to \mathbb{R}^N$ we impose the same axioms as in the preceding section. Since the definitions of a null player and of the axioms of Null Player, Zerosum, Anonymity, and the Transfer Property do not change for power indices on $\mathcal{D}$, we do not repeat these definitions here.

For $M \in P_0(N)$ and $j \in N$, let the control structure $\bar{u}^{M,j} \in \mathcal{D}$ be defined by $u_i^{M,j} = z$ for all $i \neq j$, and $u_j^{M,j}(\{i\}) = 1$ if and only $i \in M$ for all $i \in N$. Hence, $\bar{u}^{M,j}$ can be identified with the digraph that has (only) links from every player $i \in M$ to player $j$. The following lemma (Lemma 4.3 in [17]) follows directly from the definitions.

**Lemma 10.1** *Let the power index $\varphi$ on $\mathcal{D}$ satisfy Null Player, Zerosum, and Anonymity. Then there are $\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that for every $M \in P_0(N)$ and $j \in N$, with $m = |M|$:*

(a) *if $j \notin M$, then for every $i \in N$*

$$\varphi_i(\bar{u}^{M,j}) = \begin{cases} 0 & \text{if } i \notin M \cup j \\ \alpha_m/m & \text{if } i \in M \\ -\alpha_m & \text{if } i = j \end{cases}$$

(b) *if $j \in M$, then for every $i \in N$*

$$\varphi_i(\bar{u}^{M,j}) = \begin{cases} 0 & \text{if } i \notin M \\ \beta_m/m & \text{if } i \in M \setminus j \\ \beta_m/m - \beta_m & \text{if } i = j \end{cases}$$

*where $\alpha_0 = \beta_1 = 0$.*

The digraph $\bar{u}^{M,j}$ plays a role similar to that of a unanimity TU-game. By adding the Transfer Property we obtain the following result (Theorem 4.4 in [17]). Here, for $\bar{w} \in \mathcal{D}$ and $j \in N$, $M_j^{\bar{w}} = \{i \in N \mid w_j(\{i\}) = 1\}$ is the set of players who have a link to player $j$, i.e., who control player $j$.

**Theorem 10.4** *A power index $\varphi$ on $\mathcal{D}$ satisfies Null Player, Zero-sum, Anonymity, and the Transfer Property if and only if there are $\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that for each $\bar{w} \in \mathcal{D}$ we have $\varphi(\bar{w}) = \sum_{j \in N} \varphi(u^{M_j^{\bar{w}},j})$, with $\varphi(u^{M_j^{\bar{w}},j})$ as defined in* (a) *and* (b) *of Lemma 10.1.*

Denote a power index $\varphi$ as in Theorem 10.4 with parameters $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and $\beta = (\beta_2, \ldots, \beta_n)$ by $\varphi^{\alpha,\beta}$. It can be checked that the weights in Theorem 10.4 coincide with those in Theorem 10.3: More precisely, on $\mathcal{D}$, for $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n)$, $\Phi^\omega$ coincides with the power index $\varphi^{\alpha,\beta}$.

Theorem 10.4 does not put any restrictions on the parameters $\alpha, \beta$. We next present three possibly plausible further conditions which have to do with adding additional links.

The first condition says that if we add a link to a player $j$ from some player $i$, then this should not change the power of the players who already have a link to $j$. In the control parlance: if player $j$ gets additionally controlled by some player $i$, then this should not change the power of the players who were already controlling $j$.

**Link Addition 1** Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w'_j(i) = 1$. Then $\varphi_h(\bar{w}) = \varphi_h(\bar{w}')$ for all $h \in N \setminus \{j\}$ such that $w_j(h) = 1$.

**Corollary 10.2** *Let $\varphi = \varphi^{\alpha,\beta}$. Then $\varphi$ satisfies LA1 if and only if there is a $c \in \mathbb{R}$ such that $\alpha_k = kc$ for all $k = 1, \ldots, n-1$ and $\beta_k = kc$ for all $k = 2, \ldots, n$.*

Thus, under LA1 we obtain a one-parameter family of power indices of the form

$$\varphi_i^c(\bar{w}) = c\left(|\{j \in N \mid w_j(i) = 1\}| - |\{j \in N \mid w_i(j) = 1\}|\right),$$

where $c \in \mathbb{R}$. For instance, for $c = 1$ and in the control terminology, the power of player $i$ is equal to the number of players controlled by player $i$ minus the number of players controlling player $i$. This is similar to the Copeland score in social choice theory [3], under the interpretation that a link from $i$ to $j$ means that $i$ is preferred to $j$.

The second condition requires that it is player $j$ whose power does not change if a link is added from some player $i$ to $j$, provided that there was already a link from some other player to $j$. In terms of control: If a player $j$ becomes additionally controlled by some player $i$, then this should not change the power of player $j$.

**Link Addition 2** Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w'_j(i) = 1$. Also, let $w_j(\{k\}) = 1$ for some $k \in N \setminus \{j\}$. Then $\varphi_j(\bar{w}) = \varphi_j(\bar{w}')$.

**Corollary 10.3** *Let $\varphi = \varphi^{\alpha,\beta}$. Then $\varphi$ satisfies LA2 if and only if there is a $c \in \mathbb{R}$ such that $\alpha_k = c$ for all $k = 1, \ldots, n-1$ and $\beta_k = \frac{k}{k-1} c$ for all $k = 2, \ldots, n$.*

The power indices characterized in Corollary 10.3 take the form

$$\bar{\varphi}_i^c(\bar{w}) = \sum_{j \in N \setminus \{i\} \,:\, w_j(\{i\})=1} \frac{c}{|\{k \in N \setminus \{j\} \mid w_j(k) = 1\}|} - c\, 1_{\{M_i^{\bar{w}} \setminus \{i\} \neq \emptyset\}}$$

where $1_{\{P\}} = 1$ if statement $P$ is true and $1_{\{P\}} = 0$ otherwise. According to a power index $\bar{\varphi}^c$, if a player $i$ has a link to some other player $j$, then he equally shares the amount of power $c$ with the other players having a link to $j$, except possibly $j$. If player $i$ is controlled by someone other than himself, then he loses an amount $c$ of power. This power index is similar to the idea of the $\beta$-measure as in [24] or its reflexive variant in [23].

The final condition we consider says that if we add a link from a player $i$ to a player $j$, but player $j$ was already controlled by some other player (possibly by himself) then both have the same gain (or loss) in power. This condition may make sense, perhaps not so much in the control setting, but rather in a setting where players $i$ and $j$ have some common interests – for instance, they work in the same department of a university.

**Link Addition 3** Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w'_j(i) = 1$. Also, let $w_j(\{k\}) = 1$ for some $k \in N$. Then $\varphi_i(\bar{w}) - \varphi_i(\bar{w}') = \varphi_j(\bar{w}) - \varphi_j(\bar{w}')$.

**Corollary 10.4** *Let $\varphi = \varphi^{\alpha,\beta}$. Then $\varphi$ satisfies LA3 if and only if there is a $c \in \mathbb{R}$ such that $\alpha_k = \frac{2}{k+1} c$ for all $k = 1, \ldots, n-1$, and $\beta_k = 0$ for all $k = 2, \ldots, n$.*

The power indices characterized in Corollary 10.4 take the form:

$$\widetilde{\varphi}_i^c(\bar{w}) = \sum_{j \in N \,:\, w_j(\{i\})=1, w_j(\{j\})=0} \frac{\alpha_{|M_j^{\bar{w}}|}}{|M_j^{\bar{w}}|} - \alpha_{|M_i^{\bar{w}}|}\, 1_{\{M_i^{\bar{w}} \neq \emptyset,\ i \notin M_i^{\bar{w}}\}}\ ,$$

with $\alpha_k$ as in Corollary 10.4. In control terms, according to a power index $\widetilde{\varphi}^c$, if player $j$ controls himself, then no player, including player $j$, derives (positive or negative) power from controlling $j$. Further, the power (negative, if $c > 0$) from being controlled decreases as the number of controlling players increases. We note that $\widetilde{\varphi}^c$ is related to the apex power index in [22]).

## 10.6 Conclusions

In this chapter we have reviewed some of our works on power indices for situations that go beyond simple games, namely power indices for effectivity functions and power indices for control structures. We have also shown that a spatial power index like that of Owen and Shapley [15] is a special case of the former, while some power indices for digraphs, like the Copeland score [3], are special cases of the latter. Our power indices for effectivity functions are clearly based on the Shapley value: See Theorems 10.1 and 10.2. The general formula for power indices for control structures in Theorem 10.3 is based on dividends and therefore indirectly related to the Shapley value. For instance, if we take $\alpha_1 = \ldots = \alpha_{n-1} = \beta_2 = \ldots = \beta_n = \gamma \in \mathbb{R}$, then the resulting power index $\Phi^\omega$ is given by

$$\Phi_i^\omega(\bar{w}) = \gamma \left( \sum_{k \in N} Sh_i(w_k) - 1 \right)$$

for every $\bar{w} \in \mathcal{W}$ and $i \in N$. Also, for $\varphi(u^{M_j^{\bar{w}},j})$ in Theorem 10.4 we can write

$$\varphi(u^{M_j^{\bar{w}},j}) = \begin{cases} \alpha_{|M_j^{\bar{w}}|}\, Sh\left(u_{M_j^{\bar{w}}} - u_{\{j\}}\right) & \text{if } j \notin M_j^{\bar{w}} \\ \beta_{|M_j^{\bar{w}}|}\, Sh\left(u_{M_j^{\bar{w}}} - u_{\{j\}}\right) & \text{if } j \in M_j^{\bar{w}} \end{cases}$$

where $u_T$ for $T \subseteq N$ is the unanimity TU-game.

The two approaches, i.e., to effectivity functions and control structures, cannot be directly linked. Total power according to a power index for control structures (Theorem 10.3) is equal to zero: In this case, it is natural to assign both negative and positive power and to have the sum equal to zero. Total power according to a power index for effectivity functions (Theorems 10.1 and 10.2) is equal to one, and the power of every player is nonnegative; which is as in the standard case. Moreover, in the control case, if player $i$ controls player $j$, then in some sense player $i$ is 'effective' for $\{j\}$, but this is not quite consistent with the usual conditions on an effectivity function: For instance, if player $i$ controls player $j$, then this does not imply that $i$ controls every set of players containing $j$, but if $i$ is effective for $\{j\}$ in the formal meaning of this expression as defined above, then $i$ is effective for every set of players containing $j$. Hence, it is not obvious how to construct a general model encompassing the two approaches reviewed in this chapter.

## 10.7 Acknowledgments

I would like to thank the editors of this volume for their constructive comments.

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# Chapter 11

## Cost Allocation with Variable Production and the Shapley Value

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***Dedication**: Less than a month after sending the first version of this work, we were shocked by the death of our colleague and friend Juan Carlos. In addition to this chapter, we have collaborated with him countless times. In sadness, we dedicate this chapter to Juan Carlos as a tribute for his friendship.*

## 11.1 Introduction

Different advantages may arise from collaboration, so that in many cases it is appropriate to determine the contribution of each one of the interested parties. A public corporation builds a dam to provide several services in a territory (power generation, navigation irrigation, municipal supply, etc.), and the cost must be allocated among the beneficiaries. In a university, the common expenses have to be distributed among distinct departments. An airport is built and the cost of its construction and maintenance must be distributed among the users. A multidimensional firm has to allocate costs among its different units. When the members of the European Union cooperate, they have to agree how to share the budget.

In these instances, cost shares are settled either internally by agreement among the parties or externally by an administrative authority, under the condition of covering exactly the costs. The final settlement is usually reached according with some reasonable criteria of "fairness" (although it is true that there are still no such simple or obvious criteria). This makes cost allocation an area specially suitable for the application of the cooperative game theory. Actually both realms influence each other, to the point that some cost allocations methods foreshadowed key solution concepts of the game theory.[1]

A fruitful way to find solutions in cooperative game theory has been the axiomatic approach. Initiated by Nash (1950), this approach consists in formulating some reasonable principles or properties, also called *axioms*, that are only satisfied by one specific solution concept. Then it is said that the solution is characterized by these axioms, and its plausibility may be examined in the light of the sensibility of these properties, together with the experience for particular applications. The Shapley value (1953) is arguably one of the most prominent cooperative solutions concepts proposed in this way. Since then, in addition to Shapley's one, other characterizations of this value have been proposed; to name a few of them, Myerson (1980), Young (1985a), and Hart and Mas-Colell (1989). On the other hand, Aumann and Shapley (1974) developed a theory for non-atomic games using an axiomatic approach to define a value for these games.[2] The resulting value is called the Aumann-Shapley value and can be considered as a generalization of the Shapley value from finite games to the class of non-atomic games.

These and other characterizations have been translated from the cooperative games to cost allocation problems. The purpose of the translation is precisely to make easier to judge the reasonableness of the application of the Shapley value also from an economic point of view. The aim in this chapter

[1] The Tennessee Valley Authority problem is an example of this (Section 2 below).

[2] Actually these authors employed in addition two other approaches for the value: The random order and the asymptotic approaches. Both can be considered complementary of the axiomatic one, since the three lead to the same value for a wide collection of non-atomic games.

is featuring a brief survey of the characterizations of the Shapley value found in the cost allocation literature.

In this survey, we will consider a basic model consisting of three elements: A set of agents involved in a problem (beneficiaries, users, departments, etc.); a specification of the quantity of the good demanded by each agent; and finally a joint cost function. Then we will distinguish three different cases depending on the nature of the demands. First we consider the case in which the demand of each agent may take only the values 0 or 1, that is they can decide to participate or not in the joint project. This case corresponds to the classical model studied first by Shubik (1962). The Shapley-Shubik method will be the cost allocation rule examined for this case. In the other two cases, the agents' demands can take any value. We differentiate the continuum case and the discrete case, depending on wether the goods demanded by the agents are divisible or not. In both cases our attention will be focused on the Aumann-Shapley method, the cost allocation method inspired by the value of non-atomic games.

The chapter is organized as follows. Section 11.2 is devoted to three classical examples of the cost allocation literature. These examples serve as an illustration of the theoretical model presented in Section 11.3 together with the notation. Sections 11.4, 11.5 and 11.6 are dedicated respectively to the three cases mentioned in the above paragraph. Some conclusions close the chapter.

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## 11.2 Three Motivating Examples

In this section three classical examples of the literature on the cost allocation problem are presented. They serve respectively as illustration for the formal models considered later on.

### 11.2.1 The Tennessee Valley Authority

The Tennessee Valley Authority (TVA) is a Federal corporation founded when President Roosevelt signed the TVA Act on 1933. This corporation was created to address some major problems facing the valley, such as control flooding, power generation, navigation improvements along the Tennessee River, as well as other subsidiary responsibilities for national defense and assist in the economic development of the region. Thus, the TVA carried out large-scale projects and constructed dams and reservoirs along the Tennessee River basin, and today TVA is the largest public energy company in the United States.

In some of the sections of the TVA Act, it is decreed that the cost of TVA projects should be specifically allocated among the issues involved. This resulted in a considerable work to analyze different cost allocation methods. A survey of this task is offered in Ransmeier (1942). The concepts and criteria devised in this work can be considered as game theory "*avant la lettre*". For instance a "preliminary criterion of a satisfactory allocation" (Ransmeier, 1942, p. 220) is:

> *The method should have a reasonable logical basis. It should not result in charging any objective with a greater investment than the fair capitalized value of the annual benefit of this objective to the consumer. It should not result in charging any objective with a greater investment than would suffice for its development at an alternate single purpose site. Finally, it should not charge any two or more objectives with a greater investment than would suffice for alternate dual or multiple purpose improvement.*

| Subset $S$ of Purposes: | Cost: $c(S)$ |
|---|---|
| $\emptyset$ | 0 |
| $\{n\}$ | 163 520 |
| $\{f\}$ | 140 826 |
| $\{p\}$ | 250 096 |
| $\{n, f\}$ | 301 607 |
| $\{n, p\}$ | 378 821 |
| $\{f, p\}$ | 367 370 |
| $\{n, f, p\}$ | 412 584 |

**TABLE 11.1**: Cost function for the TVA cost allocation problem.

In most of the TVA projects, there were basically three purposes: Navigation (n), flood control (f) and power generation (p). The cost figures (in thousands of dollars) for the different sets (or coalitions) of projects are given in Table 11.1, which is an adaptation from Ransmeier (1942). The figures can be used to formalize the TVA problem as a transferable utility game $(N, c)$ as indicated. This allows us to make use of the solution concepts of the cooperative game theory to find a "fair" agreement to share the costs. For instance the Shapley value in the TVA problem is $(117\,829,\ 100\,756.5,\ 193\,998.5)$ that satisfies the criteria stated in the above quotation. It is worth mentioning that the last sequences of this quotation just say that the allocation should be in the core of the corresponding transferable utility game.

### 11.2.2 Internal Telephone Billing Rates

The next example is a compact summary of a work due to Billera, Heath and Raanan (1978), when they developed a procedure to determine the rates of the telephone calls at Cornell University. Telephone service can be purchased in bulk, and it is used by a large number of small ("infinitesimal") users, so that the total demand may be high enough to make acquiring the service profitable. The question is how to charge each user a "fair" share of the cost, assuming that the total costs have to be fully covered.

The problem turns out to be quite involved because there are several ways of offering the service: First the Direct Distance Dialing (DDD) that depends on the length of the call and the distance to the destination only; second the Foreign Exchange (FX), if the user has to be connected to an exchange in another area; and, finally, five Wide Area Telecommunication Service (WATS) covering the continental U.S. excluded the user's state and divided in five concentric zones, whose service can be obtained from the company under two different plans that consist of an initial fee plus an incremental charge per hour. The telephone system at the university includes a computerized device that routes each call onto the best sequence of WATS or FX lines, and if they fail, the call is sent DDD. The costs cannot be distributed among the calls in a straightforward way due to several reasons: Some costs are not directly associated with the calls (maintenance, operator's salary, etc.), the timing of the calls accumulates on a monthly basis, the first few calls have not any incremental charges, etc.

To solve the problem Billera, Heath and Raanan (1978) construct a non-atomic game as follows. Each calling instant is a "player", so that the underlying space of players, denoted by $I$, is the monthly collection of calls. These calls can be classified according to three criteria: (a) the time of the day when they are made (midnight to 1 A.M., 1–2 A.M.,...); (b) their destination (number of WATS bands and FX lines, etc., in the system); and (c) the type of day when they are made (business day or weekend). If there are $k$ destinations, then we have $n = 24 \times k \times 2$ types of calls. Now define $n$ measures on $I$, denoted $\mu_j$ $(j = 1, \dots, n)$, where for any subset of calls $S \subseteq I$ the real number $\mu_j(S)$ represents the total duration of telephone calls of type $j$ in $S$. On the other hand, an optimization routine calculates the least cost of serving a given load $X = (x_1, \dots, x_n)$ on the system. This least cost is denoted $f(X) = f(x_1, \dots, x_n)$. Now it can be defined a non-atomic game $v$ on $I$ by $v(S) = f\big(\mu_1(S), \dots, \mu_n(S)\big)$, that is the minimal cost of servicing the demands given by $S$. To solve the rates problem, the authors applied the Aumann and Shapley (1974) method to this game.

### 11.2.3 Aircraft Landing Fees

Littlechild and Thompson (1977) applied a game theoretical model to analyze the problem of sharing the common costs of the construction and maintenance of a runway at Birmingham Airport (U.K.) among its users in the period 1968–1969. As they noticed the costs of an airport have a simple but interesting structure: "*The cost of building a runway depends essentially upon the largest aircraft for which the runway is designed, while the cost of subsequently using the runway is proportional to the number of movements of each type of air-*

*craft*". After determining the optimal size of the runway, and assuming that the total costs have to be recovered, the question is how to introduce some basic notions of fairness in the final distribution.

For solving this question Littlechild and Thompson (1977) modeled this situation with a transferable utility game. A player is a potential aircraft movement (take off or landing). Let $N = \{1, \ldots, n\}$ be the set of players or movements. Suppose there are $m$ different types of aircrafts that use the runway. Let $N_i$ be the set of movements by planes of type $i$, $i = 1, \ldots, m$. For every type $i$ denote

$b_i$ = benefit of movement of type $i$,

$c_i$ = cost of movement of type $i$,

$g_i$ = cost of building a runway to accommodate movement of type $i$.

Take any subset $S \subset N$ of movements. The incurred cost to accommodate all movements in $S$ is equal to the cost of accommodating the largest type of aircraft in this subset; that is,

$$G(S) = \max \{g_i \,:\, S \cap N_i \neq \emptyset\} \tag{11.1}$$

is the capital cost of a runway for subset $S$ of movements.

Thus, the profits obtained by coalition $S$ are given by

$$v(S) = \max_{R \subseteq S} \Big\{ \Big( \sum_{j \in R} (b_j - c_j) - G(R) \Big)_+ \Big\},$$

that is, the members of $S \subseteq N$ will carry out the most profitable runway that is feasible for it, whenever this runway generates a non-negative surplus.[3] This is the characteristic function of a cooperative game and gives us access to the cooperative game theory solution concepts.

Yet it may be also interesting to determine prices based on the more objective elements of cost and number of movements, and not in the less ascertainable element as the benefits. Consequently, Littlechild and Thompson (1977) also consider the "*airport cost game*" $G$. These authors paid specially attention to the Shapley value and the nucleolus of the games $v$ and $G$. Some of their data are collected in Table 11.2.

Airport games have led to an extensive literature. To name a few, we cite the seminal papers by Littlechild and Owen (1973, 1976) and Littlechild (1974). Later on Kuipers, Mosquera and Zarzuelo (2013) extended the model for studying the problem of sharing costs in more general situations such as highway toll pricing.

[3] $a_+$ denotes $\max\{a, 0\}$.

| Aircraft type | Number of movements $n_i$ | Capital cost contribution $g_i$ | Benefit for movement $b_i$ | Maintenance cost for movement $c_i$ | Shapley value (limit approx.) | Shapley value (based on capital cost only) |
|---|---|---|---|---|---|---|
| Fokker Friendship 27 | 42 | 65 899 | 91.18 | 5.23 | 6.12 | 4.86 |
| Viscount 800 | 9555 | 76 725 | 100.77 | 6.09 | 6.74 | 5.66 |
| Hawker Siddeley Trident | 288 | 95 200 | 110.95 | 7.55 | 7.36 | 10.30 |
| Britannia | 303 | 97 200 | 172.11 | 7.71 | 11.71 | 10.85 |
| Caravelle VIR | 151 | 97 436 | 139.15 | 7.73 | 9.36 | 10.92 |
| BAC 111 (500) | 1315 | 98 142 | 160.61 | 7.79 | 10.88 | 11.13 |
| Vanguard 953 | 505 | 102 496 | 314.88 | 8.13 | 21.84 | 13.40 |
| Comet 413 | 1128 | 104 849 | 105.44 | 8.32 | 6.91 | 15.07 |
| Britannia 300 | 151 | 113 322 | 205.47 | 8.99 | 40.34 | 44.80 |
| Corvair Corronado | 112 | 115 440 | 204.84 | 9.16 | 40.17 | 60.61 |
| Boeing 707 | 22 | 117 676 | 251.70 | 9.34 | 101.64 | 162.24 |

**TABLE 11.2**: Birmingham airport data, 1968–1969: Movements, benefits, costs and the Shapley value.

## 11.3 Notation and Preliminaries

Denote $\mathbb{R}$ the set of real numbers, $\mathbb{N}$ the set of non-negative integers, and $\mathcal{N}$ the set of non-empty finite subsets of $\mathbb{N}$.

If $N \in \mathcal{N}$, denote by $|N|$ the cardinality of $N$, and let $\mathbb{R}^N$ be the $|N|$-dimensional Euclidean space with coordinates indexed by the elements of $N$. If $\boldsymbol{x} = (x_i)_{i\in N} \in \mathbb{R}^N$ and $M \subseteq N$ we write $x_M = (x_i)_{i\in M}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ write $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for all $i \in N$. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$, write $[\boldsymbol{a}, \boldsymbol{b}]$ for the set $\{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}\}$. Some distinguished vectors in $\mathbb{R}^N$ are: The origin $\mathbf{0} = (0, \ldots, 0)$; and for every $M \subseteq N$ its indicator $\mathbf{1}^M$, defined by $1_i^M = 1$ if $i \in M$ and $1_i^M = 0$, otherwise.

A *cost sharing problem* is a triple $(N, \bar{\boldsymbol{q}}, C)$, where $N \in \mathcal{N}$ represents a set of agents, $\bar{\boldsymbol{q}} = (\bar{q}_i)_{i\in N} \in \mathbb{R}^N$ is the list of their consumptions or demands, and $C(\boldsymbol{q})$ is a cost function that represents the cost of jointly producing $\boldsymbol{q} = (q_i)_{i\in N} \in [0, \bar{\boldsymbol{q}}]$. We will assume throughout $C(\mathbf{0}) = 0$.

Depending on the nature of the total demand and the cost function we have classified these problems in three different kinds:

1. *The classical case:* When $\bar{\boldsymbol{q}} = \mathbf{1}^N$ and $C : [\mathbf{0}, \bar{\boldsymbol{q}}] \cap \{0,1\}^N \to \mathbb{R}$.
2. *The continuum case:* When $\bar{\boldsymbol{q}} \in \mathbb{R}^N$ and $C : [\mathbf{0}, \bar{\boldsymbol{q}}] \to \mathbb{R}$.
3. *The discrete case:* When $\bar{\boldsymbol{q}} \in \mathbb{N}^N$ and $C : [0, \bar{\boldsymbol{q}}] \cap \mathbb{N}^N \to \mathbb{R}$.

In the following sections, we will consider these three types of problems.

## 11.4 The Classical Case and the Shapley-Shubik Method

Since the pioneering work of Shubik (1962) most of the applications of game theory to cost allocation have been based upon the characteristic function form of the games.

A cost allocation situation can be modeled with a transferable utility (TU) game, called *cost game*, by assigning to a group of agents, or coalition, the cost that would incur by satisfying exactly the demands of its members. In our setting every cost allocation problem $(N, \mathbf{1}^N, C)$ can be univocally associated with the TU game whose characteristic function $c$ is defined by

$$c(S) = C(\mathbf{1}^S, \mathbf{0}_{N\setminus S}).$$

This is the approach that was followed to solve the Tennessee Valley Authority example of Subsection 11.2.1 (see Table 11.1).

Along this section we will fix the set $N$ of agents, and therefore we will speak of the cost allocation problem $c$ instead of $(N, \mathbf{1}^N, C)$. We will denote by $\mathcal{P}^B$ the class of classical problems.

A *cost allocation rule* is a mapping $\phi : \mathcal{P}^B \to \mathbb{R}^N$ prescribing the cost share of every agent in such a way the costs are precisely satisfied, that is

$$\sum_{i\in N} \phi_i(c) = c(N).$$

With the cost game at hand we are enabled to access cooperative game theory solution concepts in order to address the cost allocation problem. In particular, the Shapley value of the problem $c$ allocates the total cost $c(N)$ for every $i \in N$ according to the well-known expression

$$Sh_i(c) = \sum_{S\subseteq N\setminus i} \frac{|S|!\big(|N| - |S| - 1\big)!}{|N|!} \big(c(S \cup i) - c(S)\big).$$

This expression admits a probabilistic interpretation. Imagine that in a random way, an agent is chosen, and until the coalition $N$ is completed, agents are randomly joined one-by-one. Assume that when an agent is chosen, it pays its *marginal cost contribution*, i.e., $c(S \cup i) - c(S)$. Then $|S|!\big(|N| - |S| - 1\big)!$ is the number of possible ways in which this process can occur, that is $S$ is the coalition formed immediately before agent $i$ is incorporated. As $|N|!$ is the total number of possible ways to perform this process, $Sh_i(c)$ is the expected cost of agent $i$ if the payments take place in this manner.

In view of the interpretation above, the Shapley value can be formulated in an alternative way. Let $\mathcal{R}(N)$ be the set of all orderings of $N$, and for each $R \in \mathcal{R}(N)$ and each $i \in N$, denote by $R[i]$ to the set of agents prior to

$i$ according to $R$. The *marginal cost contribution* of agent $i$ relative to $R$ is defined by

$$MC_i(c, R) = c\big(R[i] \cup i\big) - c\big(R[i]\big).$$

Thus, the Shapley value for $i$ is just the average of $MC_i(c, R)$ over the $|N|!$ orderings of $N$.

### 11.4.1 Axiomatic Characterizations of the Shapley-Shubik Method

Similarly to the Nash (1950) approach to define the cooperative bargaining solution, Shapley (1953) formulated a group of axioms that unambiguously characterizes the Shapley value. Next we describe these axioms that are the formalization of some reasonable principles of fairness.

Let $\phi$ be a cost allocation rule.

The first axiom is that we can add the cost shares of different problems. So, if a given problem can be decomposed into two different problems, the result of the original problem should be equal to the sum of the results of the other two problems.

*Additivity*: Let $c, c' \in \mathcal{P}^B$, then $\phi(c + c') = \phi(c) + \phi(c')$.

The second one says that the method should be independent of the names of the agents.

Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection. If $N \in \mathcal{N}$, write $\pi N = \{\pi(i) : i \in N\}$. If $x \in \mathbb{R}^N$, define $\pi x \in \mathbb{R}^{\pi N}$ by $(\pi x)_{\pi(i)} = x_i$ for all $i \in \pi N$. Moreover, if $c \in \mathcal{P}^B$, define $\pi c\big(\pi(S)\big) = c(S)$.

*Symmetry*: Let $c \in \mathcal{P}^B$, $i \in N$, and $\pi : \mathbb{N} \to \mathbb{N}$ a bijection, then $\phi_{\pi(i)}\pi(\pi c) = \phi_i(c)$.

The last one requires that if the marginal cost contribution of an agent is always zero, then its cost share must be zero as well.

*Dummy Agent*: Let $c \in \mathcal{P}^B$ and $i \in N$. If $c(S \cup i) = c(S)$ for all $S \subseteq N$, then $\phi_i(c) = 0$.

**Theorem 11.1 (Shapley, 1953)** *There exists a unique cost allocation rule that satisfies Additivity, Symmetry and Dummy Agent, and it is the Shapley value.*

The Additivity axiom, though mathematically attractive, is less innocent than what it seems at first sight. For we cannot expect that in general $c$ and $c'$ are always separate problems, whilst $c + c'$ appears in this axiom as composed by two problems.

Young (1985a) proposed an alternative characterization of the Shapley value using neither Additivity nor Dummy Agent. Instead, this author suggested the *Strong Monotonicity* axiom below. The rationale for this property is that a method should charge a lower cost to an agent when its marginal cost decreases.

*Strong Monotonicity*: Let $c, c' \in \mathcal{P}^B$ and $i \in N$. If $c(S \cup i) - c(S) \leq c'(S \cup i) - c'(S)$ for all $S \subseteq N$, then $\phi_i(c) \leq \phi_i(c')$.

**Theorem 11.2 (Young, 1985b)** *There exists a unique cost allocation rule that satisfies Symmetry and Strong Monotonicity, and it is the Shapley value.*

## 11.5 The Continuum Case and the Aumann-Shapley Method

Aumann and Shapley (1974) in their celebrated book *Values of Non-atomic Games*, extended the Shapley value to games with a continuum of agents. Their ideas are particularly suitable for cost allocation problems in which the output demanded by the agents can vary in a continuous way. They were firstly applied by Billera, Heath and Raanan (1978) to compute equitable telephone billing rates among users at Cornell University (see the Internal Telephone Billing Rates example above). Later on, Billera and Heath (1982) and Mirman and Tauman (1982) redefined the axioms of Aumann and Shapley (1974) for values of non-atomic games to cost allocation problems.

In this section, we consider cost allocation problems $(N, \bar{\boldsymbol{q}}, C)$, where the demands are real numbers, i.e., $\bar{\boldsymbol{q}} \in \mathbb{R}^N_+$, and the cost function is defined on $[0, \bar{\boldsymbol{q}}] \subset \mathbb{R}^N_+$, i.e., the cost $C(q)$ depends on the different levels of the demanded goods $q = (q_i)_{i \in N}$. In addition, we shall assume that the joint cost function $C(q)$ has continuous first partial derivatives on its domain (one sided on the boundary). We will denote by $\mathcal{P}$ the class of these problems.

A *cost allocation method* is a mapping $\Phi : \mathcal{P} \to \mathbb{R}^N$ that specifies the price per unit that every agent has to pay in order to satisfy exactly the total cost, that is

$$\sum_{i \in N} \bar{q}_i \Phi_i(N, \bar{\boldsymbol{q}}, C) = C(\bar{\boldsymbol{q}}).$$

In a similar way to the classical case, every problem $(N, \bar{\boldsymbol{q}}, C)$ can be associated with a non-atomic game. For this, we consider the collection or set $I$ containing together the various types of output demanded by the agents, so that $I$ includes exactly a quantity $\bar{q}_i$ of output $i$. For each subset $S \subseteq I$, denote $c(S)$ the cost of producing $S$. Then $(I, c)$ is a non-atomic game, and the Aumann-Shapley value of a unity of the output of agent $i$ is the Aumann-Shapley price of $i$ in the cost allocation problem. That is, using the diagonal formula for the value on $p$NAD, we have that the Aumann-Shapley method assigns to each $i \in N$

$$AS_i(N, \bar{\boldsymbol{q}}, C) = \int_0^1 \frac{\partial C(t\bar{\boldsymbol{q}})}{\partial q_i} dt.$$

In view of this formula, the Aumann-Shapley price of an agent can be interpreted as its average marginal cost along the diagonal $t\bar{\boldsymbol{q}} \in [\mathbf{0}, \bar{\boldsymbol{q}}]$, $0 \leq t \leq 1$.

The Internal Telephone Billing Rates example of Subsection 1.2.2 serves to illustrate the model considered in this section.

### 11.5.1 Axiomatic Characterizations of the Aumann-Shapley Method

Denote by $\Phi$ a cost allocation method on $\mathcal{P}$.

As in the classical case, the first axiom says that the cost shares of different problems can be added together.

*$\mathcal{C}$-Additivity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}$, then $\Phi(N, \bar{\boldsymbol{q}}, C + C') = \Phi(N, \bar{\boldsymbol{q}}, C) + \Phi(N, \bar{\boldsymbol{q}}, C)$.

The next axiom resembles the Symmetry axiom of the classical case. It was introduced by Young (1985b), who used the following example to illustrate it. Suppose that $C(y_1, \ldots, y_m)$ is the joint cost of producing $m$ types of gasoline, and the quantity $y_i \geq 0$ of each type is a blend of $n$ refinery grades $x_1, \ldots, x_n$. Say $y_i = \sum_{j=1}^n a_{ij}x_j$, all $a_{ij} \geq 0$. By writing $A = (a_{ij})$, the costs could be written as well in terms of $\boldsymbol{x} = (x_1, \ldots, x_n)$, as follows: $C'(\boldsymbol{x}) = C(A\boldsymbol{x})$. The procedure $\Phi$ is said to be aggregation invariant if prices are aggregated in the same manner as product quantities. Formally

*$\mathcal{C}$-Aggregation Invariance*: Let $(N, \bar{\boldsymbol{q}}, C), (M, \bar{\boldsymbol{q}}', C') \in \mathcal{P}$, and $A$ an $m \times n$ nonnegative matrix such that $\bar{\boldsymbol{q}} = A\bar{\boldsymbol{q}}'$ and $C'(\boldsymbol{q}') = C(A\boldsymbol{q}')$ for every $\boldsymbol{q}' \in [0, \bar{\boldsymbol{q}}']$. Then $\Phi(M, \bar{\boldsymbol{q}}', C') = \Phi(N, \bar{\boldsymbol{q}}, C)A$.

Another quite compelling condition is that if the costs are nondecreasing, then the prices cannot be negative.

*$\mathcal{C}$-Positivity*: Let $(N, \bar{\boldsymbol{q}}, C) \in \mathcal{P}$, such that $C$ is non-decreasing on $[0, \bar{\boldsymbol{q}}]$. Then $\Phi(N, \bar{\boldsymbol{q}}, C)A \geq 0$.

**Theorem 11.3 (Billera and Heath, 1982; Mirman and Tauman, 1982)** *There exists a unique cost allocation method that satisfies $\mathcal{C}$-Additivity, $\mathcal{C}$-Aggregation Invariance and $\mathcal{C}$-Positivity, and it is the Aumann-Shapley method.*[4]

[4] In fact Billera and Heath (1982) employed a weaker version of aggregation invariance in which the matrix $A$ consists of one row only. On the other hand, Mirman and Tauman (1982) substituted Aggregation Invariance by another two weaker versions of this axiom: The first one, called *Rescaling*, requiring this property only for the case in which the matrix $A$ is diagonal; and the second one, *Weaker Consistency*, asking this condition only when $A$ consists of one row formed by ones.

In addition, Young (1985b) also characterized the Aumann-Shapley method with a a property that is the counterpart of the Strong Monotonicity axiom of the classical case.

*$\mathcal{C}$-Strong Monotonicity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}$, and $i \in N$. If $\partial C(\boldsymbol{q})/\partial q_i \leq \partial C'(\boldsymbol{q})/\partial q_i$ for every $\boldsymbol{q} \in [0, \bar{\boldsymbol{q}}]$. Then $\Phi_i(N, \bar{\boldsymbol{q}}, C) \leq \Phi_i(N, \bar{\boldsymbol{q}}, C)$.[5]

**Theorem 11.4 (Young, 1985b)** *There exists a unique cost allocation method that satisfies $\mathcal{C}$-Aggregation Invariance and $\mathcal{C}$-Strong Monotonicity, and it is the Aumann-Shapley method.*

Furthermore, Young (1985b) proposed another property of monotonicity even stronger, by letting one compare the marginal contributions of two different agents, in contrast with the property above that refers to the cost share of one agent only.

*$\mathcal{C}$-Symmetric Monotonicity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}$, and $i, j \in N$. If $\partial C(\boldsymbol{q})/\partial q_i \leq \partial C'(\boldsymbol{q})/\partial q_j$ for every $\boldsymbol{q} \in [0, \bar{\boldsymbol{q}}]$. Then $\Phi_i(N, \bar{\boldsymbol{q}}, C) \leq \Phi_j(N, \bar{\boldsymbol{q}}, C)$.

**Theorem 11.5 (Young, 1985b)** *There exists a unique cost sharing method that satisfies $\mathcal{C}$-Symmetric Monotonicity, and it is the Aumann-Shapley method.*

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## 11.6 The Discrete Case and the Aumann-Shapley Method

In the discrete case, agents can demand several units of the good, and these units are indivisible. Now, we turn to describe the Aumann-Shapley method for this case.

In a *discrete cost allocation* problems $(N, \bar{\boldsymbol{q}}, C)$ the demands are nonnegative integer numbers, i.e., $\bar{\boldsymbol{q}} \in \mathbb{N}_+^N$, and the cost function is defined on $[0, \bar{\boldsymbol{q}}] \cap \mathbb{N}_+^N$, i.e., the cost $C(\boldsymbol{q})$ depends on the different levels of the demanded goods $\boldsymbol{q} = (q_i)_{i \in N}$ that can take only nonnegative integer values. We will denote by $\mathcal{P}^D$ the class of these problems.

Similar to the continuum case, we have the following definition. A *discrete cost allocation method* $\psi : \mathcal{P}^D \to \mathbb{R}^N$ specifies the price per unit of every agent to balance the total cost, that is

$$\sum_{i \in N} \bar{q}_i \psi_i(N, \bar{\boldsymbol{q}}, C) = C(\bar{\boldsymbol{q}}).$$

[5] The partial derivatives are considered one-sided in the boundary throughout this section.

As in the classical and continuum cases, a cooperative game is also built starting from a discrete cost allocation problem $(N, \bar{\boldsymbol{q}}, C)$. To do this, the demand of each agent $i$ is regarded as formed by $\bar{q}_i$ elements, one by each unit. Thus, with each agent $i$, there is associated a set or coalition $N_i$ with $|N_i| = \bar{q}_i$ players. The grand coalition turns to be $N^{\bar{\boldsymbol{q}}} = \bigcup_{i \in N} N_i$. Then each coalition $S \subseteq N^{\bar{\boldsymbol{q}}}$ of players has associated a vector of demands $\boldsymbol{q}(S) = \big( |S \cap N_i| \big)_{i \in N}$, and then the cost of producing $S$ is

$$v^{\bar{\boldsymbol{q}},C}(S) = C\big(\boldsymbol{q}(S)\big).$$

The discrete Aumann-Shapley method, denoted $\mathcal{AS}$, assigns to every agent $i \in N$ in the problem $(N, \bar{\boldsymbol{q}}, C)$, precisely the Shapley value of any one of its representatives in the TU game $\left(N^{\bar{\boldsymbol{q}}}, v^{\bar{\boldsymbol{q}},C}\right)$. That is, for each $i \in N$

$$\mathcal{AS}_i(N, \bar{\boldsymbol{q}}, C) = Sh_j\left(N^{\bar{\boldsymbol{q}}}, v^{\bar{\boldsymbol{q}},C}\right) \qquad \text{for every } j \in N_i.$$

Thus, the discrete Aumann-Shapley method takes into account more information about cost in different consumption profiles than the Shapley-Shubik method. Indeed, while the Shapley-Shubik method takes the whole demand for every agent, the discrete Aumann-Shapley method deems all the intermediate consumptions between 0 and $\bar{\boldsymbol{q}}$.

**Remark 11.1** *A remark is in order about modeling the cost allocation problem to determine the aircraft landing fees of Subsection 1.2.3. In expression (11.1), where the airport cost game $G$ is defined, each movement is considered to be a player, and then the set of players is $N$.*[6] *Alternatively, we can consider that every set $N_i$ of movements of a certain type $i$ is represented by the same agent, w.l.o.g. denoted also by $i$ $(i = 1, \ldots, m)$. Then we have a discrete allocation problem $(M, \bar{\boldsymbol{q}}, C)$. Indeed, the set $M = \{1, \ldots, m\}$ is the set of agents, that is every type corresponds to an agent. The demands are $\bar{q}_i = |N_i|$, that is the number of movements of each type. And the cost function defined on $[0, \bar{\boldsymbol{q}}]$ is*

$$C(\boldsymbol{q}) = \max\{g_i \,:\, \bar{q}_i \neq 0\} \;\; \textit{if } \boldsymbol{q} \neq 0, \quad \textit{and} \quad C(0) = 0.$$

*Notice that $\mathcal{AS}_i(M, \bar{\boldsymbol{q}}, C) = Sh_j(G)$ for every $j \in N_i$, and every $i \in M$.*

### 11.6.1 Axiomatic Characterizations of the Discrete Aumann-Shapley Method

The discrete Aumann-Shapley method has not received too much attention yet. Here, four characterizations are presented.

In the sequel, let $\psi$ be a discrete cost allocation method.

[6] So all the players (movements) of the same type are symmetric in this game $G$.

The first axiomatization of the discrete Aumann-Shapley method is due to Calvo and Santos (2000). Actually these authors characterize the Aumann-Shapley value for multichoice games (firstly studied in Hsiao and Raghavan, 1992 and 1993), but every discrete cost allocation problem can be obviously identified with a multichoice game. The characterization is based on a translation of the "balanced contributions" property of Myerson (1980).

*$\mathcal{D}$-Balanced Contributions*: Let $(N, \bar{\boldsymbol{q}}, C) \in \mathcal{P}^D$, and $i, j \in N$ such that $\bar{q}_i, \bar{q}_j > 0$. Then,

$$\psi_i (N, \bar{\boldsymbol{q}}, C) - \psi_i \left(N, \bar{\boldsymbol{q}} - \mathbf{1}^j, C\right) = \psi_j (N, \bar{\boldsymbol{q}}, C) - \psi_j \left(N, \bar{\boldsymbol{q}} - \mathbf{1}^i, C\right).$$

**Theorem 11.6 (Calvo and Santos, 2000)** *There exists a unique discrete cost allocation method that satisfies $\mathcal{D}$-Balanced Contributions, and it is the discrete Aumann Shapley method.*

Next characterization is due to Sprumont (2005) with three axioms. The first two are the well-known Additivity and Dummy Agent axioms translated to the new model. But first let us introduce some additional notation. Let $(N, \bar{\boldsymbol{q}}, f) \in \mathcal{P}^D$, and $i \in N$:

a) If $\boldsymbol{q} \in [0, \bar{\boldsymbol{q}}]$ and $q_i < \bar{q}_i$, write: $\partial_i^+ C(\boldsymbol{q}) = C(\boldsymbol{q} + 1^i) - C(\boldsymbol{q})$.

b) If $\boldsymbol{q} \in [0, \bar{\boldsymbol{q}}]$ and $q_i > 0$, write: $\partial_i^- C(\boldsymbol{q}) = C(\boldsymbol{q}) - C(\boldsymbol{q} - 1^i)$.

*$\mathcal{D}$-Additivity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}^D$, then $\psi(N, \bar{\boldsymbol{q}}, C + C') = \psi(N, \bar{\boldsymbol{q}}, C) + \psi(N, \bar{\boldsymbol{q}}, C)$.

*$\mathcal{D}$-Dummy Agent*: Let $(N, \bar{\boldsymbol{q}}, C) \in \mathcal{P}^D$ and $i \in N$. If $\partial_i^- C(q) = 0$ for all $q \in [0, \bar{\boldsymbol{q}}]$, then $\phi_i (N, \bar{\boldsymbol{q}}, C) = 0$.

The third one, called *no merging or splitting* was proposed by Sprumont (2005) and resembles the $\mathcal{D}$-Aggregation Invariance axiom of the former section. It requires an agent to obtain the same if it splits into several agents. This axiom avoids splitting manipulations.

*$\mathcal{D}$-No Merging or Splitting*: Let $(N, q, C) \in \mathcal{P}^D$, $i \in N$, and $I \subseteq \mathbb{N}$ finite such that $N \cap I = \{i\}$. Let $(N', \bar{\boldsymbol{q}}', C') \in \mathcal{P}^D$, such that: (1) $N' = (N \backslash i) \cup I$; (2) $\bar{q}'_j = \bar{q}_j$ for all $j \in N \backslash i$; (3) $\sum_{k \in I} \bar{q}'_k = \bar{q}_i$; and (4) $C'(q) = C\left(q_{N \backslash i}, \sum_{k \in I} q_k\right)$. Then: $\psi_k (N', \bar{\boldsymbol{q}}', C') = \psi_i (N, \bar{\boldsymbol{q}}, C)$ for every $k \in I$.

Notice that when agent $i$ splits into several agents $k \in I$, then its demand $\bar{q}_i$ also splits into several demands $\bar{q}'_k$. Moreover, when agent $i$ splits into several agents, one of those split agents is $i$, and hence it is supposed that agent $i$ is present and there are new agents, all of them with the split demands. Observe also that this axiom does not impose any requirement on the cost shares of agents different from $i$.

**Theorem 11.7 (Sprumont, 2005)** *There exists a unique discrete cost allocation method that satisfies $\mathcal{D}$-Additivity, $\mathcal{D}$-Dummy Agent and $\mathcal{D}$-No Merging or Splitting, and it is the discrete Aumann Shapley method.*

Next characterization parallels the one in Theorem 11.4 of the continuum case. Now Additivity and Dummy Agent axioms are replaced by $\mathcal{D}$-Monotonicity axiom below. This new axiom requires that when moving from cost function $C'$ to $C$, if the marginal contribution of agent $i$ does not increase, then the rule cannot assign a higher price to $i$ either.

*$\mathcal{D}$-Monotonicity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}^D$, and $i \in N$. If for all $q \in [0, \bar{\boldsymbol{q}}]$ such that $q_i > 0$ it holds $\partial_i^- C(\boldsymbol{q}) \leq \partial_i^- C'(\boldsymbol{q})$, then $\psi_i(N, \bar{\boldsymbol{q}}, C) \leq \psi_i(N, \bar{\boldsymbol{q}}, C')$.

**Theorem 11.8 (Albizuri, Díez and Sarachu, 2014)** *There exists a unique discrete cost allocation method that satisfies $\mathcal{D}$-No Merging or Splitting and $\mathcal{D}$-Monotonicity, and it is the discrete Aumann Shapley method.*

There is still a further characterization that parallels Theorem 11.5 by Young (1985b). In order to do that, we have to translate $\mathcal{C}$-Symmetric Monotonicity axiom to the discrete setup.

*$\mathcal{D}$-Symmetric Monotonicity*: Let $(N, \bar{\boldsymbol{q}}, C), (N, \bar{\boldsymbol{q}}, C') \in \mathcal{P}^D$, and $i, j \in N$. If for all $q \in [0, \bar{\boldsymbol{q}}]$ it holds $\partial_i^+ C(\bar{\boldsymbol{q}}) \leq \partial_j^+ C'(\bar{\boldsymbol{q}})$ and $\partial_i^- C(\bar{\boldsymbol{q}}) \leq \partial_j^- C'(\bar{\boldsymbol{q}})$, then $\psi_i(N, \bar{\boldsymbol{q}}, C) \leq \psi_j(N, \bar{\boldsymbol{q}}, C')$.

**Theorem 11.9 (Albizuri and Zarzuelo, 2017)** *There exists a unique discrete cost allocation method that satisfies $\mathcal{D}$-Symmetric Monotonicity, and it is the discrete Aumann Shapley method.*

## 11.7 Conclusions

Several characterizations of the counterparts of the Shapley value and the Aumann-Shapley value for cost allocation problems have been reviewed. Mainly two kinds of characterizations have been examined, whose main difference is that they are based upon two different principles (see Table 11.3). The first one is based on the property of additivity, saying that the payments of different cost allocation problems can be aggregated. And the second one on the principle of monotonicity requiring that a lower cost share must be attributed when the marginal cost decreases. These characterizations endorse the Shapley value and the Aumann-Shapley value as plausible rules to be taken into consideration in many applications.

Despite the good properties of these values, some cautions are in order. The first one is that cost allocation refers to practical problems and not to abstract ones. So when facing a concrete cost allocation problem, a major obstacle to be saved is modeling, which requires to take special care. For

| Case | With *Additivity* | With *Monotonicity* | |
|---|---|---|---|
| Classical | **Th. 1.1**<br>*Add, Sym, Dum* | **Th. 1.2**<br>*Sym, StMon* | |
| Continuous | **Th. 1.3**<br>$\mathcal{C}$-*Add*, $\mathcal{C}$-*AInv*, $\mathcal{C}$-*Pos* | **Th. 1.4**<br>$\mathcal{C}$-*AInv*, $\mathcal{C}$-*StMon* | **Th. 1.5**<br>$\mathcal{C}$-*SymMon* |
| Discrete | **Th. 1.7**<br>$\mathcal{D}$-*Add*, $\mathcal{D}$-*NoMS*, $\mathcal{D}$-*Dum* | **Th. 1.8**<br>$\mathcal{D}$-*NoMS*, $\mathcal{D}$-*Mon* | **Th. 1.9**<br>$\mathcal{D}$-*SymMon* |

**TABLE 11.3**: Summary of the axiomatic characterizations.

instance when invoking a “fairness” principle, the shares of the involved agents are automatically being compared, so it is crucial to determine the identity of these agents. They may be municipalities, aircraft movements, telephone calls, products of a firm, countries, etc. Other significant factors when building a model are the amount of available information—especially when defining the joint cost function—and the difficulties of computation when the number of agents is very large.

A different kind of caveat refers to the specific model deemed in this survey. When agents collaborate in a joint project, they not only incur costs but also obtain benefits. However, in the model used here only costs have been considered and the possible benefits of the agents were ignored, despite the fact that benefits are decisive to determine an efficient joint collaboration, and should be taken into account to fix the final assignment. In this respect, it is worth mentioning that Littlechild and Thompson (1977) in the Airport Landing Fees example of Section 11.2 consider the benefits of the movements in their model. This is reflected in Table 11.2 of the Aircraft Landing Fees example, where the fifth column contains the Shapley value for the case in which benefits are included, and the last one in which they are not taken in consideration. In the fifth column only an approximation of the Shapley value is given due to the computational difficulties when benefits are included in the model. Perhaps this is the reason for which there are very few models that include costs and benefits simultaneously.[7]

[7] Brânzei *et al.* (2006) propose an algorithm for computing the nucleolus of airport problems with benefits. Finding an effective algorithm for the Shapley value in these problems is still an open problem.

Although attention has been paid in this chapter exclusively to the Shapley value and its properties, there are many other methods of allocating joint costs. To cite some of them: The separable cost remaining benefits (SCRB) method,[8] the (pre)nucleolus (Schmeidler, 1969), Ramsey prices (1927), and Gately (1974), among others. All these methods are backed by some rationale and their applicability depends on the specific context. Some readers may feel uncomfortable because of this abundance of methods. Nevertheless, agreements or arbitrators may be based on different principles that sometimes overlap but yield different results. Needless to say, no method can claim to be uniquely the "best". As Straffin and Heaney (1981) state: " ...rationality seems to demand a multiplicity of viewpoints, and narrow insistence on the virtues of one method is a vice rather than a virtue".

To conclude the chapter, it is worth mentioning other interesting surveys that can be found in the literature on cost allocation and similar or overlapping disciplines. A remarkable survey of the Aumann-Shapley method in the continuum case can be found in Tauman (1988), where in addition to the axiomatic aspects, one can find other developments less related to game theory. Another excellent overview on cost allocation is Young (1994), which highlights the tight relationship between game theory and cost allocation. Finally Moulin (2002) is an outstanding and extensive overview on the cost allocation methods that pay attention to most of the models treated in the literature on cost allocation.

## 11.8 Acknowledgments

This research has been supported by Spanish Ministerio de Economía y Competitividad (ECO2015-66803-P), and by Universidad del País Vasco/Euskal Herriko Unibertsitatea (GIU17/051).

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# Chapter 12

## Pure Bargaining Problems and the Shapley Rule: A Survey

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## 12.1 Introduction

In this chapter we review two seminal articles of a new research line closely related to the Shapley value [25] (also in [24] and [23]). They are included as items [8] and [10] in the Bibliography.[1] So as not to enlarge the contents too much, all proofs—most of which are stimulating and by no means straightforward—will be omitted here, so the reader interested in them is invited to look at the aforementioned articles.

Initially, a great motivation for [8] was that of establishing, once and for ever, a detailed criticism against the proportional rule as a sharing method in collective affairs, while at the same time giving evidence of the great advantages of the Shapley value. In cooperative games, the contrast between both methods is quite obvious, since the proportional rule[2] disregards most coalitional utilities: Precisely, all those corresponding to intermediate coalitions (i.e., coalitions with cardinality $s$ such that $1 < s < n$, where $n$ is the number of players). This becomes more and more critical as the number of players increases, and it implies an absolute lack of sensitivity with regard to the data defining any given problem (game). On the contrary, the Shapley value is always concerned with all marginal contributions without exception and enjoys therefore a nice sensitivity.

Thus, we decided to consider a framework "as advantageous as possible" to the proportional rule, so as not to be accused of any a priori tendency in favor of the Shapley value. Therefore, we introduced the notion of *pure bargaining problem* (PBP, for short), where only the total utility and the individual utilities of the agents are given: The temptation to use the proportional rule in such a situation is clear. We also had in mind that, often, the simpler the framework, the easier it is to check the relevance of a concept.

By identifying in a one-to-one linear way each PBP with a quasi-additive cooperative game, we translated the notion of Shapley value and obtained the *Shapley rule* for PBPs. This rule coincides with the well-known equal surplus sharing rule, but our procedure emphasizes its close relationship to the Shapley value. The Shapley rule represents a consistent alternative to the proportional rule as a much more satisfactory solution concept for PBPs (we do not use here the term "consistency" in the specific sense introduced in [13] with regard to a "reduced game" notion). Utilities are assumed to be completely transferable, so that the class of problems considered here differs from the class of "bargaining problems" more commonly analyzed in the literature.

The second part of the survey refers to [10]. In this second paper, we considered a more complicated setup, that of *pure bargaining problems endowed with a coalition structure* (PBPCS, for short). This notion is not completely

[1] Item [8] was selected by editors Manfred J. Holler and Hannu Nurmi to be reproduced in a commemorative volume [9]. We feel honored by this distinction.

[2] Or, more formally, the *proportional value* introduced in [21].

equivalent to that of (quasi-additive) cooperative game with a coalition structure since we assume that a *quotient* PBP, also exogenously given, is attached to the coalition structure. Thus, the theory for this new framework needs to be built in a somewhat different form. We were led to consider four behavioral options for the involved agents, to discuss individual preferences, and to introduce and study a *modified Shapley rule* and its natural domain.

The organization of the survey is as follows. In Section 12.2, the notion of *pure bargaining problem* (PBP) is provided and the concept of *sharing rule* is stated and exemplified with the classical *proportional rule* and the *equal surplus sharing rule*. In Section 12.3, we attach to each PBP a quasi-additive game (*closure*), thus reducing any PBP to a cooperative game. By using this idea, in Section 12.4 we analyze the core notion, introduce the *Shapley rule* for PBPs, compare it with the proportional rule, and characterize those PBPs for which the Shapley rule and the proportional rule coincide. Section 12.5 is devoted to giving two axiomatic characterizations of the Shapley rule on the space of all PBPs, and also on several subsets of interest: Among them, the domain of the proportional rule, the open positive and negative orthants, and the cone of (strict) superadditive PBPs. In Section 12.6, we present a criticism on the proportional rule, mainly highlighting its inconsistency in related cost-saving problems and added cost problems.

In Section 12.7 we introduce the model of *pure bargaining problem with a coalition structure* (PBPCS) and discuss the main options available to the agents: Individual behavior (I), cooperative behavior (C), isolated unions behavior (U), and bargaining unions behavior (B). Essentially, the former two are the options in a pure bargaining problem, whereas the latter two respectively recall the treatment given by Aumann-Drèze [5] and Owen [22] (also in [23]) to cooperative games with a coalition structure. A numerical example is presented and discussed in Section 12.8. In Section 12.9, a main result characterizes all agents' and unions' preferences on the four options to act. In Section 12.10 we introduce the *modified Shapley rule* for pure bargaining problems with a coalition structure, provide an axiomatic characterization of this rule, and determine its natural domain, that is, the set of pure bargaining problems with a coalition structure where the bargaining unions behavior is the best option for all agents. Section 12.11 collects the conclusions of the survey. Section 12.12 suggests future work in this research line.

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## 12.2 Pure Bargaining Problems and Sharing Rules

Let $N = \{1, 2, \ldots, n\}$ (with $n \geq 1$) be a set of agents and assume that they are given: (a) a set of utilities $u_1, u_2, \ldots, u_n$ available to the agents individually and (b) a total utility $u_N$ that, alternatively, the agents can jointly get if all of them agree—utilities denoting costs will be represented by negative numbers.[3] Of course, if $n = 1$ then we impose $u_N = u_1$. A vector $u = (u_1, u_2, \ldots, u_n | u_N)$ collects all this information and we will say that it represents a *pure bargaining problem* (*PBP*, or simply *problem*, in the sequel) on $N$. The *surplus* of $u$ is defined as

$$\Delta(u) = u_N - \sum_{j \in N} u_j.$$

We will say that $u$ is *additive* if $\Delta(u) = 0$ and *superadditive* if $\Delta(u) > 0$. The latter is the most interesting case since the problem consists in sharing $u_N$ among the agents in a rational way, i.e., in such a manner that all of them should agree and feel (more or less) satisfied with the outcome. Of course, the individual utilities $u_1, u_2, \ldots, u_n$ should be taken into account, so when $\Delta(u) > 0$ there is something to gain by cooperating. The transferable utility assumption means that, in principle, any vector $x = (x_1, x_2, \ldots, x_n)$ with $x_1 + x_2 + \cdots + x_n = u_N$ is feasible if the $n$ agents agree.

**Example 12.1** *(A cost allocation problem) Assume that three consumers, A, B and C, wish to get some kind of supply (electricity, water, gas) from a distributor D. The locations are* $A(2, 2)$, $B(-2, 2)$, $C(-2, -2)$ *and* $D(2, 0)$, *the distances given in kilometers (see Fig. 12.1). The connection cost amounts to 100 monetary units per km.*

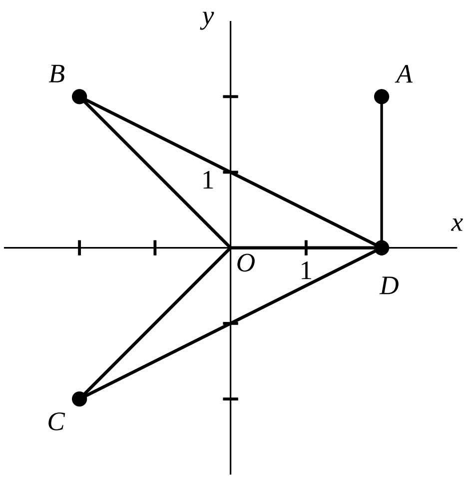


**FIGURE 12.1**: Consumer and distributor positions.

*For individual connections, the distributor offers lines DA, DB, and DC. For A, B, and C together, the offer consists in using DA, DO, OB, and OC. The question is how to share the joint connection cost. Then we have a (rounded) superadditive cost problem* $u^c = (-200, -448, -448 | -966)$ *that*

[3]This avoids introducing *subadditivity* as a desirable property for cost problems.

*describes the individual and joint costs and is defined in* $N = \{1, 2, 3\}$*, where 1 is A, 2 is B, and 3 is C. Assume that the three consumers sign a joint contract with the distributor. How should they share the total cost of 966?*

*An equivalent approach is obtained when considering the saving PBP* $u^s = (0, 0, 0|130)$*, which gives the savings derived from agreeing or not the joint contract. Now the question is: How should the three consumers share the net savings of 130 for a whole contract? Of course, there should exist a consistent solution for both cost and saving (related) problems.*

Let $E_{n+1} = \mathbb{R}^n \times \mathbb{R}$ denote the $(n+1)$-dimensional vector space formed by all PBPs on $N$. In order to deal with, and solve, all possible PBPs on $N$, one should look for a *sharing rule*, i.e., a function $f : E_{n+1} \longrightarrow \mathbb{R}^n$. Given $u \in E_{n+1}$, for each $i \in N$ the $i$-coordinate $f_i[u]$ will provide the share of $u_N$ that corresponds to agent $i$ according to $f$. Of course, there are infinitely many such functions: For example, $f_1[u] = u_N$ and $f_i[u] = 0$ for $i \neq 1$ would define one of them. More interesting ideas are given by the proportional rule, often used in practice and denoted here by $\pi$, and the equal surplus sharing rule, denoted by $\varepsilon$.

**Definition 12.1** *(a) The proportional rule* $\pi$ *is defined by*

$$\pi_i[u] = \frac{u_i}{u_1 + u_2 + \cdots + u_n} u_N \quad \textit{for each } i \in N.$$

*For further purposes, we notice that this expression is equivalent to*

$$\pi_i[u] = u_i + \frac{u_i}{\sum_{j \in N} u_j} \Delta(u). \tag{12.1}$$

*However, a main problem is that the domain of the proportional rule is not* $E_{n+1}$ *but the subset*

$$E^{\pi}_{n+1} = \{u \in E_{n+1} : u_1 + u_2 + \cdots + u_n \neq 0\}, \tag{12.2}$$

*that is, the complement of a hyperplane.*

*(b) Instead of this, the equal surplus sharing rule* $\varepsilon$ *is defined by*

$$\varepsilon_i[u] = u_i + \frac{\Delta(u)}{n} \quad \textit{for each } i \in N,$$

*its domain being the entire space* $E_{n+1}$ *without restriction.*

---

## 12.3 Closures and Quasi-Additive Games

The Shapley value [25], denoted here by $\varphi$, cannot be directly applied to PBPs as a sharing rule. We will therefore associate a TU cooperative game with each

PBP in a natural way. Let $\mathcal{G}_N$ be the vector space of all TU cooperative games with $N$ as set of players and let us define a map $\sigma : E_{n+1} \longrightarrow \mathcal{G}_N$ as follows. If $u = (u_1, u_2, \ldots, u_n|u_N)$, then $\overline{u} = \sigma(u)$ is given by

$$\overline{u}(S) = \begin{cases} \sum_{i\in S} u_i & \text{if} \quad S \neq N, \\ u_N & \text{if} \quad S = N. \end{cases}$$

The idea behind this definition is simple. Since, given a PBP $u$, nothing is known about the utility available to each intermediate coalition $S \subset N$ with $|S| > 1$, a reasonable assumption is that such a coalition can get the sum of the individual utilities of its members. Game $\overline{u}$ will be called the *closure* of $u$. It is easy to check that $\sigma$ is a linear map and it is one-to-one, i.e., $\ker\{\sigma\} = \{0\}$.

Let us recall that a cooperative game $v$ is *additive* iff $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$. If we drop this condition just for $S = N$ and give the name *quasi-additive* to the games that fulfill it for all $S \subset N$, it follows that these games precisely form the image set $\mathrm{Im}(\sigma)$, and hence a game is quasi-additive iff it is the closure of a PBP, which is unique. The dimension of the subspace of quasi-additive games is $n + 1$. (If $n = 2$, then $\sigma$ is onto and therefore *any* cooperative 2-person game is the closure of a PBP.)

As $\sigma$ is an embedding of $E_{n+1}$ into $\mathcal{G}_N$, reasonable restrictions for games can be adapted to PBPs after identifying each PBP with the corresponding closure. Then, we call a PBP $u \in E_{n+1}$

- *additive* iff $u_1 + u_2 + \cdots + u_n = u_N$, i.e., iff $\Delta(u) = 0$
- *superadditive* (strictly) iff $u_1 + u_2 + \cdots + u_n < u_N$, i.e., iff $\Delta(u) > 0$
- *symmetric* iff $u_i = u_j$ for all $i, j \in N$
- *positive* iff $u_1, u_2, \ldots, u_n, u_N > 0$
- *negative* iff $u_1, u_2, \ldots, u_n, u_N < 0$

## 12.4 Core and the Shapley Rule

Now, let us first apply the Shapley value to any quasi-additive game, i.e., to $\overline{u}$ for any $u \in E_{n+1}$, and obtain an explicit formula. For each $i \in N$ we have

$$\varphi_i[\overline{u}] = \sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!} \left[\overline{u}(S) - \overline{u}(S\backslash\{i\})\right] =$$

$$= \sum_{\substack{S\ni i\\ S\neq N}} \frac{(s-1)!(n-s)!}{n!} u_i + \frac{1}{n}\left[u_N - \sum_{j\neq i} u_j\right] = u_i + \frac{\Delta(u)}{n},$$

since, for all $i \in N$,

$$\sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} = 1.$$

A quasi-additive game $\overline{u} = \sigma(u)$ is convex [26], that is, it satisfies

$$\overline{u}(S) + \overline{u}(T) \leq \overline{u}(S \cap T) + \overline{u}(S \cup T) \quad \text{for all } S, T \subseteq N,$$

iff $\Delta(u) \geq 0$. The core [11], given for a general cooperative game $v$ by

$$C(v) = \{x = (x_1, \dots, x_n) : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\},$$

takes here the much simpler form

$$C(\overline{u}) = \{x = (x_1, \dots, x_n) : \sum_{i \in N} x_i = u_N \;\text{ and }\; x_i \geq u_i \;\text{ for all } i = 1, \dots, n\}.$$

$C(\overline{u})$ is nonempty iff $\Delta(u) \geq 0$, and then $\varphi[\overline{u}] \in C(\overline{u})$ according to [26]. In fact, for a quasi-additive game the nonnegativity of the surplus is equivalent to, and not only sufficient for, the nonemptiness of the core.

Fig. 12.2 describes the core geometrically for $n = 2$. In the interesting case, when $\Delta(u) > 0$, the core is the closed segment $AB$ on the line $x_1 + x_2 = u_N$. The Shapley value $\varphi[\overline{u}]$ is the intersection of this line with $x_1 - u_1 = x_2 - u_2$, the orthogonal line from the *disagreement point* $D$. In other words, the Shapley value is the orthogonal projection of the disagreement point onto the core. As a limiting case, if $\Delta(u) = 0$, then the line $x_1 + x_2 = u_N$ reduces to the line $x_1 + x_2 = u_1 + u_2$, $A$ and $B$ coincide with $D$, and hence the core reduces to this disagreement point, which coincides with the Shapley value of the game. Finally, if $\Delta(u) < 0$ the core of $\overline{u}$ is empty.

The generalization of these ideas to arbitrary $n$ is straightforward. The disagreement point $D$ is given by $x_1 = u_1,\ x_2 = u_2,\ \dots,\ x_n = u_n$. The core is the simplex defined by $x_1 \geq u_1,\ x_2 \geq u_2,\ \dots,\ x_n \geq u_n$ in the hyperplane $x_1 + x_2 + \cdots + x_n = u_N$. It becomes empty if $\Delta(u) < 0$ and reduces to the disagreement point if $\Delta(u) = 0$. Otherwise, that is, whenever $\Delta(u) > 0$, *the Shapley value* $\varphi[\overline{u}]$ *is the orthogonal projection of the disagreement point onto the core* or, in other words, the intersection of the core with the orthogonal line $x_1 - u_1 = x_2 - u_2 = \cdots = x_n - u_n$.

We find here, thus, a particular case of Nash's classical bargaining problem [18]. The feasible set $S$ is defined by $\sum_{i \in N} x_i \leq u_N$, the Pareto frontier is given by $\sum_{i \in N} x_i = u_N$, and the disagreement point is $D = (u_1, u_2, \dots, u_n)$, which may lie above the Pareto frontier (just in case that $\Delta(u) < 0$). Moreover, $\varphi[\overline{u}]$ coincides with the Nash solution.

We are now ready to introduce the Shapley rule for PBPs.

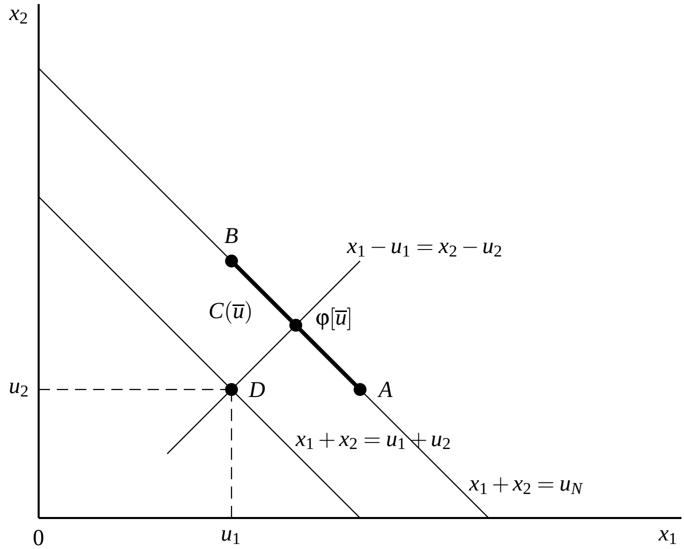


**FIGURE 12.2**: Core and Shapley value of a quasi-additive game $\overline{u}$ for $n = 2$.

**Definition 12.2** *By setting*

$$\overline{\varphi}[u] = \varphi[\overline{u}] \quad \textit{for all } u \in E_{n+1} \tag{12.3}$$

*we obtain a function* $\overline{\varphi} : E_{n+1} \longrightarrow \mathbb{R}^n$. *Function* $\overline{\varphi}$ *will be called the Shapley rule (for PBPs). It is given by*

$$\overline{\varphi}_i[u] = u_i + \frac{\Delta(u)}{n} \quad \textit{for each } i \in N \textit{ and each } u \in E_{n+1}. \tag{12.4}$$

Thus, the Shapley rule solves each problem in the following way: (a) first, each agent is allocated his individual utility; (b) once this has been done, the remaining utility—the surplus—is equally shared among all agents. The Shapley rule shows therefore an "egalitarian flavor" in the sense of [6]. Indeed, this rule is a mixture consisting of a "competitive" component, which rewards each agent according to the individual utility, and a "solidarity" component that treats all agents equally. It satisfies standardness for two-agent problems in the sense of [12].

Notice that $\overline{\varphi}$ is linear, coincides with $\varepsilon$, the equal surplus sharing rule introduced in Definition 12.1(b), and it has the property that, for *congruent* PBPs (in the sense of [20] for games), it is covariant with respect to utility transformations: If there are $\alpha, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that

$$u_i = \alpha v_i + \beta_i \quad \text{for each } \ i \in N \qquad \text{and} \qquad u_N = \alpha v_N + \sum_{i \in N} \beta_i,$$

then $\overline{\varphi}_i[u] = \alpha \overline{\varphi}_i[v] + \beta_i$.

**Example 12.2** *In Example 12.1,* $\Delta(u^c) = 130 = \Delta(u^s)$ *so the Shapley rule saves one third of this to each consumer and yields*

$$\overline{\varphi}[u^c] = (-156.67, -404.67, -404.67)$$

*for the cost PBP* $u^c$*, and*

$$\overline{\varphi}[u^s] = (43.33, 43.33, 43.33)$$

*for the saving PBP* $u^s$*. This reflects the fairness of the Shapley rule: Equal sharing of savings or, in other words,* $\overline{\varphi}_i[u^c] = u_i + \overline{\varphi}_i[u^s]$ *for all* $i \in N$*.*

To close this section we determine the set of PBPs where the Shapley rule and the proportional rule coincide and discuss individual rationality.

**Proposition 12.1** *The Shapley rule and the proportional rule coincide on a PBP* $u \in E^{\pi}_{n+1}$ *iff* $u$ *is additive or symmetric.*

**Remark 12.1** *For any sharing rule* $f$*, the property of individual rationality states that*

$$f_i[u] \geq u_i \quad \text{for all } i \in N \text{ and all } u \in E_{n+1}.$$

*It is easy to verify that the Shapley rule satisfies this property just in the domain of all additive or superadditive PBPs. Indeed, using Eq. (12.4) it follows that, for all* $i \in N$*,*

$$\overline{\varphi}_i[u] \geq u_i \quad \text{iff} \quad \Delta(u) \geq 0.$$

*In* $u = (-2, 1|1)$*, a troubling example where* $\Delta(u) = 2$ *and hence superadditivity holds, we get* $\overline{\varphi}[u] = (-1, 2)$*, an individually rational and quite reasonable result. Instead, the proportional rule gives* $\pi[u] = (2, -1)$*, a completely counterintuitive output. The domain where the proportional rule satisfies individual rationality, not difficult but cumbersome to describe, covers only a fraction of the set of superadditive PBPs.*

## 12.5 Axiomatic Characterizations of the Shapley Rule

When looking for a function $f : E_{n+1} \longrightarrow \mathbb{R}^n$, some reasonable properties should be imposed. To state a set of them, we previously define dummy agent, null agent, and symmetric agents in a PBP. Agent $i \in N$ is a *dummy* in a PBP $u$ iff $u_N = u_i + \sum_{j \neq i} u_j$, and *null* if, moreover, $u_i = 0$. Agents $i, j \in N$ are *symmetric* in a PBP $u$ iff $u_i = u_j$. A PBP with a dummy agent must be additive and therefore all agents are dummies. Conversely, if a PBP is additive, then all agents are dummies.

### 12.5.1 Main Theorem

Let us consider the following properties, stated (for a function $f$ defined) on $E_{n+1}$:

(i) *Efficiency*: $\sum_{i \in N} f_i[u] = u_N$ for every $u$.

(ii) *Dummy agent property*: If $i$ is a dummy in $u$, then $f_i[u] = u_i$.

(iii) *Symmetry*: If $i$ and $j$ are symmetric in $u$, then $f_i[u] = f_j[u]$.

(iv) *Additivity*: $f[u + v] = f[u] + f[v]$ for all $u, v$.

These properties deserve to be called "axioms" because of their elegant simplicity. It is hard to claim that they are not compelling. Efficiency, also called *group rationality*, means that the agents are going to share the total amount available to them. The dummy agent property essentially says that if a PBP is additive, then each agent should receive his individual utility. Symmetry establishes that two agents that are equally powerful individually should receive the same payoff. Finally, additivity implies that the allocation in a sum of PBPs must coincide with the sum of allocations in each PBP.

As to the logical independence of this axiomatic system, it suffices to find four rules that satisfy all axioms but one. Only a problem that shows the failure is needed in each case (a counterexample). This is not difficult to find.

The question is the following: Is there some function satisfying properties (i)–(iv)? If so, is it unique? The positive answers are given in the next result.

**Theorem 12.1** *(First main axiomatic characterization of the Shapley rule) There is one and only one function $f : E_{n+1} \longrightarrow \mathbb{R}^n$ that satisfies properties (i)–(iv). It is the Shapley rule $\overline{\varphi}$.*

Readers aware of Shapley's seminal work for cooperative games [25] will not be greatly surprised by Theorem 12.1. It may be noticed that it is equivalent to an axiomatic characterization of (the restriction of) the Shapley value on the subspace of quasi-additive games. In fact, in the existence proof of Theorem 12.1, all properties would follow from Eq. (12.3), the additivity of $\sigma$, and the corresponding properties of the classical Shapley value on games. A similar remark would apply to Theorem 12.3 below with regard to Young's work [28].

### 12.5.2 Other Domains

Several subsets of $E_{n+1}$ deserve special attention, and it would be therefore of interest to have axiomatic characterizations for (the restriction of) the Shapley rule on each one of these domains. So as not to enlarge the analysis too much, we will restrict it to the following ones:

- $E^{\pi}_{n+1}$, the domain of the proportional rule. We wish to contrast below (Subsection 12.6.3) the Shapley rule and the proportional rule strictly in this domain, in order to give "all advantages" to $\pi$ (if any) in our discussion.

- $E^{++}_{n+1}$, the open orthant of positive PBPs. It is mapped by $\sigma$ into the subset of positive games considered in [21]. A similar reason applies to $E^{--}_{n+1}$, the open orthant of negative PBPs.

- $\overline{E}^{(s)a}_{n+1}$, the closed cone of problems $u$ such that $\Delta(u) \geq 0$.

- $E^{sa}_{n+1}$, the open cone of superadditive PBPs. These PBPs, where the surplus is $\Delta(u) > 0$, are the most interesting ones since in each one of them there is something to gain by cooperation.

- The intersection of $\overline{E}^{+}_{n+1}$, the closed orthant of nonnegative PBPs, with $E^{sa}_{n+1}$. And also the intersection with this cone of $\overline{E}^{-}_{n+1}$, the closed orthant of nonpositive PBPs. The former includes all profit PBPs, while the latter includes all cost PBPs. By combining with the positivity of the surplus in both cases, we obtain the two most appealing types of PBPs in practice.

If $E$ denotes any of the subsets of $E_{n+1}$ mentioned just above, properties (i)–(iv) make sense for $f : E \longrightarrow \mathbb{R}^n$ if we state them only for $u, v \in E$. The sole exception is property (iv) for $E^{\pi}_{n+1}$ since it is the only one of these domains not closed under addition of PBPs. Therefore, in this case we will assume that the property is:

(iv) *Additivity*: if $u,\ v,\ u+v \in E^{\pi}_{n+1}$, then $f[u+v] = f[u] + f[v]$.

**Theorem 12.2** *(Additional axiomatic characterizations of the Shapley rule) If $E$ is any of the domains*

$$E^{\pi}_{n+1},\quad E^{++}_{n+1},\quad E^{--}_{n+1},\quad \overline{E}^{(s)a}_{n+1},\quad E^{sa}_{n+1},\quad E^{sa}_{n+1} \cap E^{+}_{n+1} \quad \textit{or} \quad E^{sa}_{n+1} \cap E^{-}_{n+1},$$

*there is one and only one function $f : E \longrightarrow \mathbb{R}^n$ that satisfies properties (i)–(iv). In all cases it is (the restriction of) the Shapley rule $\overline{\varphi}$.*

### 12.5.3 Discussing Monotonicity

In the literature on cooperative games, several monotonicity conditions have been suggested for solution concepts. Here we will recall some of the most relevant ones and will adapt them to the PBP setup, i.e., for sharing rules.

Let $\overline{u}, \overline{v} \in \mathcal{G}_N$ and $g : \mathcal{G}_N \longrightarrow \mathbb{R}^n$ be a solution concept. *Coalitional monotonicity* [27] states that if $\overline{u}(T) \geq \overline{v}(T)$ for some $T \subseteq N$ and $\overline{u}(S) = \overline{v}(S)$ for all $S \neq T$, then $g_i[\overline{u}] \geq g_i[\overline{v}]$ for all $i \in T$. In the particular case where

$T = N$, we obtain *aggregate monotonicity* [15]. *Strong monotonicity* [28] refers to marginal contributions and states that if $\overline{u}(S)-\overline{u}(S\backslash\{i\}) \geq \overline{v}(S)-\overline{v}(S\backslash\{i\})$ for all $S \subseteq N$, then $g_i[\overline{u}] \geq g_i[\overline{v}]$ for that $i$. The Shapley value satisfies all these conditions [28]. However, on quasi-additive games, coalitional monotonicity makes sense only for $T = N$, thus reducing to aggregate monotonicity which, in turn, becomes a consequence of strong monotonicity. Then we will translate to PBPs only this last property.

From a different approach, new interesting monotonicity conditions, among which we find again strong monotonicity, have been proposed in [7]. They are based on: (a) the *desirability relation* $D$, introduced in [14] for the players of any game $\overline{u}$ and given by

$$iDj \text{ in } \overline{u} \quad \text{iff} \quad \overline{u}(S \cup \{i\}) - \overline{u}(S) \geq \overline{u}(S \cup \{j\}) - \overline{u}(S) \text{ for all } S \subseteq N\backslash\{i, j\},$$

which compares *the positions of two players in a common game*; and (b) a similar relation $B$, introduced in [7] and given by

$$\overline{u}\, B\, \overline{v} \text{ for } i \quad \text{iff} \quad \overline{u}(S \cup \{i\}) - \overline{u}(S) \geq \overline{v}(S \cup \{i\}) - \overline{v}(S) \text{ for all } S \subseteq N\backslash\{i\},$$

which compares *the positions of a common player in two games.* We recall only:

- *monotonicity*: If $iDj$ in $\overline{u}$, then $g_i[\overline{u}] \geq g_j[\overline{u}]$
- *strong monotonicity*: If $\overline{u}\, B\, \overline{v}$ for $i$, then $g_i[\overline{u}] \geq g_i[\overline{v}]$.

The Shapley value satisfies these conditions [7]. Moreover, for any quasi-additive game $\overline{u} = \sigma(u)$, where $u = (u_1, u_2, \ldots, u_n|u_N) \in E_{n+1}$, the marginal contribution to $S \subseteq N$ of a player $i \in S$ in $\overline{u}$ is given by

$$\overline{u}(S) - \overline{u}(S\backslash\{i\}) = \begin{cases} u_i & \text{if} \quad S \neq N, \\ u_N - \displaystyle\sum_{j \neq i} u_j & \text{if} \quad S = N, \end{cases}$$

and it follows that

- $iDj$ in $\overline{u}$ iff $u_i \geq u_j$
- $\overline{u}\, B\, \overline{v}$ for $i$ iff $u_i \geq v_i$ and $\Delta(u) + u_i \geq \Delta(v) + v_i$.

Thus, relations $D$ and $B$, as well as both monotonicity conditions stated above, make sense in PBPs (just replacing each quasi-additive game $\overline{u}$ with the corresponding PBP $u$ given by $\sigma^{-1}$), and the Shapley rule $\overline{\varphi}$ satisfies these conditions.

Once within the PBP framework, we obtain a new main axiomatic characterization of the Shapley rule on $E_{n+1}$ that is quite different from Theorem 12.1 and is reminiscent of Young's [28] characterization of the Shapley value without using additivity. Its proof is not at all trivial.

**Theorem 12.3** *(Second main axiomatic characterization of the Shapley rule) There is one and only one function $f : E_{n+1} \longrightarrow \mathbb{R}^n$ that satisfies efficiency, symmetry and strong monotonicity. It is the Shapley rule $\overline{\varphi}$.*

An interesting suggestion made by the reviewer is addressed to consider alternative characterizations of the Shapley value or the Equal Surplus solution for games, try to adapt them to PBPs, and reach new axiomatizations of the Shapley rule. For example, during the revision we have checked that the well-known axiomatic characterization of the Shapley value by means of efficiency and the balanced contributions property can be translated to the Shapley rule.

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## 12.6 Criticism on the Proportional Rule

We shall discuss here several aspects of the proportional rule, most of which are far from being satisfactory, and will contrast them with the behavior of the Shapley rule.

### 12.6.1 Restricted Domain

As was already mentioned in Definition 12.1, the domain of the proportional rule $\pi$ is not the entire space $E_{n+1}$ but the subset defined by Eq. (12.2):

$$E_{n+1}^{\pi} = \{u \in E_{n+1} : u_1 + u_2 + \cdots + u_n \neq 0\}.$$

Thus, in Example 12.1 $\pi$ applies to $u^c$ but it cannot be applied to $u^s$. Instead, the Shapley rule $\overline{\varphi}$ applies to all PBPs without restriction (cf. Example 12.2).

### 12.6.2 Doubly Discriminatory Level

Within its domain, the proportional rule coincides with the Shapley rule just on additive or symmetric PBPs. However, these are very particular cases and, in general, the two rules differ. As a matter of comparison, note that the expression of $\pi_i[u]$ given in Eq. (12.1),

$$\pi_i[u] = u_i + \frac{u_i}{\sum_{j \in N} u_j} \Delta(u),$$

shows that the proportional rule (a) allocates to each agent his individual utility (as the Shapley rule does) but (b) it shares the remaining utility proportionally to the individual utilities. In other words, no solidarity component exists in the proportional rule, as both components are of a competitive nature. Instead, in this second step, the Shapley rule acts equitably.

Then the proportional rule is, conceptually, more complicated than the Shapley rule (incidentally, note that the calculus for the Shapley rule is easier than for the proportional rule) and it may include a *doubly discriminatory level* since, when comparing any two agents, it rewards twice the agent that individually can get the highest utility on his own. This discriminatory level arises, for example, in the case of nonnegative superadditive PBPs in $E^{\pi}_{n+1}$.

### 12.6.3 The Axiomatic Framework

In its restricted domain $E^{\pi}_{n+1}$, where the Shapley rule has been axiomatically characterized by Theorem 12.2, the proportional rule satisfies the properties of efficiency, dummy agent and symmetry. It fails to satisfy additivity (otherwise, it would coincide with the Shapley rule by Theorem 12.2) and also strong monotonicity.

Now, in spite of its simplicity and mathematical tradition, it may be that additivity is, in principle, the least appealing property and might seem to practitioners only a "mathematical delicatessen": The reason is that one does not easily capture the meaning of the sum of PBPs in practice. This will be illustrated in the next subsections.

### 12.6.4 Inconsistency: Cost-Saving Problems

In Example 12.1, where related costs and savings arise, the proportional rule cannot be applied to the saving PBP and no kind of consistency can then be discussed. Instead, the consistency of the Shapley rule is clear since

$$\overline{\varphi}_i[u^c] = u_i^c + \overline{\varphi}_i[u^s] \quad \text{for each } i \in N.$$

The conclusion is that, using the Shapley rule, all consumers are indifferent between sharing costs and sharing savings (as it should be).

**Example 12.3** *(A purchasing pool) Here the proportional rule will apply to all PBPs, but it will show inconsistency. Let* $N = \{1, 2, 3\}$ *be a purchasing pool of three firms and assume that, periodically, its members make to a common supplier orders of 1500, 2400 and 3000 units, respectively, of a product with unit cost 1. The supplier offers the following discounts:*

- *nothing for units from 1 to 1000*
- *9% off for units from 1001 to 2000*
- *15% off for units from 2001 to 3000*
- *24% off for units from 3001 upwards*

*Table 12.1 provides the full data for this purchasing pool. The members of the pool do not form a joint venture. They join just to get discounts for* *accumulated orders. Two alternatives are offered: (a) sharing the actual joint cost of* $-5724$*; (b) sharing the joint saving of 1176 after assuming that, previously, all members have individually deposited in a joint bank account the cost of their respective orders without discounts and the supplier's bill has been already paid from this account (a usual procedure in practice).*

| firms/ pool | order cost $u^0$ | discounts applied | actual cost $u^c$ | saving $u^s$ |
|---|---|---|---|---|
| $\{1\}$ | $-1500$ | 9% | $-1455$ | 45 |
| $\{2\}$ | $-2400$ | 9% and 15% | $-2250$ | 150 |
| $\{3\}$ | $-3000$ | 9% and 15% | $-2760$ | 240 |
| $\{1,2,3\}$ | $-6900$ | 9%, 15% and 24% | $-5724$ | 1176 |

**TABLE 12.1**: Purchasing pool data.

| $i$ | $\pi_i[u^0]$ | $\pi_i[u^c]$ | $\pi_i[u^s]$ | $\overline{\varphi}_i[u^0]$ | $\overline{\varphi}_i[u^c]$ | $\overline{\varphi}_i[u^s]$ |
|---|---|---|---|---|---|---|
| 1 | $-1500$ | $-1288.23$ | 121.66 | $-1500$ | $-1208$ | 292 |
| 2 | $-2400$ | $-1992.11$ | 405.52 | $-2400$ | $-2003$ | 397 |
| 3 | $-3000$ | $-2443.66$ | 648.83 | $-3000$ | $-2513$ | 487 |
| sums | $-6900$ | $-5724.00$ | 1176.00 | $-6900$ | $-5724$ | 1176 |

**TABLE 12.2**: Purchasing pool allocations.

*Table 12.2 yields the result of applying the proportional and Shapley rules to each alternative. Notice that we have three PBPs: An additive PBP* $u^0$ *of costs without discount, a PBP* $u^c$ *of actual costs (i.e., with discount), and a PBP* $u^s$ *of savings. They are obviously related by* $u^0 + u^s = u^c$*. While the Shapley rule is consistent in the sense that* $\overline{\varphi}[u^0]+\overline{\varphi}[u^s] = \overline{\varphi}[u^c]$*, which follows from additivity, this is not the case for the proportional rule, which does not satisfy this property as can be checked in Table 12.2. Notice, moreover, that* $\pi[u^0] = \overline{\varphi}[u^0]$ *because* $u^0$ *is an additive PBP.*

*Therefore, when using the Shapley value all members of the pool are indifferent between sharing costs with discount and sharing savings. Instead, this is not the case if the proportional rule is applied: Firms 1 and 2 prefer sharing costs, whereas firm 3 prefers sharing savings; the inconsistency (or lack of fairness) of the procedure is obvious.*

### 12.6.5 Inconsistency: Added Costs Problems

Let us consider a second example where additivity is crucial.

**Example 12.4** *(Added costs) We slightly modify Example 12.1 and assume that the consumers are interested in two goods (say, water and gas) carried by the same supplier. The costs are given in Table 12.3.*

*Again, we consider three PBPs: $u^w$, which describes the water costs; $u^g$, which gives the gas costs; and the sum $u^w + u^g$ which yields the added costs. Table 12.4 provides the result of applying the proportional and Shapley rules to each one of these PBPs. While the Shapley rule is consistent in the sense that $\overline{\varphi}[u^w + u^g] = \overline{\varphi}[u^w] + \overline{\varphi}[u^g]$, as follows from additivity, this is not the case for the proportional rule, which does not satisfy this property and fails therefore to be consistent in added costs problems.*

| group | water costs $u^w$ | gas costs $u^g$ | water + gas added costs $u^w + u^g$ |
|---|---|---|---|
| $\{1\}$ | $-300$ | $-150$ | $-450$ |
| $\{2\}$ | $-200$ | $-500$ | $-700$ |
| $\{3\}$ | $-100$ | $-250$ | $-350$ |
| $\{1,2,3\}$ | $-540$ | $-720$ | $-1260$ |
| $\{1\}+\{2\}+\{3\}$ | $-600$ | $-900$ | $-1500$ |

**TABLE 12.3**: Water and gas supply.

| $i$ | $\pi_i[u^w]$ | $\pi_i[u^g]$ | $\pi_i[u^w + u^g]$ | $\overline{\varphi}_i[u^w]$ | $\overline{\varphi}_i[u^g]$ | $\overline{\varphi}_i[u^w + u^g]$ |
|---|---|---|---|---|---|---|
| 1 | $-270$ | $-120$ | $-378$ | $-280$ | $-90$ | $-370$ |
| 2 | $-180$ | $-400$ | $-588$ | $-180$ | $-440$ | $-620$ |
| 3 | $-90$ | $-200$ | $-294$ | $-80$ | $-190$ | $-270$ |
| sums | $-540$ | $-720$ | $-1260$ | $-540$ | $-720$ | $-1260$ |

**TABLE 12.4**: Water and gas allocations.

*In this case, if the proportional rule is applied, consumer 1 prefers to share the payment of a water + gas joint bill, whereas consumers 2 and 3 prefer to share the payment of separate bills. Once more, the inconsistency (or lack* *of fairness) of the procedure is evident. Instead, using the Shapley rule, all consumers are indifferent between sharing separate bills or a joint bill.*

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## 12.7 Pure Bargaining Problems with a Coalition Structure

We consider here a more complicated framework, that of pure bargaining problems endowed with a coalition structure, and discuss the application of the Shapley rule $\overline{\varphi}$ to this new setup.

The general model is as follows. Let $N = \{1, 2, \ldots, n\}$ (with $n \geq 1$) be a set of agents and $u = (u_1, u_2, \ldots, u_n|u_N)$ be a PBP in $N$. Now let us also assume that a coalition structure $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ (with $m \geq 1$) and utilities $u_1^*, u_2^*, \ldots, u_m^*$ and $u_M^*$ are given, where $M = \{1, 2, \ldots, m\}$ represents the set of unions understood as supra-agents. Each $u_k^*$ is the utility that the agents of union $B_k$ can jointly obtain if all of them cooperate to this end, independently of the remaining agents—those of $N \backslash B_k$. If $B_k = \{i\}$ we assume that $u_k^* = u_i$. Finally, $u_M^*$ is the total utility that the unions can jointly obtain by acting as supra-agents. Thus, we do not necessarily assume that $u_k^* = \sum_{i \in B_k} u_i$ for each $k$ nor that $u_M^* = u_N$.

Vector $u^* = (u_1^*, u_2^*, \ldots, u_m^*|u_M^*)$ will be called the *quotient PBP* (in $M$), and $[u, \mathcal{B}, u^*]$ will be called a *pure bargaining problem with a coalition structure* or, for short, a *PBPCS*. In the particular case where $u_M^* = u_N$, the model might be viewed as an intermediate step between a PBP, where only the individual utilities and the total one are given, and a cooperative game, where a utility $u(S)$ for each $S \subseteq N$ is given.

In order to be consistent with the PBP model considered in the previous sections, for trivial coalition structures we assume that no new information is provided. Hence, if $\mathcal{B} = \mathcal{B}^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ then $M = N$ and $B_i = \{i\}$ for each $i \in N$; so, in addition to $u_i^* = u_i$ for all $i$, we impose $u_M^* = u_N$. And if $\mathcal{B} = \mathcal{B}^N = \{N\}$ then $M = \{1\}$ and $B_1 = N$, and we impose that $u_1^* = u_M^* = u_N$. Therefore, in both cases $[u, \mathcal{B}, u^*]$ essentially reduces to $u$.

In a PBP $u$, the agents have only two options: Agreeing all together in cooperating to obtain the total utility $u_N$ and share it or, otherwise, and even if just one of them disagrees, acting individually and merely getting the individual utilities. Instead, given a PBPCS $[u, \mathcal{B}, u^*]$, four main options are available to the agents:

- *Individual behavior* (I). The agents decide to act individually and obtain $u_1, u_2, \ldots, u_n$, respectively. This is a sort of "disagreement point" to which they can always go back if the next options are not successful.

- *Cooperative behavior* (C). All agents agree to cooperate in order to obtain $u_N$ and share it using the Shapley rule, disregarding the coalition structure and hence the next possibilities to act within or via unions.

- *Isolated unions behavior* (U). All agents of each union $B_k$ agree to cooperate in order to obtain $u_k^*$ and share it using the Shapley rule. (Maybe this will be the behavior only in some unions, in which case the agents of the remaining ones will be forced to act individually.) If $B_k = \{i_1, i_2, \ldots, i_{b_k}\}$, the *local* PBP in $B_k$ is

$$u^k = (u_{i_1}, u_{i_2}, \ldots, u_{i_{b_k}} | u_k^*).$$

  Notice that if $B_k = \{i\}$, then $u_k^* = u_i$.

  This behavior recalls Aumann and Drèze's approach [5] when discussing the extension of the classical Shapley value to cooperative games with a coalition structure, which leads to a solution that consists in applying the Shapley value to the subgame played in each union. Here, the Shapley rule is applied to each local problem.

- *Bargaining unions behavior* (B). This is a two-step procedure that requires the agreement of all agents at two levels: Forming the union they belong to, and then allowing it to agree with the other unions. Then all unions bargain first among themselves in the quotient problem

$$u^* = (u_1^*, u_2^*, \ldots, u_m^* | u_M^*)$$

  and share $u_M^*$ by using the Shapley rule. Thus, each union $B_k$ gets $\overline{\varphi}_k[u^*]$. Next, within each $B_k$ its agents agree to cooperate for sharing $\overline{\varphi}_k[u^*]$ using again the Shapley rule. If $B_k = \{i_1, i_2, \ldots, i_{b_k}\}$ and $b_k > 1$, these agents act in the *alternative* local PBP given by

$$\overline{u}^k = (u_{i_1}, u_{i_2}, \ldots, u_{i_{b_k}} | \, \overline{\varphi}_k[u^*])$$

  and apply the Shapley rule to this problem. If $b_k = 1$ the unique agent in $B_k$ directly gets $\overline{\varphi}_k[u^*]$.[4] This behavior recalls Owen's approach [22] when introducing the coalition value for cooperative games with a coalition structure: This value consists in applying, first, the Shapley value to the quotient game played by the unions and applying, then, the Shapley value again to an internal game within each union to share among its members the payoff obtained by that union in the quotient game. Here, the procedure is very similar, but using at both levels the Shapley rule.

As expected, for trivial structures these four options reduce to the standard ones in a PBP. Indeed, we have that (for all agents): (a) if $\mathcal{B} = \mathcal{B}^n$, then U reduces to I and B to C; (b) if $\mathcal{B} = \mathcal{B}^N$, then U and B reduce to C.

[4] In fact, if $B_k = \{i\}$, then $(u_i | \overline{\varphi}_k[u^*])$ might not be a PBP, but it is clear that, acting as "supra-agent" $k$ representing himself, agent $i$ can obtain $\overline{\varphi}_k[u^*]$ if all unions choose the behavioral option B.

## 12.8 A Numerical Example

To see how this model works, we will consider a 9-person PBPCS. (It is difficult to see how a smaller problem would allow us to illustrate this, since we should, as a practical matter, have at least three unions, each with approximately three agents.) We are grateful to Fabrice Valognes, of the Université of Caen-Basse Normandie, for a suggestion he made to us.

**Example 12.5** *(A cost allocation problem with a coalition structure) Suppose that 9 manufacturers, located at points $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, and $J$, need to get some particular supply from a distributor located at point $S$. Note that the 9 manufacturers are not competitors, as they produce different consumer goods.*[5] *They are located in three different cities, and those in each city have generally good relations, working (perhaps) through the local chambers of commerce, each one of which has legal ability to represent its members jointly and also to negotiate with the other chambers in dealing with the distributor.*

*The locations of the agents (manufacturers) and the distributor are shown in Fig. 12.3. We will be interested only in transportation costs (i.e., connection costs) and will assume, in order to make calculations easier, that the cost of connecting any two points is proportional to the square of the distance between them.*[6]

*If some set of manufacturers sign a joint contract with the distributor to obtain a common connection, the procedure chosen by the distributor will consist in all cases in establishing a connection from $S$ to the barycenter of the locations of these manufacturers and connecting then this barycenter to each manufacturer. Thus, for the cooperative behavior $C$ we will use the "total" barycenter* **N** *(for the nine manufacturers), the location of which is given in Fig. 12.3. The dashed lines represent the total connection with all manufacturers via* **N**.

*However, the manufacturers are associated in the local chambers of commerce of each city. These are*

$$B_1 = \{A, B, C\}, \quad B_2 = \{D, E, F, G\} \quad \textit{and} \quad B_3 = \{H, J\},$$

*and this is the coalition structure $\mathcal{B} = \{B_1, B_2, B_3\}$ that will be considered here. Then, whenever the unions effectively matter, i.e., for the isolated unions behavior $U$ and the bargaining unions behavior $B$, we will also use the "local" (city) barycenters* **P**, **Q** *and* **R**, *referred to (the agents of) unions $B_1$, $B_2$ and $B_3$, respectively. The locations of these three points are also indicated in Fig. 12.3, and the thick lines represent the local connections using the*

[5] For example, we might assume that the manufacturers produce machines and vehicles of various types and constitute an oligopoly, while the distributor is a monopolist of a raw material such as steel.

[6] Since the Shapley rule is linear and hence homogeneous, and since we are only interested in comparing costs, we may take the factor of proportionality equal to 1.

*corresponding barycenters. Besides, we will consider the "unions barycenter"* $\mathbf{M}$*, the barycenter of* $\mathbf{P}$*,* $\mathbf{Q}$ *and* $\mathbf{R}$ *also described in Fig. 12.3, to compute* $u^*_M$*, which will be used for the bargaining unions behavior* $B$*.*

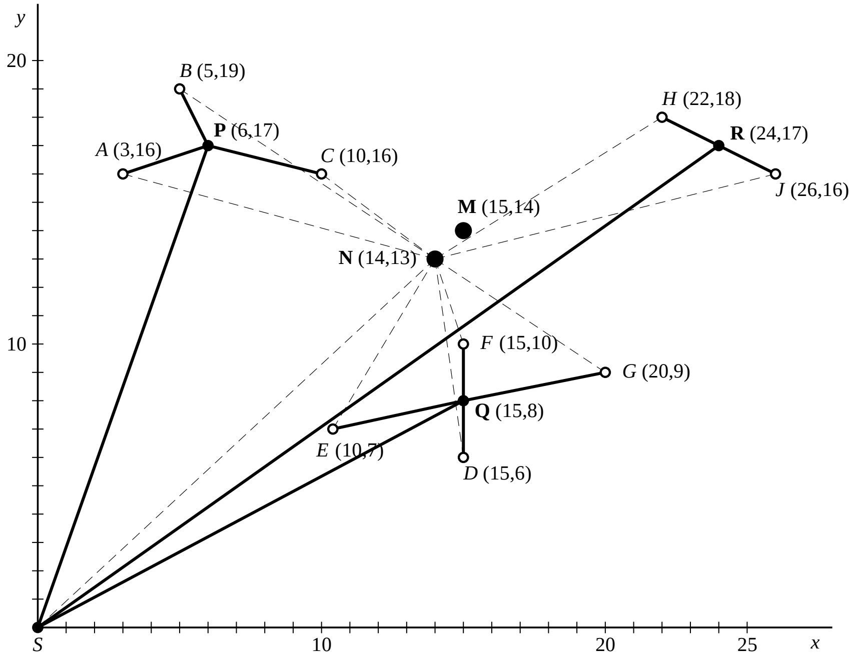


**FIGURE 12.3**: Manufacturers, distributor, barycenters, and connections.

*Then we will analyze the cost PBPCS* $[u, \mathcal{B}, u^*]$ *given by*

$$u = (-265, -386, -356, -261, -149, -325, -481, -808, -932| - 1043),$$
$$\mathcal{B} = \{B_1, B_2, B_3\}, \quad \textit{and} \quad u^* = (-357, -349, -875| - 739),$$

*where*

$$u^*_M = d(S, \mathbf{M})^2 + \Big[d(\mathbf{M}, \mathbf{P})^2 + \sum_{X \in B_1} d(\mathbf{P}, X)^2\Big] +$$
$$\Big[d(\mathbf{M}, \mathbf{Q})^2 + \sum_{Y \in B_2} d(\mathbf{Q}, Y)^2\Big] + \Big[d(\mathbf{M}, \mathbf{R})^2 + \sum_{Z \in B_3} d(\mathbf{R}, Z)^2\Big].$$

*We will consider the four options available to the agents. Table 12.5 will show the sharing in each case as well as agents' and unions' preferences over the behavior options.*

***Individual behavior (I).*** *If the manufacturers act all individually, then each one of them signs an individual contract with the distributor and pays the amount indicated by the corresponding component of* $u$*: Thus,* $A$ *pays 265,* $B$ *pays 386, and so on until* $J$*, who pays 932.*

***Cooperative behavior (C).*** *If all manufacturers agree to sign a joint contract with the distributor, and collectively pay therefore 1043, the Shapley rule applies to* $u$*, where* $\Delta(u) = 2920$*, and yields the (rounded) sharing of the joint cost among them:*

$$\overline{\varphi}[u] = \\ = (59.44,\ -61.56,\ -31.56,\ 63.44,\ 175.44,\ -0.56,\ -156.56,\ -483.56,\ -607.56).$$

***Isolated unions behavior (U).*** *If in each city all manufacturers decide to act together through their chamber of commerce, sign a contract concerning themselves, and disregard therefore the cooperation with the remaining manufacturers, then there are three local problems, to which we will apply the Shapley rule:*

$$\begin{array}{lll} B_1: & u^1 = (-265, -386, -356| - 357), & \Delta(u^1) = 650 \\ B_2: & u^2 = (-261, -149, -325, -481| - 349), & \Delta(u^2) = 867 \\ B_3: & u^3 = (-808, -932| - 875), & \Delta(u^3) = 865 \end{array}$$

| | I | C | U | B | preference |
|---|---|---|---|---|---|
| $A$ | −265.00 | 59.44 | −48.33 | 45.22 | I < U < B < C |
| $B$ | −386.00 | −61.56 | −169.33 | −75.78 | I < U < B < C |
| $C$ | −356.00 | −31.56 | −139.33 | −45.78 | I < U < B < C |
| $B_1$ | −1007.00 | −33.67 | −357.00 | −76.33 | I < U < B < C |
| $D$ | −261.00 | 63.44 | −44.25 | 25.92 | I < U < B < C |
| $E$ | −149.00 | 175.44 | 67.75 | 137.92 | I < U < B < C |
| $F$ | −325.00 | −0.56 | −108.25 | −38.08 | I < U < B < C |
| $G$ | −481.00 | −156.56 | −264.25 | −194.08 | I < U < B < C |
| $B_2$ | −1216.00 | 81.78 | −349.00 | −68.33 | I < U < B < C |
| $H$ | −808.00 | −483.56 | −375.50 | −235.17 | I < C < U < B |
| $J$ | −932.00 | −607.56 | −499.50 | −359.17 | I < C < U < B |
| $B_3$ | −1740.00 | −1091.11 | −875.00 | −594.33 | I < C < U < B |
| $N$ | −3963.00 | −1043.00 | −1581.00 | −739.00 | I < U < C < B |

**TABLE 12.5**: Sharing and behavioral preferences.

***Bargaining unions behavior (B).*** *The first step is defined by the quotient problem concerning the unions:*

$$u^* = (-357, -349, -875| - 739)$$

*with* $\Delta(u^*) = 842$*. The Shapley rule yields* $\overline{\varphi}[u^*] = (-76.33, -68.33, -594.33)$*. This gives rise to the alternative local problems, where the Shapley rule will be applied again:*

$$\begin{array}{lll} B_1: & \overline{u}^{\,1} = (-265, -386, -356| - 76.33), & \Delta(\overline{u}^{\,1}) = 930.67 \\ B_2: & \overline{u}^{\,2} = (-261, -149, -325, -481| - 68.33), & \Delta(\overline{u}^{\,2}) = 1147.67 \\ B_3: & \overline{u}^{\,3} = (-808, -932| - 594.33), & \Delta(\overline{u}^{\,3}) = 1145.67 \end{array}$$

*Summing up, the manufacturers' preferences as to the four options are*

- I < U < B < C *for A, B and C*
- I < U < B < C *for D, E, F and G*
- I < C < U < B *for H and J*

*The conclusion is that, since options C and B require the agreement of all agents, they will not be chosen and all unions will follow what we have called the isolated unions behavior (U).*

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## 12.9 A General Result on Preferences

Some properties of the general model will be established in this section. Most of them are illustrated by Example 12.5. Given a pair of distinct behavioral options (X,Y), any agent will have one and only one preference of the form X < Y, X = Y, or X > Y, in accordance with the payoffs that the agent obtains under each one of these options. In Theorem 12.4, each possibility will be characterized in terms of surpluses, or surpluses per capita, of the different problems involved in the considered PBPCS.

Since all conditions follow from solving a numerical inequality, the reader should be warned that, to avoid making the statement too cumbersome, we will use $\gtreqless$. This implies that an equivalence like, e.g., C $\gtreqless$ I iff $\Delta(u) \gtreqless 0$ will mean that the following conditions hold:

(a) C > I iff $\Delta(u) > 0$.

(b) C = I iff $\Delta(u) = 0$.

(c) C < I iff $\Delta(u) < 0$.

Given a PBPCS $[u, \mathcal{B}, u^*]$, let us recall that $n = |N|$, $m = |M|$ and $b_k = |B_k|$ denote cardinalities, the original and quotient PBP are, respectively, $u = (u_1, u_2, \ldots, u_n | u_N)$ and $u^* = (u_1^*, u_2^*, \ldots, u_m^* | u_M^*)$, and, for each $k \in M$, $u^k = (u_{i_1}, u_{i_2}, \ldots, u_{i_{b_k}} | u_k^*)$ is the local PBP in union $B_k$ for option U.

**Theorem 12.4** *Let $[u, \mathcal{B}, u^*]$ be a PBPCS in $N$. Then:*

*(a)* $\mathrm{C} \gtreqqless \mathrm{I}$ *for all $i \in N$ iff* $\Delta(u) \gtreqqless 0$.

*(b)* $\mathrm{U} \gtreqqless \mathrm{I}$ *for all $i \in B_k$ iff* $\Delta(u^k) \gtreqqless 0$.

*(c)* $\mathrm{B} \gtreqqless \mathrm{I}$ *for all $i \in B_k$ iff* $\Delta(u^k) + \dfrac{\Delta(u^*)}{m} \gtreqqless 0$.

*(d)* $\mathrm{U} \gtreqqless \mathrm{C}$ *for all $i \in B_k$ iff* $\dfrac{\Delta(u^k)}{b_k} \gtreqqless \dfrac{\Delta(u)}{n}$.

*(e)* $\mathrm{B} \gtreqqless \mathrm{C}$ *for all $i \in B_k$ iff* $\dfrac{\Delta(u^k)}{b_k} + \dfrac{\Delta(u^*)}{mb_k} \gtreqqless \dfrac{\Delta(u)}{n}$.

*(f)* $\mathrm{B} \gtreqqless \mathrm{U}$ *for all $i \in N$ iff* $\Delta(u^*) \gtreqqless 0$.

**Remark 12.2** *We collect here some consequences of Theorem 12.4.*

- *Any preference on* (I,C) *is common for all agents $i \in N$ and depends only on $u$.*
- *The same happens for* (U,B)*, and the preference depends only on the quotient problem $u^*$.*
- *The remaining four preferences, i.e., on* (I,U), (I,B), (C,U) *and* (C,B)*, are common for at least all agents of each union $B_k$. By combining this with the previous items it follows that all agents of each union $B_k$ order the four options equally, so we can speak of "unions' preferences".*

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## 12.10 The Modified Shapley Rule and its Natural Domain

Let $N = \{1, 2, \ldots, n\}$ be the set of agents, with $n \geq 1$. A PBPCS $[u, \mathcal{B}, u^*]$ in $N$ is defined by three objects:

- $u = (u_1, u_2, \ldots, u_n | u_N)$, which is a PBP in $N$
- $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$, which is a coalition structure in $N$, with $M = \{1, 2, \ldots, m\}$ and $m \geq 1$
- $u^* = (u_1^*, u_2^*, \ldots, u_m^* | u_M^*)$, which is a PBP in $M$

with the following restrictions:

(r1) If $\mathcal{B} = \mathcal{B}^n$ or $\mathcal{B} = \mathcal{B}^N$ then $u^*_M = u_N$

(r2) if $n = 1$, then $u_N = u_1$

(r3) if $m = 1$, then $u^*_M = u^*_1$

(r4) if $b_k = 1$ for some $k \in M$, i.e., if $B_k = \{i\}$ is a singleton, then $u^*_k = u_i$

$E_N$ will denote the set of all PBPCS defined in $N$, and we put $E = \bigcup_{n=1}^{\infty} E_N$. These two sets do not have any structure. Instead, the set $E_N^{\mathcal{B}}$, formed by all PBPCS defined in $N$ with a *fixed* coalition structure $\mathcal{B}$, becomes a vector space under the natural linear operations given by

- $[u, \mathcal{B}, u^*] + [v, \mathcal{B}, v^*] = [u + v, \mathcal{B}, u^* + v^*]$
- $\lambda[u, \mathcal{B}, u^*] = [\lambda u, \mathcal{B}, \lambda u^*] \quad$ for all $\lambda \in \mathbb{R}$

A *coalitional sharing rule* on $E_N$ means a map $g : E_N \longrightarrow \mathbb{R}^n$. Given a PBPCS $[u, \mathcal{B}, u^*]$ in $N$, for each $i \in N$ the $i$-coordinate $g_i[u, \mathcal{B}, u^*]$ of vector $g[u, \mathcal{B}, u^*]$ gives the utility that is allocated to agent $i$ according to $g$.

**Definition 12.3** *The modified Shapley rule, denoted by $\overline{\psi}$, is the coalitional sharing rule on $E_N$ that allocates utilities, to all agents of a PBPCS $[u, \mathcal{B}, u^*]$ in $N$, according to the bargaining unions behavior $B$. An explicit expression for $\overline{\psi}$ is as follows: Given $[u, \mathcal{B}, u^*]$ in $N$, if $i \in B_k$ then*

$$\overline{\psi}_i[u, \mathcal{B}, u^*] = u_i + \frac{\Delta(u^k)}{b_k} + \frac{\Delta(u^*)}{mb_k}. \tag{12.5}$$

Some first properties of $\overline{\psi}$ are stated in the following result.

**Proposition 12.2** *The modified Shapley rule satisfies the following elementary properties:*

*(a) Trivial coalition structures: If $\mathcal{B} = \mathcal{B}^n$ or $\mathcal{B} = \mathcal{B}^N$, then $\overline{\psi}[u, \mathcal{B}, u^*] = \overline{\varphi}[u]$.*

*(b) General coalition structure: For any $\mathcal{B}$, $\sum_{i \in B_k} \overline{\psi}_i[u, \mathcal{B}, u^*] = \overline{\varphi}_k[u^*]$ for all $k \in M$.*

*(c) Coalitional symmetry: If $u^*_k = u^*_h$, then*

$$\sum_{i \in B_k} \overline{\psi}_i[u, \mathcal{B}, u^*] = \sum_{j \in B_h} \overline{\psi}_j[u, \mathcal{B}, u^*].$$

In order to characterize axiomatically the modified Shapley rule, let us consider the following properties for a coalitional sharing rule $g : E_N \longrightarrow \mathbb{R}^n$.

(i) *Group rationality*: $\sum_{i \in N} g_i[u, \mathcal{B}, u^*] = u^*_M$ for every $[u, \mathcal{B}, u^*]$ in $N$.

(ii) *Individual rationality*: If $\Delta(u^*) \geq 0$ and $\Delta(u^k) \geq 0$ for all $k \in M$, then $g_i[u, \mathcal{B}, u^*] \geq u_i$ for all $i \in N$.

(iii) *Coalitional rationality*: If $\Delta(u^*) \geq 0$, then $\sum_{i \in B_k} g_i[u, \mathcal{B}, u^*] \geq u^*_k$ for all $k \in M$.

(iv) *Symmetry*: If $u_i = u_j$ and $i, j \in B_k$ for some $k \in M$, then $g_i[u, \mathcal{B}, u^*] = g_j[u, \mathcal{B}, u^*]$.

(v) *Additivity*: For all $\mathcal{B}$ and all $[u, \mathcal{B}, u^*]$ and $[v, \mathcal{B}, v^*]$ in $E_N$,

$$g[u + v, \mathcal{B}, u^* + v^*] = g[u, \mathcal{B}, u^*] + g[v, \mathcal{B}, v^*].$$

(vi) *Singletons*: If $B_k = \{i\}$, then $g_i[u, \mathcal{B}, u^*] = \overline{\varphi}_k[u^*]$.[7]

We have checked the logical independence of this axiomatic system. It characterizes the modified Shapley rule $\overline{\psi}$, defined by Eq. (12.5). The statement is as follows.

**Theorem 12.5** *There is one and only one coalitional sharing rule on $E_N$ that satisfies properties (i)–(vi). It is the modified Shapley rule $\overline{\psi}$.*

Now we are mainly interested in those PBPCS where the bargaining unions behavior B is the best option for all agents. These problems constitute the natural domain of the modified Shapley rule $\overline{\psi}$. The basic question is: Under which conditions will all agents prefer B to I, C and U, in a given PBPCS $[u, \mathcal{B}, u^*]$ in $N$? If $\mathcal{B} = \mathcal{B}^n$, then B $=$ C $>$ U $=$ I iff $\Delta(u) > 0$. If $\mathcal{B} = \mathcal{B}^N$, then B $=$ U $=$ C $>$ I iff $\Delta(u) > 0$. For any other coalition structure $\mathcal{B}$, since B requires more agreements than the other three options, we will consider *strict* preferences only.[8]

**Theorem 12.6** *Let $[u, \mathcal{B}, u^*]$ be any PBPCS in $N$ with a nontrivial $\mathcal{B}$. Therefore* B $>$ I, B $>$ C *and* B $>$ U *for all agents in $[u, \mathcal{B}, u^*]$ iff the following conditions simultaneously hold:*

*(i)* $\Delta(u^k) + \dfrac{\Delta(u^*)}{m} > 0$ *for all* $k$

*(ii)* $\dfrac{\Delta(u^k)}{b_k} + \dfrac{\Delta(u^*)}{m b_k} > \dfrac{\Delta(u)}{n}$ *for all* $k$

*(iii)* $\Delta(u^*) > 0$

---

[7] In Remark 12.3 we will write this equation in an alternative form that avoids mentioning the Shapley rule.

[8] For example, it is clear that, in any PBP $u$, all agents prefer C to I (C $>$ I) iff $u$ is superadditive, i.e., $\Delta(u) > 0$. We might include the case where $u$ is additive, i.e., $\Delta(u) = 0$, but then C $=$ I for all agents and very probably they would choose I since this behavior does not require any agreement. This follows directly or from Theorem 12.4(a).

**Example 12.6** *In particular, any PBPCS* $[u, \mathcal{B}, u^*]$ *such that*

*(1)* $u^*$ *satisfy* $\Delta(u^*) > 0$

*(2) all* $u^k$ *satisfy* $\Delta(u^k) \geq 0$*, and*

*(3)* $\Delta(u^*)$ *is "large enough" in the sense that* $\Delta(u^*) > \dfrac{mb_k}{n}\Delta(u)$ *for all* $k$

*satisfies conditions (i)–(iii) of Theorem 12.6. Conditions (1)–(3) are only sufficient but quite reasonable restrictions to obtain that behavior B is the best option for all agents. Under (1) and (2), condition (3) can be dropped if* $u$ *is not superadditive, or otherwise it needs only to be checked for the largest union (maximum* $b_k$*).*

Thus, for any PBPCS $[u, \mathcal{B}, u^*]$ such that B is the best option for all agents, the modified Shapley rule $\overline{\psi}$ is the natural way to share $u^*_M$ among all agents. There are two possibilities: Following the two-step procedure detailed in Section 12.7 or applying Eq. (12.5) directly.

**Example 12.7** *Let* $N = \{1, 2, 3, 4, 5, 6\}$ *(so* $n = 6$*) and*

$$u = (5, 4, 4, 2, 1, 3|25).$$

*Let* $\mathcal{B} = \{B_1, B_2, B_3\}$*, with* $B_1 = \{1, 2, 3\}$*,* $B_2 = \{4, 5\}$ *and* $B_3 = \{6\}$*. Thus,* $b_1 = 3$*,* $b_2 = 2$ *and* $b_3 = 1$*. Finally, let*

$$u^* = (19, 5, 3|45)$$

*be the quotient problem in* $M = \{1, 2, 3\}$*, that represents the set of unions as entities (so* $m = 3$*). Then* $[u, \mathcal{B}, u^*]$ *is a PBPCS in* $N$*. The local problems are*

$$u^1 = (5, 4, 4|19), \quad u^2 = (2, 1|5) \quad \textit{and} \quad u^3 = (3|3).$$

*Then, the sufficient conditions of Example 12.6 are satisfied and we conclude that option B is the best behavior in this PBPCS for all agents* $i \in N$*.*[9] *The application of the modified Shapley rule, using for example Eq. (12.5) and*

$$\Delta(u^1) = 6, \quad \Delta(u^2) = 2, \quad \Delta(u^3) = 0, \quad \textit{and} \quad \Delta(u^*) = 18,$$

*yields*

$$\overline{\psi}[u, \mathcal{B}, u^*] = (9, 8, 8, 6, 5, 9),$$

*a sharing that satisfies all agents, and more than any other behavioral option.*

[9] However, notice that the preferences on the remaining three options are not common. Indeed, we find B > U > C > I for $B_1$, B > U = C > I for $B_2$, and B > C > U = I for $B_3$.

**Remark 12.3** *The characterization of the modified Shapley rule $\overline{\psi}$ given by Theorem 12.5 refers, in principle, to $E_N$. But, if we define coalitional sharing rule as any map*

$$g : E = \bigcup_{n=1}^{\infty} E_N \longrightarrow \bigcup_{n=1}^{\infty} \mathbb{R}^n$$

*such that if $[u, \mathcal{B}, u^*] \in E_N$, then $g[u, \mathcal{B}, u^*] \in \mathbb{R}^n$, where $n = |N|$, then the axiomatic system makes sense and Theorem 12.5 also holds for $E$. We will use the same symbols $g$ and $\overline{\psi}$ for all domains on which they are considered.*

*Then, having in mind this extension, we see that, for any $[u, \mathcal{B}, u^*]$ in a given $N$, we can write in $M$, by Proposition 12.2(a),*

$$\overline{\psi}[u^*, \mathcal{B}^m, u^{**}] = \overline{\varphi}[u^*].$$

*The singletons property (vi) may then be written as follows:*

*(vi) Singletons: If $B_k = \{i\}$, then $g_i[u, \mathcal{B}, u^*] = g_k[u^*, \mathcal{B}^m, u^{**}]$,*

*which avoids mentioning the Shapley rule.*

**Remark 12.4** *Let $E^*$, $E_N^*$ and $(E_N^{\mathcal{B}})^*$ be the respective subsets of $E$, $E_N$ and $E_N^{\mathcal{B}}$ formed by the PBPCS where $B$ is a better behavioral option than $U$, $C$ and $I$ for all agents, i.e., PBPCS that satisfy the conditions of Theorem 12.6. $(E_N^{\mathcal{B}})^*$ is an open cone, and Theorem 12.5 gives rise to an axiomatic characterization of the modified Shapley rule on this cone and hence on $E_N^*$ and $E^*$.*

Indeed, we have:

**Theorem 12.7** *There is one and only one coalitional sharing rule defined on $(E_N^{\mathcal{B}})^*$ that satisfies properties (i)–(vi) in this cone. It is (the restriction of) the modified Shapley rule $\overline{\psi}$.*

## 12.11 Conclusions

Pure bargaining problems, introduced in [8], constitute a natural framework and, at the same time, a simple case of both Nash's bargaining model [18] and the cooperative game model (as they can be identified with quasi-additive games). Their simplicity allows us to better capture the meaning of certain notions, most of which are translated from the cooperative game theory.

The Shapley rule, also introduced in [8], is a well-founded solution notion for any PBP, which enjoys satisfactory properties similar to those of the

Shapley value. It can be clearly distinguished from other previous notions like, e.g., some proportionality rules including the classical proportional rule.

The axiomatic viewpoints established by Shapley when defining the value notion for cooperative games, and by Young when replacing the dummy/null player and additivity properties with strong monotonicity, which have been adapted to PBPs, allow us to evaluate any sharing rule and, in particular, to compare the proportional rule and the Shapley rule. The relevant points are:

1. A first essential failure of the proportional rule is its restricted domain, defined by Eq. (12.2). Instead, the Shapley rule applies without any restriction to all PBPs.

2. When putting together Eqs. (12.1) and (12.4), the procedures look somewhat similar: First, each agent $i$ is allocated his individual utility $u_i$; next, the surplus is shared among all agents. However, the proportional rule includes a (hidden) doubly discriminatory level hard to justify.

3. In which cases do these two allocation rules coincide? As has been shown, the Shapley rule and the proportional rule coincide on a PBP $u$ satisfying Eq. (12.2) iff this PBP is additive or symmetric—the most trivial cases.

4. As to the Shapley axioms, the proportional rule satisfies in its restricted domain the properties of efficiency, dummy and null agent, and symmetry, but not the strong monotonicity property in Young's sense.

5. This leaves us with the lack of additivity: The proportional rule is homogeneous but not additive. Let us raise the question: Is this failure important or, on the contrary, is additivity simply a standard mathematical property, just of a technical nature, without special relevance in practice? The answer is quite surprising. From the lack of additivity, serious inconsistencies of the proportional rule follow when dealing with e.g., related cost-savings problems and added cost problems.

In summary, we have analyzed the proportional rule, from an axiomatic viewpoint but also from a practical viewpoint. Several properties and failures of the proportional rule have been remarked. We therefore contend that the Shapley rule should replace in practice the proportional rule in PBPs, that is, in cooperative affairs where the coalitions of intermediate size ($1 < |S| < n$) do not matter.

The introduction of a coalition structure in a PBP by means of the general model presented in [10] constitutes a novelty in the literature. Four behavioral options for the agents of any PBPCS are suggested: Two of them (I and C) are equivalent to agents' options in a PBP, while the other two (U and B) respectively recall the treatment given by Aumann-Drèze and Owen to cooperative games with a coalition structure.[10] The Shapley rule is intensively used

[10]Both approaches are interesting topics currently: See e.g. recent references [1] and [2], where the proportional rule and the Shapley value are combined for monotonic games with a coalition structure following, respectively, Owen's and Aumann-Drèze's viewpoints.

for evaluating the results of each option and gives rise to agents' individual preferences on the issues. A numerical example illustrates all these ideas.

A main result gives complete and easy conditions (eighteen in all) to determine all agents' preferences on I, C, U and B. They are stated in terms of surpluses and surpluses per capita. Finally, a modified Shapley rule for all PBPCS has been defined and axiomatically characterized, and even on its natural domain: The cone of PBPCS where the bargaining unions' behavior B is the best option for all agents, which has been described previously.

## 12.12 Suggestions

Future work on this research line might consider several aspects. We propose the following: (a) a first refinement of the analysis of a PBPCS when there is no unanimous behavioral preference (as in Example 12.5) and a subsequent bargaining, only among some unions, is feasible; (b) a second refinement of this analysis, based on a cost assessment when accepting a not preferred option and the use of side payments to compensate (partially, at least) this cost; (c) the introduction of a coalition structure in a PBP trying to improve the final payoff; (d) more generally, the possibility of speaking, within this model, of endogenous coalition formation and discussing the stability of coalition structures in the sense of the strong Nash equilibrium (cf. [4] and [19]); (e) additional theoretical work on the behavioral options in a PBPCS; and (f) procedures for introducing restrictions to cooperation in a PBP.

Suggestions (a), (c) and (d) will make sense when there exists a general mechanism to define the quotient PBP, as it happens in Example 12.1 (based on distributor's offers) and in Example 12.5 (using the barycenter method). Next we provide more or less detailed hints for each one of our proposals.

(a) Complementary analysis of a PBPCS: Alternative 1

The conclusion obtained in Example 12.5 strictly follows from the general model established in Section 12.7. Using Remark 12.2, we are allowed to consider each union's preferences. In that model, once one or more unions have chosen the isolated unions behavior (U), each remaining union is forced to adopt this same option or allow its members to act individually (I). Thus, in Example 12.5, $B_1$ and $B_2$ cannot impose the cooperative behavior (C) to $B_3$, which in turn cannot impose the bargaining unions behavior (B) to them. Therefore, in principle, each union prefers to deal on its own with the distributor (U) according to Table 12.5.

However, it may be interesting to pay attention to several differences arising from Table 12.5. These are: (a) the loss to each union when adopting option U instead of its preferred option; (b) the loss to $B_1$ and $B_2$ under option B with regard to their preferred option (C) and the loss to $B_3$ under option C with regard to B but also to U. If $\Pi_k(\mathrm{X})$ denotes the *joint* allocation to union $B_k$ under option X, Table 12.6 yields these differences.

| | |
|---|---|
| $\Pi_1(\mathrm{U}) - \Pi_1(\mathrm{C}) = -323.33$ | $\Pi_1(\mathrm{B}) - \Pi_1(\mathrm{C}) = -42.67$ |
| $\Pi_2(\mathrm{U}) - \Pi_2(\mathrm{C}) = -430.78$ | $\Pi_2(\mathrm{B}) - \Pi_2(\mathrm{C}) = -150.11$ |
| $\Pi_3(\mathrm{U}) - \Pi_3(\mathrm{B}) = -280.67$ | $\Pi_3(\mathrm{C}) - \Pi_3(\mathrm{B}) = -496.78$ |
| | $\Pi_3(\mathrm{C}) - \Pi_3(\mathrm{U}) = -216.11$ |

**TABLE 12.6**: Comparing joint allocations under different behavioral options.

The first column of Table 12.6 shows that each union suffers a great damage when it follows option U. The second column suggests that, maybe, option B for $B_1$ and $B_2$, or even option C for $B_3$, would not be so bad as they seem. So let us go further in our analysis and study a first alternative.

This alternative is feasible in Example 12.5 because there exists a general rule for cost formation, based on connecting via barycenters. Then it makes sense to imagine $B_1$ and $B_2$ starting a second negotiation step to discuss a new, "restricted" PBPCS, trying to find a common best option that gives them a result better than option U. Let us consider this possibility.

The total barycenter for $N' = B_1 \cup B_2 = \{A, B, C, D, E, F, G\}$ and the barycenter of $\mathbf{P}$ and $\mathbf{Q}$ are, respectively, $\mathbf{N}' = (11.14, 11.86)$ (rounded) and $\mathbf{M}' = (10.50, 12.50)$. Using now these barycenters when necessary, the *restricted PBPCS* $[u', \mathcal{B}', (u')^*]$ is defined in $N'$ by

$$u' = (-265, -386, -356, -261, -149, -325, -481 | -369.72),$$
$$\mathcal{B}' = \{B_1, B_2\}, \quad (u')^*_1 = -357, \quad (u')^*_2 = -349 \quad \text{and} \quad (u')^*_{M'} = -439.50.$$

By applying Theorem 12.4 we find a common preference I $<$ U $<$ B$'$ $<$ C$'$ for all agents of $N'$. The allocation vector under option C$'$ is

$$(-0.25, -121.25, -91.25, 3.75, 115.75, -60.25, -216.25),$$

so $\Pi_1(\mathrm{C}') = -212.75$ and $\Pi_2(\mathrm{C}') = -157.00$. Then, $\Pi_1(\mathrm{C}') - \Pi_1(\mathrm{U}) = 144.25$ (with a common individual saving of 48.08) and $\Pi_2(\mathrm{C}') - \Pi_2(\mathrm{U}) = 192.00$ (with a common individual saving of 48.00). Hence, the possibility of discussing a *restricted PBPCS*, concerning $B_1$ and $B_2$ only, gives rise to a new issue: These unions agree in choosing the cooperative behavior for themselves (C$'$) once $B_3$ has been left aside. Thus, we modify our primitive conclusion (all unions would follow option U) and contend that unions $B_1$ and $B_2$ would follow option C$'$ whereas $B_3$ would (necessarily) follow option U. Notice, however, that

$$\Pi_1(\mathrm{C}') - \Pi_1(\mathrm{C}) = -179.08 \quad \text{and} \quad \Pi_2(\mathrm{C}') - \Pi_2(\mathrm{C}) = -238.78.$$

In words, C′ is an intermediate option for $B_1$ and $B_2$ since U < C′ < C for all their members.

(b) Complementary analysis of a PBPCS: Alternative 2

The basic discrepancy in Example 12.5 is that $B_1$ and $B_2$ prefer option C, or otherwise option B, whereas $B_3$ prefers option B, or otherwise option U. Then we discuss this question mainly as a problem among unions. Table 12.7 recalls the allocations to unions described in Table 12.5.

| $B_1$ | $B_2$ | $B_3$ | $N$ |
|---|---|---|---|
| $\Pi_1(\text{B}) = -76.33$ | $\Pi_2(\text{B}) = -68.33$ | $\Pi_3(\text{B}) = -594.33$ | $-739.00$ |
| $\Pi_1(\text{C}) = -33.67$ | $\Pi_2(\text{C}) = 81.78$ | $\Pi_3(\text{C}) = -1091.11$ | $-1043.00$ |
| $\Pi_1(\text{U}) = -357.00$ | $\Pi_2(\text{U}) = -349.00$ | $\Pi_3(\text{U}) = -875.00$ | $-1581.00$ |

**TABLE 12.7**: Allocations to unions under different behavioral options.

According to the last column of Table 12.7, the most logical attempt to achieve a full agreement on one option seems to be that $B_3$ tries to convince $B_1$ and $B_2$ to choose option B. To this end, $B_3$ should offer a compensatory side payment to the other unions. Clearly, the reference must be the allocations $\Pi_1(\text{C})$ and $\Pi_2(\text{C})$. The result of such an agreement would be given by "compromise" allocations of the form

$$(-76.33 + \varepsilon_1, -68.33 + \varepsilon_2, -594.33 - \varepsilon_1 - \varepsilon_2).$$

Which conditions should satisfy $\varepsilon_1, \varepsilon_2 > 0$? Essentially, $\varepsilon_1 + \varepsilon_2 < 280.67$, since this precisely equals $\Pi_3(\text{B}) - \Pi_3(\text{U})$ and, otherwise, $B_3$ would prefer option U instead of B. In principle, $\varepsilon_1$ and $\varepsilon_2$ might depend on the bargaining ability of each union, but we wish to mention two particular possibilities.

(1) $\varepsilon_1 = 43$ and $\varepsilon_2 = 151$ are satisfactory for $B_1$ and $B_2$ because they get allocations slightly better than under C; but not so much for $B_3$, which would be allocated $-594.33 - 194.00 = -788.33$ that is twice closer to $\Pi_3(\text{U})$ than to $\Pi_3(\text{B})$.

(2) A reasonable possibility is given by side payments $\varepsilon_1 = 21.33$ and $\varepsilon_2 = 75.05$, which take $B_1$ and $B_2$ to the middle point of their respective allocations under C and B, while $B_3$ is allocated in all $-594.33 - 96.38 = -690.71$, that is twice closer to $\Pi_3(\text{B})$ than to $\Pi_3(\text{U})$.[11]

[11] Instead, if $B_1$ and $B_2$ were successful when trying to convince $B_3$ to choose option C, we would have "compromise" allocations of the form $(-33.67-\varepsilon_1, 81.78-\varepsilon_2, -1091.11+\varepsilon_1+\varepsilon_2)$

Finally, all side payments (positive for $B_1$ and $B_2$ and negative for $B_3$) would be equally shared among the owners of each union, giving rise to individual "compromise" allocations derived from the agreement to choose B. In case of possibility (2) above these allocations are

$$(52.33, -68.67, -38.67, 44.68, 156.68, -19.32, -175.32, -283.36, -407.35).$$

(c) Coalition structure as a bargaining tool in a PBP

Let us revisit Example 12.1. Maybe some readers do not feel satisfied with the costs allocated to consumers $B$ and $C$ under cooperation in Example 12.2. They are aware that consumer $A$ does not contribute to any saving in the joint contract. Therefore, perhaps some agreement between $B$ and $C$, in order to form a legal association able to deal with the distributor as a single consumer and to sign a joint contract (with or without $A$), could improve the savings obtained by these two consumers if a special connection were designed for them. How to introduce this argument in the model?

Let us consider the structure $\mathcal{B} = \{B_1, B_2\} = \{\{A\}, \{B, C\}\}$. Then two basic possibilities exist for defining the quotient PBP $u^* = (u_1^*, u_2^* | u_M^*)$.

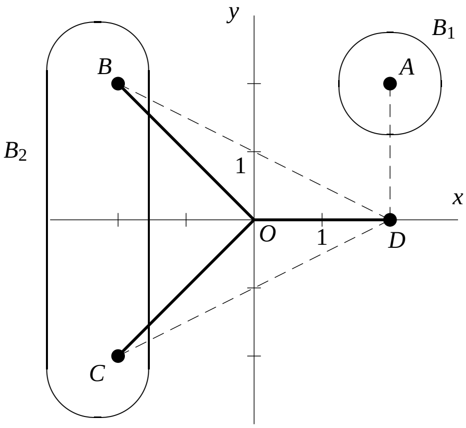


**FIGURE 12.4**: Consumers, distributor, and a new connection for $B_2$.

1. If $D$ takes into account the existence of $B_2$ and offers to its members the "Y" joint connection defined by $DO$, $OB$ and $OC$ (thick line in Fig. 12.4), then we have $u^* = (-200, -766| -966)$. Using Theorem 12.4,

with $\varepsilon_1, \varepsilon_2 > 0$. However, in this case, satisfying completely $B_3$ without being $B_1$ and $B_2$ worse than under option B requires

$$\varepsilon_1 < 42.67, \qquad \varepsilon_2 < 150.11 \qquad \text{and} \qquad \varepsilon_1 + \varepsilon_2 > 216.11,$$

a set of incompatible conditions.

it follows that I = U = B < C for consumer $A$, but I < C < U = B for consumers $B$ and $C$. Union $B_2$ (and hence, forcedly, also $B_1$) will then adopt option U, and the allocations will be $(-200, -383, -383)$, a good result for $B$ and $C$ (but not for $A$) since, according to Example 12.2, the allocations in the original PBP, that is, under option C, were $(-156.67, -404.67, -404.67)$.

2. If, instead, $D$ does not offer to $B$ and $C$ the "Y" connection but only the individual ones, then we have $u^* = (-200, -894|-966)$. Using Theorem 12.4, we find that I = U < C < B for consumer $A$ but I = U < B < C for consumers $B$ and $C$. Then all consumers will adopt option U as in case 1, but the allocations will be here $(-200, -447, -447)$, a bad result for $B$ and $C$ (and also for $A$) in comparison with Example 12.2.

In this example, the exogenous information given by the alternative quotient PBPs is a consequence of the distributor's decision as to the connections that he offers to the consumers.

(d) Endogenous coalition formation and stability in a PBP

We wish to consider here, by means of a simple example, the possibility of endogenous coalition formation in a PBP. The essential point is the existence of a procedure to construct the quotient PBP for *every* coalition structure.

Let $N = \{1, 2, 3\}$ and $u = (1, 2, 3|12)$ be a PBP in $N$. We will use in this example a rather loose notation, not so formal as in the general model. There are five possible coalition structures in $N$:

$$\mathcal{B}^n,\ \mathcal{B}_1 = \{\{1\}, \{2, 3\}\},\ \mathcal{B}_2 = \{\{2\}, \{1, 3\}\},\ \mathcal{B}_3 = \{\{3\}, \{1, 2\}\} \text{ and } \mathcal{B}^N.$$

Let us consider $\mathcal{B}_k = \{\{k\}, \{i, j\}\}$ for $k = 1, 2, 3$ and assume that the procedure to construct the quotient PBP $u^*$ in case $k$ is as follows:

$$u^*_k = u_k, \quad u^*_{ij} = k(u_i + u_j) \quad \text{and} \quad u^*_M = u_N + k^2 - 1.$$

$\pi(\text{X}, \mathcal{B})$ will denote here the payoff vector under behavioral option X and coalition structure $\mathcal{B}$. For $\mathcal{B}^n$ we find that I = U < C = B for all agents, so C = B is the preferred option and $\pi(\text{C}, \mathcal{B}^n) = (3, 4, 5)$. Analogously, for $\mathcal{B}^N$ we find I < C = U = B for all agents, so C = U = B is the preferred option and $\pi(\text{C}, \mathcal{B}^N) = (3, 4, 5)$. Now, let us consider, e.g., $\mathcal{B}_1$. The quotient PBP is $u^* = (1, 5|12)$, and $\overline{\varphi}[u^*] = (4, 8)$. The local PBPs in $\{2, 3\}$ are $u^{23} = (2, 3|5)$ for option U and $\overline{u}^{23} = (2, 3|8)$ for option B. Here I = U < C < B for agent 1 but I = U < B < C for agents 2 and 3, with $\pi(\text{C}, \mathcal{B}_1) = (3, 4, 5)$ and $\pi(\text{B}, \mathcal{B}_1) = (4.0, 3.5, 4.5)$. After proceeding in a similar way for $\mathcal{B}_2$ and $\mathcal{B}_3$, we collect the relevant results in Table 12.8.

It is clear that the only stable coalition structure (in the sense of a Nash strong equilibrium) arises when the agents organize themselves in the coalition structure $\mathcal{B}_3 = \{\{3\}, \{1, 2\}\}$, adopt the bargaining unions behavior (option

| coalition structure $\mathcal{B}$: | $\mathcal{B}^n$ | $\mathcal{B}_1$ | $\mathcal{B}_1$ | $\mathcal{B}_2$ | $\mathcal{B}_3$ | $\mathcal{B}^N$ |
|---|---|---|---|---|---|---|
| ↓ payoff / optimal option X: | C=B | C | B | B | B | C=U=B |
| $\pi_1(\mathrm{X}, \mathcal{B})$ | 3 | 3 | 4.0 | 4.25 | 6 | 3 |
| $\pi_2(\mathrm{X}, \mathcal{B})$ | 4 | 4 | 3.5 | 4.50 | 7 | 4 |
| $\pi_3(\mathrm{X}, \mathcal{B})$ | 5 | 5 | 4.5 | 6.25 | 7 | 5 |

**TABLE 12.8**: Payoffs under different CS and optimal behavioral options.

B), and share therefore $u_M^* = 20$ among them according to the payoff vector $\pi(\mathrm{B}, \mathcal{B}_3) = (6, 7, 7)$.

(e) Additional theoretical work for PBPCS

The characterization of the PBPCS where option U (resp., I or C) is the behavior strictly preferred for all agents, and the definition of the corresponding modified Shapley rules, would be interesting and probably easier than the work for option B described in Section 12.10.

(f) Restrictions to cooperation in a PBP

Finally, the introduction of different affinity degrees between agents (even incompatibilities and partnership formation), due to ideological, strategic or other reasons, would allow us to determine the effects of this new exogenous information on the issues at stake, that is, on the behavior of the agents and hence on the expected payoffs to each one of them. Standard models of restricted cooperation in game theory, which consist in modifying the Shapley value and might therefore be translated to PBPs, can be found in, e.g., [16], [17] and [3].

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## 12.13 Acknowledgments

The authors wish to thank the editors Encarnación Algaba, Vito Fragnelli and Joaquín Sánchez-Soriano for their invitation to contribute a chapter to this volume on the Shapley value. They are also grateful to an anonymous reviewer for stimulating comments and suggestions.

Research partially supported by Grant MTM2015–66818–P of the Economy and Competitiveness Spanish Ministry and the European Regional Development Fund, which sponsored Owen's visit to Terrassa in June 2017.

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# Chapter 13

## The Shapley Value as a Tool for Evaluating Groups: Axiomatization and Applications

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### 13.1 Introduction

We review in this chapter an extension of the Shapley value as a priori evaluation of the prospects of a group of players in a multi-person game. Our approach is inspired by the question originally addressed by Shapley in his seminal paper (Shapley, 1953): "*How would one evaluate the prospects of a player in a multiperson interaction, that is, in a game?*" (see Hart 1987).

Following this interpretation, we propose to use the *Generalized Shapley value*, introduced by Marichal *et al.* (2007), as a priori evaluation of the prospects of a group of players when acting as a group without imposing on the other agents any concrete coalition structure. Mathematically, the *Generalized Shapley value* is related with the Shapley value of certain quotient games (Owen, 1977) (also called merging games by Derks and Tijs (2000)) which capture the situation when all the members of a group are committed in some way to bargain with the others as a unit. It is remarkable, however, that in their approach they do not deal with the problem of assigning values to groups, but individual ones.

The evaluation of the joint importance, performance or relevance of a group of active agents is an interesting issue that deserves special attention, due to the new behavior that can appear when some subset of agents decide to act together in a coordinated way. The characteristics and attributes of each agent do not explain, in general, the emergent behavior of the group; instead, it is necessary to take into account their redundancy, complementarity and possible interactions. Because of the same reason that makes $v(i)$ different from $\phi_i(N, v)$ in the individual case, the value of a group $C$ given by the characteristic function $v(C)$ does not capture the emergent behavior. Thus, a direct use of $v(C)$ to measure the a priori value of group $C$ is not in general the best approach to solve this problem. Note that following the Von Neumann original approach, that value corresponds to the worth of the coalition in the most unfavorable setting by considering the maximum benefit that the coalition can guarantee for their members. Anyway, it evaluates a unique scenario.

In many cases this problem arises in the context of a social network that reflects the affinities between the agents involved. In this framework, as Wasserman and Faust (2004) point out, one of the primary issues is the identification of the "most important" or "prominent" actors in a social network. Definitions of *individual importance* have been proposed by many authors. For example, classical measures of *centrality –degree*, *betweenness* or *closeness*– that rely only upon the information given by the structure of the social network, were defined and widely studied (see the review of Freeman, 1979). Other individual measures, such as the *individual game theoretic centrality* index of Gomez *et al.* (2003), which also takes into account the achievements that agents can reach through their interactions in the network, have been described. However, despite the relevance of defining an appropriate *group centrality* measure which has been pointed out by different authors in a variety of contexts – mainly in the framework of information diffusion models (see Kempe *et al.*, 2005), but also in the social networks context (Everett and Borgatti (1999); Borgatti (2006); Latora and Marchiori (2007); Kolaczyk *et al.* (2009)) – this issue has not received too much attention so far. Following Gomez *et al.* (2003), in which a TU game $(N, v)$ is considered in order to incorporate to the problem information about the *functionality* of the network, i.e., the interests that motivate the interactions among the actors of the social network, we proposed in Flores *et al.* (2016) to adapt the classical approach of Myerson (1977) and we applied the *Generalized Shapley value* as a group centrality measure.

To summarize, we first consider a framework in which all the agents in a certain group $C$ join a game as an *alliance* by a common decision, or perhaps following a common external signal. This is the case addressed in Examples 13.1 and 13.3, where the agreement of investiture between the two parties *PSOE* and *Ciudadanos* in the Spanish 11th *Cortes Generales* is evaluated in two scenarios, depending on whether the political affinities at that time are considered or not. However, this does not necessarily happen; in some situations there is an external decision agent whose job is the selection of the most appropriate group of agents that will undertake a previously selected goal. This is the case of Example 13.2, in which a commercial agent wants to select the best group of firms to make a proposal for coordinating their orders. Anyway, in both situations the group valuation is intended to evaluate the prospects of a group if their members act jointly.

The second situation in which it is assumed the existence of an external agent, the decision maker, that is able to coordinate the actions of the members of the group, and whose objective is the selection of the best evaluated group accomplishing some specifications, is treated in detail in Flores *et al.* 2016 and 2018; in those papers, some relevant and realistic examples exploring the potential application of the *Generalized Shapley value* as a tool for the assessment of groups are analyzed in detail. To be specific, we considered two examples in which we illustrate the application of the Shapley value as a group valuation in the following scenarios:

- The analysis of criminal or terrorist organizations, where the police want to identify a small group of criminals or terrorists to neutralize in order to break up the criminal organization. To illustrate this case, we analyzed in Flores *et al.* (2018) the use of the *Generalized Shapley value* as an evaluation tool to two terrorist networks which have been considered in Lindelauf *et al.* (2013), where the authors introduce a game-theoretic approach to identify the key players in a terrorist network. The two cases were the operational network of Jemaah Islamiyah's Bali bombing and the network of hijackers of Al Qaeda's 9/11 attack.

- The analysis of formal and informal social networks in an organization, as well as the employee's participation in virtual communities of practice for seeking knowledge. Here, organizations are interested in using these social networks for their own interests: To promote collaborative working groups, to diffuse innovations and ideas, to force the approval of a proposal, to foster the sharing of information and knowledge to meet their business needs, etc. (see for instance Cross and Parker (2004) and Chiu *et al.* (2006)). To illustrate this case, we analyzed in Flores *et al.* (2016) the use of the *Generalized Shapley value à la* Myerson in a network within a consulting company as that analyzed in Borgatti (2006), which consists of advice-seeking ties among members of a global company.

It should be noted that in general these two problems involve the solution of a combinatorial selection problem that merits a more careful study and needs the development of heuristics in order to apply effectively the Shapley value to the group selection problem.

The present chapter is devoted to review some approaches to the evaluation of groups in a cooperative game theory setting. In Section 13.2 we introduce the main tool, the Shapley group value, providing a specific axiomatic characterization for it. We apply the Shapley group value in a real and hot context, by evaluating the prospects of the Spanish political parties in a voting game that models the 2015 Spanish elections. The question of the profitability of a group is also addressed, and we end up by describing an inventory cost game which shows how the Shapley group value can be used to determine the profitability of a group. In Section 13.3 we analyze the issue in the context of a cooperative situation limited by a graph of relations. In this case, the potential power of a group must take into account the position of its members in the network and the way in which they can interact, be redundant or contribute to intermediation. We also go back to the voting game of the first section, and in particular introduce a graph that models the political affinities between parties and allows a more precise analysis.

## 13.2 The Generalized Shapley Value: A Tool for Evaluating Groups

In this section we will recover the definition of the *generalized Shapley value* (Marichal et al., 2007), whose application as a measure to evaluate the prospects of a group of players in a TU game is analyzed afterwards. First of all, let us recall the general concept of *generalized value*.

**Definition 13.1 (Marichal et al., 2007)** *A* generalized value *is regarded as a* valuation *mapping $\xi^g$ that assigns for every game $v \in \mathcal{G}_N$ and every $C \subset N$ a real number $\xi^g(C; N, v) \in \mathbb{R}$ that reflects the power of the coalition $C$ in the game $v$, and such that $\xi^g(\emptyset; N, v) = 0$.*

Now, in order to introduce the definition of the main generalized value that we are going to discuss, the *generalized Shapley value*, also defined by Marichal et al. in 2007, we need to remember the definition of a *merging game* of Derks and Tijs (2000). These authors analyze the *profitability* of group formation in a more general setting[1] by means of considering Lehrer's (1988) type of merging. The merging game models the situation in which the agents of a given selected group are substituted by a "proxy player" that acts on their behalf. So, let us consider a game $v \in \mathcal{G}_N$ and a non-empty coalition

[1]But not, as said above, the problem of assigning values to groups.

$C \subset N \in \mathcal{N}$. We denote by $\mathcal{P}_C$ the *C-partition* given by $C$ and the single-person coalitions of players not in $C$. In these conditions, the *merging game* with respect to $\mathcal{P}_C$ is defined as the $(n - c + 1)$-person cooperative game $(N_C, v_C)$, being the agent set $N_C = (N \setminus C) \cup \{\mathbf{c}\}$, $\mathbf{c}$ a single proxy player $\mathbf{c} \equiv C$, and $v_C$ given by:

$$v_C(S) = \begin{cases} v(S), & \text{if } \mathbf{c} \notin S, \\ v(S \cup C) & \text{if } \mathbf{c} \in S, \end{cases} \quad \forall\, S \subset N_C. \tag{13.1}$$

Note that, in order to consider an accurate valuation, we must describe what group integration means for the applications we have in mind. In this framework, group integration does not necessarily imply that agents in $C$ make an agreement to act jointly. For instance, in our first previous example, there exists an external agent, the police, who selects a group of terrorists to turn back into double agents, or to misinform in order to spread their misinformation through the criminal organization network. The selected terrorists are not in general aware about the other selected terrorists' identities. The same occurs in the second example, in which the organization can select a group of employees to be used as seeds to diffuse innovations and ideas through the employees' network. Therefore, when measuring group $C$'s expectations, we will evaluate them like a *unit* anyway, adopting the *merging of players* approach of Derks and Tijs (2000).

Formally, the generalized Shapley value is defined as follows:

**Definition 13.2 (Marichal et al., 2007)** *The* Shapley group value *is the group value that assigns for every $v \in \mathcal{G}_N$, $N \in \mathcal{N}$, the valuation mapping $\phi^g(\cdot; N, v)$ given by:*

$$\phi^g(C; N, v) = \phi_{\mathbf{c}}(N_C, v_C), \text{ for each group } \emptyset \neq C \subset N,$$

*where $(N_C, v_C)$ is the merging game with respect to $C$.*

From now on, we will refer to the generalized Shapley value as "Shapley group value", which is more appropriate in this context.

We remark that for each group with at least two players the corresponding merging game is different, and therefore each $\phi^g(C; N, v)$ is obtained by applying the Shapley value to a different game. Note that only for the trivial case in which $C_1$ and $C_2$ are both singletons the two merging games, $(N_{C_1}, v_{C_1})$ and $(N_{C_2}, v_{C_2})$, are the same for different $C_1$ and $C_2$ (and in fact the same as the original game $(N, v)$).

Our goal is to describe properties of the Shapley group value which are important from the point of view of group valuation and its applications, and to use them to obtain an axiomatic characterization. Unlike the previous axiomatic proposed by Marichal et al., we have proposed an alternative set of axioms for it which does not include a direct formulation of the classical efficiency axiom (see Flores *et al.*, 2018). As a matter of fact, the efficiency axiom makes sense when the goal is to distribute a fixed amount; however, as we remarked in the introduction, we are not allocating goods, nor distributing benefits. Moreover, for each group evaluation we are using a different game. Consequently, we think that an axiomatization without a direct formulation of the efficiency axiom is relevant for the application we have in mind.

We will recall some definitions first. It is said that a game $(N, v)$ is a *unanimity game* if there is a coalition $S \subset N$ such that for every $T \subset N$, $v(T) = 1$ if $S \subset T$, and $v(T) = 0$ otherwise. The usual notation for this game is $(N, u_S)$. Unanimity games are a basis of the vector space $\mathcal{G}_N$ of all games with player set $N$. Also, given $v \in \mathcal{G}_N$, $i \in N$ is a *dummy player* if $v(S \cup i) = v(S) + v(i)$ for all $S \subset N$. In particular, a dummy player with $v(i) = 0$ is said to be *null* in $v$. A game $v \in \mathcal{G}_N$ is *monotonic* if $v(T) \geq v(S)$ for all $S \subset T \subset N$.

The properties that will appear in our axiomatic will be the following:

**Properties**

Let $\xi^g$ be a real mapping defined over the set of all games $\mathcal{G}$. Then, $\xi^g$ verifies:

1. *G-null player*, if $\xi^g(C \cup i; N, v) = \xi^g(C; N, v)$ for all $C \subsetneq N \in \mathcal{N}$, $i \in N \setminus C$ and $v \in \mathcal{G}_N$, when $i$ is a null player;
2. *G-linearity*, if $\xi^g(C; N, \alpha_1 v + \alpha_2 w) = \alpha_1 \xi^g(C; N, v) + \alpha_2 \xi^g(C; N, w)$ for all $C \subset N \in \mathcal{N}$, $\alpha_1, \alpha_2 \in \mathbb{R}$, and games $v, w \in \mathcal{G}_N$, where $\alpha_1 v + \alpha_2 w \in \mathcal{G}_N$ is given by $(\alpha_1 v + \alpha_2 w)(S) = \alpha_1 v(S) + \alpha_2 w(S)$ for all $S \subset N$;
3. *G-coalitional balanced contributions* (or $G$-CBC for short), if for all $C \subsetneq N \in \mathcal{N}$, $i, j \in N \setminus C$ and $v \in \mathcal{G}_N$, we have
$$\Big[\xi^g(C \cup i; N, v) - \xi^g(C; N, v)\Big] - \Big[\xi^g(C \cup i; N\backslash j, v_{-j}) - \xi^g(C; N\backslash j, v_{-j})\Big] = \Big[\xi^g(C \cup j; N, v) - \xi^g(C; N, v)\Big] - \Big[\xi^g(C \cup j; N\backslash i, v_{-i}) - \xi^g(C; N\backslash i, v_{-i})\Big], \quad (13.2)$$
where $v_{-i} \in \mathcal{G}_{N\backslash i}$ stands for the restriction of the characteristic function $v$ to the set of players $N\backslash i$;
4. *G-symmetry over pure bargaining games* (or $G$-SPB for short), if $\xi^g(C; N, u_N) = \frac{1}{n-c+1}$ for each non-empty $C \subset N \in \mathcal{N}$, where $(N, u_N)$ is the unanimity game with respect to the grand coalition.

Let us comment on the properties. The $G$-linearity was already used by Marichal *et al.* (2007) in their axiomatizations and its meaning is clear, as well as the necessity of a null player property. $G$-coalitional balanced contributions is a generalization of the balanced contribution property that was used by Myerson (1977) in his characterization of the Shapley value, and it is a symmetry property.

The property $G$-SPB deserves a better explanation. It leads to regard each group as one representative, independent of the number of original agents that compose it, when he/she and all the remaining players are strictly necessary. For example, in a voting game in which all the votes are needed to pass a bill, and all of them are equally powerful regardless of the number of seats they originally have. This is precisely the property which better reflects the fact that the group is replaced by a unique new player, its representative, that acts on behalf of the whole group.

The following result, whose proof can be found in Flores *et al.* (2018), establishes an alternative axiomatic for the Shapley group value that does not rely on efficiency.

**Theorem 13.1** *The unique group value over the set of all games $\mathcal{G}$ verifying $G$-null player, $G$-linearity, $G$-CBC, and $G$-SPB is the Shapley group value $\phi^g$. Moreover, the axioms are logically independent.*

Next, we show a real application of the Shapley group value for evaluating the prospects of a group in case their members sign an agreement to act together. To be specific, in the following example, we evaluate the agreement of investiture between the two parties *PSOE* and *Ciudadanos* in the Spanish 11th *Cortes Generales* without taking into consideration any political affinity between the involved parties. For the case in which those affinities are considered the reader is referred to Example 13.3 (page 272).

**Example 13.1** *Let us consider the distribution of seats in the Congress of Deputies obtained as a result of 2015 Spanish general elections held on Sunday, 20 December 2015, to elect the 11th* Cortes Generales*, that is shown in Table 13.1.*

| Party | Seats |
|---|---|
| PP | 123 |
| PSOE | 90 |
| Podemos | 69 |
| Ciudadanos | 40 |
| ERC | 9 |
| DyL | 8 |
| PNV | 6 |
| Unidad Popular (UP) | 2 |
| Bildu | 2 |
| Coalición Canaria | 1 |

**TABLE 13.1**: Congress of Deputies in the 11th Spanish *Cortes Generales.*

*Those results produced a fragmented Parliament and Mariano Rajoy, the most voted party's leader, declined the King's proposal to form government. In that situation, Pedro Sánchez, as the leader of* PSOE*, assumed the formation of government, and signed an agreement of investiture with* Ciudadanos*, one* *of the emergent parties on the right field. Did the* PSOE*'s leader make the correct decision in terms of voting power? If we use the Shapley value as a measure of voting power, we obtain the figures in Table 13.2 in the prior scenario, before any agreement was signed. Note that in this case, the voting game that models the Congress of Deputies is the weighted voting game with quota* $q = 176$ *and vector of weights* $\mathbf{w} = (123, 90, 69, 40, 9, 8, 6, 2, 2, 1)$.

| Party | Shapley value |
|---|---|
| PP | 0.4024 |
| PSOE | 0.2198 |
| Podemos | 0.2198 |
| Ciudadanos | 0.0690 |
| ERC | 0.0302 |
| DyL | 0.0254 |
| PNV | 0.0198 |
| Unidad Popular (UP) | 0.0056 |
| Bildu | 0.0056 |
| Coalición Canaria | 0.0024 |

**TABLE 13.2**: Congress of Deputies in the 11th Spanish *Cortes Generales*.

*Note that no party in the Congress of Deputies has a dummy role in the voting game that models the prior scenario, and therefore all of them enter in at least one minimal winning coalition. On the contrary, if the two parties PSOE and Ciudadanos act as a group, the situation changes dramatically and this fact is reflected in its group value. The merging game corresponding to the alliance* $\{PSOE - Ciudadanos\}$ *is the weighted majority game with the same quota* $q = 176$ *and weight vector* $\mathbf{w}_{PSOE-Ciudadanos} = (123, \mathbf{130}, 69, 9, 8, 6, 2, 2, 1)$*, being 130 the weight of the proxy player. In this case the number of minimal winning coalitions reduces to three:* $\{PP, PSOE-Ciudadanos\}$, $\{PP, Podemos\}$ *and* $\{PSOE-Ciudadanos, Podemos\}$. *Thus, this alliance turns all parties but* PP, Podemos *and the allied ones in dummies, whereas the three blocks – PP, PSOE – Ciudadanos and Podemos – turn equally powerful. The group value of the two allied parties PSOE and Ciudadanos is* $\frac{1}{3}$, *which exceeds the sum of their individual values in the original situation* $0.2888 = 0.2198 + 0.0690$.

*If we analyze the potential alliance that was proposed at that moment by Pablo Iglesias, leader of Podemos, i.e., PSOE-Podemos-Unidad Popular, we observe that its valuation as a group is greater. Now, the merging game corresponding to the alliance* $\{PSOE - Podemos - UP\}$ *is the weighted majority game with quota* $q = 176$ *and weight vector* $\mathbf{w}_{PSOE-Podemos-UP} = (123, \mathbf{161}, 40, 9, 8, 6, 2, 1)$*, and the following holds:*

$$\phi^g(\{PSOE - Podemos - UP\}; N, \mathbf{w}, 176) = 0.4607 > \frac{1}{3} = \phi^g(\{PSOE - Ciudadanos\}; N, \mathbf{w}, 176).$$

*However, as we will see in Example 13.3, the situation is different if we take into consideration political affinities.*

The remaining of this review will be devoted to study some features of the value which are relevant for its use as a tool to assess groups.

### 13.2.1 Profitability of a Group

This section is devoted to discuss the *profitability* of a group in a game. After some definitions, we first undertake a review of the literature concerning this property, and then we introduce an interesting decomposition of the marginal contribution of a player to the Shapley group value (Theorem 13.2). We conclude with an example in which the profitability of groups in an *inventory cost game* (Meca *et al.*, 2003) is analyzed according to their sizes and the characteristics of the agents' individual optimal number of orders per unit of time.

We will start by showing that superadditivity guarantees that the expected value of a group $C$ at least equals the maximum value that the members can get acting individually. This in particular implies that big groups are desirable in monotonic games.

**Proposition 13.1** *If $N \in \mathcal{N}$ is a finite set of players and $v$ is a game in $\mathcal{G}_N$, the following properties hold for the Shapley group value $\phi^g$:*

1. Group Rationality*: $\phi^g(C;N,v) \geq v(C)$ for every $C \subset N \in \mathcal{N}$ if the game $v \in \mathcal{G}_N$ is superadditive, and*
2. Monotonicity*: if the game $v \in \mathcal{G}_N$ is monotonic, $\phi^g(C;N,v) \leq \phi^g(D;N,v)$ for every pair of coalitions $C \subset D \subset N \in \mathcal{N}$.*

Following this approach, the profitability of the integration of a group $C$ can be described as the difference

$$Prof(C;N,v) := \phi^g(C;N,v) - \sum_{i \in C} \phi_i(N,v),$$

between the Shapley value of a coalition $C$ and the sum of the classical individual Shapley values of the corresponding players. A related point of view was developed by Derks and Tijs (2000) and Segal (2003). They obtained the following:

**Proposition 13.2 (Derks and Tijs, 2000)** *For any finite set of players $N \in \mathcal{N}$ and every game $v$ in $\mathcal{G}_N$, the group $C \subset N \in \mathcal{N}$ is* profitable *(or* mergeable*, in their terminology) if all coalitions whose Harsanyi dividend is positive are either contained in $C$ or have at most one player in common with $C$.*

The approach of Segal (2003) is different and is based on the difference operators, which we now recall.

Given a pair of players $i, j \in N$, the *second-order difference operator* is the following composition of marginal contribution operators (i.e., first-order difference operators):

$$\Delta^2_{ij}(S; N, v) = v(S \cup \{i, j\}) - v(S \cup j) - v(S \cup i) + v(S), \quad \forall\, S \subset N \setminus \{i, j\}.$$

Here $\Delta^2_{ij}(S; N, v)$ stands for the player $i$'s effect over the marginal contribution of player $j$ (or vice versa). Note that $v(S \cup \{i, j\}) - v(S) = \Delta^2_{ij}(S; N, v) + \Delta_i(S; N, v) + \Delta_j(S; N, v)$, and thus $\Delta^2_{ij}(S; N, v) > 0$ implies that the marginal contribution of players $i, j$ as a group exceeds the sum of the individual marginal contributions of each player.

The players $i$ and $j$ are usually called *strategic complements* whenever $\Delta^2_{ij}(S; N, v) \geq 0$, for all $S \subset N \setminus \{i, j\}$. In turn, they are said to be *strategic substitutes* whenever $\Delta^2_{ij}(S; N, v) \leq 0$, for all $S \subset N \setminus \{i, j\}$. Hence, the operator $\Delta^2_{ij}(S; N, v)$ admits an interpretation as a measure of the *interaction* of the players $i$ and $j$ with respect to the players in $S$.

In a similar way, the *third-order difference operator* for players $i, j, k \in N$ is defined as $\Delta^3_{ijk}(\cdot; N, v) = \Delta_i(\Delta^2_{jk}(\cdot; N, v))$, for all $S \subset N \setminus \{i, j, k\}$. Observe that $\Delta^3_{ijk}(S; N, v)$ codifies the effect of player $k$ over the complementarity between players $i$ and $j$ with respect to the players in $S$. The operator is again independent of the order of the differences. See also Fujimoto *et al.* (2006).

Based on these definitions, Segal (2003) showed that the merging of two players $i, j \in N$ is profitable (respectively unprofitable) whenever the presence of the outside players reduces (respectively increases) the complementarity between the players that are colluding. If a new member $j \in N \setminus C$ is incorporated to an integrated group $C$, profitability is measured with respect to the situation in which the players of group $C$ are colluding. That is, profitability can be interpreted as

$$\phi^g(C \cup j; N, v) \geq \phi^g(C; N, v) + \phi_j(N_C, v_C). \tag{13.3}$$

**Proposition 13.3 (Segal, 2003)** *Let $N \in \mathcal{N}$ be any finite set of players, and $v$ be any game in $\mathcal{G}_N$. Then:*

1. *A group $C = \{i, j\} \subset N$ of two players is* profitable *(*unprofitable*) if $\Delta^3_{ijk}(S; N, v) \leq (\geq)\ 0$, for every coalition $S \subset N \setminus \{i, j, k\}$, and for all $k \in N \setminus C$. If the reverse inequalities hold, then group $C$ is unprofitable.*

2. *The union of the integrated group $C \subset N$ and the player $j \notin C$ is profitable (unprofitable) if $\Delta^3_{ijk}(S; N, v) \leq (\geq)\ 0$, for every coalition $S \subset N \setminus \{i, j, k\}$, and for all $i \in C$, $k \in N \setminus (C \cup i)$.*

It is a direct consequence of these results that complementarity and substitutability are not related in principle to profitability. However, the Shapley group value takes account of these kinds of relations among the players when evaluating the value of a group (Theorem 13.2), making use of the *average complementarity*:

**Definition 13.3** *For any finite set $N \in \mathcal{N}$ of players and for every game $v$ in $\mathcal{G}_N$, the* average complementarity *of players $i, j \in N$ is defined as the following average of second-order differences:*

$$\psi_{ij}(N,v) := \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-s-1)!}{n!} \Delta^2_{ij}(S;N,v), \quad \textit{for all } i \neq j \in N. \tag{13.4}$$

Here $\psi_{ij}(N,v)$ is taken over all possible orders of $N = \{1,\ldots,n\}$, considered equiprobable. The operator $\Delta^2_{ij}(S;N,v)$ is taken over all orders in which coalition $S$ contains all players arriving between $i$ and $j$, and $i$ comes before $j$. In fact, it can be seen as an *interaction index* (see Grabisch and Roubens, 1999).

**Theorem 13.2** *For any finite set $N \in \mathcal{N}$ of players, for every game $v$ in $\mathcal{G}_N$, for every group $C$ in $N$, and for $i \notin C$, the* marginal contribution *of the player $i \in N \setminus C$ to the Shapley group value of $C$ equals:*

$$\begin{aligned} MC^g_i(C;N,v) &:= \phi^g(C \cup i;N,v) - \phi^g(C;N,v) = \\ &= \phi_i(N \setminus C, v|_{N \setminus C}) + \psi_{\mathbf{c}i}(N_C, v_C). \end{aligned} \tag{13.5}$$

We conclude from the previous results that the value of a group will be given by a very subtle and involved combination of complementary and independence between the agents of the groups, and that the group with the most valuable agents and the most valuable group can be very different. In Flores *et al.* (2018) we describe an easy example that shows clearly this fact.

Now, we will illustrate by means of an example how the ideas above can be used to get a deep understanding on the features that determine the profitability of a group.

**Example 13.2** *Let us consider an* inventory cost game *(introduced in Meca et al., 2003) in which there is a fixed cost per order and the agents cooperate by ordering simultaneously (as a cooperative) their orders. In this case, if $N$ is the set of firms, $a > 0$ the fixed cost per order, and $m_i$ is the individual optimal number of orders per unit of time of firm $i$ for each $i \in N$, the corresponding inventory cost game $(N,c)$ is given by:*

$$c(S) := 2a\sqrt{\sum_{i \in S} m_i^2}, \quad \textit{for all } \emptyset \neq S \subseteq N, \tag{13.6}$$

*and* $c(\emptyset) = 0$. *Since* $(N, c)$ *is a cost game, the profitability of group* $T$ *should be obtained as*

$$Prof(T; N, c) := \sum_{i \in T} \phi_i(N, c) - \phi^g(T; N, c), \qquad \text{for all } \emptyset \neq T \subseteq N. \quad (13.7)$$

*In this setting, a commercial agent could be interested in selecting the best group of k clients to manage their orders. In that sense, the commercial agent should select the group of size k more* profitable, *i.e., with the highest profit margin, in order to maximize its own benefit assuring also a benefit to each of his clients.*

*Note that applying Proposition 13.3, and taking into account that the inventory cost game 13.6 is a particular case of a* production game *(Shapley and Shubik, 1967)*

$$c(S) = g(b(S)), \text{ where } b(S) = \textstyle\sum_{i \in S} b_i, \text{ and being } b_i = m_i^2, \text{ for all } i \in N,$$

*with* $g(\cdot) = 2a\sqrt{\cdot}$, *and whose third derivative is non-negative, it follows straightforward that every possible group* $T \subseteq N$ *is profitable as a cost game (i.e., is unprofitable in terms of Proposition 13.3). Beyond the fact that every possible group of k potential clients is profitable, the question that remains is which one of them is the most profitable. It is easy to check that for a given group of size k, the group with the highest group value is that of the k firms with greatest values of* $m_i$. *However, this is not always the case for the profitability. The most profitable group of size* $k = 2$ *is always* $T_2 = \{m_{(n-1)}, m_{(n)}\}$, *where* $(m_{(1)}, \ldots, m_{(n)})$ *is the vector of optimal orders arranged in non-decreasing order, but as the group size k increases this is not always the case. It could be the case in which the most profitable group is* $T_k = \{m_{(n-k+1)}, \ldots, m_{(n)}\}$ *or* $T_k^{-max} = \{m_{(n-k)}, \ldots, m_{(n-1)}\}$ *depending on the differences among the optimal number of orders* $\{m_i\}_{i \in N}$ *and also on the number* $n = |N|$ *of firms involved in the inventory situation.*

*We will illustrate this effect by means of an example. Let us consider for instance two kinds of inventory situations, which we respectively refer to as* linear *and* quadratic, *with n firms, the same* $a > 0$ *fixed cost per order, and vectors of optimal orders* $\mathbf{m}^\ell = (m_1^\ell, \ldots, m_n^\ell) = (1, 2, \ldots, n)$ *and* $\mathbf{m}^{quad} = (1^2, 2^2, \ldots, n^2)$, *respectively.*

*We observe that for each size* $k \geq 3$, *and for each inventory situation, there exists a threshold* $\widetilde{n}_k^\ell$ *and* $\widetilde{n}_k^{quad}$ *such that the most profitable group of size k is* $T_k$ *for every inventory situation with* $n \geq \widetilde{n}_k^\ell$ *in the linear case, or* $n \geq \widetilde{n}_k^{quad}$ *in the quadratic one, and the most profitable group is* $T_k^{-max}$ *otherwise. In Table 13.3 these thresholds are obtained for k up to 10.*[2]

[2]The required Shapley group values of all the groups that were necessary to determine the figures of the previous table have been obtained by means of exact calculations for all $|N| \leq 8$. For greater cardinals, the group values have been estimated via Monte Carlo simulation following Castro, Gomez and Tejada (2009) approach.

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $\widetilde{n}_k^{\ell}$ | 7 | 10 | 13 | 16 | 19 | 22 | 25 | 28 |
| $\widetilde{n}_k^{quad}$ | 10 | 15 | 20 | 24 | 28 | 32 | 37 | 41 |

**TABLE 13.3**: Inventories.

*Note that the minimum number of agents $\widetilde{n}_k^{\ell}$ and $\widetilde{n}_k^{quad}$ that is needed in order to be $T_k$ the most profitable group is always greater for the quadratic case and increases in both cases with the size of the group. It seems that a certain critical mass of the remaining agents is needed in order to be more profitable.*

*To end the example, and in order to get an insight of the observed behaviour, we will compare the marginal contribution of the two firms $m_{(n-k)}$ and $m_{(n)}$ to the profitability of each intermediate group $\{m_{(n-k+1)}, \ldots, m_{(n-1)}\}$ decomposing those marginal contributions in the two components considered in Theorem 13.2. It follows from* (13.5) *that the marginal contribution of firm $i \in N \setminus T$ to the profitability of group $T$ is given by:*

$$\begin{aligned} MP_i(T; N, c) &:= Prof(T \cup i; N, c) - Prof(T; N, c) = \\ &= \phi_i(N, c) - \phi_i(N \setminus T, c|_{N \setminus T}) - \psi_{\mathbf{t}i}(N_T, c_T), \end{aligned} \quad (13.8)$$

*where the merging game $(N_T, c_T)$ is given by:*

$$c_T(S) = \begin{cases} 2a\sqrt{\sum_{i \in S} m_i^2} & \text{if } \mathbf{t} \notin S, \\ 2a\sqrt{\sum_{i \in S \setminus \{\mathbf{t}\}} m_i^2 + \sum_{i \in T} m_i^2} & \text{if } \mathbf{t} \in S, \end{cases} \quad \forall\, S \subset N_T. \quad (13.9)$$

*Thus, the marginal contribution of firm $i$ to the profitability of the already formed group $T$ is a combination of the price increase for firm $i$ (due to the abandonment of group $T$ from the cooperative $(\phi_i(N \setminus T, v|_{N \setminus T}) - \phi_i(N, c))$), and the extent of the substitutability between group $T$ and firm $i$ in the corresponding merging game $(-\psi_{\mathbf{t}i}(N_T, c_T))$. The price increase has a negative impact, whereas the increment of the substitutability has a positive effect. Thus, since both terms increase with the value of $m_i$, a relevant critical mass in the amount of orders of the remaining firms in $N \setminus (T \cup i)$ is needed in order to keep the cost increasing for the firm $i$ with the highest optimal order under its increment in substitutability.*

*For instance, in the linear case, with $a = 1$, in Table 13.4 we observe the profitability and substitutability, where the value of the increment in the profitability of group $T$ due to the presence of firm $i$ in the last column is obtained as the substitutability degree (in the 3rd column) minus the rise in price (in the 2nd column).*

| $n = 6$, $k = 3$, $T = \{4, 5\}$ | | | |
|---|---|---|---|
| Firm $i$ | Rise in price $\phi_i(N \setminus T, c\|_{N\setminus T}) - \phi_i(N, c)$ | Substitutability deg. $-\psi_{\mathbf{t}i}(N_T, c_T)$ | Profitability increm. $MP_i(T; N, c)$ |
| 3 | 0,8092 | 1,2989 | 0,4897 |
| 6 | 2,0600 | 2,4985 | 0,4385 |
| $n = 7$, $k = 3$, $T = \{5, 6\}$ | | | |
| Firm $i$ | Rise in price $\phi_i(N \setminus T, c\|_{N\setminus T}) - \phi_i(N, c)$ | Substitutability deg. $-\psi_{\mathbf{t}i}(N_T, c_T)$ | Profitability increm. $MP_i(T; N, c)$ |
| 4 | 0,9257 | 1,5754 | 0,6497 |
| 7 | 2,0419 | 2,7078 | 0,6659 |

**TABLE 13.4**: Profitability and substitutability.

## 13.3 Assessment of Groups in a Social Network

In this section we are still interested in evaluating groups, but adapting the classical approach of Myerson (1977), we apply the Shapley group value to a situation in which the communication and coalition formation between the players is restricted by a graph.

So far, our efforts have been focused in evaluating groups, without assuming any restrictions in the communications between the players of the game. However, there are many situations in which the relations between agents are modeled by means of a graph. This is for instance the case of many organizations where the use of formal and informal social networks is a new reality, which fosters the use of methods to evaluate the ability of each group to achieve the organization goals inside the social network. This issue, i.e., the assessment of groups in a network organization, is approached in Flores *et al.* (2016) by means of what we have called *Myerson group value*. The reader is referred to that paper and all the references therein for information about many other interesting applications.

In this context, there are several choices of the game. As we do not want to alter the communication structure by a contraction of the graph, and taking into account that the agents of the group are not necessarily connected by the graph (and even they do not act by agreement, although they can follow an external sign), we propose to select groups by means of their value in the graph-restricted game $(N, v_\Gamma)$; in this way, is introduced the *Myerson group value* as the Shapley group value of this game. Remark that the formation of coalitions is restricted by the graph, but not the formation of the groups we are interested in evaluating. We also show some interesting properties of the proposed group measures and go back to Example 13.1, but taking into account the restrictions in cooperation due to political affinities.

We start by rigorously defining the restriction of the game when the group $C$ has been formed. Previously, we introduced some notation.

A *undirected graph* or simply a *graph* $(N, \Gamma)$ consists of a finite set $N = \{1, \ldots, n\}$ of nodes and a set $\Gamma$ of *edges* whose elements are unordered pairs of distinct nodes. A graph $(N', \Gamma')$ is a *subgraph* of $(N, \Gamma)$ if $N' \subseteq N$ and $\Gamma' \subseteq \Gamma$, where the edge $\{i, j\}$ can be in $\Gamma'$ only if $i$ and $j$ are in $N'$. A *path* between two nodes $i$ and $j$ in a graph $(N, \Gamma)$ is a subgraph of $(N, \Gamma)$ consisting of a sequence of nodes and edges $P(i, j) = \{i = i_1, i_2, \ldots, i_{k-1}, i_k = j\}$, with $k \geq 2$ satisfying the property that for all $1 \leq r \leq k-1$, $\{i_r, i_{r+1}\} \in \Gamma$. A *cycle* is a path $P = \{i = i_1, i_2, \ldots, i_{k-1}, i = i_k\}$. A graph is *connected* if every pair $i, j \in N$ of its nodes is connected, i.e., if there is a path in the graph from node $i$ to node $j$; otherwise, the graph is *disconnected*. The maximal connected subgraphs of a disconnected graph are called its *connected components*, or components for short. Let $S \subseteq N$ be a subset of nodes, then $con_\Gamma(S)$ will denote the set of connected components of the subgraph $(S, \Gamma_S)$ induced by $S$. We will refer to $con_\Gamma(S)$ as the set of connected components of $S$ in $\Gamma$.

Now, we can define the restriction of the game when the group $C$ has been formed.

**Definition 13.4** *Denote by $(N, \Gamma)$ be a social network, and by $(N, v)$ a TU game. Then for every group $C \subseteq N$ the* graph-restricted game $(N_C, v_{\Gamma,C})$ *is defined as:*

$$v_{\Gamma,C}(S) = v_\Gamma(S) = \sum_{T_k \in con_\Gamma(S)} v(T_k),$$

$$v_{\Gamma,C}(S \cup \mathbf{c}) = v_\Gamma(S \cup C) = \sum_{T_k \in con_\Gamma(S \cup C)} v(T_k),$$

*for every coalition $S \subseteq N \setminus C = N_C \setminus \{\mathbf{c}\}$.*

In this context, we propose the following value:

**Definition 13.5** *With the previous notation, the* Myerson value of the group *$C \subseteq N$ is defined to be $\phi^g_C(N, v, \Gamma) := \phi^g(C; N, v_\Gamma) = \phi_{\mathbf{c}}(N_C, v_{\Gamma,C})$, for every group $C \subseteq N$.*

Following our previous interpretation of the Shapley group value, the Myerson value of $C$ can be seen as a priori valuation of the expectation of group $C$ in the game $(N, v)$ when communications between the players are restricted by $\Gamma$.

It is interesting to account for the variations in the value of a group due to the position of the players in the graph. In this context, we follow the point of view of Gomez *et al.* (2003) to approach the idea of *centrality* in Social Networks. Not only the topology of the network should determine the centrality of an agent, but also the purpose of the organization does, which is problem-specific and can be modeled by a TU game $(N, v)$.

**Definition 13.6** *In the previous notation, the* integrated centrality of the group $C \subseteq N$ *is defined to be:*

$$\gamma_C^g(N, v, \Gamma) := \phi^g(C; N, v_\Gamma) - \sum_{i \in C} \phi_i(N, v), \quad \forall\, C \subseteq N. \tag{13.10}$$

We have defined in this way several differences which represent the change in the valuation of group $C$ due to two factors: The *positional effect* that codifies the importance of the position of the agents in the graph, and the *integration effect* that depends on the worth of each group of agents for the purpose of the organization, and models the relations and synergies between them. In fact, the integrated centrality of group $C$ can also be expressed as

$$\gamma_C^g(N, v, \Gamma) = \Big(\underbrace{\phi^g(C; N, v_\Gamma) - \phi^g(C; N, v)}_{\textit{positional effect}}\Big) + \underbrace{\Big(\phi^g(C; N, v) - \sum_{i \in C} \phi_i(N, v)\Big)}_{\textit{integration effect}}.$$

Here $\phi^g(C; N, v_\Gamma) - \phi^g(C; N, v)$ is a measure of the change in the value of group $C$ due to their position in the network, and the second difference measures the benefits derived from their agreement to act both jointly and taking into account the purpose of the organization. Concrete examples where these differences are computed can be found in Flores *et al.* (2016).

Next, we recover the analysis of the agreement of investiture signed by *PSOE* and *Ciudadanos* in the 11th Spanish *Cortes Generales*, but now taking into account the natural restrictions that arise from the political affinities among the involved parties.

**Example 13.3** *Let us consider the following communication graph which captures the aforementioned affinities among the parties at the beginning of the 11th term.*

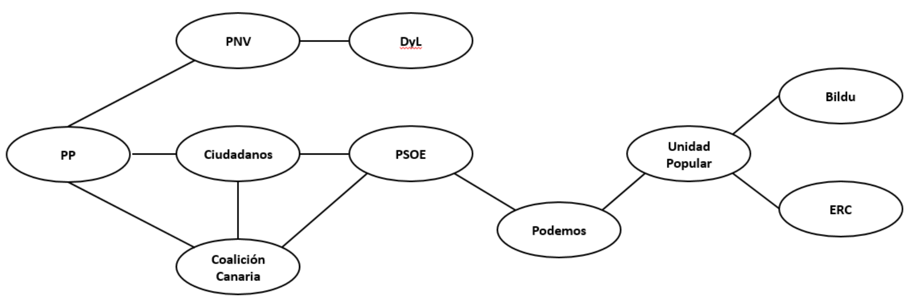


**FIGURE 13.1**: Graph of political affinities among the parties after the 2015 election.

| Party | Myerson value |
|---|---|
| PP | 0.2165 |
| PSOE | 0.2999 |
| Podemos | 0.0833 |
| Ciudadanos | 0.2167 |
| ERC | 0.0000 |
| DyL | 0.0501 |
| PNV | 0.0500 |
| Unidad Popular (UP) | 0.0000 |
| Bildu | 0.0000 |
| Coalición Canaria | 0.0833 |

**TABLE 13.5**: Congress of Deputies in the 11th Spanish *Cortes Generales.*

*Now, if we reevaluate the situation analyzed in Example 13.1, we obtain the following figures:*

*It is remarkable that, according to the new distribution of power, the most powerful party is PSOE, which was in fact responsible for trying to form the government, and the second one is Ciudadanos.*

*Now, if we consider the graph-restricted merging game that models the scenario in which PSOE and Ciudadanos act as a group* $(N_{PSOE-Ciudadanos}, (\mathbf{w}_{PSOE-Ciudadanos}, 176)_\Gamma)$ *all the remaining parties stand being dummies, since they were dummies in the corresponding merging game* $(N_{PSOE-Ciudadanos}, (\mathbf{w}_{PSOE-Ciudadanos}, 176))$ *(see Example 13.1) and they do not have any betweenness power in the graph of Figure 13.1. Thus, the original graph can be reduced to the next one (Figure 13.2), which is useful for the analysis of the graph-restricted merging game that models the agreement of investiture PSOE – Ciudadanos, taking into account the political affinities:*


**FIGURE 13.2**: Reduced graph of political affinities when *PSOE-Ciudadanos* act as a group.

*In this new situation, the value of the group PSOE-Ciudadanos is* $\frac{2}{3}$, *whereas the power of other two non-dummy parties, PP and Podemos, falls to a half when compared with the same situation when no political affinities are considered: From* $\frac{1}{3}$ *to* $\frac{1}{6}$. *In that case, the group occupies a central position in the reduced graph depicted in Figure 13.2 that allows them to arrange with the party at its right,* PP, *as well as with the party at its left,* Podemos, *at their convenience.*

*On the contrary, the position of the alternative alliance PSOE-Podemos-IU is not so central. In fact, in the corresponding graph-restricted merging game that models the scenario in which PSOE, Podemos and UP act as a group* $(N_{PSOE-Podemos-UP}, (\mathbf{w}_{PSOE-Podemos-UP}, 176)_\Gamma)$*, only ERC and Bildu become dummies when political affinities are considered. In that case, the value of the group is* 0.3833*, and*

$$\phi^g(\{PSOE - Podemos - UP\}; (N, \mathbf{w}, 176)_\Gamma) = 0.3833 < \frac{2}{3} = \phi^g(\{PSOE - Ciudadanos\}; (N, \mathbf{w}, 176)_\Gamma).$$

*The integrated centrality of each group is:*

$$\gamma^g_{PSOE-Ciudadanos}(N, (\mathbf{w}, 176), \Gamma) = \frac{2}{3} - 0.2198 - 0.0690 = 0.3779.$$

$$\gamma^g_{PSOE-Podemos-UP}(N, (\mathbf{w}, 176), \Gamma) = 0.3833 - 0.2198 - 0.2198 - 0.0056 = = -0.0619.$$

*The outcome of the above calculations is that the agreement of investiture led by PSOE was the best possible option for them, although PP and Podemos blocked Pedro Sánchez's investiture. In this way, new elections were forced, results for PP improved, and this party was then able to form government.*

In the next section, we give a general decomposition of the Myerson group value in two kinds of value: *communication value* and *betweenness value.* As a result, we obtain that the Myerson value of a group evaluates its contribution in order to enable the formation of coalitions, either through mediation or connection. We also use this decomposition to elaborate on the concept of *redundancy* between groups of agents.

### 13.3.1 Myerson Group Value Decomposition: Communication and Betweenness

In this section we recover two results that appear in Flores *et al.* (2016) and establish, in the spirit of Gomez *et al.* (2003), a general decomposition of the Myerson group value into two different values: *communication* and *betweenness.*

Let us first formalize the notion of being an *intermediary.* For a general connected graph $(N, \Gamma)$, let $\mathcal{M}_\Gamma(S) = \{S_1, \ldots, S_r\} \neq \emptyset$ be the set of *minimal connection sets* of $S$ in $\Gamma$, and $\mathcal{AM}_\Gamma(S) = \bigcup_{\ell=1}^{r} S_\ell$ be the set of agents in $\mathcal{M}_\Gamma(S)$. That is, every subgraph $(S_\ell, \Gamma_{S_\ell})$, $\ell = 1, \ldots, r$, is connected and contains the subgraph $(S, \Gamma_S)$ and, for every other connected subgraph $(T, \Gamma_T)$

containing $(S, \Gamma_S)$, there exists $\ell \in \{1, \ldots, r\}$ such that $S_\ell \subseteq T$. Note that when $(N, \Gamma)$ is a *tree* (i.e., a connected graph that contains no cycle), then for every $S \subseteq N$ there exists a unique smallest connected subgraph in $\Gamma$ which contains the subgraph $(S, \Gamma_S)$; we will call this subgraph the *connected hull* $(S, \Gamma_S)$, and denote it by $H_\Gamma(S)$.[3] The same occurs when $(N, \Gamma)$ is a *cycle-complete* connected graph; recall that the term *cycle-complete* graph was introduced by van den Nouweland and Borm (1971) to refer to those graphs in which for every cycle of distinct elements $N' = \{i_1, i_2, \ldots, i_{k-1}, i_k\}$ holds that the subgraph induced by $N'$ is a *complete graph* (i.e., all nodes are adjacent to the remaining ones).

Given a social network $(N, \Gamma)$ and a coalition $S \subseteq N$, the set of *intermediaries of* $S$ *in* $\Gamma$ is determined by

$$Bet_\Gamma(S) = \mathcal{AM}_\Gamma(S) \setminus S = \{j \notin S \,/\, \exists\, S_\ell \in \mathcal{M}_\Gamma(S) \text{ with } j \in S_\ell\}.$$

Depending on the group $C$, two kinds of coalitions $S \subseteq N$ with $C \cap \mathcal{AM}_\Gamma(S) \neq \emptyset$ can be distinguished: The coalitions $S \subseteq N$ that incorporate agents of $C$, and the ones that do not do it, but in which some members of $C$ may be needed to be connected.

Now it follows from (13.11) below that the Myerson value of group $C$ can be decomposed in the *communication value*, given by the portion of value corresponding to those payoffs received as agents of different coalitions $S$ with $S \cap C \neq \emptyset$; and *betweenness value*, which corresponds to the payoffs received as intermediaries between agents of coalitions $S$ that do not intersect $C$.

**Proposition 13.4** *Again with the previous notation, if* $(N, \Gamma)$ *is a connected graph, then* $\phi^g(C; N, v_\Gamma)$ *is given by*

$$\begin{aligned} &\phi^{Com}(C; N, v_\Gamma) + \phi^{Bet}(C; N, v_\Gamma) = \\ &= \overbrace{\sum_{\substack{S \subseteq N \\ S \cap C \neq \emptyset}} \Delta^N(v, S)\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C})}^{\text{communication value}} + \overbrace{\sum_{\substack{S \subseteq N \setminus C \\ Bet_\Gamma(S) \cap C \neq \emptyset}} \Delta^N(v, S)\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C})}^{\text{betweenness value}}. \end{aligned} \tag{13.11}$$

An explicit expression for $\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C})$ can be found in Gomez *et al.* (2003) (formula (11) on page 36). Moreover, if $(N, \Gamma)$ is a cycle-complete connected graph, (13.11) simplifies to (13.12):

[3]Formally, the subgraph which contains $(S, \Gamma_S)$ is the subgraph induced by $H_\Gamma(S)$.

**Proposition 13.5** *In the previous notation, if* $(N,\Gamma)$ *is a cycle-complete connected graph, then* $\phi^g(C;N,v_\Gamma)$ *is given by*

$$\phi^{Com}(C;N,v_\Gamma)+\phi^{Bet}(C;N,v_\Gamma)=$$

$$=\overbrace{\sum_{\substack{S\subseteq N\\ S\cap C\neq\emptyset}}\frac{\Delta^N(v,S)}{|H_\Gamma(S)\setminus C|+1}}^{\textit{communication value}}+\overbrace{\sum_{\substack{S\subseteq N\setminus C\\ H_\Gamma(S)\cap C\neq\emptyset}}\frac{\Delta^N(v,S)}{|H_\Gamma(S)\setminus C|+1}}^{\textit{betweenness value}} \tag{13.12}$$

### 13.3.2 Communication and Betweenness Redundancy

In this last section, we will discuss the concept of *redundancy* between groups of agents. We will use the crucial decomposition of Proposition 13.4 to describe, by means of an example, how the proposed group measure accounts for the two kinds of *redundancy* considered by Borgatti (2006): Redundancy with respect to adjacency and distance, and redundancy with respect to bridging. Observe that this notion of redundancy is very close to the profitability of groups defined in Section 13.2.1; however, it receives a different name in the context of networks, as the presence of a graph makes more pertinent to ask if the players are redundant than to understand if they win or lose by staying in the same group (or not). Let us state the main definition:

**Definition 13.7** *Denote by* $(N,\Gamma)$ *be a social network, and by* $(N,v)$ *a TU game. Then, the* redundancy *between two players* $i$ *and* $j$ *is defined as:*

$$Red(i,j,N,v,\Gamma):=\gamma_i^g(N,v,\Gamma)+\gamma_j^g(N,v,\Gamma)-\gamma_{i\cup j}^g(N,v,\Gamma),$$

*for every pair of groups* $i,j\subseteq N$.

In particular, we remark that negative redundancy can be interpreted as profitability between groups, and then can be seen as a positive feature. Let us explain now the aforementioned two kinds of redundancy with the aid of an example which can be found in Flores *et al.* (2016).

**Example 13.4** *Let* $(N,v)$ *be a symmetric TU game, and consider the following social network depicted in Figure 13.3.*

*Consider now the group* $C=\{5,6\}$*; the redundancy of its agents (see Definition 13.7) is given by* $Red(\{5\},\{6\},N,v,\Gamma)=\phi_5(N,v,\Gamma)+\phi_6(N,v,\Gamma)-\phi^g_{\{5,6\}}(N,v,\Gamma)$.

*Applying* (13.11) *to the computation of* $\phi_5(N,v,\Gamma)$, $\phi_6(N,v,\Gamma)$ *and* $\phi_{\{5,6\}}(N,v,\Gamma)$*, the difference between the two kinds of redundancy of 5 and 6 are apparent. The change of the communication value when agents 5 and 6 form a group implies a redundancy with respect to adjacency and distance*

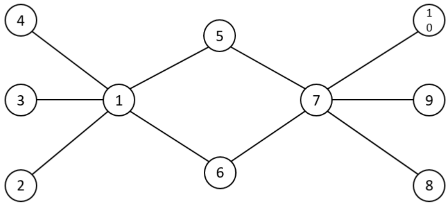


**FIGURE 13.3**: Social Network. Example 1 in Flores *et al.* (2016).

*(*communication-redundancy*), whereas the change of betweenness implies redundancy with respect to bridging (*betweenness-redundancy*), i.e.:*

$$Red(\{5\},\{6\},N,v,\Gamma) = ComRed(\{5\},\{6\},N,v,\Gamma)+BetRed(\{5\},\{6\},N,v,\Gamma).$$

*With this nomenclature, using the first summand of* (13.11) *and after some calculations, the* communication-redundancy *of agents 5 and 6 is denoted by* $ComRed(\{5\},\{6\},N,v,\Gamma)$ *and given by:*

$$\frac{1}{6}\Delta^N(v,\{5,6\}) + \sum_{\substack{S\subseteq N\\ \{5,6\}\subset S}} \underbrace{\left(\frac{|H_\Gamma(S)|-2}{|H_\Gamma(S)|(|H_\Gamma(S)|-1)}\right)}_{\geq 0} \Delta^N(v,S). \qquad (13.13)$$

*In the same way, the second summand of formula* (13.11) *gives the* betweenness-redundancy *of these players, denoted by* $BetRed(\{5\},\{6\},N,v,\Gamma)$ *and given by:*

$$\sum_{\substack{\emptyset\neq S_1\subseteq\{1,2,3,4\}\\ \emptyset\neq S_2\subseteq\{7,8,9,10\}}} \underbrace{\left(\frac{1}{k_1+k_2+1} - \frac{2}{k_1+k_2+2}\right)}_{\leq 0} \Delta^N(v,S_1\cup S_2), \qquad (13.14)$$

*where* $k_1 = |H_\Gamma(S_1\cup\{1\})|$ *and* $k_2 = |H_\Gamma(S_2\cup\{7\})|$*. Observe that the coefficients in* (13.13) *are positive and the coefficients in* (13.14) *are negative. Hence, it can be established that in this case the agents are redundant for spreading purposes, although both are necessary if the goal is to break the communications. It should be pointed out that the above coefficients deal with the structure of the network, but not with the interest in forming coalitions. The latter is measured through the Harsanyi dividends of coalition* $S$ *in* (13.13)*, while those of* $S_1\cup S_2$ *in* (13.14) *also determine the amount of positive and negative redundancy.*

*In Flores et al. (2016), it is considered a particular game that models the case in which the organization is interested in transmitting information through bilateral channels, and it is shown how agents 5 and 6 are redundant for spreading purposes, but they are complementary with respect to bridging.*

It should be remarked that our previous decomposition results only deal with *connected* social networks. Nevertheless, this is not an important handicap, since the Myerson value of a group in a disconnected graph is the sum of the Myerson value of the maximal subgroups in the connected components of the graph.

**Proposition 13.6** *Denote again by* $(N, \Gamma)$ *a social network, and let* $(N, v)$ *be a TU game. If* $(N, \Gamma)$ *is a disconnected graph, call* $(N^k, \Gamma^k)$, $k = 1, \ldots, r$, *its connected components. Then* $\phi^g_C(N, v, \Gamma) = \sum_{k=1}^r \phi^g_{C^k}(N^k, v^k, \Gamma^k)$, *where* $v^k$ *stands for the restriction of* $v$ *to* $N^k$ *and* $C^k = C \cap N^k$.

## 13.4 Conclusions

In this review we have shown the potential of the Generalized Shapley value introduced by Marichal *et al.* (2007) as a tool for evaluating the prospects of a group of agents that act in a coordinated way in a multi-person interaction situation. Some relevant examples of situations illustrating the importance of this kind of evaluation have been described. Considering that in many cases this necessity arises in the context of social networks, its generalization à la Myerson is also reviewed as a key concept for the assessment of groups when the affinities between the agents involved in the problem are considered.

In order to show the validity and the interest of this extension of the Shapley value as a tool for evaluating groups, we have focused on those properties and features of the value which are important from the point of view of group valuation and we have shown their applicability by means of two simple and interesting examples. In this sense, for the Shapley group value we have recovered an axiomatic characterization that does not include a direct formulation of the classical efficiency axiom. We have also shown how the ideas of complementarity and substitutability relate to profitability of acting as a group, and the two components that determine the marginal contribution of a new player to a group that is already formed, which combine interaction and independence. With respect to the Myerson group value, we have focused on its usefulness to distinguish between two kinds of value, for connecting and for intermediating, as well as between two kinds of redundancy of agents, with respect to adjacency and to distance.

This review is based mainly on the work of Marichal *et al.* (2007) about generalized values and the works of Flores *et al.* of 2016 and 2018, in which the application of the generalized Shapley value and the Myerson group value as a tool to evaluate groups is analyzed in detail and motivated with the aid of some relevant applications.

## 13.5 Acknowledgments

This research has been supported by I+D+i research project MTM2015-70550 from the Government of Spain.

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# Chapter 14

## *A Value for $j$-Cooperative Games: Some Theoretical Aspects and Applications*

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## 14.1 Introduction

Shapley, in his seminal article [50], introduced a value for cooperative games which is uniquely characterized by some natural axioms; since then, the value has had a great impact both theoretically and practicaly. As conspicuous examples of theoretical works we mention [25, 28, 43], and [27, 45] which contain compilations of practical results in the cooperative and voting contexts.

The restriction of the Shapley value to simple games is known as the Shapley-Shubik power index [51]. As pointed out by Felsenthal and Machover [31], the index can be interpreted as measure of power as a payoff (e.g., when dividing a cake or a unit of a divisible object among players). The index as a measure of influence is more questionable because the axiom of efficiency has no interest, a remarkable characterization of the index without using efficiency can be found in [26]. In addition, the valuable probabilistic model considered for the Shapley value in cooperative games loses its interest for simple games in which the players do not necessarily wish to vote in favor of the proposal in their turn of vote. As observed in [29] and in [9], an alternative model based on roll-calls also extends to the cooperative framework and it is the key for finding, as done in this paper, an explicit formula for a value on the class of multichoice games (here denoted as $j$-cooperative games for coherence) that respects the original model by Shapley. We do not call it "Shapley value" because as explained below there are already many different values with such denomination, which can cause confusion. The idea of this value is based on the player gain capacity and on the blocking capacity in her turn to vote.

In the context of cooperative games, players decide whether or not to cooperate and this is their only possible action. Several more general models have been considered with more than two actions for players. Just to recall some of them, Bolger [14, 15, 16, 17] considered games with $n$ players and $r$ alternatives, not necessarily ordered or comparable among them, Amer et al. [5, 6] considered games with multiple alternatives and called them $r$-games, closely related with Bolger's model. Bolger defines and axiomatically characterizes an extension of the Shapley value to games with alternatives, whereas the index due to Penrose [47], Banzhaf [7], Coleman [22] is extended by Amer et al. [5]. Nevertheless, all these values refer to the value of a player for a particular alternative.

Bicooperative games are introduced in [10]. In these games ordered pairs of disjoint coalitions of players are considered. Each such pair yields a partition of the set of all players in three groups. Players in the first coalition are in favor of the proposal, and players in the second coalition object to it. The remaining players are not convinced of its benefits, but they have no intention of objecting to it. The characteristic function can be interpreted as a positive maximal gain or as a negative minimal loss. A value of zero is assigned to the tripartition in which everybody is indifferent. Thus, the value zero plays a central position in the characteristic function of a bicooperative game and the game can be regarded as a balance between two opposite forces. A notion of the Shapley value in this context is provided in [12, 13].

Multi-choice games are considered by Hsiao and Raghavan [40, 41]. These authors consider games in which the actions of the players are ordered in the sense that, for every pair of different actions one action carries more weight than the other action. In their model, they reserve an action for those who are not active at any level. Hsiao and Raghavan also define a (matricial) notion of the Shapley value in multi-choice games that depends on actions. Some variants of their value are proposed in [39, 49].

Extensions of simple games are mainly proposed in [30], for voting games including abstention as an intermediate input level, and in [34] where $(j, k)$-simple games are considered a class of games in which voters may choose any of $j$ ordered levels of approval and $k$ stands for the number of aggregated ordered results. The last work provides a notion of weighted game endorsed by characterizations of the property of trade-robustness. Other important notions as those of the desirability relation, transitivity, acyclicity, and hierarchies, are extended in this broader context in [36, 37, 46, 48, 53].

In this paper we propose a value that has nothing to do with those cited above, with all the ingredients for both $j$-cooperative games (a trivial more convenient adaptation of multi-choice games) and $j$-simple games (i.e., $(j, 2)$-simple games as defined in [34]). As shown below, the proposed value is consistent in both frameworks and it gives a numerical evaluation for each player independently of the input alternatives for players. The probabilistic model used to create this value is that of roll-calls, which shows to be the correct one for both $j$-cooperative games and $j$-simple games. This feature is opposed to the original probabilistic model used by Shapley [50, 52] and Shapley-Shubik [51] and which has been rightly criticized by several authors as highly artificial (see, for instance, [42] or [18]) when referring to simple games.

The rest of the chapter is organized as follows. In Section 14.2 some motivating examples are presented. Section 14.3 introduces some preliminaries and the contexts of $j$-cooperative games and $j$-simple games. A value with its explicit formula is proposed for the class of $j$-cooperative games in Section 14.4. Section 14.5 provides a probabilistic model as a justification of the value. Section 14.6 proves that the value for two input alternatives coincides, as expected, with the Shapley value. An alternative formula to compute the proposed value is given in Section 14.7. Section 14.8 proposes an axiomatic characterization following the seminal ideas of Shapley's axiomatization for his value; the main contribution lies on the introduction of a fifth axiom for unanimity games. After defining the meaning of constant sum game for $j$-cooperative games, we compute their proposed value in Section 14.9. The method of generating functions for computing the value for weighted 3-simple games is shown in Section 14.10 and used to compute the value for the voting system of the United Nations Security Council and for a variant of it that avoids the veto-right of permanent nations. The examples are revisited in Section 14.11 and the value is computed for them. A brief Conclusion ends the paper in Section 14.12.

## 14.2 Some Motivating Examples

In this section, we present some examples to illustrate the versatility of the kind of games we consider. We start with a very simple example of a ternary voting game already considered by Felsenthal and Machover [31]. Then we continue with an example of economic nature, another of academic activities, the description of the United Nations Security Council voting system and also a new modified version for it that avoids the veto-right of the permanent members without harming these five nations too much. In describing these examples, we use some intuitive terminology which is concisely defined in the next section. A value that captures the idea of Shapley's value for cooperative games is proposed in Section 14.4 for a more general context. Such value will be computed in Section 14.11 for all the examples described in the rest of this section.

**Example 14.1 (A ternary voting game)** *Consider Example 8.3.7, page 288 in [31]. The set of voters is $N = \{a, b, c\}$ and the bill is passed* if *voter $a$ votes for it and at least one of the other two does not oppose it. From the* 27 *possible ways to vote for members in $N$, there are only* 8 *that pass the bill.*

**Example 14.2 (A team of workers)** *A team of three workers have to perform a task. All three can carry out their task at three different levels: Full involvement, medium involvement and lack of involvement. Only one of them, called $a$, is qualified to operate a machine that is essential to achieve a satisfactory execution of the work to be done. The other two workers, called $b$ and $c$, play a symmetrical role and also turn out to be indispensable together and a lack of involvement on the part of the two would be fatal for the execution of the task. Other combinations for these two workers with at least a medium involvement by worker $a$ lead to more or less satisfactory results depending on the degree of involvement for these two workers. Full involvement by the three suppose a win of* 4 *thousand euros. The following characteristic function specifies the gain for all combinations*

$$v(S) = \begin{cases} 4 - |S_2| - 2|S_3| & \textit{if} \quad a \in S_1 \\ \max\{0, |S_1| - |S_3|\} & \textit{if} \quad a \in S_2 \\ 0 & \textit{if} \quad a \in S_3 \end{cases}$$

*where $S = (S_1, S_2, S_3)$ and $S_1$ contains the workers with full involvement, $S_2$ contains the workers with an intermediate involvement, and $S_3$ contains the rest of the workers with the lowest level of involvement.*

*If we do not have any information about workers' attitude and the workers, how should the total gain be distributed among them? The value we propose in this paper assigns to them:* $(2, 1, 1)$ *where the payment* 2 *is for the qualified worker $a$.*

**Example 14.3 (A two-part test)** *A test has two parts, a and b, which consist of ten questions each. Each question is binary scored by:* 1 *if it is correct and* 0 *otherwise. Thus, the result for each part is the number of corrected answers which is a number from* 0 *to* 10*. The aggregated result for the test is a weighted mean of the well-answered questions. Part a is weighted as a* 60% *and part b is weighted as a* 40%*.*

*Let* $N = \{a, b\}$ *be the set of parts of the test. Let* $S = (S_1, S_2, \ldots, S_{10}, S_{11})$ *be a* 11*-partition of* $N$ *in which* $S_i$ *contains the parts of the test with a score of* $11 - i$ *for* $i = 1, 2, \ldots, 10, 11$*. If* $a \in S_h$ *and* $b \in S_i$*, then the aggregated score is given by* $V(S) = 6(11 - h) + 4(11 - i) = 110 - 6h - 4i$*, which scales the student's test mark between* 0 *and* 100*.*

*If we do not have any information about possible differences, if they would exist, between both tests, which is the importance of each test for the exam? The value we propose in this paper assigns the intuitive answer:* $(60, 40)$ *which preserves the relative importance between the two parts.*

**Example 14.4 (The UNSC voting system)** *As noted by [31], the United Nations Security Council (UNSC) can be modeled as a* 3*-simple game: A resolution is approved if there are at least nine members in favor and permanent members are not against it. This means that also if some of the permanent members abstain, without explicitly imposing the veto, a resolution can be carried on. The resulting game* $v$ *has* 15 *players, with the subset* $P$ *of the five permanent members, and a tripartition* $S = (S_1, S_2, S_3)$ *is winning (i.e.,* $v(S) = 1$*)* if and only if

$$|S_1| \geq 9 \quad \text{and} \quad S_3 \cap P = \emptyset.$$

*where* $S_1$ *contains the members in favor of the resolution,* $S_3$ *the members against it, and* $S_2$ *the abstainers. For further discussion on this significant system, see for example [23].*

**Example 14.5 (A modified voting system for the UNSC)** *The UNSC is critical to global peace and security, yet more than twenty years of negotiations over its reform have proved fruitless; see in [38] a survey on several proposed reforms that have not been implemented.*

*A simple modified version of the UNSC voting game is proposed here that does not involve changes in the world countries forming it, would consist in just modifying the possibility of approval of a resolution if one permanent member is against it but all the other members are in favor of it. This means that for any permanent member* $p \in P$*, the five losing tripartitions* $(N \setminus \{p\}, \emptyset, \{p\})$ *of the current system convert into winning tripartitions, and this is the only difference between the current and the proposed UNSC voting system. The inclusion of these five tripartitions in the set of winning tripartitions prevents the permanent members to have veto-right, but this situation only occurs when the other fourteen countries agree to vote in favor of the resolution at hand.*

The next section is devoted to formally introduce the class of games we deal with in this chapter.

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## 14.3 Preliminaries: $j$-Cooperative Games

Let $N$ be a finite set of *players*. A *$j$-partition* of $N$ is a collection of $j$ mutually disjoint subsets of $N$, $S_1, \ldots, S_j$ such that $\bigcup_{k=1}^{j} S_k = N$. Note that any $S_i$ may be empty. Any subset $S$ of $N$ is called a *coalition* and we denote its cardinality by $s$.

A $j$-partition describes a division of players among $j$ alternatives or $j$ levels of voting approval or $j$ possible actions or choices players can realize or choose. We assume that these $j$ different alternatives are ordered and convey that level 1 corresponds to the highest level of performance, while the last, level $j$, corresponds to the lowest level. Thus, players in $S_1$ are those who work at the highest level, while those in $S_j$ work at the lowest level of activity. In a voting context, voters in $S_1$ are those who vote for the highest level of approval, whereas those in $S_j$ are those who vote for the lowest level of approval. Thus, the convention chosen is *ordinal* rather than numerical.

From now on we denote with $J^N$ the set of all $j$-partitions on $N$ endowed with an (strict) order from the first (highest) order of performance or activity to the last (lowest) one. Although we assume an order of the levels of activity, we do not do any assumption over the quantification of these levels. Thus, acting at the second level just means that such level of activity is lower than in level 1 but greater than in level 3.

A partial order $\subseteq^j$ on the set $J^N$ is considered. If $S, T \in J^N$, then $S \subseteq^j T$ means $S_k \subseteq^j \bigcup_{i=1}^{k} T_i$ for any $k = 1, \ldots, j-1$. In words, $S$ is contained in $T$ if players in $T$ are working or voting in the same or in a higher level than in $S$. We use $S \subset^j T$ if $S \subseteq^j T$ and $S \neq T$. The $j$-partitions $\mathcal{N} = (\emptyset, \ldots, \emptyset, N)$ and $\mathcal{M} = (N, \emptyset, \ldots, \emptyset)$ are respectively the minimum and maximum for the order $\subseteq^j$.

A binary voting situation in which voters (we use the term voters instead of the term players in the voting context) can vote among several ordered alternatives can be formalized by a $(j, 2)$-simple game, i.e., voters can vote in $j$ different ordered ways to approve or reject a resolution and the aggregate output is binary. As previously said, we refer to $(j, 2)$-simple game as $j$-simple games throughout this article.

**Definition 14.1** *[[34]] Let $N$ be a finite set and $J^N$ be the set of all totally ordered $j$-partitions on $N$. A* $j$-simple game *is a function $v : J^N \to \{0, 1\}$ such that:* (i) *it is monotonic: If $S \subset^j T$, then $v(S) \leq v(T)$;* (ii) $v(\mathcal{N}) = 0$ *and* $v(\mathcal{M}) = 1$.

We denote with $\mathcal{SJ}_N$ the space of all $j$-simple games on the finite set $N$. Note that $(2, 2)$-simple games are simple games since for any bipartition $S = (S_1, S_2)$ the first component $S_1$ is identified with the set of 'yes'-voters and $S_2 = N \setminus S_1$ with the set of 'no'-voters. Thus, any bipartition is in one-to-one correspondence with coalition $S_1$. Note also that $(3, 2)$-simple games can be interpreted as ternary voting games, as considered by [30], if the first level of approval corresponds to voting 'yes', the second level to abstain and the third level to voting 'no'.

In any $j$-simple game, the aggregated output set is binary and represented by $\{0, 1\}$, where these two numbers have the respective meaning that the submitted proposal is either defeated or passed.

**Definition 14.2** *Let $N$ be a finite set and $J^N$ be the set of all totally ordered $j$-partitions on $N$. A* $j$-cooperative game *is a function $v : J^N \to \mathbb{R}$ such that $v(\mathcal{N}) = 0$.*

We denote by $\mathcal{J}_N$ the space of $j$-cooperative games on the finite set $N$. Note that a 2-cooperative game corresponds to a cooperative game in which the bipartition $S = (S_1, N \setminus S_1)$ is identified with the coalition $S_1$ formed by players who decide to cooperate.

The previous definition is almost equivalent to that of a multi-choice game as defined in [40, 41]. A distinction is that in the multi-choice setting an input level is distinguished from the others and reserved for lack of activity. In our context the last input level does not necessarily mean a total lack of activity and this becomes clear in the voting context, for $j$-simple games. For instance, for ternary voting games ($j = 3$ with three input choices: Voting 'yes', 'abstain' or voting 'no') the last input level means voting against the submitted proposal, which would not be coherent with the multi-choice model and the same happens for other choices of $j$. Moreover, the restriction from $j$-cooperative games to $j$-simple games becomes natural.

There are many interesting subclasses of cooperative games that can easily be extended to $j$-cooperative games for $j > 2$. Here we just refer to monotonicity.

A $j$-cooperative game is *monotonic*, if for any pair of $j$-partitions $S$ and $T$, such that $S \subset^j T$ then $v(S) \leq v(T)$.

Clearly, $\mathcal{J}_N$ is a vectorial space of dimension $j^n - 1$ and a basis formed by monotonic $j$-cooperative games is the one of *unanimity games* defined as:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq^j T \\ 0, & \text{otherwise,} \end{cases}$$

for all $j$-partition $S \neq \mathcal{N}$.

## 14.4 A Value for $j$-Cooperative Games

Let us introduce the following notation. From a given $j$-partition $S$, we define the $j$-partition $S_{a\uparrow_k}$ in which player $a$ has moved from the lowest level $j$ to the superior level $k$ $(k < j)$, and the $j$-partition $S_{a\downarrow_k}$ in which player $a$ has moved from the highest level of activity 1 to the inferior level $k$ $(k > 1)$. If $a \in S_j$ :

$$S_{a\uparrow_k} = (S_1, \ldots, S_k \cup \{a\}, \ldots, S_j \setminus \{a\})$$

for any $k = 1, \cdots, j-1$; and if $a \in S_1$:

$$S_{a\downarrow_k} = (S_1 \setminus \{a\}, \ldots, S_k \cup \{a\}, \ldots, S_j)$$

for any $k = 2, \cdots, j$.

The idea we pursue with these two definitions is to consider two special types of *marginal contributions* for $j$-partitions in a given game $v$:

$$\begin{array}{ll} m^k(v, S, a) = v(S_{a\uparrow_k}) - v(S) & \text{if} \quad a \in S_j \\ m_k(v, S, a) = v(S) - v(S_{a\downarrow_k}) & \text{if} \quad a \in S_1 \end{array}$$

In the next definition, we propose a value for $j$-cooperative games inspired with the ideas of the Shapley value, [50], for cooperative games. The explicit formula for the proposed value depends on the marginal contributions $m^k(v, S, a)$ and $m_k(v, S, a)$. Before showing its explicit formulation, we give an intuitive idea that later will be justified.

In her turn, player $a$ can achieve in choosing the input $k$ an additional gain of $m^k(v, S, a)$ with respect to the gain obtained from her predecessors with the choice of the input each made. But, with the choice of input $k$, player $a$ also prevents her predecessors from obtaining the extra gain of $m_k(v, S, a)$. Thus, in some sense, player $a$ has a double capacity: That of direct gain and that of blocking extra gain.

**Definition 14.3 (A value for $j$-cooperative games)** *For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $\mathcal{F}$-*value *is defined as*

$$\mathcal{F}_a(v) = \frac{1}{j^n n!} \left[ \sum_{\substack{S \in J^N: \\ a \in S_j}} \sum_{k=1}^{j-1} \gamma_j^n(s_j - 1) m^k(v, S, a) + \sum_{\substack{S \in J^N: \\ a \in S_1}} \sum_{k=2}^{j} \gamma_j^n(s_1 - 1) m_k(v, S, a) \right] \tag{14.1}$$

*where*

$$\gamma_j^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n-t-1+i)!}{j^i i!}, \tag{14.2}$$

*for $t = 0, 1, \ldots, n-1$.*

We show the coefficients in (14.2) in the next three tables for small values of $n$, $n \leq 6$ and for: $j = 2$ (Table 14.1), $j = 3$ (Table 14.2), and $j = 4$ (Table 14.3).

| $n \downarrow \mid t \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 1 | 3 | | | | |
| 3 | 2 | 4 | 14 | | | |
| 4 | 6 | 10 | 22 | 90 | | |
| 5 | 24 | 36 | 64 | 156 | 744 | |
| 6 | 120 | 168 | 264 | 504 | 1368 | 7560 |

**TABLE 14.1**: Numerical coefficients $\gamma_2^n(t)$ for 2-cooperative games up to 6 players.

| $n \downarrow \mid t \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 1 | 4 | | | | |
| 3 | 2 | 5 | 26 | | | |
| 4 | 6 | 12 | 36 | 240 | | |
| 5 | 24 | 42 | 96 | 348 | 2904 | |
| 6 | 120 | 192 | 372 | 984 | 4296 | 43680 |

**TABLE 14.2**: Numerical coefficients $\gamma_3^n(t)$ for 3-cooperative games up to 6 players.

| $n \downarrow \mid t \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 1 | 5 | | | | |
| 3 | 2 | 6 | 42 | | | |
| 4 | 6 | 14 | 54 | 510 | | |
| 5 | 24 | 48 | 136 | 672 | 8184 | |
| 6 | 120 | 216 | 504 | 1752 | 10872 | 163800 |

**TABLE 14.3**: Numerical coefficients $\gamma_4^n(t)$ for 2-cooperative games up to 6 players.

## 14.5 Probabilistic Justification of the $\mathcal{F}$-Value

In the following we mainly use the notation from [32] and also refer to [29, 30, 31] for precise definitions when the number of input alternatives is 3. We consider a probabilistic model in which two relevant data for each player $a \in N$ are taken: The ordering in the queue for $a$ and the input alternative chosen for $a$ in her turn. A *roll-call* specifies these two data for each player, so that the number of roll-calls is $n!j^n$. Let $\mathcal{R}_j^N$ be the set of all roll-calls and $\mathcal{R} \in \mathcal{R}_j^n$.

When we are restricted to $j$-simple games the notion of pivotal voter is crucial and extendable to $j$-cooperative games.

Voter $a$ is *pivotal* in the $j$-simple game if she is the only one who decides the (binary) outcome after her election of the input, no matter how the others following her in the queue will vote. The idea of a value that has all the ingredients of the Shapley-Shubik power index for $j$-simple games is based on the definition given in [32].

For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $f$-power index

$$\mathfrak{f}_a(v) = \frac{|\{\mathcal{R} \in \mathcal{R}_j^n \ : \ a = piv(\mathcal{R}, v)\}|}{j^n n!}.$$

This formula measures the probability of being a pivotal voter in the space of all roll-calls with the uniform distribution. It has the disadvantage that does not depend on the characteristic function $v$.

Although there is a single pivotal player in a roll-call, we can distinguish between two types of being a pivotal player in a $j$-simple game. A player $a$ is *positively* pivotal if after voting for the $k$-input the $j$-partition of those who voted before her with the rest of the players voting for the lowest level $j$ is winning. Instead, a player $a$ is *negatively* pivotal if after voting for the $k$-input the $j$-partition of those who voted before her with the rest of players voting for the first level of approval is losing, i.e., although all voters following $a$ in the queue were to vote for the first level of approval, the result of the vote would still be ‘losing’.

This idea of pivotal player and its two versions for a roll-call is easily extendible to $j$-cooperative games. Apart of doing such extension, we also wish to express the proposed value for a $j$-cooperative game in terms of the marginal contributions $m^k(v, S, a)$ and $m_k(v, S, a)$ that involve $j$-partitions rather than roll-calls. Thus, the idea is to associate a set of roll-calls with each $j$-partition with the idea described above when adapting from a positively pivotal player (for $j$-simple games) to the marginal contribution $m^k(v, S, a)$ (for $j$-cooperative games). Similarly, a set of roll-calls is associated with each $j$-partition when adapting from a negatively pivotal player (for $j$-simple games) to the marginal contribution $m_k(v, S, a)$ (for $j$-cooperative games). This is collected by the coefficient $\gamma_j^n(t)$ given in Equation (14.2).

Given a subset $T$ of $N$ with cardinality $t$ and a player $a \notin T$, the coefficient $\gamma_j^n(t)$ counts the number of roll-calls such that:

- all players in $N \setminus (T \cup \{a\})$ precede $a$ in the queue and thus have already chosen the input level;
- players in $T$ either precede or follow $a$ in the queue:
  - if they precede $a$ in the queue, they have already chosen the input level, while
  - if they follow $a$ in the queue, they have not yet chosen the input level and thus all $j$ input alternatives are counted.

Let us call this set the $T$-*free* set of roll-calls for $a$, since no matter if players in $T$ precede or not $a$ in the queue. Players preceding $a$ are the only ones who have already chosen their input alternative.

**Lemma 14.1** *The cardinality of the $T$-*free *set of roll-calls for a given player $a \notin T$ is $\gamma_j^n(t)$.*

*Proof.* Consider the $T$-*free* set of roll-calls for a given player $a \notin T$. Let $i$ be the number of players in $T$ preceding $a$, thus $i$ can be any number between 0 and $t$.

The number of players preceding $a$ in the queue are $n - t - 1 + i$ since $a \in N \setminus T$. As all orderings for these players are allowed, we have for them $(n - t - 1 + i)!$ possible orderings. Any subset of $i$ players in $T$ may precede $a$, thus $\binom{t}{i}$ is the number of elections for them.

The number of players following $a$ in the queue are then $t - i$, again as all orderings for these players are allowed we have for them $(t - i)!$ possible orderings. Moreover, these players can choose any input alternative, so that we have for them $j^{t-i}$ choices.

By applying the multiplication principle, it follows that the $T$-*free* set of roll-calls for a given player not belonging to $T$ is:

$$\sum_{i=0}^{t} (n - t - 1 + i)! \binom{t}{i} (t - i)! j^{t-i}$$

and after taking out common factors

$$\gamma_j^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n - t - 1 + i)!}{j^i i!}$$

as stated. ■

**Theorem 14.1** *The value based on marginal contributions under the uniform probability scheme for roll-calls is the $\mathcal{F}$-value.*

*Proof.* The marginal contribution $m^k(v, S, a)$ for player $a \in S_j$ is the gain that player $a$ can assure to $j$-partition $S$ when the player in her turn chooses the $k$-level of activity instead of the lowest level $j$, i.e., it is the gain capacity for $a$ in her turn. Such gain capacity after choosing the $k$-level is quantified as $v(S_{a\uparrow k}) - v(S)$ with $a \in S_j$.

The multiplication factor of $m^k(v, S, a)$ only depends on the number of players in $S_j \setminus \{a\}$, $s_j - 1$, and counts all $(S_j - \{a\})$-free roll-calls that can be formed according to Lemma 14.1, i.e., the number of roll-calls for which $a$ adds the value $m^k(v, S, a)$. As $k$ is any number in between 1 and $j - 1$ and $a \in S_j$ is the only requirement for $S$, we consider the two former addends in the first part of Equation (14.1). After dividing by the total number of roll-calls $j^n n!$ we obtain the total gain capacity for $a$.

Similarly, the marginal contribution $m_k(v, S, a)$ for player $a \in S_1$ is the lost gain that player $a$ causes to $j$-partition $S$ when the player in her turn chooses the $k$-level of activity instead of the highest level of activity 1, i.e., it is the blocking capacity for $a$ in her turn. Such blocking capacity after choosing the $k$-level is quantified as $v(S) - v(S_{a\downarrow k})$ with $a \in S_1$.

The multiplication factor of $m_k(v, S, a)$ only depends on the number of players in $S_1 \setminus \{a\}$, $s_1 - 1$, and counts all $(S_1 - \{a\})$-free roll-calls that can be formed according to Lemma 14.1, i.e., the number of roll-calls for which player $a$ causes a loss of $m_k(v, S, a)$. As $k$ is any number in between 2 and $j$ and $a \in S_1$ is the only requirement for $S$, we consider the two last addends in the second part of Equation (14.1). After dividing by the total number of roll-calls $j^n n!$ we obtain the total blocking capacity for $a$. ∎

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## 14.6 The $\mathcal{F}$-Value Restricted to Cooperative Games Is the Shapley Value

The purpose of this section is to prove that the $\mathcal{F}$-value for 2-cooperative games is the Shapley value for cooperative games. Cooperative games are 2-cooperative games in our context and therefore the value of coalition $S \subseteq N$ in a cooperative game is the value of the bipartition $(S, N \setminus S)$. Thus, we can indistinctly write $v(S)$ or $v(S, N \setminus S)$.

Thus, to prove our claim, we need to demonstrate the coincidence of the value in (14.1) with the Shapley value.

The well-known formula of the Shapley value in terms of the marginal contributions of the characteristic function is given by:

$$\phi_a(v) = \sum_{S \subseteq N \setminus \{a\}} \rho^n(s)[v(S \cup \{a\}) - v(S)], \tag{14.3}$$

where $s = |S|$ and

$$\rho^n(s) = \frac{s!(n-s-1)!}{n!}.$$

Less known is the equivalent expression for the Shapley value [9]. For any $a \in N$:

$$\phi_a(v) = \sum_{S \subseteq N \smallsetminus \{a\}} \Gamma^n(s)[v(S \cup \{a\}) - v(S)], \tag{14.4}$$

where $s = |S|$ and for any $s = 0, \ldots, n-1$:

$$\Gamma^n(s) = \frac{1}{2^n n!}\left[s! \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k + (n-s-1)! \sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-1-k)!} 2^k\right] \tag{14.5}$$

By using the coefficients: $\lambda^n(s) = s! \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k$ for $s = 0, 1, \ldots, n-1$ then equation (14.5) can be expressed as:

$$\Gamma^n(s) = \frac{1}{2^n n!}\left[\lambda^n(s) + \lambda^n(n-s-1)\right]$$

**Theorem 14.2** *The $\mathcal{F}$-value for 2-cooperative games coincides with the Shapley value.*

*Proof.* We need to prove the equivalence of formulas (14.1) and (14.3). Formula (14.1) for $j = 2$ becomes

$$\Phi_a(v) = \frac{1}{2^n n!}\left[\sum_{\substack{S \in 2^N: \\ a \in S_2}} \gamma_2^n(s_2 - 1) m^1(v, S, a) + \sum_{\substack{S \in 2^N: \\ a \in S_1}} \gamma_2^n(s_1 - 1) m_2(v, S, a)\right]$$

which is equivalent to

$$\Phi_a(v) = \frac{1}{2^n n!}\left[\sum_{S_1 \subseteq N \setminus \{a\}} (\gamma_2^n(n - s_1 - 1) + \gamma_2^n(s_1 - 1))(v(S_1 \cup \{a\}) - v(S_1))\right]$$

where in the last expression the characteristic function $v$ is applied to coalition $S_1$ instead of the 2-partition $(S_1, N \setminus S_1)$.

By rearranging properly the subscripts, we obtain the two next equalities:

$$\lambda^n(n - s_1 - 1) = \gamma_2^n(n - s_1 - 1) \quad \text{and} \quad \lambda^n(s_1) = \gamma_2^n(s_1)$$

This shows the equivalence of the $\mathcal{F}$-value with the value in (14.4). The proof of Corollary 3 in [9] shows the equality of the coefficients $\rho^n(s)$ and $\Gamma^n(s)$ for every $0 \leq s \leq n-1$ and therefore the equivalence of the $\mathcal{F}$-value for 2-cooperative games with the Shapley value for cooperative games. ■

## 14.7 Another Formulation for the $\mathcal{F}$-Value

The value $\mathcal{F}$ for $j$-cooperative games proposed in the previous section is given in terms of some marginal contributions as shown in (14.1), but it also can be expressed as a linear combination of the different values of the characteristic function on each $j$-partition.

Indeed, the next proposition is directly obtained from (14.1) by conveniently grouping the coefficients of $v(S)$ for each $j$-partition $S \neq \mathcal{N}$. Therefore, we omit its simple proof.

**Proposition 14.1** *For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $\mathcal{F}$-*value *admits the expression*

$$\mathcal{F}_a(v) = \sum_{S \in J^N} b_j^n(s_1, s_j) v(S) \tag{14.6}$$

*where*

$$b(s_1, s_j) = \begin{cases} \dfrac{\gamma_j^n(s_j) + (j-1)\gamma_j^n(s_1 - 1)}{j^n n!}, & \text{if } a \in S_1 \\[2ex] \dfrac{\gamma_j^n(s_j) - \gamma_j^n(s_1)}{j^n n!}, & \text{if } a \in S_i,\ 1 < i < j \\[2ex] -\dfrac{\gamma_j^n(s_1) + (j-1)\gamma_j^n(s_j - 1)}{j^n n!}, & \text{if } a \in S_j \end{cases} \tag{14.7}$$

*and $s_1 \geq 0$, $n > s_j \geq 0$ and $s_1 + s_j \leq n$.*

Note that $b(s_1, s_j) = 0$ for every $S$ with $s_1 = s_j$ (with $a \in S_i$ for some $1 < i < j$).

The next equation shows Formula (14.6) for $j = 3$ and player set $N = \{a, b\}$.

$$\begin{aligned} \mathcal{F}_a(v) \quad = \quad & \frac{1}{2}v(\{a,b\},\emptyset,\emptyset) + \frac{1}{6}v(\{a\},\{b\},\emptyset) + \frac{1}{3}v(\{a\},\emptyset,\{b\}) - \frac{1}{6}v(\{b\},\{a\},\emptyset) \\ & + \frac{1}{6}v(\emptyset,\{a\},\{b\}) - \frac{1}{3}v(\{b\},\emptyset,\{a\}) - \frac{1}{6}v(\emptyset,\{b\},\{a\}). \end{aligned}$$

As a simple illustration on the different types of computing the value proposed, we revisit the first example described in Section 14.2.

The voting system in Example 14.1 is a 3-simple game and it can be described by the set of minimal winning tripartitions (i.e., minimal winning tripartitions with respect to the inclusion $\subseteq^3$) trivially defined from the characteristic function $v$:

$$W^m(v) \ = \ \{(\{a\},\{b\},\{c\})\ ,\ (\{a\},\{c\},\{b\})\}$$

by monotonicity it is easy to generate the six remaining winning tripartitions.

We start with this example by showing three ways to compute the $\mathfrak{f}$-power index. We will see that these three successive methods are becoming simpler since the first involves all roll-calls, the second all tripartitions, whereas the third only winning tripartitions. Thus, the gain in each step is significant.

The first procedure described in [31] involves all roll-calls and is based on the definition of pivotal player

$$\mathfrak{f}_a(v) = \frac{|\{\mathcal{R} \in \mathcal{R}_j^n \ : \ a = piv(\mathcal{R}, v)\}|}{j^n n!}, \tag{14.8}$$

for $j = 3$. Following [31] to compute (14.8), it follows that:

1. $a$ votes first and does not vote 'yes'. This probability is $\frac{2}{9}$.
2. $a$ votes second, the first voter voted 'no' and $a$ does not vote 'yes'. This has probability $\frac{2}{27}$.
3. $a$ votes second and the first voter did not vote 'no'. The probability is $\frac{6}{27}$.
4. $a$ votes last, and the other two did not vote 'no'. This has probability $\frac{8}{27}$.

Thus, $\mathfrak{f}_a(v) = \frac{22}{27}$; and by anonymity and efficiency $\mathfrak{f}_b(v) = \mathfrak{f}_c(v) = \frac{5}{54}$.

The second procedure uses (14.1) directly for $n = j = 3$ so that the coefficients are: $\gamma_3^3(0) = 2$, $\gamma_3^3(1) = 5$, $\gamma_3^3(2) = 26$ (see the third row in Table 14.2) which need to be accounted only for the marginal contributions being equal to 1 and for tripartitions $S$ with either $a \in S_1$ or $a \in S_3$:

1. 2 of these marginal contributions for $a$ have coefficient 26,
2. 12 of these marginal contributions for $a$ have coefficient 5, and
3. 12 of these marginal contributions for $a$ have coefficient 2.

Thus, we obtain, as expected, the same result. As shown in this example, in general it becomes simpler to deal with $j$-partitions, with a total number of $j^n$ elements, than roll-calls that count $n!j^n$ elements.

The third procedure involves only winning tripartitions since we apply Equation (14.6) and its coefficients in (14.7). As the number of winning tripartitions in this example is 8, the expression in (14.6) is just the sum of the coefficients in (14.7) corresponding to winning tripartitions. All these coefficients have as a denominator the number of roll-calls: $3!3^3 = 162$. Thus, we just need to compute the numerators in (14.7) for the winning tripartitions (we ignore the superscript and subscript since for this game $n = j = 3$). These numerators, $b'(s_1, s_j) = 162 \cdot b(s_1, s_j)$, are shown in Table 14.4. The sum of these coefficients is, as expected, 132 so that $\mathfrak{f}_a(v) = \frac{22}{27}$.

It is important to note that some existing values that are called 'Shapley value' for some extensions of cooperative games do not coincide with the $\mathcal{F}$-value. For instance, we have checked for this simple example the values given

| winning tripartitions | $b'(s_1, s_j)$ |
|---|---|
| $(\{a,b,c\},\emptyset,\emptyset)$ | $b'(3,0)=54$ |
| $(\{a,b\},\{c\},\emptyset)$ | $b'(2,0)=12$ |
| $(\{a,c\},\{b\},\emptyset)$ | $b'(2,0)=12$ |
| $(\{a,b\},\emptyset,\{c\})$ | $b'(2,1)=15$ |
| $(\{a,c\},\emptyset,\{b\})$ | $b'(2,1)=15$ |
| $(\{a\},\{b,c\},\emptyset)$ | $b'(1,0)=6$ |
| $(\{a\},\{b\},\{c\})$ | $b'(1,1)=9$ |
| $(\{a\},\{c\},\{b\})$ | $b'(1,1)=9$ |

**TABLE 14.4**: Numerators of the coefficients $b(s_1, s_j)$ in (14.7) for this game.

by Hsiao and Raghavan for multi-choice games [41], by Bolger [14, 15, 17] for games with $r$-alternatives or by Bilbao et al. [12] for bicooperative games. We have obtained different results. Note that the $\mathcal{F}$-value coincides with the power index $\mathfrak{f}$ when we are restricted to the ternary case ($j = 3$ with abstention as intermediate input).

## 14.8 Axiomatization

The first idea that comes to mind is whether Shapley's classic axioms or their adaptation to $j$-cooperative games serves to characterize the considered value.

It is considerably simple to verify that these axioms are met for the considered value (see the list in next subsection) and it is also quite simple to verify that these are not enough to uniquely characterize it. In cooperative games the axioms of efficiency, anonymity and that of null player determine the Shapley value of the unanimity games, which by induction and the axiom of additivity (or transfer for simple games) uniquely extend the value to the rest of the games.

Thus, if we search for an axiomatic set including these axioms, it seems reasonable to add a conclusive property for determining the value on unanimity games.

### 14.8.1 Classical Axioms for $j$-Cooperative Games

In the following, $\psi : \mathcal{J}_N \to \mathbb{R}^n$ is a value for $j$-cooperative games.

**Anonymity** (briefly denoted by An) The value $\psi$ satisfies *anonymity* if for all game $v \in \mathcal{J}_N$, any permutation $\pi$ of $N$ and any $a \in N$

$$\psi_a(v) = \psi_{\pi(a)}(\pi v)$$

where $(\pi v)(S) = v(\pi(S))$.

**Null Player** (N) The value $\psi$ satisfies the *null player* axiom if given a null player[1] $a$ in the game $v$, then

$$\psi_a(v) = 0.$$

**Efficiency** (E) The index $\psi$ satisfies *efficiency* if for any $v \in \mathcal{J}_N$

$$\sum_{a \in N} \psi_a(v) = v(N).$$

**Additivity** (Ad) The value $\psi$ satisfies *additivity* if for any $v, w \in \mathcal{J}_N$

$$\psi(v + w) = \psi(v) + \psi(w).$$

**Transfer** (T) The index $\psi$ satisfies *transfer* if for any $v, w \in \mathcal{S}J_N$

$$\psi(v) + \psi(w) = \psi(v \wedge w) + \psi(v \vee w),$$

where $(v \wedge w)(S) = \min\{v(S), w(S)\}$ and $(v \vee w)(S) = \max\{v(S), w(S)\}$ for all $S \in J^N$.

We remark that in the characterization we provide in Theorem 14.3, the weaker condition that can replace anonymity is symmetry. Two players $a$ and $b$ are *equivalent* if for every $S$ such that $\{a, b\} \subseteq S_j$ it holds $m^k(v, S, a) = m^k(v, S, b)$ for all $k = 1, \ldots, j-1$. The value $\psi$ satisfies *symmetry* if for any $a, b \in N$ and game $v \in \mathcal{J}_N$ it holds: $\psi_a(v) = \psi_b(v)$ if $a$ and $b$ are equivalent.

A particular case, for 3-simple games, has been proven in detail in [8] and its extension to arbitrary $j$-cooperative games does not represent any difficulty so that the tedious but simple proof is omitted. The following trivial result is left for the reader.

[1] Player $a$ is *null* in the $j$-cooperative game $v \in \mathcal{J}_N$ if $m^1(v, S, a) = 0$ for all $a \in S_j$.

**Lemma 14.2** $(i)$ *The $\mathfrak{f}$-power index for $j$-simple games satisfies the axioms of: Anonymity, transfer, efficiency and null player.*

$(ii)$ *The $\mathcal{F}$-value for $j$-cooperative games satisfies the axioms of: Anonymity, additivity, efficiency and null player.*

The basic idea of the classical proof for the Shapley value for cooperative games or the Shapley-Shubik power index for simple games is that the axioms of anonymity, null player and efficiency uniquely characterize the value or index on unanimity games, and as these games form a basis or a lattice of the set of games the value or index uniquely extends to the rest of the games by additivity for cooperative games and transfer for both types of games.

If we intend to follow the same thread as in the original respective proofs by Shapley [50] and Dubey [24], we must ascertain how the $\mathcal{F}$-value works on unanimity games. The following lemma establishes the case for which anonymity, null-player and efficiency axioms are sufficient to determine a value on unanimity games.

**Lemma 14.3** *Let $u_S$ be the unanimity $j$-simple game. A value on $u_S$ is uniquely determined by the axioms of anonymity, efficiency and null player* if and only if *there is a unique $i < j$ such that $S_i \neq \emptyset$.*

*Proof.* ($\Leftarrow$) It is clear that all players in $S_j$ are nulls in $u_S$, while all players in $S_i$ are anonymous in $u_S$ and as $S_i \cup S_j = N$ by efficiency follows that all players in $S_i$ receive $1/s_i$, while the players in $S_j$ receive 0 for the value.

($\Rightarrow$) We proceed by the way of contradiction. Assume that for at least two indices $i < i' < j$ we have $S_i \neq \emptyset$ and $S_{i'} \neq \emptyset$ in the unanimity game $u_S$. Consider the value $\psi$ which assigns $1/s_i$ to all players in $S_i$ and zero to the others. Consider the value $\psi'$ which assigns $1/s_{i'}$ to all players in $S_{i'}$ and zero to the others. These two different values satisfy anonymity, efficiency and null player axioms, a contradiction with the uniqueness assumption. ■

The need of a new axiom to uniquely characterize the value on unanimity games is now clear. Indeed, according to Lemma 14.3 only if $j = 2$ (i.e., for cooperative games) the three axioms uniquely determine the value on unanimity games.

We propose a new axiom on unanimity games that together with the other four uniquely characterize the $\mathcal{F}$-value and the $\mathfrak{f}$-power index for $j$-cooperative games and $j$-simple games, respectively.

### 14.8.2 An Axiom on Unanimity Games

Assume now $j \geq 3$. Let $S$ be any $j$-partition with $a \in S_1$. When player $a$ shifts her vote to the lower input level $i$ ($i = 2, \ldots, j-1$), we have the following expression:

$$\mathcal{F}_a(u_S) - g_a(u_S) = \frac{j-1}{j-i}(\mathcal{F}_a(u_S) - g_a(u_S)) \tag{14.9}$$

where $g_a(u_s)$ is the value derived by $\mathcal{F}$ in $u_S$ when $a \in S_1$ is the last non-null player in the queue of the roll-calls. Thus, it just lacks to find the value of $g_a(u_S)$ which is the proportion of roll-calls in which $a$ is pivotal for $u_S$ and occupies the last position among the non-null players in $u_S$. Thus, $g_a(u_S)$ is the product of the following three numbers:

1. The proportion of roll-calls for which $a$ is the last non-null player in the queue. This number is
$$\frac{(s_1 + s_2 + \cdots + s_{j-1} - 1)!}{(s_1 + s_2 + \cdots + s_{j-1})!} = \frac{1}{(s_1 + s_2 + \cdots + s_{j-1})} = \frac{1}{n - s_j}.$$

2. The proportion of roll-calls in which $a$ is pivotal in the last non-null player position in the queue. To be pivotal in the last position, it is necessary that the rest of non-null players, who all precede her in the queue, have chosen the same or a better input level than in $S$. Thus, in her turn, player $a$ can decide either to make a partition $T$ winning by choosing level 1 or losing by choosing levels $2, \ldots, j$. Thus, in order for $a$ to be pivotal, any $j$-partition $T$ in which the non-null players in $u_S$ different from $a$ have already chosen the input level, with $S \subseteq^j T$ must be pivotal. In fact, $T$ is winning in $u_S$ but it could be losing if $a$ changes her mind to vote for an inferior input level. Consider
$$\frac{|W(u_S)|}{j^n} = \delta(u_S).$$
in which $\delta(v)$ is the structural decisiveness index of the game $v$ which gives the proportion of winning $j$-partitions in the game, this extension to $j$-simple games leads to the *structural decisiveness* index. The structural decisiveness index for simple games was introduced by Coleman [22] and studied in depth in Carreras [20, 21].

3. The number of input levels for which player $a$ is pivotal when she is the last non-null player in the queue is
$$j.$$

The product of these three numbers defines the unknown $g_a(u_S)$ which is

$$g_a(u_S) = j\delta(u_S)\frac{1}{n - s_j} = \frac{j\delta(u_S)}{n - s_j} \tag{14.10}$$

Thus, we can formulate the last axiom for an arbitrary value from (14.9) and the last expression. Note that from (14.10) the expression $g_a(u_S)$ can be interpreted as the decisiveness per capita with respect to non-null players of game $u_S$ multiplied by the number of available inputs for each player.

**Axiom of level change effect on unanimity games for** $j \geq 3$ (U) Let $u_S$ be a unanimity game and $a \in S_1$. Then

$$\psi_a(u_{S_{a\downarrow i}}) = \frac{1}{j-1}\left[(j-i)\psi_a(u_S) + (i-1)g_a(u_S)\right] \tag{14.11}$$

The next result gives sense what we intend to.

**Lemma 14.4** *A value $\psi$ for $j$-cooperative games that satisfies anonymity, null-player, efficiency and level change effect on unanimity games is uniquely determined on the set of unanimity games.*

*Proof.* By the Axioms of (An) and (N), the value of $\psi$ in any $u_S$, where $S$ is a $j$-partition, depends only on the numbers $s_i$ for all $i = 1, \ldots, j-1$ since $\psi_a(u_S) = 0$ if $a \in S_j$ and $\psi_a(u_S) = \psi_b(u_S)$ if $a, b \in S_i$ for some $i$. Thus, form now on the vector $\overline{s} := (s_1, s_2, \ldots, s_{j-1}, s_j)$ represents all $j$-partitions $S$ with these respective cardinalities. In particular, the vector $(n, 0, \ldots, 0)$ represents the $j$-partition $\mathcal{M}$ which assigns a value of $1/n$ to each player according to Lemma 14.3 which only assumes (An), (N) and (E). Now we consider all vectors lexicographically ordered so that $(n, 0, \ldots, 0)$ is the first in the ranking. The value $\psi$ is then uniquely determined by (An), (N), (E) and (U) on the unanimity games corresponding to the subsequent vectors in the ordering: $(n-1, 1, 0, \ldots, 0)$,...,$(n-1, 0, \ldots, 0, 1)$. From the value of $\psi$ on all these unanimity games we can obtain the value of $\psi$ for all the unanimity games whose vectors verify that $s_1 = n-2$ by applying by (An), (N), (E) and (U). If $m$ is the number of non-null components of $\overline{s}$ in between 2 and $j-1$, both included, then the Axiom (U) is applied $m$ times so that $m$ unanimity games with known $\psi$ with $n-1$ as a vectorial first component intervene. By the finiteness of the number of vectors, the process stops with the determination of $\psi$ for all the unanimity games. ∎

To clarify the preceding proof note that the value of $\psi$ on $u_S$ with $\overline{s} := (s_1, s_2, \ldots, s_{j-1}, s_j)$ $(s_1 < n)$ is determined from the values of $\psi$ on unanimity games preceding $\overline{s}$ in lexicographic ordering and with a vectorial first component of $s_1 + 1$. Assume for example that $j = 6$, $n = 8$ and $\overline{s} := (3, 0, 1, 2, 1, 0)$. By Axiom (U), which is given by the recurrence relation in (14.11), $\psi$ is determined in $u_S$ for the player in the third level from the value of $\psi$ in $u_T$ of a player in the first level of $\overline{t} := (4, 0, 0, 2, 1, 0)$. Analogously, by Axiom (U) $\psi$ is determined in $u_S$ for a player in the fourth level from the value of $\psi$ in $u_R$ of a player in the first level of $\overline{r} := (4, 0, 1, 1, 1, 0)$; and by Axiom (U) $\psi$ is determined in $u_S$ for the player in the fifth level from the value of $\psi$ in $u_X$ of a player in the first level of $\overline{x} := (4, 0, 1, 2, 0, 0)$. Finally, the value of $\psi$ for the three players in the first level of $\overline{s}$ are determined by (E) and (An). Thus, $\psi$ is determined on $u_S$.

### 14.8.3 An Axiomatization for the $\mathcal{F}$-Value

The last step for uniquely characterizing the value $\mathcal{F}$-value and the $\mathfrak{f}$-power index on $j$-cooperative and $j$-simple games respectively is the extension to all games, but this follows the same guidelines as in the seminal papers by Shapley [50] and Dubey [24], respectively. In our framework the unanimity games also form a basis of the set of $j$-cooperative games and by additivity (and transfer for $j$-simple games) the value $\psi$ uniquely extends to the rest of games. We also refer to [8] for the proof for 3-simple games and whose extension to the broader case of multiple input alternatives becomes tedious but simple. The following just states the result.

**Theorem 14.3** (*i*) *A value $\psi$ on $j$-cooperative games satisfies anonymity, null player, efficiency, level change effect on unanimity games and additivity* if and only if $\psi = \mathcal{F}$.

(*ii*) *A value $\psi$ on $j$-simple games satisfies anonymity, null player, efficiency, level change effect on unanimity games and transfer* if and only if $\psi = \mathfrak{f}$.

We conclude by pointing out that these five axioms are independent as shown in [8] for 3-simple games. The examples used there easily extend to greater values for $j$.

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## 14.9 The $\mathcal{F}$-Value on Constant-Sum $j$-Cooperative Games

Given a $j$-cooperative game $(N, v)$, we consider

$$a(k) := v(\emptyset, \dots, \emptyset, \underbrace{\{a\}}_{k}, \emptyset, \dots, \emptyset, N \setminus \{a\})$$

which is the value that player $a$ can obtain by choosing input level $k$ and without any degree of collaboration by the others. As $v$ is requested to be monotonic, it holds $a(1) \geq a(2) \geq \cdots \geq a(j-1) \geq a(j) = 0$. Thus, the maximum achievement player $a$ can obtain by herself without the collaboration of the others is $a(1)$.

A $j$-cooperative game $(N, v)$ is of *constant-sum* if

$$v(S) := \sum_{i=1}^{j} \sum_{a \in S_i} a(i)$$

for all $S \in J^N$.

The players do not take advantage of cooperation in this type of games, cooperation does not provide any surplus to them. The following result is quite intuitive and any reasonable value for $j$-cooperative games should give the same assignment.

**Theorem 14.4** *Let $(N, v)$ be a constant-sum $j$-cooperative game. Then, $\mathcal{F}_a(v) = a(1)$ for all $a \in N$.*

*Proof.* Observe that in a constant-sum game $m^k(v, S) = a(k)$, while $m_k(v, S) = a(1) - a(k)$. Then Equation (14.1) becomes

$$\mathcal{F}_a(v) = \frac{1}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_j}}\sum_{k=1}^{j-1}\gamma_j^n(s_j-1)a(k) + \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j}\gamma_j^n(s_1-1)(a(1)-a(k))\right]$$

As in the first addend there is one term with $a(1)$ and $a(j) = 0$ it follows:

$$\begin{aligned}\mathcal{F}_a(v) \quad = \quad & \frac{1}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_j}}\gamma_j^n(s_j-1)a(1) + \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j}\gamma_j^n(s_1-1)a(1)\right.\\ & \left. + \left(\sum_{\substack{S\in J^N:\\ a\in S_j}}\sum_{k=2}^{j-1}\gamma_j^n(s_j-1) - \sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=2}^{j-1}\gamma_j^n(s_1-1)\right)a(k)\right]\end{aligned} \tag{14.12}$$

As there is a bijection between the $j$-partitions in which $a \in S_1$ and those in which $a \in S_j$ we can group the terms in the first row of (14.12) and also deduce that the addends in the second row of (14.12) cancel. Thus, the previous expression is simplified to

$$\begin{aligned}\mathcal{F}_a(v) &= \frac{a(1)}{j^n n!}\left[\sum_{\substack{S\in J^N:\\ a\in S_1}}\sum_{k=1}^{j}\gamma_j^n(s_1-1)\right] = \frac{a(1)}{j^n n!}\left[\sum_{k=1}^{j}\sum_{\substack{S\in J^N:\\ a\in S_1}}\gamma_j^n(s_1-1)\right]\\ &= \frac{a(1)}{j^n n!}\left[j\left(\sum_{\substack{S\in J^N:\\ a\in S_1}}\gamma_j^n(s_1-1)\right)\right]\end{aligned} \tag{14.13}$$

As the last addend in (14.13) counts the total number of roll-calls such that $a \in S_1$ which is $j^{n-1}n!$, we have: $\mathcal{F}_a(v) = a(1)$. ■

## 14.10 Generating Functions for Computing the $\mathcal{F}$-Value for Weighted $j$-Simple Games

In this section we show the method of generating functions to compute the value proposed in this paper. Although everything we do is extendible to $j$-simple games, for any $j \geq 2$, we just consider, for avoiding more notation complications, the case $j = 3$ which includes ternary voting systems. We focus on this case because we are interested in computing the value for the UNSC voting system and a natural variation of it.

Formula (14.1) for ternary cooperative game reduces to

$$\begin{aligned}\mathcal{F}_a(v) &= \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_1}(\gamma_3^n(s_3)+\gamma_3^n(s_1-1))\,[v(S)-v(S_{a\downarrow 3})]\right]\\ &+ \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_1}\gamma_3^n(s_1-1)\,[v(S)-v(S_{a\downarrow 2})]\right]\\ &+ \tfrac{1}{3^n n!}\left[\sum_{S:a\in S_2}\gamma_3^n(s_3)\,[v(S)-v(S_{a\downarrow 3})]\right].\end{aligned} \tag{14.14}$$

As in $j$-simple games, all marginal contributions are either 1 or 0, it is convenient to use the two sets:

$$\mathcal{C}_a^{YA}(v) = \{S \in 3^N :\ a \in S_1, S \in W, S_{a\downarrow 2} \notin W\}$$
$$\mathcal{C}_a^{AN}(v) = \{S \in 3^N :\ a \in S_1, S_{a\downarrow 2} \in W, S_{a\downarrow 3} \notin W\}$$

and then compute the power index as

$$\begin{aligned}\mathcal{F}_a(v) &= \tfrac{1}{3^n n!}\left[\sum_{S\in\mathcal{C}_a^{YA}(v)}(\gamma_3^n(s_3)+2\gamma_3^n(s_1-1))\right.\\ &+ \left.\tfrac{1}{3^n n!}\sum_{S\in\mathcal{C}_a^{AN}(v)}(2\gamma_3^n(s_3)+\gamma_3^n(s_1-1))\right].\end{aligned} \tag{14.15}$$

The delay in the development of a convincing theory for simple games with ordered alternatives is possibly due to the lack of a consistent notion of weighted game in this context. This important issue was solved with the concept of weighted $j$-simple game provided in [34]. A characterization for it in terms of trade robustness was provided there, since then several alternative works deal with the notion of weighted $j$-simple game, among others [35, 36, 37].

Such definition for binary voting systems reduces to the existence of $j$ ordered weights, that respect monotonicity, for each voter and a quota such that a $j$-partition $S$ is winning if the sum of the weights of voters at the level of approval they choose is greater or equal than the quota. As observed in [34], one of these $j$ weights can be normalized at zero. In the context of ternary voting games where the options for voters are: Voting 'yes', 'abstaining' or voting 'no', it seems natural normalizing at the level of abstention and thus, every voter has a non-negative weight for voting yes and a non-positive weight for voting no. Thus, we can associate to each voter $a \in N$ the triple $(w_a^{yes}, w_a^{abs}, w_a^{no})$ with $w_a^{yes} \geq w_a^{abs} \geq w_a^{no}$, and after normalization at the intermediate level, we have: $w_a^{yes} \geq 0$, $w_a^{abs} = 0$ and $w_a^{no} \leq 0$.

Let us consider that a representation for the weighted game is:

$$v \equiv [q; (w_1^{yes}, w_1^{no}), \cdots, (w_n^{yes}, w_n^{no})]$$

where $q$ is the *quota*. Thus,

$$v(S) = 1 \text{ if and only if } w(S) := \sum_{i \in S_1} w_i^{yes} + \sum_{i \in S_3} w_i^{no} \geq q$$

Since we have the explicit formula (14.15), in case of a weighted game we can compute the power index by using generating functions. Generating functions for computing power indices have been used in many works among others [1], [2], [3], [4], [11], [19], [44]. Generating functions for 3-simple games have been used in [33] for computing the Banzhaf power index and some other power indices. We now introduce generating functions for computing the power index $\mathfrak{f}$ for the UNSC voting system with abstention.

**Definition 14.4** *Let* $v \equiv [q; (w_1^{yes}, w_1^{no}), \cdots, (w_n^{yes}, w_n^{no})]$ *be a representation of a weighted game with abstention. For any* $a \in N$*, the generating function is defined as*

$$F_a(x) = \prod_{p \in N, p \neq a} \left( y x^{w_p^{yes}} + 1 + t x^{w_p^{no}} \right) \tag{14.16}$$

Observe that the role of the variables $y$ and $t$ are the counting of the number of 'yes'-voters and 'no'-voters, respectively. Then, there is no need to count the number of abstainers since it can be deduced since the number of voters is known. Note also that the power of the variable $x$ is the weight, which in the case of an abstainer is zero, which explains the 1 in the middle position.

The function $F_a(x)$ can also be written as

$$F_a(x) = \sum_{k=\underline{w}}^{\overline{w}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} b_{k,i,j} y^i t^j x^k$$

where $\underline{w} = \sum_{i \in N} w_i^{no}$ and $\overline{w} = \sum_{i \in N} w_i^{yes}$.

In the previous formula, the coefficient $b_{k,i,j}$ counts the number of tripartitions $S$ of total weight $k$ such that there are $i$ players in $S_1$ and $h$ players in $S_3$. Using these coefficients, Equation (14.15) becomes

$$\mathcal{F}_a(v) = \frac{1}{3^n n!} \left[ \sum_{k=q-w_a^{yes}}^{q-1} b_{k,i,h} (2\gamma_j^n(i) + \gamma_j^n(h)) + \sum_{k=q}^{q-w_a^{no}-1} b_{k,i,h} (\gamma_j^n(i) + 2\gamma_j^n(h)) \right] \tag{14.17}$$

for any player $a$ such that $\mathcal{C}_a^{YA}(v) \neq \emptyset$ and $\mathcal{C}_a^{AN}(v) \neq \emptyset$. If voter $a$ is null, then the $\mathcal{F}$-value is zero. If voter $a$ is null in the $YA$-level (which implies $\mathcal{C}_a^{YA}(v) = \emptyset$) but not in the $AN$-level (which implies $\mathcal{C}_a^{AN}(v) \neq \emptyset$), then the first addend in (14.17) must be replaced by 0; and conversely, if voter $a$ is not null in the $YA$-level (which implies $\mathcal{C}_a^{YA}(v) \neq \emptyset$) but it is in the $AN$-level (which implies $\mathcal{C}_a^{AN}(v) = \emptyset$), then the second addend in (14.17) must be replaced by 0.

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## 14.11 Examples Revisited

**Example 14.6 (Example 14.2 revisited)** *As $n = j = 3$, the coefficients in* (14.2) *are: $\gamma_3^3(0) = 2$, $\gamma_3^3(1) = 5$ and $\gamma_3^3(2) = 26$; we then obtain $\mathcal{F}(v) = (2, 1, 1)$ after the substitution in* (14.1) *where the payment* 2 *is for the qualified worker $a$ and* 1 *is the payment for each of the other two.*

**Example 14.7 (Example 14.3 revisited)** *Each test plays the role of a player. As we did in the previous example, we could use* (14.1) *with its coefficients $\gamma_{11}^2(0) = 1$ and $\gamma_{11}^2(1) = 12$ to obtain $\mathcal{F}(v) = (60, 40)$. However, the result directly follows from Theorem 14.4 since $v$ is a constant-sum game. Thus, the importance of each test for the exam is given by the intuitive assignment* $(60, 40)$ *that preserves the relative importance between the two parts.*

**Example 14.8 (Example 14.4 revisited)** *Recall that for the UNSC voting system, the winning tripartitions $S$ satisfy*

$$|S_1| \geq 9 \ \text{ and } S_3 \cap P = \emptyset.$$

*We compute the value by using the method of generating functions. A weighted representation for this voting system, see [34], is given by a threshold of* 9 *a weight of* $(1, 0, -6)$ *for each permanent member and a weight of* $(1, 0, 0)$ *for a non-permanent member.*

*We now compute the power index by using its expression in Equation* (14.15). *It is then clear that for a permanent member $p$ it holds:*

$$\mathcal{C}_p^{YA}(v) = \{S : p \in S_1, |S_1| = 9, \text{ and } |S_3 \cap P| = \emptyset\}$$

*and*

$$\mathcal{C}_p^{AN}(v) = \{S : p \in S_1, |S_1| > 9 \text{ and } |S_3 \cap P| = \emptyset\}.$$

*So,*

$$\mathfrak{f}_p(v) = \sum_{s_3=0}^{6} \left\{ [2\gamma_3^{15}(8) + \gamma_3^{15}(s_3)] \sum_{j=\max\{0, s_3-2\}}^{4} \binom{4}{j}\binom{10}{8-j}\binom{j+2}{s_3} \right\} +$$
$$\sum_{s_1=10}^{15} \sum_{s_3=0}^{15-s_1} \left\{ [\gamma_3^{15}(s_1-1) + 2\gamma_3^{15}(s_3)] \sum_{j=\max\{0, s_1+s_3-11\}}^{4} \binom{4}{j}\binom{10}{s_1-1-j}\binom{11-s_1+j}{s_3} \right\}.$$

*On the other hand, for a non-permanent* $r$ *we have*

$$\mathcal{C}_r^{YA}(v) = \{S : r \in S_1, |S_1| = 9, \text{ and } |S_3 \cap P| = \emptyset\}$$

*and* $\mathcal{C}_r^{AN}(v) = \emptyset$. *Thus,*

$$\mathfrak{f}_r(v) = \sum_{s_3=0}^{6} \left\{ [2\gamma_3^{15}(8) + \gamma_3^{15}(s_3)] \sum_{j=\max\{0,s_3-1\}}^{5} \binom{5}{j}\binom{9}{8-j}\binom{j+1}{s_3} \right\}.$$

*Using these formulas we obtain*

$$\mathfrak{f}_p(v) = 0.16338987329859317, \qquad \mathfrak{f}_r(v) = 0.01830506335070341.$$

$$\mathfrak{f}_p(v) \approx 0.16339, \qquad \mathfrak{f}_r(v) \approx 0.018305.$$

*It is a close result to the one computed in [31] by using* (14.8), *although it differs a bit from it. Likely the difference lies in a rounding problem. Observe that the relative importance according to this index for the two types of voters is given by*

$$\frac{\mathfrak{f}_p(v)}{\mathfrak{f}_r(v)} \approx 8.93,$$

*which is still too big in favor of the permanent nations.*

**Example 14.9 (Example 14.5 revisited)** *Recall that the modification of the UNSC we have proposed converts the five losing tripartitions* $(N \setminus \{p\}, \emptyset, \{p\})$ *for all* $p \in P$ *into winning. The remaining tripartitions do not change its status.*

*This new* 3*-simple game can still be represented as a weighted game with quota* $q = 9$ *and vector of weights for the permanent members* $(1, 0, -5)$ *and* $(1, 0, 0)$ *for non-permanent members.*

*Using again the generating function method, the values we obtain for a permanent member* $p$ *and for a non-permanent member* $r$ *are:*

$$\mathfrak{f}_p(v) = 0.013958034451108942, \qquad \mathfrak{f}_q(v) = 0.030209827744455294.$$

$$\mathfrak{f}_p(v) \approx 0.013958, \qquad \mathfrak{f}_q(v) \approx 0.03021.$$

*Observe that the relative importance according to this index for the two types of voters is in this slightly modified example:*

$$\frac{\mathfrak{f}_p(v)}{\mathfrak{f}_r(v)} \approx 4.62.$$

*i.e., the relative importance has been reduced to almost half with respect to the standard model.*

*The United Nations Security Council is critical to global peace and security, yet more than twenty years of negotiations over its reform have proved fruitless. The change proposal we do for the UNSC voting system only alters* *five tripartitions over more than 14.3 million. As shown, this has two effects. On the one hand, it reduces the relative power to the half between the two types of voters and, on the other hand, it avoids veto power by permanent members in an acceptable way:*

*'if everyone thinks differently, it is that I must be wrong'.*

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## 14.12 Conclusions

The value proposed in this paper for $j$-cooperative games or multi-choice games has ingredients to be a generalization of the Shapley value and it can make stake out which is the most reasonable extension for the well-known value to the broader context considered. Among the arguments supporting the value proposed here, we can find the following: It is totally consistent in its particularization from $j$-cooperative games to $j$-simple games; it admits an explicit formula in terms of the characteristic function; it is supported by a probabilistic model; it is supported by an axiomatic characterization; it assigns to each player a single numerical value that does not depend on input alternatives.

The capacity of theoretical studies and applications of the value on the contexts described are enormous and future research is encouraged.

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## 14.13 Acknowledgments

This research was partially supported by funds from the Spanish Ministry of Economy and Competitiveness (MINECO) and from the European Union (FEDER funds) under grant MTM2015–66818-P(MINECO/FEDER).

I would like to thank an anonymous referee for the valuable report done, which has been very useful to improve the chapter.

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# Chapter 15

## The Shapley Value of Corporation Tax Games with Dual Benefactors

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### 15.1 Introduction

In recent years, as a result of an eminently globalized environment, the debate on the necessary cooperation among states and firms has been intensified. The absence of this cooperation among countries can cause both a race to the bottom tax competition in fiscal policies and opacity or financial secrecy. On the part of firms or individuals, it can cause underground economy, tax evasion or fiscal fraud. All of them are inefficient behaviors.

In particular, the underground economy is a significant problem and difficult to deal with. The causes and negative effects of the underground economy have been debated by authors as [2], [3], [9], and [10], among other authors.

The solutions to be adopted to detect and reduce the underground economy have been studied, for example, by [1], [5], [14], [15], and [16]. Three solutions of particular relevance are the design of optimal tax systems, the increase in transparency and information, and a greater severity of the punishments. These elements allow to increase the capability to detect and discourage the infringing behaviors. These efforts not only benefit the states themselves by allowing an increase in tax collection, but also benefit all the firms that act in accordance with the law, since it eliminates the competitors that acted in a submerged manner.

However, carrying out effective policies focused at combating the underground economy requires a high economic cost in human and material resources that must be faced by the countries governments. Cooperation among countries and firms could reduce these costs. For example, cooperation among countries could be based on the desire for transparency and the transfer of information in order to facilitate the detection of fraudulent behavior, allowing a reduction of costs. In addition, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation could be rewarded by a tax reduction.

Inspired by the Spanish tax system, [7] introduce a cooperative model, where the Government is considered the only benefactor, as it keeps costs at the same level, zero cost, while reducing the costs of those investors who act legally (beneficiaries). Investors may decide to cooperate or not cooperate with the Government. If they decide to cooperate, the Government will provide a framework of legal certainty, which is in their benefit. On the contrary, if investors decide not cooperate with the Government and try to defraud the system by tax evasion, they can be detected and charged with unlawful behavior. Once this irregular behavior is demonstrated, they will be punished and required to return all amount defrauded plus a penalty. This means that the costs of not cooperating with the Government would be higher than cooperating, and so all investors are willing to pay the lowest taxes under legal protection of the Government. The authors present the class of corporation tax games as an application of linear cost games to the corporate tax reduction system.

Linear cost games were introduced by [6] as a particular case of k-norm cost games with benefactor and beneficiaries, when k = 1. The authors introduce a class of cost-coalitional problems, which are based on a priori information about the cost faced by each agent in each set that it could belong to. Then, they focus on problems with decreasingly monotonic coalitional costs. Their paper studies the effects of giving and receiving, on cost-coalitional problems, when there exist players whose participation in an alliance always contributes to the savings of all alliance members (benefactors), and there also exist players whose cost decreases in such an alliance (beneficiaries).

[6] show that when there are multiple benefactors, an agent sees the same individual costs in any coalition that contains at least one benefactor and is not

all-inclusive. Thus, with a single benefactor all the members of a coalition may see their cost increase if he leaves the group; they say that he is irreplaceable. On the other hand, when there are several benefactors, the cost of a member of the coalition remains the same as long as there is another benefactor in the coalition; they say then that each benefactor in this case is replaceable. They study separately the two cases, and use linear and quadratic norm cost games to analyze the role played by benefactors and beneficiaries in achieving stability of different cooperating alliances. Different notions of stability, the core and the bargaining set, are considered there and provided conditions for stability of the grand coalition which leads to minimum value of total cost incurred by all agents.

In this chapter, we present a new model of corporate tax system with several firms and countries (multiple dual benefactors). Countries are dual in the sense they are benefactors (they reduce the cost of both firms and other countries) and beneficiaries (the information provided by other countries reduces its cost). They are also irreplaceable benefactors because all the members of a coalition may see their cost increase if one of them leaves the group. It differs from the corporate tax system given by [7] in the following three points. First, there is a single benefactor there. Moreover, the definition of benefactor given by [7] is a particular case of the definition of dual and irreplaceable benefactor given here. We can say that dual benefactors here generalize benefactors there. Second, the concept of beneficiary in [7] is less restrictive than the one considered here. We can say that a beneficiary here is a beneficiary in the corporate tax system given there (see Section 15.2 for more details). And third, we propose here the Shapley value [11] as a stable allocation rule for sharing the reduced total costs. [6] and [7] proved that the grand coalition is stable in the sense of the core, but they didn't study the Shapley value. Here we present a simple expression for the Shapley value of multiple corporation tax games that benefits all agents and, in particular, compensates the benefactors for their dual role and irreplaceable character. A recent survey on this allocation rule is [8].

The outline of the paper is as follow. First, in Section 15.2, the cost-coalitional problems with multiple dual and irreplaceable benefactors and some of their properties are described. After that, in Section 15.3, we introduce the class of cooperative cost games associated to cost-coalitional problems with multiple dual and irreplaceable benefactors, the so-called multiple corporation tax games. Section 15.4 presents a simple and easily computable expression for the Shapley value of multiple corporation tax games. An example illustrating the model and the role played by dual and irreplaceable benefactors is given in Section 15.5. Finally, some concluding remarks and highlights for further research are collected in Section 15.6.

## 15.2 Cost-Coalitional Problems with Multiple Dual and Irreplaceable Benefactors

Let $E = \{1, 2, .., e\}$ be a set of firms, and $P = \{1, 2, .., p\}$ be a set of countries, with $S_j^i \geq 0$ and $\bar{S}_j^i \geq 0$ be respectively a tax and a reduced tax that firm $j$ pays in country $i$, with $S_j^i > \bar{S}_j^i$. Let $N = E \cup P$ denote the set of all agents (firms and countries), with $|N| = n = e + p$, where $e \geq 1$ and $p \geq 2$. We define $T \subseteq N$ as an arbitrary set of agents in $N$. If two given countries are in a coalition $T$, then they cooperate and share information, which implies that they can reduce their levels of tax evasion and underground economy. The size of the reduction depends on how much information a country has and how relevant it is for the other country. Note that, for a country $i$, the more countries are in a coalition with it, the more relevant information this country gathers, and consequently, the smaller the degree of tax evasion and underground economy it has. Formally, let $w_i^T$ be a measure of the underground economy and tax evasion of country $i$ when it is in a coalition $T$, thus, given two sets $T \subseteq T' \subseteq N$, we assume that always $w_i^T > w_i^{T'}$ if $(T' \backslash T) \cap P \neq \emptyset$, and $w_i^T = w_i^{T'}$ otherwise. Therefore, always $w_i^T \geq w_i^{T'}$. We denote by $w_i$ the countrys' stand-alone measure of tax evasion, i.e., $w_i = w_i^{\{i\}}$.

Any agent $k \in T$ incurs certain non-negative cost, which depends on the subset $T$. We denote this cost by $c_k^T$, and by $c_k$ an agents' stand-alone cost, i.e., $c_k = c_k^{\{k\}}$. For any coalition $T \subseteq N$, the costs of agents are:

1. $c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \backslash (P \cap T)} S_j^i$ for all $j \in T \cap E$.

2. $c_i^T = g_i\left(w_i^T\right)$ for all $i \in T \cap P$.

Where firm $j \in T$ must pay a tax $\bar{S}_j^i$ to country $i$ if $i \in T$, and $S_j^i$ if $i \notin T$. In addition, $g_i$ is a strictly increasing function such that for all $i, i' \in P$ and for all $T \subseteq N$, where $i, i' \in P \cap T$, always it holds that $g_i\left(w_i^{T \backslash \{i'\}}\right) - g_i\left(w_i^T\right) = z_{ii'}$, with $z_{ii'} > 0$ being how much the country $i'$ reduces the cost of $i$ with the information $i'$ shares with $i$.[1]

Next, we identify two special roles that all the agents can play in the model, being benefactors and beneficiaries.

**Definition 15.1** *A* ***benefactor*** *is an agent* $\bar{k} \in N$ *such that for any set* $T \subset N \backslash \bar{k}$ *and for all* $k \in T$, $c_k^T \geq c_k^{T \cup \{\bar{k}\}}$, *in addition, for at least one agent* $k \in T$, $c_k^T > c_k^{T \cup \{\bar{k}\}}$. *The agents whose cost decreases in an alliance with a benefactor are denoted by* ***beneficiaries****.*

[1] We assume $z_{ii'} > 0$, thus, countries are always benefactors. However, $z_{ii'}$ could be as close to zero as we want, i.e., the information that a country shares with an other country can be negligible. Therefore, in the limit case in which $z_{ii'} = 0$, the results should hold. In any case, a wider generalization of this model will be considered in future research.

The following lemma characterizes the agents of the game as benefactors and beneficiaries.

**Lemma 15.1** *An agent $k$ is a benefactor if and only if it is a country. However, both firms and countries can be beneficiaries.*

*Proof.* Consider agent $k' \in N$ and any set $T \subset N\backslash\{k'\}$. To prove Lemma 15.1, we first consider that agent $k'$ is a country and compare the cost of agents in $T$ and in $T \cup \{\bar{k}\}$, and second we consider that agent $k'$ is a firm, and we do the same analysis. Note that agents in $T$ could be either countries or firms:

1. Consider that agent $k'$ is a country $i'$, then
   (a) For all $i \in T \cap P$, $c_i^T = g_i\left(w_i^T\right)$ and $c_i^{T\cup\{i'\}} = g_i\left(w_i^{T\cup\{i'\}}\right)$, where $w_i^T > w_i^{T\cup\{i'\}}$ because $T \subseteq T \cup \{i'\}$ and $i' \in P$. Consequently, as $g_i$ is increasing, $c_i^T > c_i^{T\cup\{i'\}}$.
   (b) For all $j \in T \cap E$,
   $c_j^T = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + S_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i$,
   and
   $c_j^{T\cup\{i'\}} = \sum_{i\in P\cap(T\cup\{i'\})} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + \bar{S}_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i$. Consequently, $c_j^T > c_j^{T\cup\{i'\}}$ because $S_j^{i'} > \bar{S}_j^{i'}$.
2. Consider that agent $k'$ is a firm $j'$, then,
   (a) For all $i \in T \cap P$, $c_i^T = g_i\left(w_i^T\right)$ and $c_i^{T\cup\{j'\}} = g_i\left(w_i^{T\cup\{j'\}}\right)$, where, $w_i^T = w_i^{T\cup\{j'\}}$ because $T \subseteq T \cup \{j'\}$ and $j' \in E$. Consequently, $c_i^T = c_i^{T\cup\{j'\}}$.
   (b) For all $j \in T \cap E$,
   $c_j^T = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i$, and
   $c_j^{T\cup\{j'\}} = \sum_{i\in P\cap(T\cup\{j'\})} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap(T\cup\{j'\}))} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i$. Therefore, $c_j^T = c_j^{T\cup\{i'\}}$.

Point 1 implies that countries are benefactors, and point 2 implies that firms are not benefactors. Point 1 and 2 imply that countries and firms can be beneficiaries and an agent $k \in N$ is a benefactor if and only if it is a country. ■

There are agents that are dual in the sense that they are benefactors and beneficiaries, these are the countries. However, the firms are exclusively beneficiaries.

The following definition is a relevant property of a benefactor.

**Definition 15.2** *A benefactor $\bar{k} \in T \subseteq N$ is irreplaceable if $c_k^T \neq c_k^{T\backslash \bar{k}}$ for at least an agent $k \in T\backslash \bar{k}$.*

The following lemma states that our benefactors are irreplaceable.

**Lemma 15.2** *Countries are irreplaceable benefactors.*

*Proof.* Note that by Lemma 15.1 only countries can be benefactors, then consider any $T \subset N$ such that $T \cap P \neq \emptyset$ where $i' \in T \cap P$. To prove Lemma 15.2, we compare the costs in set $T$ and in set $T\backslash\{i'\}$. Agents in $T\backslash\{i'\}$ can be either countries or firms. First, if the agent is a country, $i \in (T\backslash\{i'\}) \cap P$, then $c_i^T = g_i\left(w_i^T\right) < c_i^{T\backslash\{i'\}} = g_i\left(w_i^{T\backslash\{i'\}}\right)$ because $g_i$ is increasing, and $w_i^T < w_i^{T\backslash\{i'\}}$ because $T\backslash\{i'\} \subset T$.

Second, if the agent in $T\backslash\{i'\}$ is a firm, $j \in (T\backslash\{i'\}) \cap E$, then $c_j^T = \sum\limits_{i\in P\cap T} \bar{S}_j^i + \sum\limits_{i\in P\backslash(P\cap T)} S_j^i = \sum\limits_{i\in P\cap T\backslash\{i'\}} \bar{S}_j^i + \bar{S}_j^{i'} + \sum\limits_{i\in P\backslash(P\cap T)} S_j^i$, and $c_j^{T\backslash\{i'\}} = \sum\limits_{i\in P\cap(T\backslash\{i'\})} \bar{S}_j^i + \sum\limits_{i\in P\backslash P\cap(T\backslash\{i'\})} S_j^i = \sum\limits_{i\in P\cap(T\backslash\{i'\})} \bar{S}_j^i + S_j^{i'} + \sum\limits_{i\in P\backslash(P\cap T)} S_j^i$. Consequently, $c_j^T < c_j^{T\backslash\{i'\}}$ because $\bar{S}_j^{i'} < S_j^{i'}$. ■

We denote the vector of individual agents' costs in all possible subsets by $c^N = \left(c_k^T\right)_{k\in T, \emptyset\neq T\subseteq N}$. Thus, the set of agents $N$ and the cost coalitional vector $c^N$ define a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$.

A desirable property is that cooperation is beneficial. This can be guaranteed if the costs in large subsets do not exceed their cost in smaller ones. The following definition formalizes this idea.

**Definition 15.3** *A cost-coalitional vector $c^N$ satisfies **cost monotonicity** if $c_k^T \geq c_k^{T'}$ for all $k \in T$, with $T \subset T' \subseteq N$.*

The following lemma shows that the cost-coalitional problem with multiple dual benefactors has this property.

**Lemma 15.3** *The cost coalitional problem $\left(N, c^N\right)$ has the property of cost monotonicity.*

*Proof.* Consider two sets such that $S \subset T \subseteq N$. Any agent in $S$ has to be either a country or a firm.

First, if the agent is a country $i \in S \cap P$, then always $c_i^S = g_i\left(w_i^S\right)$ and $c_i^T = g_i\left(w_i^T\right)$, which implies that $c_i^S \geq c_i^T$. Note that, $g_i$ is an increasing function, and $w_i^S \geq w_i^T$ because $S \subset T$.

Second, if the agent in $S$ is a firm $j \in S \cap E$ , then

$c_j^S = \sum\limits_{i\in P\cap S} \bar{S}_j^i + \sum\limits_{i\in P\backslash(P\cap S)} S_j^i = \sum\limits_{i\in P\cap S} \bar{S}_j^i + \sum\limits_{i\in P\cap(T\backslash S)} S_j^i + \sum\limits_{i\in P\backslash(P\cap T)} S_j^i$,

and

$$c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \backslash (P \cap T)} S_j^i = \sum_{i \in P \cap S} \bar{S}_j^i + \sum_{i \in P \cap (T \backslash S)} \bar{S}_j^i + \sum_{i \in P \backslash (P \cap T)} S_j^i.$$

Note that, if in $T \backslash S$ there is at least a country, then $c_j^S > c_j^T$ because $S_j^i > \bar{S}_j^i$, otherwise $c_j^S = c_j^T$. ∎

We now define cost games related to our cost-coalitional problem with multiple dual benefactors and prove the cooperation is beneficial for all the agents in the model, benefactors and beneficiaries.

## 15.3 Multiple Corporation Tax Games

For a given cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$ we define the multiple corporation tax game $(N, c)$, where $c(T) = \sum_{k \in T} c_k^T$ for all $T \subseteq N$, and $c(\emptyset) = 0$.

We consider now the following issue. Is it profitable for the agents in N to form the grand coalition to pay lower taxes and to reduce the degree of tax evasion? Here, we prove that the answer to this question is positive because $(N, c)$ is a subadditive game, in the sense that $c\left(T \cup T'\right) \leq c\left(T\right) + c\left(T'\right)$, for any $T$, $T' \subset N$, and $T \cap T' = \emptyset$. Notice that the superadditivity condition implies that if N is partitioned into disjoint coalitions (whose integrants reduce the degree of tax evasion), the corresponding cost will not decrease.

In fact we prove that $(N, c)$ is not only subadditive but also concave, in the sense that for all $k \in N$ and all $T, T' \subset N$ such that $T \subset T' \subset N$ with $k \in T$, then $c(T) - c(T \setminus \{k\}) \geq c(T') - c(T' \setminus \{k\})$. It is a well-known result in cooperative game theory that every concave game is subadditive. Moreover, the concavity property provides us with additional information about the game: The marginal contribution of an agent diminishes as a coalition grows. It is well known as the snowball effect. For more details on cooperative game theory see, for example, [4].

First, in Lemma 15.4, we found out which are the cost marginal contributions of the agents (firms and countries).

**Lemma 15.4** *Let $\left(N, c^N\right)$ be a cost-coalitional problem with multiple dual and irreplaceable benefactors and $(N, c)$ the associated multiple corporation tax game. Then,*

1. *For all $T \subseteq N$, for all $j \in E \cap T$,*

   $c(T) - c(T \setminus \{j\}) = c_j^T$.

2. *For all $T \subseteq N$, for all $i \in P \cap T$,*

$$c(T) - c(T \setminus \{i\}) = c_i^T - \sum_{j \in E \cap T} (S_j^i - \bar{S}_j^i) - \sum_{i' \in P \cap (T \setminus \{i\})} \left( g_{i'}(w_{i'}^{T \setminus \{i\}}) - g_{i'}(w_{i'}^T) \right).$$

*Proof.* First, we prove (1). Take a coalition $T \subseteq N$, and a firm $j \in E \cap T$. Then,

$$c(T) - c(T \setminus \{j\}) = \sum_{k \in T} c_k^T - \sum_{k \in T \setminus \{j\}} c_k^{T \setminus \{j\}} = c_j^T + \sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}).$$

Now, we prove that $\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = 0$, and so $c(T) - c(T \setminus \{j\}) = c_j^T$.

Indeed,

$$\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = \sum_{i \in P \cap (T \setminus \{j\})} (c_i^T - c_i^{T \setminus \{j\}}) + \sum_{j' \in E \cap (T \setminus \{j\})} (c_{j'}^T - c_{j'}^{T \setminus \{j\}}).$$

We know that $c_i^T - c_i^{T \setminus \{j\}} = g_i(w_i^T) - g_i(w_i^{T \setminus \{j\}}) = 0$, since $w_i^{T \setminus \{j\}} = w_i^T$.

Moreover, $c_{j'}^T - c_{j'}^{T \setminus \{j\}} = \sum_{i \in P \cap T} \bar{S}_{j'}^i + \sum_{i \in P \setminus (P \cap T)} S_{j'}^i - \sum_{i \in P \cap (T \setminus \{j\})} \bar{S}_{j'}^i - \sum_{i \in P \setminus (P \cap T \setminus \{j\})} S_{j'}^i = 0$.

Then,

$$\sum_{i \in P \cap (T \setminus \{j\})} (c_i^T - c_i^{T \setminus \{j\}}) = 0, \text{ and } \sum_{j' \in E \cap (T \setminus \{j\})} (c_{j'}^T - c_{j'}^{T \setminus \{j\}}) = 0.$$

Hence, we conclude that $\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = 0$.

Second, we prove (2). Take a coalition $T \subseteq N$, and a country $i \in P \cap T$. Then,

$$c(T) - c(T \setminus \{i\}) = \sum_{k \in T} c_k^T - \sum_{k \in T \setminus \{i\}} c_k^{T \setminus \{i\}} = c_i^T - \sum_{k \in T \setminus \{i\}} (c_k^{T \setminus \{j\}} - c_k^T).$$

We know that,

$$\sum_{k \in T \setminus \{i\}} (c_k^{T \setminus \{i\}} - c_k^T) = \sum_{i' \in P \cap (T \setminus \{i\})} (c_{i'}^{T \setminus \{i\}} - c_{i'}^T) + \sum_{j \in E \cap (T \setminus \{i\})} (c_j^{T \setminus \{i\}} - c_j^T).$$

We prove now that

$$c_j^{T \setminus \{i\}} - c_j^T = \left( S_j^i + \sum_{i' \in P \cap (T \setminus \{i\})} \bar{S}_j^{i'} + \sum_{i' \in P \setminus P \cap (T \setminus \{i\})} S_j^{i'} \right) - \left( \bar{S}_j^i + \sum_{i' \in P \cap (T \setminus \{i\})} \bar{S}_j^{i'} + \sum_{i' \in P \setminus P \cap (T \setminus \{i\})} S_j^{i'} \right) = S_j^i - \bar{S}_j^i.$$

We know, by definition, that

$$c_{i'}^{T\backslash\{i\}} - c_{i'}^{T} = g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^{T})$$

Hence, we can conclude that

$$c(T) - c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap(T\backslash\{i\})}\left(g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^{T})\right).$$

■

In point 1, this proposition states that a firm $j$ always contributes to a coalition $T \setminus \{j\}$ exactly with its cost in coalition $T$, which is $c_j^T$. As a firm is always and exclusively a beneficiary in this model, it has no effect on the cost of other agents: Either countries or firms. However, a country is a benefactor to both firms and others countries; therefore, its marginal contribution is smaller than its cost in coalition $T$. If country $i$ is withdrawn from a coalition $T$, the individual cost of firms and other countries in coalition $T$ increases.

The following theorem states that our class of games is concave.

**Theorem 15.1** *The multiple corporation tax games* $(N, c)$ *are concave.*

*Proof.* Here we have to prove that the marginal contribution of an agent $k$ diminishes as a coalition grows. Any agent $k$ can only be either a firm or a country, and Lemma 15.4 provided its marginal contribution.

If the agent is a firm $j$, then for all $T \subseteq T', j \in T$, by Lemma 15.3, $c_j^T \geq c_j^{T'}$, and so $c_j^T = c(T) - c(T \setminus \{j\}) \geq c(T') - c(T' \setminus \{j\}) = c_j^{T'}$.

On the other hand, if the agent is a country $i$, again for all $T \subset T'$, by Lemma 15.3, $c_i^T \geq c_i^{T'}$.

In addition, $\sum\limits_{j\in E\cap T}(S_j^i - \bar{S}_j^i) \leq \sum\limits_{j\in E\cap T'}(S_j^i - \bar{S}_j^i)$ because all the countries in $T$ are also in $T'$, and if $T'$ there is at least one more than in $T$, then the inequality is strict.

Finally, for the same reason $\sum\limits_{i'\in P\cap T\backslash\{i\}} z_{i'i} \leq \sum\limits_{i'\in P\cap T'\backslash\{i\}} z_{i'i}$.

Hence, we can conclude that for all $T \subset T'$ and for all $i \in P \cap T$,

$$c(T) - c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap T\backslash\{i\}} z_{i'i} \geq c_i^{T'} - \sum_{j\in E\cap T'}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap T'\backslash\{i\}} z_{i'i} = c(T) - c(T' \setminus \{i\}).$$

■

So we proved that in a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$ it is efficient that all firms pay lower taxes and all countries manage to jointly reduce their degrees of tax evasion. In that case, the reduced total cost is given by $c(N) = \sum\limits_{i\in P} c_i^N + \sum\limits_{j\in E} c_j^N$.

An allocation rule for multiple corporation tax games is a map $\psi$ which assigns a vector $\psi(N, c) \in R^N$ to every $(N, c)$, satisfying that $\sum\limits_{k\in N} \psi_k(N, c) =$

$c(N)$. Each component $\psi_k(N,c)$ indicates the cost allocated to $k \in N$, so an allocation rule for multiple corporation tax games is a procedure to allocate the reduced total cost among the agents in N when they cooperate. An allocation rule should have good properties from the following points of view:

1. Computability. For a particular game, the rule should be computable in a reasonable CPU time, even when the number of agents is large.

2. Coalitional Stability. It is very convenient that the rule proposes an allocation which belongs to the core of the cost game. This means that, for every multiple corporation tax game $(N,c)$, $\psi$ should satisfy the following:

   $\sum_{k\in T} \psi_k(N,c) \le c(T)$, for every $T \subseteq N$.

   This condition assures that no group of agents $T$ is disappointed with the proposal of the rule, because the cost allocated to it is less than or equal to the cost it would support if its members formed a coalition to pay lower taxes, and reduce the levels of tax evasion, independently of the agents in $N \setminus T$.

3. Acceptability. The rule must be understandable and acceptable by the agents.

A very natural allocation rule for multiple corporation tax games is $\psi_k(N,c) = c_k^N$, for all $k \in N$. It has good properties at least with respect to computability and coalitional stability. Notice that, for every $T \subseteq N$, $\sum_{k\in T} \psi_k(N,c) = \sum_{k\in T} c_k^N \le \sum_{k\in T} c_k^T = c(T)$.

Nevertheless, the benefactors will have serious difficulties accepting the above allocation rule that rewards the beneficiaries excessively while they do not receive enough compensation for their dual role of giving and receiving.

Since the multiple corporation tax games are concave, cooperative game theory provides allocation rules for them with good properties at least with respect to coalitional stability and acceptability. We highlight the Shapley value and the nucleolus, which always provide core allocations in this context (see [4] for details on them). Both allocations are, in general, hard to compute when the number of agents increases.

Next, we present a simple and easily calculated expression for the Shapley value of multiple corporation tax games that compensates the benefactors for their dual role and irreplaceable character.

## 15.4 The Shapley Value

One of the most important allocation rules for cost games is the Shapley value (see [11]). As we already mentioned, the Shapley value is specially convenient for concave games: It is the barycenter of its core (see [13]).

We denote by $\phi\,(N,c)$ the shapley value of multiple corporation tax games $(N,c)$, where for each agent $k \in N$, $\phi_k\,(N,c) = \sum\limits_{T\subseteq N; k\in T} \gamma(T)\,[c(T) - c(T\setminus\{k\})]$, with $\gamma(t) = \frac{(n-t)!(t-1)!}{n!}$, $|T| = t$.

The following theorem states that the Shapley value can be easily computed in the class of multiple corporation tax games. Moreover, it shows that the Shapley value provides an acceptable allocation for multiple corporation tax games: It increases the cost of a beneficiary in half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in half of the benefits it provided to the beneficiaries.

**Theorem 15.2** *For any multiple corporation tax game* $(N,c)$*, the Shapley value is*

1. *For all* $j \in E$*,* $\phi_j\,(N,c) = c_j^N + \frac{1}{2}\sum\limits_{i\in P}(S_j^i - \bar{S}_j^i)$
2. *For all* $i \in P$*,* $\phi_i\,(N,c) = c_i^N - \frac{1}{2}\sum\limits_{j\in E}(S_j^i - \bar{S}_j^i) + \frac{1}{2}\sum\limits_{i'\in P\setminus\{i\}}(z_{ii'} - z_{i'i})$

*Proof.* (1) We prove that for all $j \in E$, $\phi_j\,(N,c) = c_j^N + \frac{1}{2}\sum\limits_{i\in P}(S_j^i - \bar{S}_j^i)$.

Take $j \in E$. By Lemma 15.4, we know that

$\phi_j\,(N,c) = \sum\limits_{T\subseteq N; j\in T} \gamma(t) c_j^T.$

We can separate coalitions $j \in T \subseteq N$ into mixed coalitions ($j \in T \subseteq N, T\cap P \neq \emptyset, T\cap E \neq \emptyset$) and coalitions with only firms ($j \in T \subseteq N, T\cap P = \emptyset, T\cap E \neq \emptyset$).

Then,

$$\phi_j\,(N,c) = \sum_{j\in T\subseteq N, T\cap P=\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P} S_j^i) + \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P\setminus P\cap T'} S_j^i + \sum_{i\in P\cap T'} \bar{S}_j^i).$$

Taking into account that $\sum\limits_{T\subseteq N; j\in T} \gamma(t) = 1$, we have that

$$\sum_{j\in T\subseteq N, T\cap P=\emptyset, T\cap E\neq\emptyset} \gamma(t) = 1 - \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t),$$

and then,

$$\begin{aligned}\phi_j(N,c) =& (1-\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t))(\sum_{i\in P} S_j^i)\\ &+\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)(\sum_{i\in P\setminus P\cap T} S_j^i+\sum_{i\in P\cap T}\bar{S}_j^i)\\ =&\sum_{i\in P} S_j^i+\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)(\sum_{i\in P\setminus P\cap T} S_j^i+\sum_{i\in P\cap T}\bar{S}_j^i-\sum_{i\in P} S_j^i)\\ =&\sum_{i\in P} S_j^i-\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)\sum_{i\in P\cap T}(S_j^i-\bar{S}_j^i).\end{aligned}$$

Now, we prove that for all coalitions that contain $j\in T\cap E$ and a particular country $i\in T\cap P$,

$$\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)=1/2,$$

and then,

$$\phi_j(N,c)=\sum_{i\in P} S_j^i-\frac{1}{2}\sum_{i\in P\cap T}(S_j^i-\bar{S}_j^i)=\frac{1}{2}\sum_{i\in P}(S_j^i+\bar{S}_j^i).$$

Indeed,

$$\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)=\sum_{t=2}^{n}\begin{pmatrix} n-2\\ t-2\end{pmatrix}\gamma(t)=$$

$$\sum_{t=2}^{n}\frac{(t-1)}{n(n-1)}=\frac{\sum_{k=1}^{n}k-n}{n(n-1)}=1/2,$$

where $\begin{pmatrix} n-2\\ t-2\end{pmatrix}$ is the number of coalitions in which there is $j$ and a particular country $i'$.

Finally, doing some algebra, we have that

$$\frac{1}{2}\sum_{i\in P}(S_j^i+\bar{S}_j^i)=c_j^N+\frac{1}{2}\sum_{i\in P}(S_j^i-\bar{S}_j^i),$$

and so, we conclude that

$$\phi_j(N,c)=c_j^N+\frac{1}{2}\sum_{i\in P}(S_j^i-\bar{S}_j^i).$$

(2) We demostrate that for all $i\in P$,

$$\phi_i(N,c)=c_i^N-\frac{1}{2}\sum_{j\in E}(S_j^i-\bar{S}_j^i)+\frac{1}{2}\sum_{i'\in P\setminus\{i\}}(z_{ii'}-z_{i'i}).$$

Take $i\in P$. By Lemma 15.4, we know that

$$\begin{aligned}\phi_i(N,c)=&\sum_{i\in T\subseteq N}\gamma(t)\\ &\times\left(c_i^T-\sum_{j\in E\cap T}(S_j^i-\bar{S}_j^i)-\sum_{i'\in P\cap T\setminus\{i\}}\left(g_{i'}(w_{i'}^{T\setminus\{i\}})-g_{i'}(w_{i'}^{T})\right)\right).\end{aligned}$$

Let's calculate each of the addends separately.

(2.1) First, taking into account that $c_i^T = c_i - \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'}$, for all $T \in N$, and $\sum_{t=2}^{n} \begin{pmatrix} n-2 \\ t-2 \end{pmatrix} \gamma(t) = \frac{1}{2}$,

we obtain that

$$\sum_{i \in T \subseteq N} \gamma(t) c_i^T = c_i - \sum_{t=2}^{n} \begin{pmatrix} n-2 \\ t-2 \end{pmatrix} \gamma(t) \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'} = c_i^N + \frac{1}{2} \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'},$$

where $\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}$ is now the number of coalitions that contain $i$ and a particular country $i'$.

(2.2) Second, by a similar argument,

$$\sum_{i \in T \subseteq N} \gamma(t) \sum_{j \in E \cap T} (S_j^i - \bar{S}_j^i) = \sum_{t=2}^{n} \begin{pmatrix} n-2 \\ t-2 \end{pmatrix} \gamma(t) \sum_{j \in E} (S_j^i - \bar{S}_j^i) =$$
$$\frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i).$$

(2.3) Third, by the same argument,

$$\sum_{i \in T \subseteq N} \gamma(t) \sum_{i' \in P \cap T \setminus \{i\}} \left( g_{i'}(w_{i'}^{T \setminus \{i\}}) - g_{i'}(w_{i'}^{T}) \right) =$$
$$\sum_{t=2}^{n} \begin{pmatrix} n-2 \\ t-2 \end{pmatrix} \gamma(t) \sum_{i' \in P \setminus \{i\}} z_{i'i} = -\frac{1}{2} \sum_{i' \in P \setminus \{i\}} z_{i'i} \ .$$

Finally, adding the above three expressions, we obtain that

$$\phi_i(N,c) = c_i^N + \frac{1}{2} \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'} - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) - \frac{1}{2} \sum_{i' \in P \setminus \{i\}} z_{i'i} =$$
$$c_i^N - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) + \frac{1}{2} \sum_{i' \in P \setminus \{i\}} (z_{ii'} - z_{i'i}) \,. \quad \blacksquare$$

From Theorem 15.2 can be derived that Shapley value compensates benefactors. Note first that, the cost of a firm $j$ in the grand coalition is $c_j^N$. This firm $j$ is benefited from a country $i$ in an amount which is $S_j^i - \overline{S}_j^i$. The Shapley value reduces this benefit exactly in half, and consequently this is the amount in which the cost of firm $j$ is increased, see point 1 of Theorem 15.2. In addition, the country $i$ is compensated exactly in this amount, and consequently its cost is reduced, see point 2 of Theorem 15.2. However, a country in its relation with others countries is simultaneously benefactor and beneficiary. Let's first look at the role as beneficiary of $i$, in any coalition, the country $i$ is benefited from country $i'$ in a cost reduction of $z_{ii'}$, in this case, country $i$ plays the role of beneficiary and $i'$ of benefactor. Thus, the Shapley value reduces the benefit $z_{ii'}$ of country $i$ in half; in others words, it increases its cost by this amount. Nevertheless, at the same time, the country $i$ benefits country $i'$ in an amount equal to $z_{i'i}$. Now, country $i$ is the benefactor and $i'$ the beneficiary. In this case, the Shapley value works in the same way, it compensates the benefactor increasing the cost of the beneficiary in half of $z_{i'i}$. Therefore, in the relation between two countries, both are simultaneously benefactors and beneficiaries; however, if $z_{ii'} - z_{i'i} > 0$, then country $i$ could be seen as a "net"-beneficiary and $i'$ as a "net"-benefactor, on the contrary if $z_{ii'} - z_{i'i} < 0$. Thus, country $i$ can be a "net"-benefactor with some countries and a "net"-beneficiary with others.

In conclusion, regarding the individual cost in the grand coalition, the Shapley value increases the cost of a beneficiary in half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in half of the benefits it provided to the beneficiaries. As in our model, there are dual agents (benefactors and beneficiaries); the final effect on these agents depends on which role is stronger.

## 15.5 An Example

In this example, we propose a simple situation with two countries $A$ and $B$, and two firms 1 and 2 with activity in both countries. These countries are very concerned about their own levels of underground economy, tax evasion, and fraud. To fight against this illegal behavior, these countries must face a high economic cost in human and material resources. However, this cost can be reduced if both countries decide to cooperate and, for example, they share resources and/or information in their fight.

On the other hand, firms have to pay in each country a certain amount of taxes. Nevertheless, these firms can choose to cooperate with a particular country. For example, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation is rewarded by a tax reduction. In particular, country $A$ will fix a reduction of 10%, and $B$ will do it of 15%. Thus, each firm must pay either a tax $(S_j^i)$ or a reduced tax $(\bar{S}_j^i)$ as it is given in Table 15.1.

| | | | |
|---|---|---|---|
| $S_1^A = 2$ | $S_1^B = 4$ | $S_2^A = 5$ | $S_2^B = 8$ |
| $\overline{S}_1^A = 1.80$ | $\overline{S}_1^B = 3.40$ | $\overline{S}_2^A = 4.50$ | $\overline{S}_2^B = 6.80$ |

**TABLE 15.1**: Tax and reduced tax of each firm (in millions of euros).

We consider that the cost function of any country $c_i^T = g_i\left(w_i^T\right)$ has two terms. The first term does not depend on the type of coalition the country belongs to. In other words, it does not depend on the information other countries could provide. This is a kind of fixed cost. The second term does depend on which coalition the country is. In particular, $g_A\left(w_A\right) = 4 + w_A^T$

and $g_B(w_B) = 8 + 2w_B^T$. In addition, the levels of underground economy or tax evasion are normalized to 1 in any coalition with only one country, i.e., without the help of other countries. Thus, $w_i^T = 1$ for any $i \in P$, $T \subset N$ such that $P \cap T \backslash \{i\} = \emptyset$. However, in any coalition $T' \subset N$ such that $A, B \in T'$, $w_A^{T'} = 0.50$ and $w_B^{T'} = 0.60$.

Table 15.2 shows the cost-coalitional vector (columns 2-5) and corresponding cost game (last column); i.e., for any coalition $T \subseteq N$, the cost of each agent $c_k^T$, and the cost of this coalition $c(T)$

| Coalition \ Agent | $A$ | $B$ | 1 | 2 | $c(T)$ |
|---|---|---|---|---|---|
| $\{A\}$ | 5 | | | | 5 |
| $\{B\}$ | | 10 | | | 10 |
| $\{1\}$ | | | 6 | | 6 |
| $\{2\}$ | | | | 13 | 13 |
| $\{A,B\}$ | 4.5 | 9.2 | | | 13.70 |
| $\{A,1\}$ | 5 | | 5.80 | | 10.80 |
| $\{A,2\}$ | 5 | | | 12.50 | 17.50 |
| $\{B,1\}$ | | 10 | 5.40 | | 15.40 |
| $\{B,2\}$ | | 10 | | 11.80 | 21.80 |
| $\{1,2\}$ | | | 6 | 13 | 19 |
| $\{A,B,1\}$ | 4.50 | 9.20 | 5.20 | | 18.90 |
| $\{A,B,2\}$ | 4.5 | 9.20 | | 11.30 | 25 |
| $\{A,1,2\}$ | 5 | | 5.80 | 12.50 | 23.30 |
| $\{B,1,2\}$ | | 10 | 5.40 | 11.80 | 27.20 |
| $\{A,B,1,2\}$ | 4.50 | 9.20 | 5.20 | 11.30 | 30.20 |

**TABLE 15.2**: Cost-coalitional vector and cost game.

From the previous table, it is straightforward to obtain $z_{ii'}$, where $z_{ii'} = c_i^{T\backslash\{i'\}} - c_i^T$ for all $T \subseteq N$ such that $i, i' \in P \cap T$. Therefore, $z_{AB} = 0.50$ and $z_{BA} = 0.80$, i.e., country $B$ reduces the cost of country $A$ in 0.50 and country $A$ reduces the cost of country $B$ in 0.80. Consequently, country $A$ is a net-benefactor with country $B$, and country $B$ a net-beneficiary with country $A$.

We can calculate now the Shapley value by using the expressions from Theorem 15.2. Note that, in this case, we only need the values of Table 15.1, the last row of Table 15.2 ($c_A^N$, $c_B^N$, $c_1^N$ and $c_2^N$), and both values $z_{AB}$ and $z_{BA}$. Therefore, Theorem 15.2 allows to reduce significantly the amount of information and time to compute Shapley value.

In Table 15.3, it is shown for any agent its individual cost, the cost in the grand coalition, the Shapley value, and the difference between the last two values.

Notice that costs in the grand coalition reduce the costs of each player. Regarding the cost in the grand coalition, Shapley value decreases the cost of benefactors in half of the benefits that it provided to the beneficiaries.

| Agent \ Value | $c(\{k\})$ | $\psi_k(N,c)$ | $\phi_k(N,c)$ | $\psi_k(N,c)-\phi_k(N,c)$ |
|---|---|---|---|---|
| $A$ | 5 | 4.50 | 4 | 0.50 |
| $B$ | 10 | 9.20 | 8.45 | 0.75 |
| 1 | 6 | 5.20 | 5.60 | $-0.40$ |
| 2 | 13 | 11.30 | 12.15 | $-0.85$ |

**TABLE 15.3**: Comparsison among individual costs, cost in the grand coalition and the Shapley value.

Additionally, it increases the cost of beneficiaries in half of the benefits that they obtain from benefactors. For example, for country $A$, $\phi_A(N,c) = c_A^N - \frac{1}{2}\left((S_1^A - \bar{S}_1^A) + (S_2^A - \bar{S}_2^A)\right) + \frac{1}{2}(z_{AB} - z_{BA})$. As $z_{AB} - z_{BA} = -0.30$, country $A$ is a net-benefactor. Thus, Shapley value decreases its cost in half of this difference. However, for country $B$, the cost is increased in the same amount because it is a net-beneficiary. In this example, there are only two countries; however, if there were more countries, a given country could be a net-benefactor with some countries and a net-beneficiary with others; this depends on the sign of $z_{ii'} - z_{i'i}$.

## 15.6 Conclusions

Corporation tax games were introduced by [7] as an application of linear cost games (see [6]) to a corporate tax reduction system. Motivated by the Spanish tax system, the authors considered that the Government, as benefactor, provided different group investment options which reduced the costs of those investors who acted legally (beneficiaries).

In this chapter, we have presented a new model of cooperation in corporate tax systems with several firms and countries (multiple dual and irreplaceable benefactors). Countries are dual in the sense that they are benefactors (they reduce the costs of both firms and other countries), and beneficiaries (cost is reduced by the information provided by other countries). They are also irreplaceable benefactors because all the members of a coalition may see their cost increase if one of them leaves the group.

The class of TU cooperative games corresponding to this model is called multiple corporation tax games. We have proved that these games are concave, i.e., the marginal contribution of a firm and a country diminishes as a coalition grows (snowball effect). Hence, the grand coalition is stable in the sense of the core. This means that firms have strong incentives to cooperate with the countries instead of being fraudsters. Then, we propose the Shapley value as an easily computable core-allocation that benefits all agents and, in particular, compensates the benefactors for their dual and irreplaceable role.

Our model here distinguishes two groups of agents: Dual benefactors (countries) and beneficiaries (firms), while the original model presented by [6], considered two disjoint groups of agents, benefactors and beneficiaries. A natural extension would be to consider that all agents can be dual (benefactors and beneficiaries). We believe that similar results to those obtained here could be achieved.

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## 15.7 Acknowledgments

Authors acknowledge the financial support of Ministerio de Ciencia e Innovación through project MTM2014-54199-P and Junta de Andalucía through project SEJ2011-8065.

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# Chapter 16

## The Shapley Value in Telecommunication Problems

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### 16.1 Introduction

Game Theory has been applied to a multitude of problems of very diverse fields of knowledge. Aumann and Hart [6] provide a list of knowledge fields in which game theory has been successfully applied. Apart from the applications to Economics, for which several game theorists have been awarded with the Nobel Prize in Economics, the list includes fields such as Evolutionary Biology and Computer Science. However, since then, Game Theory has also been applied to other fields such as Genetics, Telecommunications or Multi-agent Systems in Engineering. In [83], some applications of Game Theory to engineering problems are reviewed. This review includes various applications of cooperative game theory to telecommunications problems.

In general, Game Theory can be applied to almost any situation in which there is more than one individual, in the broadest sense of the term, and that there is some kind of interaction between them that leads to a conflict of interest. Nowadays, since most systems from very different scientific disciplines usually consist of more than one element, researchers are considering the game theoretical approach to analyze them in order to obtain new information and answers about them. Therefore, every day, Game Theory becomes a more transversal discipline useful to analyze problems from very diverse and far fields [82]. One of these fields is Telecommunications. However, in order to apply Game Theory to a real-life problem, the essential elements that define the problem must be taken into account, simplifying where necessary, but not in excess to avoid that the game does not describe the problem. Some suggestions on how to apply Game Theory to real problems are found in [60]. Depending on the type of interaction among the individuals, we will find, roughly speaking, either with non-cooperative problems, if individuals compete for something and must take actions that involve strategic behavior, or with cooperative problems, if individuals collaborate to improve their results and subsequently the benefits of this cooperation should be shared among them. The first problems give rise to non-cooperative games, while the second ones give rise to cooperative games, and the individuals are called players.

In particular, the Shapley value [86] has been sucessfullly applied to many different fields. Moretti and Patrone [63] provide a survey about several applications of the Shapley value to very diverse fields, showing the transversality of the Shapley value to address real-life problems. Therefore, the Shapley value has proven to be a good tool for the so-called Game Practice [12, 73]. The Shapley value is a concept of solution for cooperative games, so it provides a distribution of the gain obtained by collaboration between the individuals involved in the problem. As Moretti and Patrone [63] pointed out, the Shapley value tries to answer the following question: *How to convert information about the worth that subsets of the player set can achieve, into a personal attribution (of payoff) to each of the players?* The Shapley value is one of the concepts of Game Theory that has attracted the most attention among researchers, so we can find numerous works in the literature about it (see, for example, [81, 96]).

Despite its great interest, the Shapley value also has a couple of drawbacks to be applied. The first objection is that the Shapley value is not always stable, in the sense that there may be a subset of players who are not satisfied with the solution proposed by the Shapley value because they themselves can obtain more. The second flaw is that its calculation can be almost impossible from the point of view of its computation in a reasonable time. Therefore, when the structure of the problem does not allow an easy computation of the Shapley value, an alternative is to resort to compute it approximately by sampling techniques [13, 14, 58]. In the case of applications in the field of Telecommunications, perhaps, the second drawback is much more important than the first, since in many problems in this field it is necessary to obtain a result in a very short time. Nevertheless, we can find many possible applications of the Shapley value in telecommunications problems as will be shown throughout this chapter.

The growth of the telecommunications industry, particularly wireless communications, has exceeded all expectations. Today, there are very few people who do not have at least one smart device, which means an important volume of business and income for the companies that offer these services. However, the communication infrastructures have a finite capacity. On the one hand, this implies that the owner of the communication infrastructure must optimize (scarce) resources, and carefully manage the available resources in order to guarantee an adequate level of quality of service (QoS) to the users. On the other hand, the users of the communication infrastructure "compete" for the resources in order to guarantee themselves the best possible service. But it must also be taken into account that a communication infrastructure consists of several entities that can operate in a centralized or decentralized manner, and therefore, there may be coordination or competition between the entities when using the (possibly scarce) resources that serve to provide the communication service to users. Game Theory deals with situations of conflict of interest, and in this kind of systems such conflicting situations can arise. Han et al. [31] present a well-structured collection of game theoretical concepts with their respective applications in the literature to wireless and communications networks. The Shapley value is one of the game theoretical concepts they include in their book, for which they point out suitable fairness criteria for allocating resources or data rates in communication networks.

Therefore, in light of the above, it makes perfect sense to include a chapter on telecommunications in a book dedicated to the Shapley value.

## 16.2 Some Uses of the Shapley Value in Mobile Communication Management

Mobile communications make remote communication possible without the need for a physical link, which contributes to a reduction in installation and maintenance costs. The above are advantages of this communication system, but the absence of a physical link entails some disadvantages. This technology faces non-controllable factors, such as weather conditions or user movements and, more importantly, the radio spectrum, on which these communications are based, is finite. The latter means that the range of the spectrum is a limited resource in the system. Consequently, a careful management of the available resources is absolutely necessary to guarantee an adequate level of quality of service (QoS) to the users. Roughly speaking, there are three elements that must be taken into account to provide optimal performance of a wireless communications system: Power control, link adaptation and channel assignment.

The objectives of the wireless communication system and its users go in the same direction but, at the same time, are opposite. On the one hand, users want to obtain the highest possible performance with a high QoS, while, on the other hand, the communication system has limited resources to meet the demands of users and its goal is to serve as many users as possible with an acceptable QoS. According to the above, users would compete for scarce resources of the system while the system tries to optimize its performance by managing the resources at its disposal at any time. Game Theory deals with conflict of interest situations, and in this type of system, as we can easily deduce from the above, conflicting situations may arise. Therefore, it seems reasonable to think that using game theory can improve the management of system resources given that the radio spectrum is a limited resource. In this regard, seeking solutions from game theory to obtain a reasonable overall performance while maintaining QoS good enough seems an interesting idea. The common approach is from non-cooperative game theory, but one can also find approaches from cooperative game theory in the literature, in particular using the Shapley value. [31, 36].

What is meant by the allocation of resources in a wireless communication system? When a user requests service, a base station (BTS) responds essentially with the following parameters: Channel, transmission mode and power. In the next two subsections, we will present a cooperative game to allocate power and transmission mode and two cooperatives games to allocate channel.

### 16.2.1 Resource Management in Wireless Networks

Roughly, the physical elements involved in wireless communications are the power transmission ($P$), the signal-to-interference-plus-noise ratio ($SINR$) and the throughput ($T$). The throughput depends on the SINR and the transmission mode, and the SINR depends on the power transmission, the path-losses between the user and BTSs, the interference and the (thermal) noise. The received SINR at user $i$ is usually given by

$$SINR_i = \frac{P_i L_{ii}}{\sum_{j \in Q_i} P_j L_{ji} + n_i}, \tag{16.1}$$

where $P_k$ is the transmission power assigned to user $k$, $L_{ji}$ is the path loss between the BTS serving user $j$ and user $i$, $Q_i$ is the set of all users using the same channel as user $i$ and $n_i$ is the thermal noise. [43] showed that the effective throughput can be written as a sigmoidal function of the SINR,

$$T(x) = \frac{A}{1 + e^{-\lambda(x-\delta)}}, \tag{16.2}$$

where the SINR, $x$, is in dB and the throughput $T(x)$ is in Kbps. The parameters $A$, $\lambda$ and $\delta$ depend on the transmission mode.

Now we have to answer the question of how to assign power, transmission mode and channel to users. In [43, 79] network-assisted resource management (NARM) to enhance the resource management in wireless networks is proposed. In NARM, interbase signaling is used to increase the information about the system for enhancing the power, transmission mode and channel assignment to users in order to increase the performance of the communication system. [44] provides some game theoretic formulations for NARM in GSM/GPRS/EGPRS-based networks, and in [76] the same technical framework is considered but they use bankruptcy problems [68] to address the problem.

Consider that when a number of users request service to a BTS, this has a limited service capacity in terms of throughput. Each user will request a throughput according to their needs and the station will have to give them the best possible service. If the available throughput is not enough to meet the total requested throughput, then we have a situation which can be described as a bankruptcy problem (see [93, 94] for a survey).

A bankruptcy problem is defined by a triple $(N, E, c)$, where $N$ is the set of claimants, $E$ is the available estate, and $c$ is the vector of claims, such that $E \leq \sum_{i \in N} c_i$, i.e., the estate is not sufficient to fully meet the demands of the claimants. In the context of a BTS in a wireless network, we consider as in [76] that the estate is given by the available throughput $(T)$, which may not be all available but a reasonable amount considering that the system is dynamic and new service requests can reach the BTS, the set of claimants $N$ is the set of users requesting for service within a time window, and the vector of claims $t$ is given by the throughput requested by the users. Therefore, we describe a resource management wireless problem as $(N, T, t)$. Now the problem is how to allocate the available throughput among the users requesting for service. An alternative is to apply the recursive completion rule, also known as run-to-the-bank rule, which coincides with the Shapley value of the associated bankruptcy game [68]. This cooperative game, which we will denote by $\left(N, v^{(T,t)}\right)$, is given by

$$v^{(T,t)}(S) = max\{0, T - \sum_{j \notin S} t_j\}, \ \forall S \subseteq N. \tag{16.3}$$

Therefore, $v^{(T,t)}(S)$ measures the throughput available after serving users out of $S$. This game describes a pessimistic viewpoint of the users regarding the throughput allocation problem.

The Shapley value of the game can be computed by considering all possible orders in which users may have requested service to the system. Assuming a First-In-First-Out (FIFO) service system, the first user to request service would obtain everything requested up to a maximum of the total throughput available, the second user requesting service would obtain the maximum between her request and the available throughput after serving the first user, so

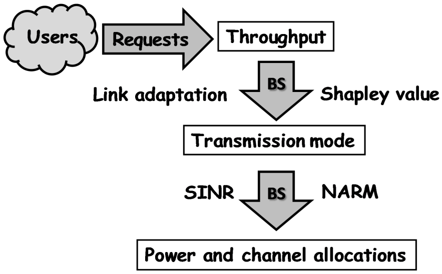


**FIGURE 16.1**: Resource management mechanism based on the Shapley value.

on and so forth. If we now consider that the service request order is not relevant within the same time window, then it seems reasonable to consider the average throughput of all possible orders. Precisely, these average throughputs coincide with the Shapley value of the bankruptcy game $\left(N, v^{(T,t)}\right)$ associated to the resource management wireless problem $(N, T, t)$.

How would the resource management algorithm work using the Shapley value? The BTS assigns to the users requesting service in a time window throughputs by using the Shapley value, with these throughputs and link adaptation, the BTS assigns to the users transmission modes, then by using the relationship between throughput and SINR, and the information obtained by interbase signaling, i.e., the NARM system, the BTS assigns to the users powers and channels. This resource management mechanism based on the Shapley value (RMMS) is depicted in Figure 16.1.

In order to illustrate how the RMMS works, we consider the following simple example.

**Example 16.1** *In a time window, four users request services to a BTS of 40, 30, 20 and 15 Kbps, respectively, and the BTS fixes an available throughput of 80 Kbps. First, the BTS computes the Shapley value. This computation involves twelve possible orders. The allocation of throughput for every order is shown in Table 16.1. The Shapley value is calculated as the average of all those allocations, that is*

$$\Phi_1 = 32.5\,Kbps;\ \Phi_2 = 22.5\,Kbps;\ \Phi_3 = 14.2\,Kbps;\ \Phi_4 = 10.8\,Kbps.$$

*After determining the throughput allocated to each user, a transmission mode must be assigned to each user. For doing this, we use the sigmoidal relationship between throughputs and transmission modes. To assign a transmission mode to user $i$, we take its allocated throughput $\Phi_i$ and look for the first sigmoidal curve intersecting with $y = \Phi_i$. Thus, in our example, when an EGPRS system is considered, user 1 would be assigned transmission mode 7, user 2 mode 6, user 3 mode 5, and user 4 also mode 5 (for the transitioning between two modes, see [44]).*

| $t_i$ | 40 | 30 | 20 | 15 | $t_i$ | 40 | 30 | 20 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| order | 1 | 2 | 3 | 4 | order | 1 | 2 | 3 | 4 |
| 1234 | 40 | 30 | 10 | 0 | 3124 | 40 | 20 | 20 | 0 |
| 1243 | 40 | 30 | 0 | 10 | 3142 | 40 | 5 | 20 | 15 |
| 1324 | 40 | 20 | 20 | 0 | 3214 | 30 | 30 | 20 | 0 |
| 1342 | 40 | 5 | 20 | 15 | 3241 | 15 | 30 | 20 | 15 |
| 1423 | 40 | 25 | 0 | 15 | 3412 | 40 | 5 | 20 | 15 |
| 1432 | 40 | 5 | 20 | 15 | 3421 | 15 | 30 | 20 | 15 |
| 2134 | 40 | 30 | 10 | 0 | 4123 | 40 | 25 | 0 | 15 |
| 2143 | 40 | 30 | 0 | 10 | 4132 | 40 | 5 | 20 | 15 |
| 2314 | 30 | 30 | 20 | 0 | 4213 | 35 | 30 | 0 | 15 |
| 2341 | 15 | 30 | 20 | 15 | 4231 | 15 | 30 | 20 | 15 |
| 2413 | 35 | 30 | 0 | 15 | 4312 | 40 | 5 | 20 | 15 |
| 2431 | 15 | 30 | 20 | 15 | 4321 | 15 | 30 | 20 | 15 |

**TABLE 16.1**: Allocation of throughput for every possible order of the users.

*Then, knowing the throughput and mode assigned to each user, we can determine the SINR for each of them by using the relationship between the throughput and the SINR given by Eq. (16.2) as follows:*

$$T(x) = \frac{A}{1+e^{-\lambda(x-\delta)}} \Rightarrow x = -\frac{ln\left(\frac{A-T(x)}{T(x)}\right) - \lambda\delta}{\lambda}\ (in\ dB). \qquad (16.4)$$

*Now, substituting the throughput assigned to each user in Eq. (16.4) with the parameters A, $\lambda$ and $\delta$ for the corresponding transmission mode, the SINRs are obtained. In this example, the SINRs would be given by*

$$SINR_1 = -\frac{ln\left(\frac{44.8-32.5}{32.5}\right) - 0.446\times 15}{0.446} = 17.18\,dB;$$

$$SINR_2 = -\frac{ln\left(\frac{29.6-22.5}{22.5}\right) - 0.451\times 10}{0.451} = 12.56\,dB;$$

$$SINR_3 = -\frac{ln\left(\frac{22.4-14.2}{14.2}\right) - 0.505\times 8}{0.505} = 9.07\,dB;$$

$$SINR_4 = -\frac{ln\left(\frac{22.4-10.8}{10.8}\right) - 0.505\times 8}{0.505} = 7.87\,dB;$$

*where the parameters are taken from [43]. Finally, the power and channel assignment to the users is determined by using the relationship given in Eq. (16.1). To carry out this last step of the RMMS, it is necessary the interbase signaling be provided by the NARM in wireless networks.*

Some final comments about resource management in wireless networks are: First, note that if the total requested throughput is below the available throughput, then there is no problem, every user is assigned his requested throughput. When below a certain threshold of SINR, the QoS is not admissible or acceptable, then the user's request should be rejected or queued. In view of this, bankruptcy rules or other allocation rules which satisfy the property of drop-out could be interesting in this context. For example, in [76] the constrained equal awards rule is used. Other definitions of the estate and the claims can be used. For example, the estate could be measured in terms of the total SINR admissible for the system. The main technical shortcoming of this approach could be the intensity of uplink and downlink signaling between BTSs and users, because this reduces the available resources, but [79] justify that network-assisted management is an attractive option from both spectrum efficiency and implementation point of view. As mentioned in the Introduction, one of the biggest problems with the Shapley value is its computational complexity. However, the number of types of requested services is usually small; therefore, by using its random order property, together with its symmetry and efficiency properties, the computation of the Shapley value can be obtained in a very reasonable time ([81], page 7). Finally, bankruptcy games are convex games and then the Shapley value belongs to the core of these games, which means that the Shapley value is a stable allocation in the sense that no coalition can obtain a better aggregate result by itself [87]. This property is noticeably relevant when defining a fair and stable allocation of resources.

### 16.2.2 Channel Allocation in Mobile Communication Networks

The high expectations and demands of users in terms of QoS provision are characteristics associated with the evolution of mobile communication systems. These demands require the design and implementation of the necessary means to achieve an efficient use of the scarce resources available. One way to do this is through the development of radio resource management (RRM) techniques. RRM techniques include channel allocation mechanisms which are responsible for allocating, managing and distributing the available channels between users and services according to some system or QoS restrictions. Thus, channel allocation is an example in which game theory can be applied since its main purpose is to manage scarce radio resources in an environment where a group of users compete for them and their actions might affect others. In channel allocation, several issues must be taken into account, such as there are a limited number of channels or the types of services (www, video, e-mail, call) among others.

The channel allocation problem we are going to approach is the following. A Base Station (BTS) has a number of channels available and several users of the system request service to it. How should the BTS distribute the available channels among them? Some possible alternatives to answer this question are to apply a rule considering priorities of service, maximize the total throughput, maximize the minimal throughput obtained by a user or apply rationing rules. In [25, 52] the application of bankruptcy-based mechanisms for channel allocation is studied. They consider the GERAN (GSM/EDGE Radio Access Network) radio interface as a TDMA representative technology over which to test those allocation mechanisms and compare them with others.

In order to apply bankruptcy or rationing rules, certain characteristics of the problem must be taken into account. First, the estate is not perfectly divisible (discrete problem), and the BTS has a finite number of channels available. Second, the users are not identical and their happiness levels will depend on the type of requested service (www, mail, video, etc.) and the assigned resources (number of channels). Third, there are services that are more resource demanding than others. And finally, the problem is dynamic. Therefore, in the context of a mobile radio network, the estate, $R$, is the finite number of radio resources available and the set of claimants $N$ is the set of users requesting service. Regarding the claims, not only the number of radio resources requested must be taken into account but also other relevant characteristics in the problem must be considered. Thus, the claim of a user is given by the type of service $s$ requested, the minimum number of radio resources $m(s)$ to receive that service, the number of resources $M(s)$ to receive an optimal QoS, the time waiting for service $t$ and whether it is already being served, $\alpha \in \{0,1\}$. The first and the two last characteristics define a system of priorities in the service. Hence, the claim of user $i$, is given by 5-tuple $U_i = (s_i, m(s_i), M(s_i), t_i, \alpha_i)$. Now the priority system is defined as follows. Given two users $i$ and $j$, with claims $U_i = (s_i, m(s_i), M(s_i), t_i, \alpha_i)$ and $U_j = (s_j, m(s_j), M(s_j), t_i, \alpha_j)$, user $i$ has higher prority than user $j$, $i >_P j$, if one of the following conditions happens

$$\begin{array}{ll} First: & \alpha_i < \alpha_j; \\ Second: & \alpha_i = \alpha_j \; and \; s_i > s_j; \\ Third: & \alpha_i = \alpha_j \; and \; s_i = s_j \; and \; t_i > t_j. \end{array} \tag{16.5}$$

We describe a radio resource management problem by the triple $(N, R, U)$. Now the problem is how to allocate the available channels (radio resources) among the users requesting for service. As in Subsection 16.2.1, an alternative is to apply a run-to-the-bank rule adapted to this situation, which will be close related to the random order idea behind the Shapley value. But, additionally, the channel allocation is a dynamical procedure. Channel allocation is carried out every certain short period of time, even when the users in the system are the same. With this in mind, on the one hand, every time channel allocation is run, the users served in the previous round must continue being served in the new round, unless they have already completed their transmission. On the other hand, the users queued in previous rounds must be prioritized in the following rounds.

To simplify the description of the procedure, we define the cooperative game $\left(N, v^{(R,M)}\right)$ whose characteristic function is given by

$$v^{(R,M)}(S) = min\left\{\sum_{i\in S} M(s_i),\ max\left\{0,\ R - \sum_{j\notin S} M(s_j)\right\}\right\},\ \forall S \subseteq N, \tag{16.6}$$

and the Shapley value of this game is $\Phi(v^{(R,M)})$.

The idea of the channel allocation mechanism is to calculate the Shapley value of the games $v^{(R,M)}$ associated to the successive radio resource management problems $(N, R, U)$ that appear in each round. The proposed channel assignment procedure works as follows. At the beginning of a round, the BTS observes the users who are requesting service and the parameters that determine their claims are updated. The time in the system is measured as the number of rounds in which a user has participated. With this information, the number of available resources is calculated, taking into account that the users served in the previous round must continue to be served, and reserving the resources that the unattended users were not able to use because they were queued. And the users' claims of resources are also calculated as the difference between their optimal numbers of resources to receive the requested services and the resources that the system reserves for them. With the amount of available resources and the previous demands, the Shapley value of the associated game (16.6) is computed. Therefore, to calculate a first allocation of resources, two types of demands have been taken into account: Reference demands which the BTS tries to guarantee and other optimal demands, so we have used bankruptcy problems with references [77, 78]. Since the allocation of resources must be in integer numbers, the whole part of the Shapley value is calculated and the remaining resources are distributed among the users according to the system of priorities $>_P$. If the resources assigned to a user are not sufficient to provide service, those resources are redistributed among the rest of the users who can receive the service. The users not finally served are queued and others are rejected if a certain timeout threshold has been exceeded. This process is repeated every round, while there are users requesting service to the BTS. The technical details of this procedure are shown in the following algorithm.

**ALGORITHM** (Channel allocation mechanism based on the Shapley value).
**Round** $k$:
**Update users** $N^{(k)}$ //set of users requesting service in round $k$
**Update parameters**
**for** $i \in N^{(k)}$ **do**
  **if** $i \in N^{(k-1)}$ **then**
    $M_i^{(k)} \leftarrow M(s_i) - m(s_i)\alpha_i^{(k-1)} - \widehat{\Phi}_i^{(k-1)}\left(1 - \alpha_i^{(k-1)}\right)$
  **else**
    $M_i^{(k)} \leftarrow M(s_i);\ \alpha_i^{(k-1)} \leftarrow 0;\ \widehat{\Phi}_i^{(k-1)} \leftarrow 0$
  **end if**
**end for**

**do**

1. $R^{(k)} \leftarrow R - \sum_{i \in N^{(k)}} m(s_i)\alpha_i^{(k-1)} - \sum_{i \in N^{(k)}} \widehat{\Phi}_i^{(k-1)} \left(1 - \alpha_i^{(k-1)}\right)$
2. Calculate $\Phi\left(v^{(R^{(k)}, M^{(k)})}\right)$
3. $D^{(k)} \leftarrow R^{(k)} - \sum_{i \in N^{(k)}} int\left(\Phi_i\left(v^{(R^{(k)}, M^{(k)})}\right)\right)$
4. Distribute $D^{(k)}$ among the users in $N^{(k)}$ according to $>_P \rightarrow \left(d_i^{(k)}\right)_{i \in N^{(k)}}$

**for** $i \in N^{(k)}$ **do**

$\widehat{\Phi}_i^{(k)} \leftarrow m(s_i)\alpha_i^{(k-1)} + \widehat{\Phi}_i^{(k-1)}\left(1 - \alpha_i^{(k-1)}\right) + int\left(\Phi_i\left(v^{(R^{(k)}, M^{(k)})}\right)\right) + d_i^{(k)}$

**if** $\widehat{\Phi}_i^{(k)} < m(s_i)$ **then**

$r_i^{(k)} \leftarrow 0$; $\alpha_i^{(k)} \leftarrow 0$; $t_i^{(k)} \leftarrow t_i^{(k-1)} + 1$

$Q^{(k)} \leftarrow Q^{(k)} + \widehat{\Phi}_i^{(k)}$

**else**

$r_i^{(k)} \leftarrow \widehat{\Phi}_i^{(k)}$; $\alpha_i^{(k)} \leftarrow 1$; $t_i^{(k)} \leftarrow t_i^{(k-1)} + 1$

**end if**

**end for**

**do**

1. Distribute $Q^{(k)}$ among the users in $N^{(k)}$ such that $\alpha_i^{(k)} = 1$ according to $>_P \rightarrow \left(q_i^{(k)}\right)$
2. $r_i^{(k)} \leftarrow r_i^{(k)} + q_i^{(k)}$

**return** $r_i^{(k)}$, $i \in N^{(k)}$ //final allocation of radio resources in round $k$.

Note that in the algorithm there are two subroutines not described. One of them is the calculation of the Shapley value, and the other is the distribution of the radio resource among users according to the system of priorities. Regarding the computation of the Shapley value, different procedures can be used, for example, the random order procedure or an estimate of the Shapley value by using sampling [13, 14, 58]. With regards to the distribution of the radio resource among users according to the system of priorities, there are also several alternatives. A possible procedure is to assign resources to the most priority user until it reaches its optimal number of resources, if there are still resources to distribute, select the second highest priority user and allocate resources until it reaches its optimal number, and so on. We call this procedure the most priority user allocation method. A second alternative consists of uniformly assigning the radio resources from the most priority user to the least priority one. If there are still resources to start again for the most priority and so on. We call this procedure the priority uniform allocation method.

In the following example, we illustrate how the method of channel allocation based on the Shapley value works.

**Example 16.2** *Consider a BTS with a single frequency carrier with 8 time slots (channels). Now four users are requesting different types of services.*

*Two users request e-mail (users 1 and 2), one user (user 3) requests www and another user (user 4) requests a 64 Kbps video. In [52] the minimum and optimal number of radio resources for different services are determined. For e-mail, the minimum number of time slot is 1 and the optimal number is 6, for www the minimum number is 2 and the optimal number of time slots is 8, and for 64 Kbps video the minimum number of time slots is 4 and the optimal number is 8. Likewise, the following priorities on the type of service are considered 64 Kbps video>www>e-mail. An application of the algorithm based on the Shapley value is the following:*

- *Round 1:*
  - $s(1)=e-mail,\ s(2)=e-mail,\ s(3)=www,\ s(4)=64\ Kbps-video$
  - $m(1) = 1,\ m(2) = 1,\ m(3) = 2,\ m(4) = 4$
  - $t^{(0)}(1) = t^{(0)}(2) = t^{(0)}(3) = t^{(0)}(4) = 0$
  - $\alpha^{(0)}(1) = \alpha^{(0)}(2) = \alpha^{(0)}(3) = \alpha^{(0)}(4) = 0$
  - $M^{(1)}(1) = 6,\ M^{(1)}(2) = 6,\ M^{(1)}(3) = 8,\ M^{(1)}(4) = 8$
  - $R^{(1)} = 8$
  - $\Phi = (1.67, 1.67, 2.33, 2.33)$
  - $D^{(1)} = 8 - 6 = 2$
  - *Using the priority uniform allocation method,* $\widehat{\Phi}^{(1)} = (1, 1, 3, 3)$
  - *The user requesting 64 Kbps video must be queued*
  - *Using the priority uniform allocation method, the final time slot allocation in Round 1 is given by* $r = (2, 2, 4, 0)$.
- *Round 2:*
  - $s(1)=e-mail,\ s(2)=e-mail,\ s(3)=www,\ s(4)=64\ Kbps-video$
  - $m(1) = 1,\ m(2) = 1,\ m(3) = 2,\ m(4) = 4$
  - $t^{(1)}(1) = t^{(1)}(2) = t^{(1)}(3) = t^{(1)}(4) = 1$
  - $\alpha^{(1)}(1) = \alpha^{(1)}(2) = \alpha^{(1)}(3) = 1,\ \alpha^{(1)}(4) = 0$
  - $M^{(2)}(1) = 6 - 1 \times 1 - 1 \times 0 = 5,\ M^{(2)}(2) = 6 - 1 \times 1 - 1 \times 0 = 5,\ M^{(2)}(3) = 8 - 2 \times 1 - 3 \times 0 = 6,\ M^{(2)}(4) = 8 - 4 \times 0 - 3 \times 1 = 5$
  - $R^{(2)} = 8 - (1 + 1 + 2 + 0) - (0 + 0 + 0 + 3) = 1$
  - $\Phi = (0.25, 0.25, 0.25, 0.25)$
  - $D^{(2)} = 1 - 0 = 1$
  - *Using the priority uniform allocation method,* $d^{(2)} = (0, 0, 0, 1)$
  - *Therefore,* $\widehat{\Phi}^{(2)} = (1, 1, 2, 4)$
  - *The final time slot allocation in Round 2 is given by* $r = (1, 1, 2, 4)$.

*If no user ends their session, then we would always get the same allocation of time slots in each round, and new users would be queued until there were time slots available.*

Some remarks are the following. The same argument that was mentioned in Subsection 16.2.1 regarding the computational complexity of the Shapley value, can be extended to this case, adding the fact that the number of radio resources is usually very limited and this further improves computing times. Some related literature to the radio resource allocation is the following. In [59] a radio resource allocation protocol is defined seeking to fairly distribute the available resources. For that, they do not use a game theoretical approach and the resource to be allocated is perfectly divisible. In [20] the radio resource assignment problem is addressed by defining Vickrey auctions derived to satisfy user QoS requirements. [75] propose a radio resource management that bases its decisions on aspects related to network resources and user preferences. In [2] an auction framework for radio resource allocation is proposed, where the Shapley value is used to evaluate the worth of some radio resources to be assigned to the users. In [34, 35, 57], a two-level resource allocation in LTE (Long Term Evolution) networks is proposed. In the upper level, the available bandwidth (considered perfectly divisible) is distributed among all types of real time and non-real time traffic by means of the Shapley value of a bankruptcy game associated to the resource allocation problem. In the lower level, the physical resource blocks are allocated to each particular traffic, but respecting the total bandwidth assigned to each type of traffic in the upper level. The allocation methods in the lower level are based on delays, priorities or optimization problems.

### 16.2.3 Bandwidth Allocation in Heterogeneous Mobile Networks

Heterogeneous mobile networks are another field in which cooperative games could be useful to provide solutions to some of their radio resource management problems. The main characteristic of these wireless systems is the physical coexistence of several radio access technologies (RATs). These RATs have different technical characteristics and performance, but they are also complementary. Therefore, these different radio resources could be managed more efficiently if they could be used in coordination. Thus, a key point for these communication systems is the design of efficient joint radio resource management (JRRM) mechanisms in order to allocate the heterogeneous radio resources to users. For these kinds of systems, [26, 27, 51, 53] proposed a JRRM using a bankruptcy game approach but adapted to the complexity of such networks in order to manage situations of channel allocation with heterogeneous wireless networks with the ability of combining several technologies. This approach is also user-oriented as in Subsections 16.2.1 and 16.2.2. Therefore, the players are the users, the problem is once again dynamic and the estate comes from the total capacity of the network technologies, but taking into account different resources from different technologies cannot be combined in order to satisfy the demand of a particular user and the minimal needs for an acceptable quality of service in each technology. However, [67] propose a simpler and network-oriented scheme for heterogeneous wireless networks based on bankruptcy techniques. In this network-oriented scheme, the different RATs are the players, and the estate is determined by the requests of the system users. Therefore, the problem is static, in the sense that the players do not change over time, and every time a user requests service, a bankruptcy problem is solved. Consequently, a user receives a certain amount of resources from each of the different technologies. Since this approach is different from those presented in the previous sections, the Niyato and Hossain model [67] is briefly described below.

Consider a mobile communication system with several RATs which can be combined in order to provide a better performance to users. When a user requests a service to the system, then a combination of resources from different RATs is assigned to that user. The user can request different types of service, such as video, email, www, etc. Each of these types of service has different needs to obtain good QoS. These needs can be measured in terms of bandwidth. Therefore, when a user requests a certain service, a central controller determines how much bandwidth of each RAT is offered to that user. These amounts of bandwidth will depend on the type of service requested, the bandwidth available in each RAT and the level of congestion in the system. Hence, on the one hand, we have the bandwidth requested by the user ($b^{requested}$) and, on the other hand, the bandwidth offered by each RAT ($b_i^{RAT}$). Now, if $\sum b_i^{RAT} \geq b^{request}$, then the connection is accepted, otherwise the connection is rejected. Therefore, when a user is accepted in the system, we can define a bankruptcy problem $(N, b^{requested}, b^{RAT})$, where $N$ is the set of available RATs, the bandwidth requested $b^{requested}$ is the estate and the vector of offered bandwidth $b^{RAT}$ is the vector of claims, in order to determine what part of the requested bandwidth is served by each RAT. To do this, once again, the associated bankruptcy game can be defined and its Shapley value calculated. Moreover, since bankruptcy games are convex, the Shapley value will always be in the core of the games and, therefore, it will be a stable allocation [87].

Now, we consider the following simple example to show how Niyato and Hossain's model is applied.

**Example 16.3** *Consider a heterogeneous wireless system with three RATs,* $N = \{1, 2, 3\}$*. There are three types of service,* $S = a, b, c$*, with bandwidth needs of 100 Kbps, 200 Kbps and 300 Kbps, respectively. In normal traffic load situations, the bandwidth offered (in Kbps) to each type of service by each*

*RAT is shown in the following matrix:*

| $S/N$ | 1 | 2 | 3 |
|---|---|---|---|
| $a$ | 50 | 75 | 100 |
| $b$ | 100 | 125 | 150 |
| $c$ | 100 | 150 | 200 |

*Therefore, we will have a different bankruptcy problem for each type of service. For example, for the service of type a, we have that the bankruptcy problem is* $(N, 100, (50, 75, 100))$*, and the characteristic function of the associated cooperative game* $\left(N, v^{(100,(50,75,100))}\right)$ *is given as in (16.3), after some simple calculations, we have that (for the sake of simplicity, we use v instead of* $v^{(100,(50,75,100))}$*)*

$$v(\emptyset)=0; v(1)=v(2)=v(3)=0; v(12)=0; v(13)=25; v(23)=50; v(N)=100.$$

*And its Shapley value is* $\Phi^a = (20.83, 33.33, 45.83)$*. In the following matrix, we show how much bandwidth of each type of service is served by each RAT.*

| $\Phi^i$ | 1 | 2 | 2 |
|---|---|---|---|
| $a$ | 20.83 | 33.33 | 45.83 |
| $b$ | 54.17 | 66.67 | 79.17 |
| $c$ | 66.67 | 91.67 | 141.67 |

*Hence, each time a user requests a service, the Shapley value of a bankruptcy game is computed when she is accepted in the system.*

Some remarks are the following. On the one hand, Niyato and Hossain's model is network-oriented, i.e., the players are the different networks, while the models showed in Subsections 16.2.1 and 16.2.2 are user-oriented, i.e., the players of the game are the users of the communication system. On the other hand, Niyato and Hossain's model assumes the resource to be allocated is perfectly divisible, while [26, 27, 51, 53] consider that the resource to be allocated among the users is non-divisible, i.e., discrete, and they also use a user-oriented approach. From a computational complexity point of view, the application of the Shapley value is not a problem, because the number of RATs in a heterogeneous wireless system is very small.

### 16.2.4 Other Applications to Wireless Networks

The literature already mentioned in the previous subsections are just some examples of the applications of cooperative games, and in particular of the Shapley value, to wireless communications networks. However, we can find other interesting applications in the related literature. Below we present a brief review of these other applications, without the intention of being exhaustive, but trying to offer a picture of the applicability, potential and interest of the Shapley value in this field of telecommunications.

For wireless ad hoc networks, [11] design a routing protocol based on the Shapley value of a suitable cooperative game to avoid selfish behavior of nodes in this type of wireless networks. [95] also use the Shapley value for a better allocation of the radio resources and avoiding selfish attack. Moreover, the Shapley value is used in [70] to detect intrusions and reduce the number of false positives in this type of networks.

For wireless mesh networks (WMN), in [37] a mechanism based on the Shapley value is developed to ensure the fair allocation of throughput among the BSs in order to guarantee minimum throughput requirements of clients in congested WMNs. The cooperative game behind is defined taking into account throughput contributions and path congestion. In [38] a similar idea to the previous one is used to design a resource management mechanism in WMN. [50] study the problem of the collaboration between service providers by sharing radio resources in order to increase revenues or decrease costs, at the same time that a better service is provided to customers. They define a linear programming game and use its Shapley value for a fair allocation of the aggregate revenue. [33] study the distribution of subchannels in an OFDMA frame, and propose to use the Shapley value or the nucleolus [84] of a bankruptcy game as allocation mechanism of subchannels between the mesh routers. A game theoretical mechanism based on the Shapley value is designed in [5] to economically compensate mesh nodes when they collaborate with the mobile network operator in data offloading in WMN.

For Long Term Evolution (LTE) systems, the collaboration among LTE, DVB-NGH (Digital Video Broadcasting-Next Generation Handheld) systems and TV channels to offer mobile TV services is studied in [80]. Then they use the Shapley value to share the generated profit from cooperation. In [22] a power allocation mechanism in LTE systems to assign power to virtual operators (VOs) is proposed. This mechanism has two stages, the first stage is based on VCG (Vickrey-Clark-Groves) auctions, while the second stage is based on the Shapley value of a suitable cooperative game in order to guarantee a fair allocation among users of different VOs. In the same context as in the previous case, [23] consider bandwidth-power allocation to optimize system energy efficiency. The optimization problem they need to solve is extremely difficult but they use the Shapley value of a suitable bankruptcy game to obtain an easier way to solve the optimization problem. In [69] a market game is defined and its Shapley value is used as radio resource allocation mechanism in LTE networks. The problem of LTEs and WiFi systems using unlicensed spectrum bands is studied in [32]. They propose a NTU cooperative game and use its Shapley value to share the airtime in order to mitigate interferences.

In order not to make the revision of the literature too extensive and long, we conclude this subsection listing some of the literature related to the Shapley value applications to spectrum management, and energy efficiency and power control in wireless networks. Regarding spectrum management, some references in which the Shapley value is used to manage the radio spectrum are [28, 29, 41, 46, 71, 72, 74, 85]. Finally, some references in which the Shapley value has been used to approach energy efficiency and power control problems are [30, 40, 45, 48, 61, 97, 100].

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## 16.3 The Shapley Value in Internet Problems

Internet has experienced a spectacular growth since its appearance at the early seventies. The development of wireless technologies has been added to the foregoing, and both together have led to a paradigm shift in many contexts. This situation yields new and exciting problems in many different contexts. Some examples are new markets and business based on Internet, the relationship between Internet Service Providers (ISPs) and those managing different services (social networks, search engines...) offered by means of Internet, the management of Internet infrastructures to offer a better service and at the same time make the business more profitable, and also technical problems such as routing, multicast transmissions, network design, etc. Many of these problems involve more than one individual, so conflicts of interest can arise in different ways. Therefore, the use of Game Theory to analyze these problems may be more than adequate [18, 36, 66]. In particular, there will be problems in which the collaboration of the parties concerned will be important for their interests, and the benefits of this collaboration must be shared fairly so that the collaboration is effective. It is at this point, where the cooperative game theory and the Shapley value can play an essential role.

The following sections will present several examples of application of cooperative games, in particular of the Shapley value, to different problems that arise in the context of Internet.

### 16.3.1 Keyword Auctions in Search Engines on Internet

The Internet is increasingly the place where consumers search companies that offer a specific service in which they are interested. However, searches can produce irrelevant results for the users or unstructured company listings. For this reason the keyword market in search engines has experienced great growth in the last two decades. The market for keyword searches on Internet is usually based on auctions, in which companies bid to achieve a better position in the search list of certain keywords. In general, a ranking auction market (RAM) is made up of the Internet search service provider that offers prominent positions in the search listings for a keyword and a set of firms that want to occupy those positions for that keyword. Therefore, for each keyword we would have one of these markets. These prominent positions in the search listing are valuable for firms because more potential clients can click on their links, and thus to likely increase their sales. Each firm is interested in reaching a position as high as possible in the search listing, because the number of potential clicks depends on its position. There is not a limited number of positions to be shown in a search listing. Thus, all the firms will be included in the search listing. Therefore, we consider a market situation with one seller (the provider) who owns as many different objects (the positions in the search listing) as the number of buyers (the firms) who are interested in them.

Keywords auctions in search engines on Internet has attracted the attention of many researchers in recent years and there is a lot of literature on the subject. Two possible approaches from Game Theory can be considered to tackle these situations. The most often approach in the related literature involves analyzing the problem from a competitive point of view (see, for example, [1, 3, 21, 47, 99]). The second approach is to study these situations from a cooperative perspective, in which it is interesting to examine the cooperative and collusive behavior of the agents concerned [4].

Following [4], in a ranking auction market (RAM) the set of players can be divided into two disjoint sets $\{0\}$ and $B = \{1, 2, \ldots, n\}$, where 0 is the search service provider (SSP) on Internet and $B$ are the firms interested in being ranked for a particular keyword. The SSP offers the service of listing the firms according to a ranking when searching for a keyword on the Internet. To determine the position in the listing, each firm $i \in B$ bids to obtain the highest position. The number of clicks obtained depends on the position in the ranking. We assume that $f_1 \geq f_2 \geq \ldots \geq f_n \geq 0$, where $f_j$ is the number of clicks obtained in position $j$. The unitary revenue per click of firm $i$ is denoted by $r_i$ and we assume, without loss of generality, $r_1 \geq r_2 \geq \ldots \geq r_n > 0$. When firm $i$ bids $b_i$, his revenue will be $(r_i - b_i) f_{\sigma(i)}$, where $\sigma(i)$ denotes its position in the ranked search listing. The revenue for the SSP will be given by $\sum_{i \in B} b_i f_{\sigma(i)}$, i.e. the SSP receives the corresponding bid for each click received by the firms. Therefore, a RAM situation is described by $< 0, B, r, f >$. If the SSP and the firms collaborate to increase revenues, a fair distribution of income should be implemented to enable that collaboration. One way to obtain that fair allocation is to resort to cooperative games and calculate the Shapley value.

Let $< 0, B, r, f >$ be a RAM situation. The associated RAM game $(N, v^{(r,f)})$ is a cooperative game with set of players $N = 0 \cup B$ and characteristic function given by

$$v^{(r,f)}(0) = 0;\ v^{(r,f)}(S) = 0,\ \forall S \subseteq B;\ v^{(r,f)}(S \cup 0) = \sum_{i \in S} r_i f_{\tau(i)},\ \forall S \subseteq B, \tag{16.7}$$

where $\tau : S \to \{1, 2, ..., |S|\}$ is defined such that $\tau(i) \leq \tau(j)$ if $i < j$. This means that only the $|S|$ first positions are considered.

**Example 16.4** *Consider a RAM situation with the SSP and 3 firms. The revenue per click is, respectively, 7, 4 and 2. The number of clicks received on each position in the ranked search listing are 9, 6 and 3, respectively. The corresponding RAM game $(N, v^{(r,f)})$ is given by*

$$v^{(r,f)}(01) = 63;\ v^{(r,f)}(02) = 36;\ v^{(r,f)}(03) = 18;$$

$$v^{(r,f)}(012) = 87;\ v^{(r,f)}(013) = 75;\ v^{(r,f)}(023) = 48;$$
$$v^{(r,f)}(N) = 93;\ v^{(r,f)}(S) = 0,\ \textit{otherwise}.$$

*The Shapley value of this game is* $\Phi(N, v^{(r,f)}) = (50.5, 25.5, 12, 5)$. *It is easy to check that* $\Phi(N, v^{(r,f)})$ *belongs to the core of the game.*

As we can see in the definition of RAM games, these games have a nice structure. The following theorem shows that RAM games are (total) big boss games [64]. This is an interesting property because big boss games always have a non-empty core and, furthermore, the core structure of these games is a simple parallelotope.

**Theorem 16.1** *[4] Let* $< 0, B, r, f >$ *be a RAM situation. The associated RAM game* $(N, v^{(r,f)})$ *is a (total) big boss game.*

Since a RAM game $(N, v^{(r,f)})$ is a big boss game, its core is the parallelotope given by

$$Core(N, v^{(r,f)}) = \left\{ x \in \mathbb{R}^{n+1} : \begin{array}{c} 0 \le x_i \le r_i f_i - \sum_{j=i+1}^{n} r_j \left(f_{j-1} - f_j\right),\ \forall i \in B; \\ x_0 = \sum_{i \in B} r_i f_i - \sum_{i \in B} x_i \end{array} \right\}. \tag{16.8}$$

The Shapley value does not belong to the core of the RAM game, in general. For example, if we consider the RAM situation with two firms, such that $r_1 = 2$ and $r_2 = 1$, and $f_1 = 5$ and $f_2 = 1$, the Shapley value is $\Phi(N, v^{(r,f)}) = (6.17, 3.67, 1.17)$ which is easy to check that does not belong to the core of the game ($v^{(r,f)}(01) = 10 > 6.17 + 3.67 = 9.83$).

Although the structure of a RAM situation is relatively simple, the expression of the Shapley value for RAM games is the following:

$$\Phi_0(N, v^{(r,f)}) = \sum_{S \subseteq B} \gamma(S) \left( \sum_{\substack{j > 0 \\ j \in S}} r_j f_{\tau(j)} \right);$$

$$\Phi_i(N, v^{(r,f)}) = \sum_{\substack{S \cup \{0\} \subseteq N \\ i \in S \subseteq B}} \gamma(S) \left( r_i f_{\tau(i)} - \sum_{\substack{j > i \\ j \in S}} r_j \left( f_{\tau(j)-1} - f_{\tau(j)} \right) \right), \forall i \in B,$$

where $\gamma(S) = \frac{(|S|)!(n-|S|+1)!}{(n+1)!}$.

However, since the RAM game is a total big boss games, we know that the Shapley value guarantees each firm at least half of her marginal contribution to the grand coalition. Furthermore, there are two simple cases in which the Shapley value can easily be calculated and is in the core of the RAM game.

**Proposition 16.1** *[4] Let* $< 0, B, r, f >$ *be a RAM situation and* $(N, v^{(r,f)})$ *the associated RAM game.*

*a) If* $r_i = \gamma > 0$, $\forall i \in B$, *then* $\Phi_0(N, v^{(r,f)}) = \frac{\gamma}{n+1} \sum_{j=1}^{n}(n - j + 1)f_j$, $\Phi_i(N, v^{(r,f)}) = \frac{\gamma}{(n+1)n} \sum_{j=1}^{n} jf_j$, $\forall i \in B$.

*b) If* $f_j = \varphi > 0$, $\forall j = 1, 2, ..., n$, *then* $\Phi_0(N, v^{(r,f)}) = \frac{1}{2}\varphi\sum_{i\in B} r_i$, $\Phi_i(N, v^{(r,f)}) = \frac{1}{2}\varphi r_i$, $\forall i \in B$.

Proposition 16.1 describes two simple situations but realistic. The first corresponds to situations in which the differences in the earnings per click are negligible, and therefore it can be assumed that all the firms have the same profit per click. The second situation occurs when there is no significant effect on the position that is occupied in the list with respect to the number of clicks that are received, therefore, the non-existence of differences between positions can be accepted as reasonable. In both cases a great symmetry is observed in the allocation of profits among the players concerned when the Shapley value is used.

There are other two interesting issues in these situations where there are two different types of players. The first is how relevant is the position in the ranked search listing to obtain a higher payment in the allocation of the total profit generated from the cooperation. The concept associated with this question is rank-sensitiveness. This refers to how sensitive an allocation of the profit is to the distribution of clicks between the positions of a ranked search listing. As will be seen below, the used concept of rank-sensitiveness is close related to the concept of Lorenz domination. The second question is related to the possibility that firms collaborate with each other, leaving the SSP aside, to obtain a greater profit. This is related to the collusion concept, i.e. how to avoid competing between agents belonging to the same level in a competitive market.

Given two decreasing distributions of $K$ clicks among $n$ ordered positions, $f$ and $g$, it is said that $f$ is more *rank-sensitive* than $g$, if the following holds

$$\sum_{i=1}^{h} f_i \geq \sum_{i=1}^{h} g_i, \ \forall h = 1, 2, ..., n. \tag{16.9}$$

The specific question is: Which distributions of the clicks are more beneficial for the SSP and which ones for the firms? In the following proposition, this question is answered for the same cases as Proposition 16.1, when the Shapley value is used as the distribution rule.

**Proposition 16.2** *[4] Let* $< 0, B, r, f >$ *and* $< 0, B, r, g >$ *be RAM situations such that* $f$ *is more rank-sensitive than* $g$, *and* $(N, v^{(r,f)})$ *and* $(N, v^{(r,g)})$ *the associated RAM games. If* $r_i = \gamma$, $\forall i \in B$, *then* $\Phi_0(N, v^{(r,f)}) \geq \Phi_0(N, v^{(r,g)})$, $\Phi_i(N, v^{(r,f)}) \leq \Phi_i(N, v^{(r,g)})$, $\forall i \in B$.

Therefore, from a cooperative point of view, when the differences between the gains per click of the firms are negligible, the distribution of the clicks among of the positions in the ranked search listing is very relevant when the Shapley value is used as allocation rule. In the more general case, strong assumptions must be imposed on the revenues per click in order to achieve some result.

Finally, we analyze how the firms in a RAM situation can collude to enhance their position in the problem. After collaborating the firms have to distribute among them the gains obtained from cooperation and the Shapley value is then a good alternative to do it.

Let $< 0, B, r, f >$ be a RAM situation and let $(N, v^{(r,f)})$ be the associated RAM game. The dual game restricted to all players except the SSP, the firms (collusion) game, $(B, v^*)$ is given, for each $S \subseteq B$, by

$$v^*(S) = v^{(r,f)}(N) - v^{(r,f)}(N \setminus S) = \sum_{k=1}^{n-|S|} (r_k - r_{i_k}) f_k + \sum_{k=n-|S|+1}^{n} r_k f_k, \quad (16.10)$$

where $N \setminus S = i_1, i_2, \ldots, i_{n-|S|}$ such that $i_1 < i_2 < \ldots < i_{n-|S|}$.

The firms collusion game represents a pessimistic idea regarding the profits the colluders can obtain when they collaborate. Thus, the firms in $S$ assume that the firms outside $S$ will bid as much as possible, so they have to bid in order to compensate this.

In this case, the Shapley value also does not have a simple mathematical expression for its calculation. Nevertheless, if we consider the symmetric case, i.e. $r_i = \gamma$ for all $i \in B$, then the firms game is given, for each $S \subseteq B$, by

$$v^*(S) = \gamma \sum_{k=n-|S|+1}^{n} f_k. \quad (16.11)$$

Moreover, if the position in the ranking is not relevant, i.e., $f_j = \varphi$, for all $j = 1, 2, \ldots, n$, then the characteristic function form of the firms game is

$$v^*(S) = \varphi \sum_{i \in S} r_i. \quad (16.12)$$

In these two situations, the Shapley value of the game or other solutions are easy to compute. In the first case, by symmetry, each firm obtains the same, i.e. $\frac{v^*(N)}{n}$. In the second case, each firm obtains her marginal contribution to the grand coalition, i.e. $\varphi r_i$.

Some comments are as follows. Although in this case the calculation of the Shapley value is not simple, in general, in realistic situations there will be only a few firms, so its computation is not an obstacle to its use. Keyword auctions in search engines on Internet has usually been studied from a non-cooperative point of view, hence there are very few works in the literature from a cooperative perspective. In addition to the aforementioned [4], other related papers in the literature are the following. [98] propose a cooperation bid strategy for firms when a VCG mechanism is used by the SSP, then use an approximate Shapley value to distribute the obtained profit among firms involved. [89] proposes a tool for improving the profits of the firms without decreasing the profits of the SSP when a GSP (Generalized Second Price) auction is used. This tool is based on the nucleolus of a suitable cooperative game.

### 16.3.2 Collaboration among ISPs

An Internet Service Provider (ISP) is a company or organization that provides different Internet services to its customers. These services include, among many others, Internet access and Internet transit. Therefore, roughly speaking, Internet consists of many ISPs which may have connected their systems depending on bilateral agreements. A non-complete collaboration of all ISPs negatively affects the routing efficiency and the cost of providing the service, and consequently, to the provision of Internet services to users. Therefore, collaboration between ISPs should come quite naturally because all parties concerned benefit, the ISPs because they can improve their income, reduce costs and enhance their service, and the different types of users of Internet because they can receive a better service. It is clear that in order for this collaboration to take place, an appropriate and fair mechanism must be designed to distribute the revenues among the ISPs. One way to do this is through cooperative games and use the Shapley value as a distribution mechanism. [39, 54, 55, 56] study different problems of collaboration among ISPs by means of cooperative games and they then apply the Shapley value as allocation mechanism of the profits generated from cooperation.

Following the ideas in [55], we introduce the following simpler model of ISP collaboration. Let us consider that there are a finite set $N = \{1, 2, .., n\}$ of ISPs which provide Internet services. Each ISP $i$ has a set of users given by $ISP(i)$. The users of each ISP generate a traffic in the network that is attended by the ISPs. This traffic will provide an income to the ISPs for the provision of the service, but also some costs due to traffic routing. We consider that the revenue for each unit of traffic is a positive fixed amount $\alpha$, because of the competence between the ISPs [55]. The cost of routing will depend on the path followed by the traffic, so we distinguish three types of traffic for each ISP. Internal traffic in the ISP $i$, i.e. between users of that ISP. In this case the cost is given by $\beta(i)$ which corresponds to the average cost of the optimal routing within the ISP $i$. Traffic from the ISP $i$ to other ISPs, i.e. from users of the ISP $i$ to users in other ISPs. In this case the cost is given by $\gamma(i)$ that corresponds to the average cost of the routing between ISPs which depends on bilateral agreements between ISPs and the quality of the route. Finally, traffic through the ISP $i$ from other ISPs. In this case the ISP $i$ receives a fixed compensation $\delta$ for using its infrastructure and the associated cost is given by $\beta(i)$. Therefore the profit for each ISP $i$, $i \in N$, is given by

$$(\alpha - \beta(i)) \sum_{j,k \in ISP(i)} r_{jk} + (\alpha - \gamma(i)) \sum_{j \in ISP(i), k \notin ISP(i)} r_{jk} + (\delta - \beta(i)) \sum_{j \notin ISP(i)} \sum_{k \in \cup_{h \in N} ISP(h)} \eta(i, jk) r_{jk}, \quad (16.13)$$

where $r_{ij}$ is the traffic between users $i$ and $j$, and $\eta(i, jk) = 1$ if the ISP $i$ belongs to the route from user $j$ to user $k$, and 0 otherwise. For simplicity, we can consider the aggregate traffic between ISPs, so that Eq. (16.13) becomes:

$$(\alpha - \beta(i))\, r_{ii} + (\alpha - \gamma(i)) \sum_{j \in N \setminus i} r_{ij} + (\delta - \beta(i)) \sum_{j \in N \setminus i} \sum_{k \in N} \eta(i, jk) r_{jk}, \quad (16.14)$$

where $r_{ij}$ is the traffic between ISPs $i$ and $j$, and $\eta(i, jk) = 1$ if the ISP $i$ belongs to the route from ISP $j$ to ISP $k$, and 0 otherwise.

Now, if a subset $S \subseteq N$ of ISPs cooperate, the (average) optimal routing cost within the subnetwork defined by the ISPs in $S$ will be $\beta(S)$ and the (average) routing cost from users in $S$ to users outside $S$ will be $\gamma(S)$. Then the profit of coalition $S$ is given by

$$(\alpha - \beta(S)) \sum_{i,j \in S} r_{ij} + (\alpha - \gamma(S)) \sum_{j \in S, k \in N \setminus S} r_{jk} + (\delta - \beta(S)) \sum_{j \in N \setminus S} \sum_{k \in N} \eta(S, jk) r_{jk}, \quad (16.15)$$

where $r_{ij}$ is the traffic between ISPs $i$ and $j$, and $\eta(S, jk) = 1$ if some ISP in $S$ belongs to the route from ISP $j$ to ISP $k$, and 0 otherwise. Hence, an ISP collaboration situation is defined by the 7-tuple $(N, r, \alpha, \beta, \gamma, \delta, \eta)$, where $N$ is the set of ISPs, $\alpha$ is the revenue per unit of traffic, $\beta : 2^N \to \mathbb{R}_+$ is an average optimal routing cost function, $\gamma : 2^N \to \mathbb{R}_+$ is an average routing cost function, $\delta$ is a kind of toll to use network infrastructures and $\eta$ is a function defining the interconnection routing between ISPs. Therefore, we can define a cooperative game $(N, v^{ISP})$ associated with the ISP collaboration situation as follows

$$v^{ISP}(S) = (\alpha - \beta(S)) \sum_{i,j \in S} r_{ij} + (\alpha - \gamma(S)) \sum_{j \in S, k \in N \setminus S} r_{jk} + (\delta - \beta(S)) \sum_{j \in N \setminus S} \sum_{k \in N} \eta(S, jk) r_{jk}, \ \forall S \subseteq N. \quad (16.16)$$

Once we have defined the game we can divide the total profit among ISPs using the Shapley value. Nevertheless, collaboration among all the ISPs will not be possible if the distribution of the profit is not fair, in the sense that there is a group of ISPs that can obtain a greater profit by themselves by leaving the grand coalition. Therefore, the allocation must be stable in the sense that it belongs to the core of the game. In order to analyze the nonemptiness of the core of the ISP game, we need to establish the relationship between the different parameters defining the problem. The following theorem shows that, under very reasonable conditions, the ISP game is convex, and hence the Shapley value belongs to the core of the game [87].

**Theorem 16.2** *Let $(N, r, \alpha, \beta, \gamma, \delta, \eta)$ be an ISP situation such that the following conditions hold*

1. *$\alpha > \delta$ and the routing costs are negligible.*
2. *The network of ISPs is a fully connected graph.*

*Then, the associated ISP game $(N, v^{ISP})$ is convex.*

*Proof.* Condition 1 implies that $\beta(S) = 0$, $\forall S \subseteq N$ and Condition 2 implies that all routes between ISPs are direct, therefore $\gamma(S) = \delta$, $\forall S \subseteq N$. Consider now $S \subseteq T \subseteq N \setminus i$, then the marginal contribution of ISP $i$ to coalitions $T$ and $S$ are given by

$$v^{ISP}(T \cup i) - v^{ISP}(T) = \alpha \left( r_{ii} + \sum_{j \in T} r_{ij} + \sum_{j \in T} r_{ji} \right) + (\alpha - \delta) \sum_{j \in N \setminus (T \cup i)} r_{ij} + \delta \sum_{j \in N \setminus (T \cup i)} r_{ji} - (\alpha - \delta) \sum_{j \in T} r_{ji} - \delta \sum_{j \in T} r_{ij}; \quad (16.17)$$

$$v^{ISP}(S \cup i) - v^{ISP}(S) = \alpha \left( r_{ii} + \sum_{j \in S} r_{ij} + \sum_{j \in S} r_{ji} \right) + (\alpha - \delta) \sum_{j \in N \setminus (S \cup i)} r_{ij} + \delta \sum_{j \in N \setminus (S \cup i)} r_{ji} - (\alpha - \delta) \sum_{j \in S} r_{ji} - \delta \sum_{j \in S} r_{ij}. \quad (16.18)$$

If we rewrite Eq. (16.17) and Eq. (16.18), we have that

$$\alpha \left( r_{ii} + \sum_{j \in T \setminus S} r_{ij} + \sum_{j \in T \setminus S} r_{ji} + \sum_{j \in S} r_{ij} + \sum_{j \in S} r_{ji} \right) + (\alpha - \delta) \sum_{j \in N \setminus (T \cup i)} r_{ij} + \delta \sum_{j \in N \setminus (T \cup i)} r_{ji} - (\alpha - \delta) \left( \sum_{j \in T \setminus S} r_{ji} + \sum_{j \in S} r_{ji} \right) - \delta \left( \sum_{j \in T \setminus S} r_{ij} + \sum_{j \in S} r_{ij} \right); \quad (16.19)$$

$$\alpha\left(r_{ii}+\sum_{j\in S}r_{ij}+\sum_{j\in S}r_{ji}\right)+(\alpha-\delta)\left(\sum_{j\in N\setminus(T\cup i)}r_{ij}+\sum_{j\in T\setminus S}r_{ij}\right)+$$

$$\delta\left(\sum_{j\in N\setminus(T\cup i)}r_{ji}+\sum_{j\in T\setminus S}r_{ji}\right)-(\alpha-\delta)\sum_{j\in S}r_{ji}-\delta\sum_{j\in S}r_{ij}. \quad (16.20)$$

Now it is straightforward to observe that $v^{ISP}(T\cup i)-v^{ISP}(T)\geq v^{ISP}(S\cup i)-v^{ISP}(S)$, therefore the game $(N,v^{ISP})$ is convex. ■

Under the conditions of Theorem 16.2, the Shapley value has a nice and simple expression as the following theorem shows.

**Theorem 16.3** *Let* $(N,r,\alpha,\beta,\gamma,\delta,\eta)$ *be an ISP situation such that the following conditions hold*

*1.* $\alpha>\delta$ *and the routing costs are negligible.*
*2. The network of ISPs is a fully connected graph.*

*Then the Shapley value of the associated ISP game* $(N,v^{ISP})$ *is given by*

$$\Phi_i(N,v^{ISP})=\alpha\sum_{k\in N}r_{ik}+\delta\sum_{k\in N\setminus i}\left(r_{ki}-r_{ik}\right),\ \forall i\in N. \quad (16.21)$$

*Proof.* Condition 1 and Condition 2 imply that for all $S\subseteq N$

$$v^{ISP}(S)=\alpha\sum_{i,j\in S}r_{ij}+(\alpha-\delta)\sum_{i\in S}\sum_{j\in N\setminus S}r_{ij}+\delta\sum_{j\in N\setminus S}\sum_{i\in S}r_{ji},\ \forall S\subseteq N. \quad (16.22)$$

Now, we can decompose the game $v^{ISP}$ into the sum of three games as follows

$$v^{ISP,B}(S)=\alpha\sum_{i,j\in S}r_{ij};$$

$$v^{ISP,F}(S)=(\alpha-\delta)\sum_{i\in S}\sum_{j\in N\setminus S}r_{ij};$$

$$v^{ISP,T}(S)=\delta\sum_{j\in N\setminus S}\sum_{i\in S}r_{ji};$$

$$v^{ISP}(S)=v^{ISP,B}(S)+v^{ISP,F}(S)+v^{ISP,T}(S),\ \forall S\subseteq N. \quad (16.23)$$

Since the Shapley value satisfies the additivity property, then the Shapley value of $(N,v^{ISP})$ is exactly the sum of the Shapley values of the three games. By using the random order property, together with the symmetry and efficiency properties of the Shapley value, we have that

$$\Phi_i(N,v^{ISP,B})=\frac{1}{2}\alpha\sum_{k\in N}\left(r_{ik}+r_{ki}\right),\ \forall i\in N; \quad (16.24)$$

$$\Phi_i(N, v^{ISP,F}) = \frac{1}{2}(\alpha - \delta) \sum_{k \in N \setminus i} (r_{ik} - r_{ki}), \; \forall i \in N; \tag{16.25}$$

$$\Phi_i(N, v^{ISP,T}) = \frac{1}{2}\delta \sum_{k \in N \setminus i} (r_{ki} - r_{ik}), \; \forall i \in N. \tag{16.26}$$

Finally, by adding Eq. (16.24), Eq. (16.25) and Eq. (16.26), after some calculations we obtain the result. ■

Under the conditions of Theorems 16.2 and 16.3, the Shapley value allocates to each ISP all the profit generated by the traffic from its users to all other users in the system, including themselves, but this profit allocation is compensated with more or less profit depending on the balance between the in-out traffic. The games defined in Eq. (16.22) have the following interpretation. The game $(N, v^{ISP,B})$ provides the profit obtained by each coalition by means of the traffic generated *"between"* users belonging to the ISPs in the coalition. The game $(N, v^{ISP,F})$ gives the profit obtained by each coalition through the traffic generated *"from"* the users of the ISPs in the coalition to the users of other ISPs. Finally, the game $(N, v^{ISP,T})$ provides the profit obtained by each coalition through the traffic generated from the users of other ISPs *"to"* the users of ISPs in the coalition.

Some related literature is the following. The model introduced in this subsection is simpler than the model introduced in [55] although both study the benefit of collaboration between ISPs regarding traffic between end-users. [54, 56] study the problem of collaboration between ISPs but differentiating between types of ISPs, content, transit and eyeball ISPs. [39] analyze the problem of collaboration among ISPs that provide video-on-demand services. [9, 49] study how to allocate profits between content creators and Internet TV Service Providers. [65] study the income distribution in multi-domain/multi-provider networks. In all these papers, cooperative games are defined and their Shapley values are used as profit allocation mechanism.

### 16.3.3 Some Additional Applications to Internet Problems

Many of the problems included in this subsection are related to computer networks in general, but we consider Internet as the best well-known example of computer network. [7] study the problem of which subset of a network nodes may be switched off, without affecting too much data traffic, in order to save energy. They define a cooperative game to describe the problem and propose the Shapley value as a criticality index for deciding which nodes to be switched off. We can also find applications of the Shapley value to peer-to-peer (P2P) networks. Two examples of these applications are [16, 62].

## 16.4 The Shapley Value in Communication Routing Problems

Some of the problems studied in the literature mentioned above include routing, but there it is not the main purpose of the application of Game Theory. Next we consider communication problems in which Game Theory has been applied to design or control routing mechanisms in communication networks.

Multicasting is a form of routing which consists of creating a directed tree that connects the source to all the receivers, thus avoiding the sending of duplicate packets from the source to the receivers. [24] study cost-sharing mechanisms for multicast transmission. They consider two elements to analyze the problem, the utility obtained by the receivers and the cost of sending the packets through the directed tree. They propose the Shapley value as one of the most suitable solutions in this context. [15] also study the multicast routing by using cost sharing of a suitable multicast game which is related to the Steiner Tree problem. They analyze the egalitarian distribution of each edge cost among its users, this distribution coincides with the Shapley value of the game. [8] analyze the performances of a family of cost sharing methods for multicast routing games, propose some reasonable properties to be satisfied by a solution and prove that the only method satisfying all the properties is the Shapley value. However, the Shapley value does not have a good performance, for this reason other cost sharing mechanisms are evaluated.

[90] design a routing algorithm based on the Shapley value, which is used to determine the probability of each node in the next hop of the path between two nodes. For that, they define cooperative games based on the delay of edges and the business level of nodes, these games are updated every time new information is available. Then the Shapley values of these games provide the routing table for each node. [42] designs a routing algorithm for wireless networks based on the Shapley value. This routing algorithm provides a high performance for wireless networks regarding energy efficiency and load balancing. Furthermore, it is flexible enough to be adaptative to dynamic network conditions. Finally, [88] propose a routing forwarding mechanism for Information Centric Networking (ICN) based on the Shapley value of the so-called alliance game, which is defined taking into account the business degree of the nodes and the delay on the links.

We briefly conclude this section by mentioning some applications of the Shapley value to other problems that also appear within the field of Telecommunications. [17, 91] apply the Shapley value for solving power allocation problems in radar networks. [10, 92] use the Shapley value for analyzing measurements and routing protocols in wireless sensor networks. To finish with the applications, the Shapley value has also been applied to cloud computing problems in [19, 101].

## 16.5 Conclusions

In this chapter we have shown several applications of Game Theory to different problems that arise in the field of Telecommunications, in particular in wireless networks, on the Internet and in routing problems in communications networks. The non-exhaustive extension of the literature reviewed gives us an idea of the interest that the cooperative approach has to solve many telecommunications problems and, in particular, the Shapley value as a paradigm of distributive justice or fairness principle. All this despite the difficulties of computational complexity that approaches from the perspective of cooperative games have, but that in many situations those can be overcome because of the structure of the problem under analysis, as seen in the previous pages. In any case, the relevance and interest of the Shapley value is demonstrated and its utility to solve different problems in the field of Telecommunications has been highlighted.

To conclude, paraphrasing the words of Michael Maschler [60], we have tried to answer the question "How can Game Theory help telecommunication engineers and managers in pursuing their goals?" Or more precisely "How can the Shapley value help telecommunication engineers and managers in solving their technical and management problems?"

## 16.6 Acknowledgments

First of all, the author thanks one anonymous referee for her helpful comments and suggestions to improve the contents of the paper. Financial support from the Ministerio de Economia y Competitividad (MINECO) of Spain and FEDER funds under project MTM2014-54199-P and from Fundacion Seneca de la Region de Murcia through grant 19320/PI/14 are gratefully acknowledged.

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# Chapter 17

## The Shapley Rule for Loss Allocation in Energy Transmission Networks

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## 17.1 Introduction

The analysis and modeling of different aspects of energy transmission networks is a prevalent topic in papers across a wide variety of disciplines. In particular, one important aspect is the study of energy losses in these networks. This issue was recently tackled in [1], where the authors say:

> A common problem is that, in virtually any network, there are losses whose sources are normally difficult to identify. Thus, one must anticipate them so that they do not lead to deficit in the system. In many cases the transmission network is owned by different agents and, typically, the authorities that manage the network decide how much energy each agent is allowed to lose. This decision should follow some general principles, which would then appear in the relevant regulations. For instance, one would like that the loss allocated to each agent takes into account characteristics of the agents, such as the size of its subnetwork or the amount of energy managed.

Although the analysis of this paper could be applied to any energy transmission network, we develop it using a gas transmission network because our leading example is the Spanish gas transmission network. It is worth noting that the use of natural gas as a source of energy has been rapidly increasing over the past few years. According to a review by British Petroleum in 2013 ([6]), the consumption of natural gas worldwide was around the 23.9% of global primary energy consumption. A more recent report published by Enerdata in 2017 (see [8]) also reports a share over 20% of natural gas.

Going back to the issue of energy losses and, more specifically, energy losses in gas networks, [1] go on to say:

> Different networks have different estimates on the percentage of gas/electricity that is lost during transportation. In Spain, for instance, this estimate is 0.2% for the gas transported in the high pressure gas network and similar figures have been reported in other countries. In order to prevent

the ensuing monetary losses, a standard approach in energy networks is to withhold at the entry points a pre-set percentage of the gas/electricity entering the network; by doing this, the energy companies that use the network for transportation are the ones effectively assuming the associated cost in the first instance. In particular, in the Spanish high pressure gas network the pre-set percentage withhold to anticipate the estimated losses is precisely 0.2%. In monetary terms, the annual cost of the gas entering the Spanish gas network is around 12000 millions of Euro, which results in approximately 25 millions of Euro in losses in the transmission network.
It is precisely at this point where the main question we try to address in this paper arises. Since a gas network is typically owned by different agents, called haulers, it must be decided how to share the withhold gas among them. More precisely, it must be decided, for each agent, the percentage of the gas entering his subnetwork that he can lose. Note that it is not possible to let each agent lose the same percentage that has been withhold for the entire network. Since most gas entering the network crosses several subnetworks, this naive approach would result in allowing the agents to lose, in aggregate, more gas than the withhold amount.

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The Spanish regulation presents an incentive mechanism to induce haulers to reduce the losses (see [5, page 106656]). On a yearly basis the following values are computed: $A_h$ is the 'allowed' loss assigned to each hauler $h$; $L_h$ is the real loss of each hauler $h$ (it is computed as the balance between entries and exits of gas in his subnetwork); given a price $p$ per unit of gas, the haulers pay $p\,(L_h - A_h)$ when $L_h - A_h > 0$ and receive $\frac{p}{2}\,(A_h - L_h)$ when $L_h - A_h \leq 0$. Therefore, the definition of the rule to assign the 'allowed' losses is a relevant issue for the management of gas transmission networks.
Regulation (EC)(no. 55/2003, [13]), from the European Union mentions some principles that should be followed by the national and international regulations regarding the natural gas market. The analysis in [1] starts with the definition of four different allocation rules for energy losses, which are then compared conducting a thorough axiomatic analysis that builds upon the above principles. Besides, an application using data from the Spanish gas transmission network is presented, comparing the allocation proposed by the different rules. The main conclusion of that paper is that the rule that behaves worst (in terms of the EU principles) is the so-called *aggregate edge's rule.*

This rule was replaced in Spain by the *flow's rule* because of the strong opposition of the small haulers (on the grounds that it favored big haulers). The *proportional tracing rule* and the *edge's rule* behave better than the flow's rule (in terms of the EU principles), with the former seeming slightly preferable.

In this paper we present a new rule, the Shapley rule, obtained as the Shapley value of a cooperative game with transferable utility that can be associated to each gas loss problem. Then, we closely follow the analysis in [1]. We first study the axiomatic behavior of the Shapley rule with respect to the same set of axioms, finding that this new rule is not as good as those performing best in the original paper: The proportional tracing rule and the edge's rule. Second, we find that, in the application from the Spanish network, the allocation proposed by the Shapley rule is very similar to that proposed by the proportional tracing rule. Motivated by this similarity, we build upon the real data from the Spanish network to conduct a simulation analysis over 10000 randomly generated modifications of it. The analysis of the resulting loss allocations shows that the average correlation between the allocation proposed by the Shapley rule the one proposed by the proportional tracing rule is over 0.99, while the minimum correlation between these two rules found in those 10000 simulations is still over 0.9. This reinforces the idea that there must be some common mechanism underlying both rules, which should definitely be explored more deeply. This is especially so if we take into account that the second highest average correlation, although still very high, is at 0.97, whereas the second highest minimum correlation for any other pair of tariffs across the 100000 simulations is just over 0.6.

The use of the Shapley value in this kind of settings is not new. It has already been used in many allocation problems. The basic idea is always the same. One starts associating to each problem a cooperative game with transferable utility. Then, the Shapley rule for the given problem is defined as the Shapley value of the associated cooperative game. This approach has been followed, for instance, in airport problems (see [11]), queueing problems (see [12] and [7]), and minimum cost spanning tree problems (see [10] and [2]). The current paper contributes to this strand of literature by defining, and studying, the Shapley rule for energy transmission networks.

In the associated cooperative game with an energy transmission network, the agents are the haulers. The value of a coalition $T$ of haulers should be defined as the loss that haulers in $T$ can have by "themselves". Several definitions are possible. We give a definition inspired in the approach taken in [9] for flow games. In their model, there are also a set of agents who own the different edges of the network and the value of a group of agents $T$ is defined as the maximum amount of flow that can be transported (from the source to the sink) using only edges belonging to agents in $T$. We apply the same principle to our model. We define the value of coalition $T$ as loss associated with the maximum demand that can be satisfied using only edges of haulers in $T$, *i.e.*, the loss associated with the maximum amount of gas that can be transported from suppliers to consumers without exceeding the capacities and demands of suppliers and consumers, respectively.

The paper is structured as follows. In Section 17.2 we summarize the relevant characteristics of the management and operation of a gas transmission network and the formal mathematical model. In Section 17.3 we introduce the Shapley rule. In Section 17.4 we present different properties, motivated by some principles stated in EU regulations. In Section 17.5 we discuss the behavior of the Shapley rule with respect to these properties and principles. In Section 17.6 we present the application to the Spanish gas transmission network.

## 17.2 The Model

In this section we introduce the mathematical model associated with a loss energy problem. In order to make this paper self-contained, we formally introduce all the elements of the model, but we do so in a very concise way. Also, to facilitate the comparison with the analysis in [1], we closely follow the notations and formal definitions in that paper. We refer the reader to sections 2 and 3 of [1] for a more detailed explanation of all concepts introduced below.

Since our motivating example comes from the Spanish gas transmission network, the exposition is carried out for gas networks. Yet, our analysis and results may be applied to other energy transmission networks. As far as this paper is concerned, a gas network may be seen as a graph, composed of nodes and edges. There are three types of nodes: Demand nodes, in which some gas leaves the network; supply nodes, in which some gas enters the network; and the rest of the nodes, in which the gas that enters and leaves coincide. Edges represent pipes. Each pipe belongs to a hauler and a hauler may own several pipes.

In order to develop our analysis, we assume that, for each pipe, its volume and the amount of gas flowing through it are known. The flow represents the total amount of energy each pipe carries during a given period of time (which we measure in GWh/d). The Technical System Manager decides how the gas flows through the network. The first step is to obtain the demands at the different nodes. Then, following some criteria, the Technical System Manager decides the gas that should be introduced at each supply node and how the gas should be routed so that the total demand is fulfilled. The volume of a pipe just depends on its length and its diameter. It is worth noting that the total amount of gas that can flow through a pipe is not just a function of its volume. Since natural gas is a compressible fluid, the capacity of a pipe crucially depends on the construction materials and the maximum pressure they can support.

A flow configuration, based on some realistic scenario of demands, is an important part of the input to a loss allocation rule. In energy networks, it is usual to work with reference scenarios with high/peak demand. This is the case of the data of the Spanish gas network analyzed in Section 17.6. The way to choose the reference scenario, although crucial to obtain cost-reflective loss allocations, is not important for the theoretical analysis of this paper. Once a methodology is chosen to allocate the losses, it can be applied to individual scenarios and also to compute averages over sets of reference scenarios to get more representative allocations.

Given a gas network configuration, we can estimate the total loss of the system, say $L$, during a given year. This total loss $L$ has to be assigned to the haulers. Let $A_h$ be the loss assigned to hauler $h$. Let $L_h$ be the real loss measured in the subnetwork of hauler $h$ during this year. In the Spanish network, given a price $p$ per unit of gas, the haulers pay $p(L_h - A_h)$ when $L_h - A_h > 0$ and receive $\frac{p}{2}(A_h - L_h)$ when $L_h - A_h \leq 0$.

### 17.2.1 The Mathematical Model

Let $U = \{1, 2, 3, \ldots\}$ be the (infinite) set of possible *nodes*. A *graph* is a pair $g = (N, E)$ where $N \subset U$ is the (finite) set of nodes and $E$ is a set of edges, defined as ordered pairs in $N$, *i.e.*, $E \subset \{(i, j) : (i, j) \in N \times N \text{ and } i \neq j\}$. More generally, a multigraph is also a pair $g = (N, E)$, but where the set of edges is a multiset $E \subset N \times N \times \mathbb{N}$. In particular, we say that two edges $(i, j, n)$ and $(i', j', n')$ are part of a multiedge if $i = i'$, $j = j'$, and $n \neq n'$. We say that $E$ does not have multiedges if the projection of $E$ on $N \times N$ is injective.

A *path* in $g$ between $i$ and $j$ is a sequence of $l > 1$ nodes $\{k_1, \ldots, k_l\}$ such that $i = k_1$, $j = k_l$, and $(k_{q-1}, k_q) \in E$ for all $q \in \{2, \ldots, l\}$. A *simple path* in $g$ between $i$ and $j$ is a path where all nodes are different. For the sake of notation, we often identify a path with the set of edges $\{(k_{q-1}, k_q)\}_{q \in \{2,\ldots,l\}}$. A graph $g$ is *connected* if for each pair of nodes $i$ and $j$ there is a path between $i$ and $j$ in the undirected version of $g$. We omit the trivial extension of these definitions for multigraphs.

A *gas loss problem* $G$ is a 5-tuple $(g, v, f, \mathcal{H}, \alpha)$ consisting in the following elements:

1. The multigraph $g = (N, E)$ represents the *gas network*.

   We assume that $g$ is a directed and connected graph without cycles, where the directions of the edges are determined by the gas flows in the given scenario. If $e = (i, j, l) \in E$, then there may be gas flowing from $i$ to $j$.

2. $v = (v_e)_{e \in E}$ where, for each $e \in E$, $v_e > 0$ denotes the *volume* of $e$.

3. $f = (f_e)_{e \in E}$ is the *flow configuration* where, for each $e \in E$, $f_e \geq 0$ denotes the flow of gas through $e$. We assume that $\sum_{e \in E} f_e > 0$.

4. $\mathcal{H} = (H, \{E_h\}_{h \in H})$ is the *hauler structure*, where $H$ denotes the set of haulers and, for each $h \in H$, $E_h$ denotes the (possibly empty) set of edges of hauler $h$. In particular, $E = \bigsqcup_{h \in H} E_h$.

5. $\alpha \in [0, 1]$ denotes the *proportion of gas* allowed to be lost by the set of haulers.

For the sake of notation, graphs are used for most of the exposition, with multigraphs being used only when they make a difference. Further, we assume that the set $H$ is infinite, although in each given problem only a finite number of them will own edges. This is convenient in the study of some properties of allocation rules. Yet, in the examples we just mention those haulers who own some edge in the given problem.

The example below is borrowed from [1]:

**Example 17.1** *Let $G$ be the gas problem where*

1. $g = (N, E)$*, where the set of nodes is* $N = \{s_1, s_2, 1, c_1, c_2\}$ *and the set of edges is* $E = \{(s_1, 1), (1, c_1), (s_2, 1), (1, c_2)\}$.

2. $v_{(s_1,1)} = v_{(s_2,1)} = v_{(1,c_1)} = v_{(1,c_2)} = 100$.

3. $f_{(s_1,1)} = 20$, $f_{(s_2,1)} = 80$, $f_{(1,c_1)} = 60$, *and* $f_{(1,c_2)} = 40$.

4. $\mathcal{H} = (H, \{E_h\}_{h \in H})$*, where* $H = \{h_1, h_2, h_3\}$ *and* $E_{h_1} = \{(s_1, 1), (1, c_1)\}$, $E_{h_2} = \{(s_2, 1)\}$*, and* $E_{h_3} = \{(1, c_2)\}$.

5. $\alpha = 0.1$.

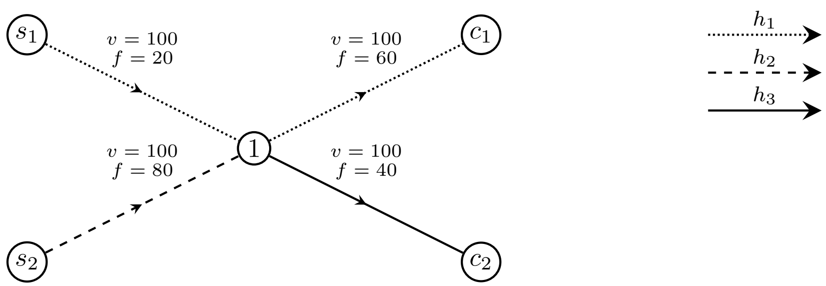


**FIGURE 17.1**: Representation of the gas problem in Example 17.1.

*This gas problem is represented in Figure 17.1 and will be used as a running example to illustrate some concepts and definitions.* ◊

We now introduce some terminology. For each $i \in N$, we denote by $Q_i$ the gas balance at node $i$, *i.e.*, the amount of gas leaving node $i$ minus the amount of gas arriving at node $i$. Formally,

$$Q_i = \sum_{(i,j)\in E} f_{(i,j)} - \sum_{(j,i)\in E} f_{(j,i)}.$$

The set of *suppliers* $S \subset N$ of the gas problem $G$ is defined as the set of nodes $s \in N$ such that $Q_s > 0$. On the other hand, the set of *consumers* $C \subset N$ is defined as the set of nodes $c \in N$ such that $Q_c < 0$. For the rest of nodes $i \in N \setminus (S \cup C)$, we have that $Q_i = 0$. We make the natural assumption that total supply and total demand are balanced, namely,

$$\sum_{s\in S} Q_s = -\sum_{c\in C} Q_c \quad \text{or, equivalently,} \quad \sum_{i\in N} Q_i = 0.$$

The total loss allowed to the haulers is $L = \alpha \sum_{s\in S} Q_s$. The *flow carried* by each hauler $h \in H$, denoted by $f_h$, is defined as the gas that reaches one of the edges of hauler $h$ from outside, that is, from some provider $s \in S$ or from an edge of another hauler. Formally, we first define, for each node $i \in N$ and each hauler $h \in H$, $Q_i^h = \max\{\sum_{(i,j)\in E_h} f_{(i,j)} - \sum_{(j,i)\in E_h} f_{(j,i)}, 0\}$; if no edge of hauler $h$ contains node $i$ we define $Q_i^h = 0$. Then, for each $h \in H$,

$$f_h = \sum_{i\in N} Q_i^h.$$

In particular, $f_h = 0$ whenever $E_h = \emptyset$.[1]

Given a gas problem $G$ and a pair $(s,c) \in S \times C$, we define $P(s,c)$ as the set of simple paths in $g$ from $s$ to $c$. We denote by $P(S,C)$ the set of all simple paths from suppliers to consumers. Namely,

$$P(S,C) = \bigcup_{(s,c)\in S\times C} P(s,c).$$

We now want to define an important notion for our analysis that we call *hauler's influence network*, which, given a hauler $h$, would contain all edges whose gas might either reach some edge in $E_h$ or come from some edge in $E_h$. Formally, for each $h \in H$, we define $\mathcal{N}^h = (g^h, v^h, f^h)$, as the subnetwork of

[1]There are alternative ways to define the notion of "flow carried by a hauler", but, as far as our analysis is concerned, they would lead to similar results. Our formulation is the one implicit in the Spanish Regulations ([3, 5]).

$(g, v, f)$ where $g^h = (N^h, E^h)$ and

$$\begin{aligned} E^h &= \{e \in E: \text{ there is } p \in P(S,C) \text{ with } e \in p \text{ and } p \cap E_h \neq \emptyset\}, \\ N^h &= \{i \in N : i \in e \text{ for some } e \in E^h\}, \\ v^h &= (v_e)_{e \in E^h}, \\ f^h &= (f_e)_{e \in E^h}. \end{aligned}$$

Sometimes we slightly abuse language and refer to an *edge's influence network*, to mean the influence network that would have a hauler who owned only that edge.

**Example 17.1** *(cont.) Going back to the gas problem in Figure 17.1, we have that $Q_{s_1} = 20$, $Q_{s_2} = 80$, $Q_1 = 0$, $Q_{c_1} = -60$, and $Q_{c_2} = -40$. Thus, $S = \{s_1, s_2\}$ and $C = \{c_1, c_2\}$. Table 17.1 contains the different $Q_i^h$ flow balances and Figure 17.2 represents the influence networks corresponding to this example.*

| $Q_i^h$ | $s_1$ | $s_2$ | 1 | $c_1$ | $c_2$ | $f_h$ |
|---|---|---|---|---|---|---|
| $h_1$ | 20 | 0 | 40 | 0 | 0 | 60 |
| $h_2$ | 0 | 80 | 0 | 0 | 0 | 80 |
| $h_3$ | 0 | 0 | 40 | 0 | 0 | 40 |

**TABLE 17.1**: Flow balances of the haulers of Example 17.1. ◊

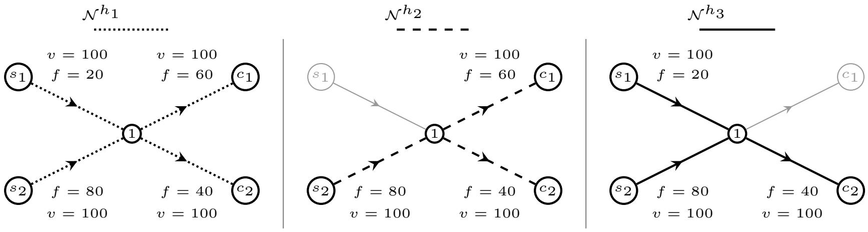


**FIGURE 17.2**: Illustration of the hauler's influence networks of Example 17.1.

## 17.3 The Shapley Rule

In [1] the authors study four rules that provide, for each gas loss problem, an allocation of the allowed loss among the different haulers. These rules, whose definitions can be seen in [1], are the following: The flow's rule, $R^{flow}$, the aggregate edge's rule, $R^{Aedge}$, the edge's rule, $R^{edge}$, and the proportional tracing rule, $R^{\Gamma^{pt}}$.

In this section, we introduce a new allocation rule: The Shapley rule. In order to do it, we first associate, to each gas loss problem a cooperative game with transferable utility, and then study the Shapley value of the associated game.

We start with some preliminaries on cooperative games. A cooperative game with transferable utility, briefly a *TU* game, is a pair $(H, l)$ where $H$ is the set of agents and, for each $T \subset H$, $l(T)$ denotes the amount that agents in $T$ can obtain by themselves. We assume that $l(\emptyset) = 0$.

The Shapley value introduced in [14] is, by far, the most studied allocation rule in cooperative game theory. It associates to each *TU* game $(H, l)$ a vector $\mathrm{Sh}(H, l) \in \mathbb{R}^H$ such that, for each $h \in H$,

$$\mathrm{Sh}_h(H, l) = \sum_{T \subset H \setminus \{h\}} \frac{|T|!\,(|H| - |T| - 1)!}{|H|!} \left( l(T \cup \{h\}) - l(T) \right).$$

In our context, $H$ represents the set of haulers and, for each $T \subset H$, $l(T)$, is the loss that haulers in $T$ can have by "themselves". Although there are several ways in which the $l(T)$ values can be defined, we present a natural one inspired in the approach taken in [9] for flow games. In their model, there is also a set of agents who own the different edges of the network and the value of a group of agents $T$ is defined as the maximum amount of flow that can be transported (from the source to the sink) using only edges belonging to agents in $T$. We apply the same principle to our model. Let $f_G(T)$ denote the maximum demand that can be satisfied using only edges of haulers in $T$, *i.e.*, the maximum amount of gas that can be transported from suppliers to consumers without exceeding the capacities and demands of suppliers and consumers, respectively. We also assume that the capacity of an edge is bounded by $f_e$, the total amount of gas flowing through that edge in the gas problem under study. Then, we define $l_G(T) = \alpha f_G(T)$; in particular, $l_G(H) = \alpha f_G(H) = \alpha \sum_{s \in S} Q_s = L$. When no confusion arises, we write $l$ instead of $l_G$.

**The Shapley rule**, $R^{\mathrm{Sh}}$. For each gas problem $G$ we define *the Shapley rule* as $R^{\mathrm{Sh}}(G) = \mathrm{Sh}(H, l_G)$.

Note that $R^{\mathrm{Sh}}(G) = \alpha \mathrm{Sh}(H, f_G)$.

Consider our running example. We first compute the associated cooperative game $l$. Hauler 1 can transport by himself 20 units. Since $\alpha = 0.1$, $l_G(1) = 0.1 \cdot 20 = 2$. Haulers 1 and 2 can transport by themselves no more than 60 units. They can do in several ways. For instance, 20 units through the path $\{(s_1, 1), (1, c_1)\}$ and 40 units through the path $\{(s_2, 1), (1, c_1)\}$. Since $\alpha = 0.1$, $l_G(1, 2) = 0.1 \cdot 60 = 6$. Analogously, we can obtain that

| $T$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|
| $l_G(T)$ | 2 | 0 | 0 | 6 | 2 | 4 | 10 |

Thus, the Shapley rule is $R^{\text{Sh}}(G) = (4, 4, 2)$. In Table 17.2 we show, for this example, the Shapley rule and the four rules defined in [1]: Although in this

| $h$ | $f_h$ | $R^{flow}$ | $R^{Aedge}$ | $R^{edge}$ | $R^{\Gamma^{pt}}$ | $R^{\text{Sh}}$ |
|---|---|---|---|---|---|---|
| 1 | 60 | 3.33 | 5 | 4 | 4 | 4 |
| 2 | 80 | 4.44 | 3.33 | 4 | 4 | 4 |
| 3 | 40 | 2.22 | 1.66 | 2 | 2 | 2 |

**TABLE 17.2**: Allocation rules of Example 17.1.

example several rules lead to the same allocation, in general the five rules are all different from one another.

## 17.4 Properties

The main objective of this chapter is to study the axiomatic behavior of the Shapley rule, and compare this behavior with that of the other rules studied in [1]. In order to do so, we focus our analysis in precisely the properties introduced in that paper, and refer the reader to the discussions therein for additional insights. Since these properties are inspired in the principles mentioned in different regulations and directives of the European Union regulation, the authors in [1] present the following discussion to provide some additional motivation to the properties and their underlying principles:

> In Directive 2003/55/EC of the European parliament and the council of 26 June 2003 ([13]), concerning common rules for the internal market in natural gas, establishes some general principles that must be pursued. Some of them are the following:

1. "tariffs are published **prior** to their entry into force".
2. "**the provision of adequate economic incentives**, using, where appropriate, all existing national and Community tools. These tools may include liability mechanisms to guarantee the necessary investment".
3. "national regulatory authorities should ensure that transmission and distribution tariffs are **non-discriminatory** and **cost-reflective**".

4. “Progressive opening of markets towards **full competition** should as soon as possible remove differences between Member States.”

   The Spanish regulation ensures that tariffs are published prior to their entry into force. Moreover, since the amount received or paid by each hauler depends monotonically on their loss (the larger is the loss, the larger is the amount the hauler pays), we can argue that it provides the adequate economic incentives.

   Regarding the principles of being non-discriminatory, cost-reflective, and foster competition, we introduce some properties related to these principles.”

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### 17.4.1 Cost-Reflective Properties

The first property requires that haulers that do not transport gas do not have any assigned loss and the second one says that if two gas problems only differ on edges without flow, then the losses assigned to each hauler should coincide.

**Null hauler** (NH). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h \in H$ be such that, for each $e \in E_h$, $f_e = 0$. Then, $R_h(G) = 0$.

**Independence of unused edges** (IUE). Let the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and, for each $h \in H$, $\bar{E}_h = E_h \setminus \hat{E}$, where $\hat{E} \subset E$ satisfies that, for each $e \in E \setminus \hat{E}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$, and, for each $e \in \hat{E}$, $f_e = 0$. Then, $R(G) = R(\bar{G})$.

A cost-reflective rule should not be sensitive to “equivalent” representations of the same network. The next two properties try to capture this idea.

**Independence of edge sectioning** (IES). Let the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and there are $\hat{h} \in H$ and $(i, j) \in E_{\hat{h}}$ satisfying

- $\bar{g} = (\bar{N}, \bar{E})$, where $\bar{N} = N \cup \{l\}$ and $l \notin N$, $\bar{E}_{\hat{h}} = (E_{\hat{h}} \setminus \{(i, j)\}) \cup \{(i, l), (l, j)\}$ and, for each $h \in H \setminus \{\hat{h}\}$, $\bar{E}_h = E_h$, and
- $\bar{f}_{(i,l)} = \bar{f}_{(l,j)} = f_{(i,j)}$, $\bar{v}_{(i,l)} + \bar{v}_{(l,j)} = v_{(i,j)}$, and, for each $e \in E \setminus \{(i, j)\}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$.[2]

Then, for each $h \in H$, $R_h(G) = R_h(\bar{G})$.

**Independence of edge multiplication** (IEM). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be such that $H = \bar{H}$ and there are $\hat{h} \in H$, $e = (i, j, m) \in E$, $\bar{e}_1 = (i, j, l_1) \in \bar{E}$, and $\bar{e}_2 = (i, j, l_2) \in \bar{E}$ satisfying

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[2] The condition $\bar{v}_{(i,l)} + \bar{v}_{(l,j)} = v_{(i,j)}$ just reflects that, when a pipe is transversely cut (orthogonally to the direction of the flow), the volume of the resulting two pipes adds up to the volume of the original pipe (and the same flow that was crossing the original pipe is crossing the two pipes in which it has been divided $\bar{f}_{(i,l)} = \bar{f}_{(l,j)} = f_{(i,j)}$).

- $\bar{g} = (N, \bar{E})$, where $\bar{E}_{\hat{h}} = (E_{\hat{h}} \setminus \{e\}) \cup \{\bar{e}_1, \bar{e}_2\}$ and, for each $h \in H \setminus \{\hat{h}\}$, $\bar{E}_h = E_h$, and

- $f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$, $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$, and, for each $e \in E \setminus \{e\}$, $\bar{f}_e = f_e$ and $\bar{v}_e = v_e$.[3]

Then, for each $h \in H$, $R_h(G) = R_h(\bar{G})$.

To prevent haulers from artificially distorting the final allocation of losses, if two haulers engage in some trades affecting their own edges, then the rest of the haulers should not be affected. This implies, in particular, that the loss allocated to a hauler does not depend on who owns the edges different from his own.

**Independence by sales** (IS). Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, $h_1$ and $h_2$ in $H$, and $e \in E$ be such that $\bar{E}_{h_1} = E_{h_1} \setminus \{e\}$, $\bar{E}_{h_2} = E_{h_2} \cup \{e\}$, and, for each $h \in H \setminus \{h_1, h_2\}$, $\bar{E}_h = E_h$. Then, for each $h \in H \setminus \{h_1, h_2\}$, $R_h(G) = R_h(\bar{G})$.[4]

**Independence of irrelevant changes** (IIC). Consider the gas problems $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ and let $h \in H \cap \bar{H}$ be such that $\mathcal{N}^h = \bar{\mathcal{N}}^h$. Then, $R_h(G) = R_h(\bar{G})$.

### 17.4.2 Non-Discriminatory Properties

The most standard non-discriminatory principle says that we should offer an equal treatment to equal agents. Some of the following properties deal with formalizations of this general notion.

**Symmetry on edges** (SE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, $f_e = f_{\bar{e}}$, and $v_e = v_{\bar{e}}$. Then, $R_h(G) = R_{\bar{h}}(G)$.

**Symmetry on paths** (SP). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, $v_e = v_{\bar{e}}$, and $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$. Then, $R_h(G) = R_{\bar{h}}(G)$.

The following properties build upon the idea that there should be some kind of proportionality on flow and volume.

**Flow proportionality on edges** (FPE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $v_e = v_{\bar{e}}$. Then, if $f_{\bar{e}} > 0$, we have

$$R_h(G) = \frac{f_e}{f_{\bar{e}}} R_{\bar{h}}(G).$$

**Volume proportionality on edges** (VPE). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and

[3] In this case, the condition $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$ just reflects that the original pipe $e$ is being replaced by two pipes identical to it: Same volume and same endpoints. The total flow in the network remains unchanged, so these two new pipes, together, carry the same flow as $e$ ($f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$).

[4] The rules satisfying IS have an interesting property, which in [1] is referred to as *edge decomposability*. Namely, these rules can be computed in a two-stage procedure. We first decide the allowed loss on each edge and later compute the allowed loss to each hauler adding the amount assigned to each of his edges.

$h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $f_e = f_{\bar{e}}$. Then,

$$R_h(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}(G).$$

**Volume proportionality on paths** (VPP). Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, and $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$. Then,

$$R_h(G) = \frac{v_e}{v_{\bar{e}}} R_{\bar{h}}(G).$$

### 17.4.3 Properties to Foster Competition

The way in which losses are allocated among haulers should not harm competition among agents. In particular, two haulers should not be better off by merging together.

**Merging proofness** (MP). Let $G = (g, v, f, \mathcal{H}, \alpha)$, $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$, $h_1, h_2 \in H$, and $h \in \bar{H}$ be such that $\bar{E}_h = E_{h_1} \cup E_{h_2}$ and, for each $\hat{h} \in H \setminus \{h_1, h_2\}$, $\bar{E}_{\hat{h}} = E_{\hat{h}}$ . Then $R_h(\bar{G}) \leq R_{h_1}(G) + R_{h_2}(G)$.

---

## 17.5 Axiomatic Behavior of the Shapley Rule

We present now the main result of this paper, which shows what properties are satisfied by the Shapley rule.

**Proposition 17.1** *1. The Shapley rule satisfies NH, IUE, IES, IEM, and SP.*

*2. The Shapley rule does not satisfy IS, SE, FPE, VPE, VPP, IIF, IIC, and MP.*

*Proof.* We start by proving statement 1.

• NH. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h \in H$ be such that, for each $e \in E_h$ $f_e = 0$. Since the edges of hauler $h$ do not carry flow, they never help to increase the total flow that can be carried between a supplier and a consumer. Thus, for each $T \subset H \backslash \{h\}$, we have that $l_G(T) = l_G(T \cup \{h\})$ and the definition of the Shapley value implies that $R_h^{\mathrm{Sh}} = 0$.

• IUE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be as in the definition of IUE, that is, there is $\hat{E} \subset E$ such that, for each $h \in H$, $\bar{E}_h = E_h \setminus \hat{E}$ and, for each $e \in \hat{E}$, $f_e = 0$.

Let $T \subset H$ be a set of players. Again, the edges that do not carry flow never help to increase the total flow that can be carried between a supplier and a consumer. Thus, they can be removed for the computation of the TU game associated with $\bar{G}$ and, therefore, for each $T \subset H$, $l_G(T) = l_{\bar{G}}(T)$. Thus, $R^{\text{Sh}}(G) = R^{\text{Sh}}(\bar{G})$.

• IES. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \bar{\mathcal{H}}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $(i,j) \in E_{\hat{h}}$ satisfying that $(i,j)$ is sectioned in two consecutive edges $(i,l), (l,j) \in \bar{E}_{\hat{h}}$.

Since $f_{(i,j)} = \bar{f}_{(i,l)} = \bar{f}_{(l,j)}$, edge sectioning does not change the maximum flow that can be carried from consumers to suppliers. Then, for each $T \subset H$, $l_G(T) = l_{\bar{G}}(T)$ and, therefore, for each $h \in H$, $R^{\text{Sh}}_h(G) = R^{\text{Sh}}_h(\bar{G})$.

• IEM. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (\bar{g}, \bar{v}, \bar{f}, \mathcal{H}, \alpha)$ be two problems that only differ because there are $\hat{h} \in H$ and $e \in E_{\hat{h}}$ satisfying that $e$ is duplicated in two multiedges $e_1, e_2 \in \bar{E}_{\hat{h}}$, with $v_e = \bar{v}_{e_1} = \bar{v}_{e_2}$.

Since $f_e = \bar{f}_{e_1} + \bar{f}_{e_2}$, edge multiplication does not change the maximum flow that can be carried from consumers to suppliers because we only have to split among $\bar{f}_{e_1}$ and $\bar{f}_{e_2}$ the maximum flow that went through $f_e$. Then, for each $T \subset H$, $l_G(T) = l_{\bar{G}}(T)$ and, therefore, for each $h \in H$, $R^{\text{Sh}}_h(G) = R^{\text{Sh}}_h(\bar{G})$.

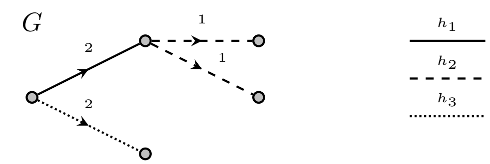
• SP. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $h, \bar{h} \in H$ be such that $E_h = \{e\}$, $E_{\bar{h}} = \{\bar{e}\}$, $v_e = v_{\bar{e}}$ and $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$.

Since $\mathcal{N}^h = \mathcal{N}^{\bar{h}}$ we have that $f_e = f_{\bar{e}}$ and, for each $p \in P(S, C)$, $e \in p$ if and only if $\bar{e} \in p$. Then, for each $T \subset H \backslash \{h, \bar{h}\}$ we have $l_G(T \cup h) = l_G(T \cup \bar{h})$. Thus, the definition of the Shapley value implies that $R^{\text{Sh}}_h(G) = R^{\text{Sh}}_{\bar{h}}(G)$.

Next, we present some counterexamples to prove statement 2.

• IS. Since IS is stronger than MP (Proposition 1 in [1]) and $R^{\text{Sh}}$ does not satisfy MP (see below), $R^{\text{Sh}}$ does not satisfy IS.

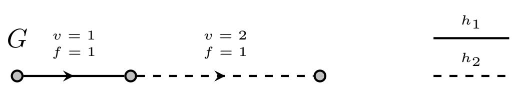
• SE. Let $G = (g, v, f, \mathcal{H}, \alpha)$ be as in the picture below.




Problem $G$ is as in the definition of SE, since $h_1 = \{e_1\}$ and $h_3 = \{e_2\}$ with $f_{e_1} = f_{e_2} = 2$ and $v_{e_1} = v_{e_2}$. However, $h_3$ can satisfy some demand on his own, while $h_1$ needs $h_2$. In particular, we get $R^{\text{Sh}}_{h_1}(G) = \alpha \neq 2\alpha = R^{\text{Sh}}_{h_3}(G)$.

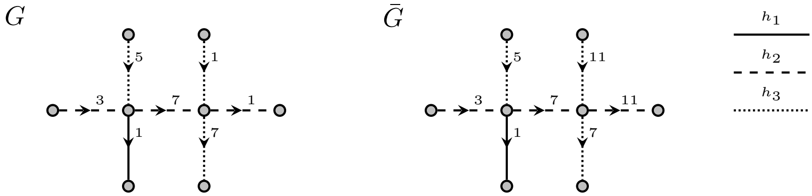
• FPE and VPE. Since FPE and VPE are stronger than SE (Proposition 1 in [1]) and $R^{\text{Sh}}$ does not satisfy SE, $R^{\text{Sh}}$ satisfies neither FPE nor VPE.

• VPP. Let $G = (g, v, f, \mathcal{H}, \alpha)$, $h_1$ and $h_2$ as in the picture below.




Clearly, $R^{\text{Sh}}_{h_2}(G) = R^{\text{Sh}}_{h_1}(G) \neq 2R^{\text{Sh}}_{h_1}(G) = \frac{v_{h_2}}{v_{h_1}} R^{\text{Sh}}_{h_1}(G)$.

• IIF. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (g, v, \bar{f}, \mathcal{H}, \alpha)$ be as in the picture below.




Problems $G$ and $\bar{G}$ are as in the definition of IIF. Note that there are two edges where the flow increases and $\mathcal{N}^{h_1} = \bar{\mathcal{N}}^{h_1}$. In this case, we get the games

– $l_G(\{h_1\}) = 0$, $l_G(\{h_2\}) = \alpha$, $l_G(\{h_3\}) = \alpha$, $l_G(\{h_1, h_2\}) = 2\alpha$, $l_G(\{h_1, h_3\}) = 2\alpha$, $l_G(\{h_2, h_3\}) = 8\alpha$, $l_G(\{h_1, h_2, h_3\}) = 9\alpha$ and

– $l_{\bar{G}}(\{h_1\}) = 0$, $l_{\bar{G}}(\{h_2\}) = 3\alpha$, $l_{\bar{G}}(\{h_3\}) = 7$, $l_{\bar{G}}(\{h_1, h_2\}) = 3\alpha$, $l_{\bar{G}}(\{h_1, h_3\}) = 8\alpha$, $l_{\bar{G}}(\{h_2, h_3\}) = 18\alpha$, $l_{\bar{G}}(\{h_1, h_2, h_3\}) = 19\alpha$.

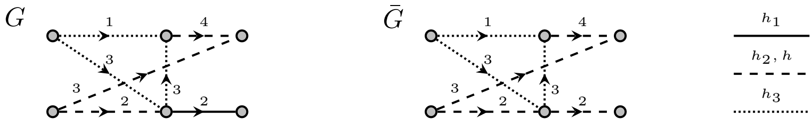
The corresponding Shapley values are so that

$$R^{\text{Sh}}_{h_1}(G) = \alpha\frac{4}{6} \neq \alpha\frac{3}{6} = R^{\text{Sh}}_{h_1}(\bar{G}).$$

The key is that the marginal contribution of hauler $h_1$ to hauler $h_2$ changes from $G$ to $\bar{G}$.

• IIC. Since IIC is stronger than IIF (Proposition 1 in [1]) and $R^{\text{Sh}}$ does not satisfy IIF, $R^{\text{Sh}}$ does not satisfy IIC.

• MP. Let $G = (g, v, f, \mathcal{H}, \alpha)$ and $\bar{G} = (g, v, f, \bar{\mathcal{H}}, \alpha)$ be as in the picture below.




Note that $H = \{h_1, h_2, h_3\}$ and $\bar{H} = \{h, h_3\}$ where $h$ is the union of $h_1$ and $h_2$. Problems $G$ and $\bar{G}$ are as in the definition of MP. In this case, we get the games

– $l_G(\{h_1\}) = 0$, $l_G(\{h_2\}) = 3\alpha$, $l_G(\{h_3\}) = 0$, $l_G(\{h_1, h_2\}) = 5\alpha$, $l_G(\{h_1, h_3\}) = 2\alpha$, $l_G(\{h_2, h_3\}) = 7\alpha$, $l_G(\{h_1, h_2, h_3\}) = 9\alpha$ and

– $l_{\bar{G}}(\{h\}) = 5\alpha$, $l_{\bar{G}}(\{h_3\}) = 0$, $l_{\bar{G}}(\{h, h_3\}) = 9\alpha$.

The corresponding Shapley values are so that

$$R^{\text{Sh}}_{h}(\bar{G}) = \alpha\frac{42}{6} > \alpha\frac{40}{6} = \alpha\frac{8}{6} + \alpha\frac{32}{6} = R^{\text{Sh}}_{h_1}(G) + R^{\text{Sh}}_{h_2}(G).$$

In Table 17.3 we compare the properties satisfied by the Shapley rule with the properties satisfied by the four rules considered in [1]. The authors then continue discussing, for each of the four rules they study, the “degree of

| **EU Principles** | Rule / Property | Flow | Aedge | Edge | Prop. Tracing | Shapley |
|---|---|---|---|---|---|---|
| Cost-Reflective | Null hauler | ✓ | ✓ | ✓ | ✓ | ✓ |
| | Ind. Unused Edges | ✓ | | ✓ | ✓ | ✓ |
| | Ind. Edge Sectioning | ✓ | ✓ | ✓ | ✓ | ✓ |
| | Ind. Edge Mult. | ✓ | | ✓ | ✓ | ✓ |
| | Ind. Sales | | | ✓ | ✓ | |
| | Ind. Irr. Changes | | | | ✓ | |
| Non-Discriminat. | Symmetry on Edges | ✓ | ✓ | ✓ | | |
| | Symmetry on Paths | ✓ | ✓ | ✓ | ✓ | ✓ |
| | Flow Prop. Edges | ✓ | ✓ | ✓ | | |
| | Volume Prop. Edges | | ✓ | ✓ | | |
| | Volume Prop. Paths | | ✓ | ✓ | ✓ | |
| | Merging Proofness | ✓ | | ✓ | ✓ | |
| Competition | Merging Proofness | ✓ | | ✓ | ✓ | |

**TABLE 17.3**: Behavior of the rules with respect to the different properties.

| **Principle\Rule** | Flow | Aedge | Edge | Prop. tracing | Shapley |
|---|---|---|---|---|---|
| Cost reflective | Normal | Low | High | Very high | Normal |
| Non-discriminatory | High | High | Very high | High | Normal |
| Foster competition | Very high | Low | Very high | Very high | Low |

**TABLE 17.4**: Degree of fulfillment of the EU principles by each rule.

fulfillment" of the three principles. Four degrees were considered: Low, normal, high, and very high. We borrow from them Table 17.4, with the addition of one last column for the Shapley rule.

Since the discussion associated to the four rules different from the Shapley value is already included in the analysis in [1], we briefly discuss now the column associated to the Shapley rule. It satisfies the same cost reflective properties as the flow's rule, thus we assign to the Shapley value the same degree in that category. Usually non-discriminatory properties are related with the principle of equal treatment of equals. Then, when comparing symmetry on edges with symmetry on paths, the later takes into account the whole structure of the network, and not just each edge on isolation. Thus, we think that focusing on paths is more reasonable and therefore we assign a normal grade to Shapley rule even though it does not satisfy most of the non-discriminatory properties. Finally, since foster competition has a unique property, the assignment is obvious.

From the table and the above discussion, it is clear that the Shapley rule does not exhibit a very good behavior with respect to the different properties and principles, being clearly outperformed by both the proportional tracing rule and the edge's rule.

There are many problems where the Shapley value of an associated cooperative game has many interesting properties compared with other rules in the same setting. We can mention, for instance, airport problems (see [11]), queueing problems (see [12] and [7]), and minimum cost spanning tree problems (see [10] and [2]). Nevertheless, in our case the Shapley value satisfies less properties than other rules. Of course it could be possible that, if we define the associated cooperative game $l_G$ in a different way, we could obtain a Shapley value with more properties.

In the next section, we take a different approach to assess the performance of the Shapley rule, which can be seen as complementary to the one developed in this section. More precisely, we study the allocations the Shapley rule proposes in different problems, a case study with real data and a set of variations of it, and comparing these allocations with the ones proposed by the other four rules.

## 17.6 Application to the Spanish Gas Transmission Network

### 17.6.1 Case Study with Real Data

In this section we apply the Shapley rule to the Spanish gas transmission network. We compare the allocation proposed by the Shapley rule with the allocations proposed by the four rules considered in [1]. We build upon the analysis there, and take as benchmark scenario one in which demands follow from reported figures for a hypothetical day of very high demand in the Spanish gas network.[5]

In Figure 17.3 we represent the Spanish gas transmission network. We have boxed the pipes belonging to each hauler, except for hauler $h_{1,}$ who owns all the remaining ones. Hauler $h_1$ is Enagás, a former public body who initially owned the whole network and still owns more that 90% of the network.

In Tables 17.5, 17.6 and 17.7, we can see the allocations proposed by the Shapley rule and the other rules. We take $\alpha = 0.002$ because is the parameter used in Spain (see [4]).

The three tables contain similar information, but measured in different ways. Moreover, the numbers they contain are the same as in [1], but where an additional column for the Shapley value has been included. Table 17.5 represents the allocated losses measured in gas units, corresponding to the direct application of the different rules to the data of the Spanish scenario under consideration. Table 17.6 represents the percentage allocated to each

[5] The computations are derived for the optimal network operation as obtained by the software GANESO$^{TM}$ (developed by researchers at the University of Santiago de Compostela and the Technological Institute for Industrial Mathematics for Reganosa Company). For further details refer to the analysis in [1].

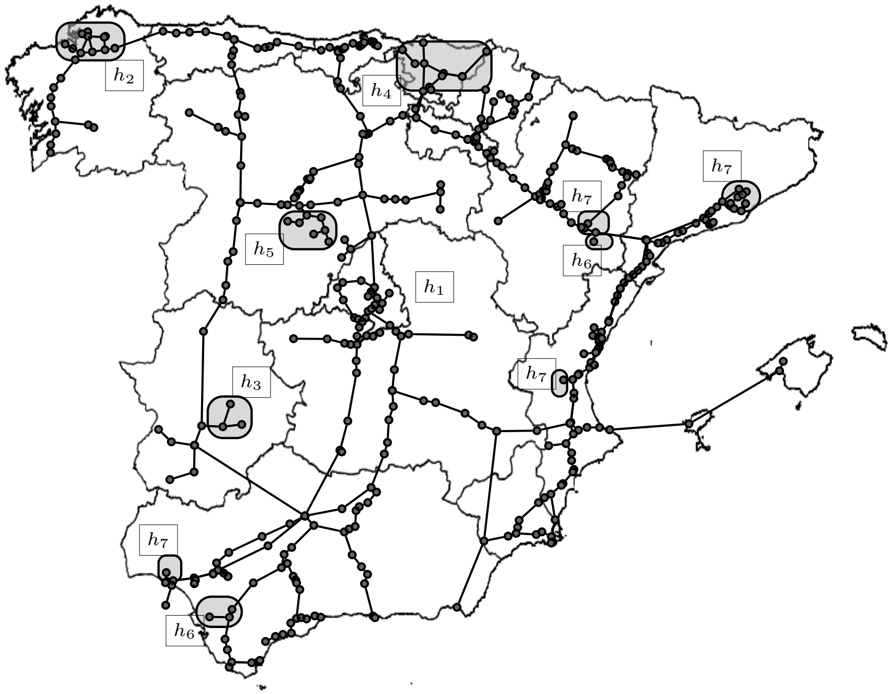



**FIGURE 17.3**: Haulers of the Spanish gas transmission network.

| **Gas losses in GWh/d** | Network Owned (%) | Flow | Aedge | Edge | Prop. Tracing | Shapley |
|---|---|---|---|---|---|---|
| Enagás ($h_1$) | 91.44 | 4.55 | 5.32 | 5.27 | 4.72 | 4.69 |
| Reganosa ($h_2$) | 1.76 | 0.21 | 0.0024 | 0.031 | 0.21 | 0.22 |
| Gas Extremadura ($h_3$) | 0.61 | 0.0071 | 0.000010 | 0.00020 | 0.000073 | 0.0038 |
| Enagás Transporte del Norte ($h_4$) | 3.54 | 0.31 | 0.0086 | 0.027 | 0.24 | 0.27 |
| Transportista Regional Gas ($h_5$) | 1.46 | 0.016 | 0.000051 | 0.0005 | 0.00052 | 0.0090 |
| Endesa Gas Transportista ($h_6$) | 0.36 | 0.0045 | 0.0000019 | 0.000029 | 0.000035 | 0.0024 |
| Gas Natural ($h_7$) | 0.82 | 0.24 | 0.00095 | 0.0062 | 0.17 | 0.14 |

**TABLE 17.5**: Gas loss allocated to the haulers (GWh/d) with $\alpha = 0.002$.

| **Percentage of gas losses (%)** | Network Owned (%) | Flow | Aedge | Edge | Prop. Tracing | Shapley |
|---|---|---|---|---|---|---|
| Enagás ($h_1$) | 91.44 | 85.19 | 99.77 | 98.77 | 88.37 | 87.88 |
| Reganosa ($h_2$) | 1.76 | 3.97 | 0.046 | 0.59 | 3.95 | 4.03 |
| Gas Extremadura ($h_3$) | 0.61 | 0.13 | 0.00019 | 0.0037 | 0.0014 | 0.072 |
| Enagás Transporte del Norte ($h_4$) | 3.54 | 5.74 | 0.16 | 0.51 | 4.44 | 5.11 |
| Transportista Regional Gas ($h_5$) | 1.46 | 0.31 | 0.00096 | 0.0094 | 0.0098 | 0.17 |
| Endesa Gas Transportista ($h_6$) | 0.36 | 0.083 | 0.000035 | 0.00055 | 0.00066 | 0.046 |
| Gas Natural ($h_7$) | 0.82 | 4.58 | 0.018 | 0.12 | 3.23 | 2.69 |

**TABLE 17.6**: Percentage of gas loss allocated to the haulers.

| Monetary equivalent in millions of € | Network Owned (%) | Flow | Aedge | Edge | Prop. Tracing | Shapley |
|---|---|---|---|---|---|---|
| Enagás ($h_1$) | 91.44 | 49.77 | 58.30 | 57.71 | 51.64 | 51.35 |
| Reganosa ($h_2$) | 1.76 | 2.32 | 0.027 | 0.34 | 2.31 | 2.36 |
| Gas Extremadura ($h_3$) | 0.61 | 0.077 | 0.00011 | 0.0022 | 0.00080 | 0.042 |
| Enagás Transporte del Norte ($h_4$) | 3.54 | 3.35 | 0.095 | 0.30 | 2.60 | 2.99 |
| Transportista Regional Gas ($h_5$) | 1.46 | 0.18 | 0.00056 | 0.0055 | 0.0057 | 0.098 |
| Endesa Gas Transportista ($h_6$) | 0.36 | 0.049 | 0.000020 | 0.00032 | 0.00039 | 0.027 |
| Gas Natural ($h_7$) | 0.82 | 2.68 | 0.010 | 0.068 | 1.89 | 1.57 |

**TABLE 17.7**: Annual monetary equivalent, assuming 1 GWh/d = 30000 €.

hauler. Finally, Table 17.7 represents the estimation of the annual monetary equivalent, provided that the same demands repeat each and every day. Since the scenario under consideration comes from a peak day, whose demand is around twice the demand of an average day, one would get more realistic estimations after dividing by two the amounts in Table 17.7. In practice, one might apply the chosen rule on a daily basis and then add up the daily allocations to get the annual loss allocation.

The aggregate edge's rule assign 99.77% of the allocated losses to Enagás, which we believe is unfair. As it was argued in [1] the aggregate edge's rule size discriminates, penalizing small haulers and favoring mergers, which hurts competition. This probably explains why most Spanish haulers strongly opposed to the aggregate edges rule until it was finally replaced by the flow's rule.

In this case, we can see that the allocation proposed by the Shapley rule is quite similar to the one proposed by the proportional tracing rule. In the next section we further explore this connection.

### 17.6.2 Simulation Study Building upon the Real Data

Given the results in the analysis above, it is natural to wonder whether or not the similarity between the allocations proposed by the Shapley rule and the proportional tracing rule is just a coincidence for the given data. In order to get additional evidence, we have run a simulation study based on the original scenario, but where relevant data of the problem are randomly modified. More precisely, we have generated 10000 scenarios from the benchmark using the following procedure:

- The only information that is modified from scenario to scenario is the ownership relation between edges and haulers, with pipes being randomly assigned to haulers.
- In order to get reasonably connected networks, the random assignment is not performed on individual pipes, but on some predetermined groups of pipes. More precisely, the pipes are divided in 16 groups, corresponding to the 16 Spanish autonomous communities (setting aside Canary Islands, which contain no pipes of the high-pressure network).

- Then, each of the 16 groups is randomly assigned to one of the 7 available haulers. We keep the same number of haulers of the Spanish network which should provide enough richness to the random generating process (note that a hauler might end up with no assigned pipes in some realizations).
- This random process is repeated 10000 times, with the goal of obtaining very diverse realizations: Homogeneous haulers, a single dominant hauler, split between medium haulers and small ones, etc.
- For each realization, we obtain the resulting loss allocation for the five rules discussed in this paper. Finally, we compute the matrix of correlations between the allocations proposed by these five rules and also with the vector of the length of pipes owned by each hauler.[6]

In Tables 17.8, 17.9 and 17.10, we summarize the information contained in those correlation matrices. Table 17.8 contains the average of the 10000 correlation matrices obtained with the above procedure. As one might expect, all correlations are relatively high, with average numbers over 0.8 between all pairs of rules. The lowest number is found between the edge's rule and Shapley's rule but, more importantly, the highest average correlation is between Shapley's rule and the proportional tracing rule, reinforcing the observation in the analysis for the benchmark scenario. Indeed, this average correlation is almost perfect, being as high as 0.9933. The next highest correlations are found when comparing the flow rule with either the proportional tracing rule or Shapley's rule, with values round 0.97. Although these correlations are also very high, they are significantly smaller than the previous one (0.9933 is just 0.007% away from perfect correlation, whereas 0.97 is more than 4 times further away).

| **Correlations** | Flow | Aedge | Edge | Prop. Tracing | Shapley | Pipe Length |
|---|---|---|---|---|---|---|
| Flow | 1.0000 | 0.9166 | 0.8717 | 0.9700 | 0.9707 | 0.6910 |
| Aedge | 0.9166 | 1.0000 | 0.8972 | 0.8988 | 0.8751 | 0.8154 |
| Edge | 0.8717 | 0.8972 | 1.0000 | 0.8776 | 0.8336 | 0.6200 |
| Prop. Tracing | 0.9700 | 0.8988 | 0.8776 | 1.0000 | 0.9933 | 0.6526 |
| Shapley | 0.9707 | 0.8751 | 0.8336 | 0.9933 | 1.0000 | 0.6446 |
| Pipe Length | 0.6910 | 0.8154 | 0.6200 | 0.6526 | 0.6446 | 1.0000 |

**TABLE 17.8**: Average of correlation matrices.

If we look now at Table 17.9, which contains, for each pair of rules, the minimum correlation between them across the 10000 realizations, we again see the strong connection between Shapley's rule and the proportional tracing rule. In the scenario where the correlation between them was smaller, it was

[6]We have also used other approaches to compare the different rules, all of them leading to the same qualitative results.

| **Correlations** | Flow | Aedge | Edge | Prop. Tracing | Shapley | Pipe Length |
|---|---|---|---|---|---|---|
| Flow | 1.0000 | −0.1037 | −0.2191 | 0.6020 | 0.5756 | −0.8766 |
| Aedge | −0.1037 | 1.0000 | −0.3963 | 0.0743 | −0.1729 | −0.4711 |
| Edge | −0.2191 | −0.3963 | 1.0000 | 0.2929 | 0.0777 | −0.6746 |
| Prop. Tracing | 0.6020 | 0.0743 | 0.2929 | 1.0000 | 0.9181 | −0.8856 |
| Shapley | 0.5756 | −0.1729 | 0.0777 | 0.9181 | 1.0000 | −0.9824 |
| Pipe Length | −0.8766 | −0.4711 | −0.6746 | −0.8856 | −0.9824 | 1.0000 |

**TABLE 17.9**: Minimum across correlation matrices.

| **Correlations** | Flow | Aedge | Edge | Prop. Tracing | Shapley | Pipe Length |
|---|---|---|---|---|---|---|
| Flow | 1.0000 | 1.0000 | 0.9998 | 0.9999 | 0.9999 | 0.9987 |
| Aedge | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9994 | 0.9997 |
| Edge | 0.9998 | 0.9999 | 1.0000 | 0.9996 | 0.9992 | 0.9995 |
| Prop. Tracing | 0.9999 | 0.9998 | 0.9996 | 1.0000 | 1.0000 | 0.9994 |
| Shapley | 0.9999 | 0.9994 | 0.9992 | 1.0000 | 1.0000 | 0.9998 |
| Pipe Length | 0.9987 | 0.9997 | 0.9995 | 0.9994 | 0.9998 | 1.0000 |

**TABLE 17.10**: Maximum across correlation matrices.

still over 0.9. For any other pair of rules, this number is at most 0.6 and in many cases it can even be negative. Finally, Table 17.10 contains the information about the maximum correlation between any pair of rules. Not surprisingly, this number is very close to one for every pair of rules.

Given the poor behavior observed by the Shapley value in the axiomatic analysis developed in Section 17.4, it is interesting to see that it exhibits such a high correlation with the proportional tracing rule which, arguably, may be considered the one performing better from the axiomatic point of view. We do not claim that the analysis we have just presented, based on numeric simulations, represents any kind of proof, but it suggests that there must be some mathematical connection between these two rules which might be the subject of future research.

## 17.7 Conclusions

In this chapter we have studied the Shapley value in the context of loss allocation in energy networks and developed an axiomatic analysis to study its behavior with respect to different axioms. The main result in the paper, Proposition 17.1 shows that the behavior of the Shapley rule is far from being as good as that of other rules studied in the literature, such as the edge's rule and the proportional tracing rule. This leaves as an open problem the issue of finding new desirable properties that the Shapley rule might satisfy and which might ultimately lead to an axiomatic characterization.

Interestingly, we then develop a comparative analysis of the different allocation rules on a set of problems originated from real data and observe that the Shapley rule has a very high correlation (over 0.99) with the proportional tracing rule. This may seem a bit contradictory with the fact that these two rules exhibit a very different behavior with respect to the set of axioms discussed in Section 17.4. Then, an open question for future research would be to understand the mechanism driving this unusually high correlation.

## 17.8 Acknowledgments

The authors acknowledge support from Ministerio de Economía y Competitividad and FEDER through projects MTM2014-60191-JIN, ECO2014-52616-R, and ECO2015-70119-REDT and from Xunta de Galicia through projects INCITE09-207-064-PR, GRC 2015/014 and ED431C 2017/38. Gustavo Bergantiños acknowledges support from Fundación Séneca de la Región de Murcia through project 19320/PI/14. Ángel M. González-Rueda acknowledges support from Ministerio de Educación through Grant FPU13/01130.

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# Chapter 18

## On Some Applications of the Shapley-Shubik Index for Finance and Politics

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## 18.1 Introduction

Certain properties of the Shapley-Shubik index are presented in this chapter ([72]), together with indications as to how such characteristics may be especially useful in predicting the outcome of bargaining in simple games. In light of this, we provide certain applications of this index to Politics and Finance undertaken at the Universities of Bergamo, Krakow, Manresa, and Monterey.

In the next section, we shall give a very short presentation of power indices. In Section 18.3 we shall present a historical overview, and then, in Section 18.4, the various indices will be compared. At this point we shall provide some applications to finance in Section 18.5, to be followed, in Section 18.6, by applications to politics.

## 18.2 Some Preliminary Definitions

Since, in this chapter, we take as granted that readers are familiar with power indices, we shall limit ourselves here to a review of basic notions without giving illustrations. Anyone wishing to know more should refer to the books by Felsenthal and Machover ([28]) and Laruelle and Valenciano ([54]), and to the papers by Turnovec, Mercik, and Mazurkiewicz ([79]), Laruelle and Valenciano ([53]), and Bertini et al. [8].

Let $N = \{1, 2, \ldots, n\}$ be a nonempty finite set. By a game on $N$ we shall mean real-valued function $v$ whose domain is the set of all subsets of $N$ such that $v(\emptyset) = 0$. We refer to any member of $N$ as a *player*, and to any subset of $N$ as a *coalition*. Game $v$ is said to be *simple* if function $v$ takes values only in the set $\{0, 1\}$: $v(S) = 0$ or $v(S) = 1$ for all coalitions $S \subseteq N$. In the first case the coalition is said to be losing; in the second case, winning. By $S_N$ we denote the set of simple games on $N$.

The $i$-th player is called *crucial* or *pivotal* for coalition $S$, if $S$ is a winning coalition, but becomes a losing coalition without the contribution of this player, i.e., $v(S) = 1$ and $v(S \backslash \{i\}) = 0$. *Cruciality* $C_i(S)$ of a player $i$ is the total number of coalitions $S \subseteq N$ for which the player $i$ is crucial. Any player who is not crucial for any coalition $S \subseteq N$ is called a *dummy player*.

A *value* for a game is function $v$ suitable with respect to sharing the total payoff $v(N)$ among the $n$ players. Suitable in terms of representing the forecasted share-out, in a predicted environment, of the total winnings of various players, or a fair division of such winnings in a normative environment.

Any function $f : S_N \rightarrow R^n$ with $|N| = n$ is a *power index*. The components $f_i(v)$ of $f(v)$ are interpreted as a measure of the power that the simple game $v$ confers to player $i$ or as $i$'s payoff expectation from playing the game $v$.

Among the several power indices that have been introduced, which we shall speak of in the following section, we find the normalized index of Banzhaf [4] and the index of Shapley-Shubik ([72]). The former assigns each player with a share of the global coalition's winnings, proportional to the number of the player's cruciality. That is:

$$\beta_i = \frac{1}{k} \sum_{S \subseteq N} C_i(S)$$

where the sum is extended to the entire coalition $S \subseteq N$ for which the $i$-th player is crucial and $k$ is a normalization coefficient.

However, the Shapley-Shubik index $\Phi$ assigns winnings to each player according to the following formula:

$$\Phi_i = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}$$

where the summation is extended to the coalitions for the $i$-th player is crucial and $s$ is the relative cardinality with respect to the coalitions under consideration.

A weighted majority game is a simple game defined by a vector of weights $w = (w_1, \ldots, w_n)$ and a majority quota $q$ that is greater than the total half-sum of weights. That is:

$$q \geq \frac{1}{2} \sum_{i=1}^{n} w_i$$

The rule that associates each weighted majority game with a corresponding simple game is the following:

$$v(S) = \begin{cases} 1 & \text{if } \sum_{\mathrm{i} \in \mathrm{S}} w_{\mathrm{i}} \geq q \\ 0 & \text{otherwise} \end{cases}$$

## 18.3 Short History

We shall give a brief historical overview of the main power indices identified up to the present. For further information on the history of power indices, please refer to the papers by Bertini et al. [8] and Gambarelli and Owen ([39]).

### 18.3.1 Power Indices Derived from Values

The Shapley-Shubik index was introduced by Shapley and Shubik [72] as a restriction to simple games of the Shapley value [71]. For further explanations see, for example, the paper by Stach [73].

The Tijs index was introduced in Tijs [78] for *quasi-balanced* simple games as a particularization of the Tijs value. For further information, see the paper by Stach [74].

The nucleolus [69] represents a particular approach to solutions for games in characteristic form in which existence and uniqueness are guaranteed. On the basis of such properties, it may also be considered a value and therefore a power index in the case of simple games.

### 18.3.2 Autonomously Generated Power Indices

The first autonomous power index is said to be that of Luther Martin in 1787 (see [67]); although its formulation is not especially mathematical, certain authors consider it as a precursor to the later indices by Penrose [65] and the normalized index of Banzhaf [4] and Coleman [19]. It should be noted that the normalized Banzhaf index is often quoted jointly with the names of the above-mentioned authors.

Later autonomous indices include the Johnston index [49], the Deegan-Packel index [22], the Public Good index [46], Holler-Packel index [48], the Public Help index [10], and the Shift index [1].

### 18.3.3 Some Other Indices with Different Derivations

There have also been indices with probabilistic derivations: For example, the Rae index [66] and König and Bräuninger's inclusiveness index [52]. For some characteristics of these indices, see the papers by Bertini and Stach [15] and Stach [75], for example. Then, there is the significant class of probabilistic indices of Gehrlein and Fishburn [44], Gehrlein, Ord, and Fishburn [45], and Mercik [58] where minority winners are considered (especially important for minority cabinets existing in politics).

A second category of indices originates with the semi-values introduced by Weber [81]. The best-known among these indices are those of Dubey, Neyman, and Weber [27] and Carreras, Freixas, and Puente [18].

## 18.4 Power Index Applications

The Banzhaf index is especially employed for normative purposes in the ambit of applications, because of its direct proportionality with regard to cruciality. On the other hand, in the context of social applications the Public Good index, the Public Help index and the nucleolus are particularly suitable. For most other applications the Shapley-Shubik index is best-suited, inasmuch as it possesses a collection of properties that no other index mentioned here may boast jointly.

There follows an outline of the main characteristics of the Shapley-Shubik index. A more formal description is to be found in Bertini et al. [8] and [9], and Dubey [25].

The first axiomatization of the Shapley-Shubik index was provided by Dubey ([25]), who uses the axioms of the Shapley value ([71]), with the exception of additivity, which needs to take a special form due to the non-linear structure of the set of simple games. The following four properties (efficiency, dummy-player, symmetry, and transfer) are the critical properties (axioms) of the Shapley-Shubik index used by Dubey ([25]).

- **Efficiency:** the players' power values add up to 1 for all non-null simple games.
- **Dummy-player (or null-player):** the payment for each dummy player is null.
- **Symmetry (anonymity):** a player's power value should not depend on her or his name. Thus, "symmetric" players should have equal power.
- **Transfer:** this axiom, while mathematically convenient, has no natural interpretation (see [55]). One of the interpretations given by Dubey, Einy and Haimanko [26] is as follows: Transfer requires that, when winning coalitions are enhanced in a game, the change in voting power depends only on the change in the game, i.e., on the set of new winning coalitions.
- **Gain-loss property:** if the power of some player increases as a result of changes in the game, the power cannot concomitantly increase for all players.

Besides the properties noted above, other properties are as follows:

- **Null player removable property:** after removing the null players from a game, non-null players' measures of power remain unchanged.
- **Non-negativity:** the players' power value should not be negative.

– **Block:** the players should have an advantage from a merger. Thus, a merger between two players should result in a greater power value than the power of a singular player.

Furthermore, the following properties are linked to majority games:

– **Dominance:** a player with a greater weight should not have a lower power index.

– **Donation:** a player who donates votes to other players should not increase her or his measure of power.

On the basis of considerations regarding the geometrical properties of the Shapley value, an algorithm was used for calculating this value in superadditive games (see [30]). The algorithm was generalized by Gambarelli [34] for subadditive games. The algorithm is linear in the number of significant coalitions and uses a theorem of early stop, based on reaching the desired degree of precision.

Other algorithms for specific applications follow: See, for instance, [60].

In majority games having a low total sum of weights, the Shapley value (which assumes in these cases the role of the Shapley-Shubik index) can be better calculated using the algorithm proposed by Mann and Shapley [56]. This algorithm was suggested to Shapley by an idea from Cantor, following which the index may be expressed in such games in a manner independent of coalitions, but directly by the weight of players; time-saving becomes exponential, because the possible coalition among $n$ player are $2^n$.

The generation of the “power” function relative to share exchanges between parties necessitates the repeated usage of Mann and Shapley’s algorithm in each of the constant power regions. A subsequent algorithm by Arcaini and Gambarelli [3] enables further savings in calculation, as it directly generates the increase in the index starting from each point of discontinuity, taking into account the information that was used to calculate the preceding value.

An algorithm is now available to generate power functions in the case of the Shapley-Shubik index for the exchange of shares between two shareholders, or between a shareholder and the ocean (see [36]).

## 18.5 Some Applications of Shapley-Shubik Power Index

Many of the properties mentioned here, in both political and financial environments, belong not only to the Shapley-Shubik index, but are also common to other indices. We shall focus here on their application in the case of the Shapley-Shubik index.

### 18.5.1 Example of Financial Applications

It is worth noting that in some cases a shareholder could own a lot more shares than another shareholder with the same decision power. He could decide then to yield some of his shares which are useless in controlling the company so as to buy other shares of other companies, thereby achieving a better power position in those companies. In general the question is whether there is a mathematical model for buying and selling shares such as to give him the maximum expectation of success in controlling various companies. The problem is very important due to the large amount of the money involved and it was solved in the 1980s by the Shapley-Shubik index.

In the following sections these issues will be examined, beginning with the easiest case of two shareholders.

### 18.5.2 Shares Shift between Two Shareholders

A further model was studied so as to predict changes in power relationships that follow a shift by a subset of the shares from one shareholder to another (see [32]).

Let us assume that the initial distribution of 100 shares among shareholders $A$, $B$ and $C$ in a 3-person weighted majority game is (51, 40, 9) (see Table 18.1). Given simply majority voting ($q \geq 51$) a transfer of shares between $B$ and $C$ will not change the situation, as $A$ will remain the majority shareholder. However, let us now analyze what happens if shares are exchanged between $A$ and $C$. If $C$ receives one share from $A$, the distribution becomes $(50, 40, 10)$ and the power distribution (according to the Shapley-Shubik index) becomes $(2/3, 1/6, 1/6)$. If $C$ receives 2 shares from $A$, the distribution of shares becomes $(49, 40, 11)$ and the power distribution is $(1/3, 1/3, 1/3)$. The division of power remains the same even if $C$ obtains 40 shares from $A$, as in this case the shares distribution becomes $(11, 40, 49)$ and each player is in the same position as the others. The situation changes only if $C$ receives 41 shares from $A$: In this case the distribution becomes $(10, 40, 50)$ and the power of $C$ increases to $2/3$. With one more share, $C$ acquires the majority and its power increases to 100%.

Table 18.1 shows that the power of $C$ is a monotonic step function of the number of shares acquired by $A$. The critical stocks which allow $C$ to pass from one position of power to another are 9, 10, 11, 50 and 51. It has been proved that by using the Shapley-Shubik index the sequence of critical stocks corresponding to shares transferred between two players $i$ and $j$ is always the same (see [32]).

The formulae generating these critical stocks $d_s$ in a company with $n$ shareholders are:

$$d_S = q - \sum_{h=1}^{n} b_h w_h \quad \text{and} \quad d_S = t - q + 1 - \sum_{h=1}^{n} b_h w_h$$

| Company | Number of shares $C$ receives from $A$ | Resulting distribution of shares | Resulting Shapley-Shubik index | Power increase of $C$ |
|---|---|---|---|---|
| A | | 51 | 1 | |
| B | | 40 | 0 | |
| C | 0 | 9 | 0 | 0 |
| A | | 50 | 2/3 | |
| B | | 40 | 1/6 | |
| C | 1 | 10 | 1/6 | 1/6 |
| A | | 49 | 1/3 | |
| B | | 40 | 1/3 | |
| C | 2 | 11 | 1/3 | 1/3 |
| A | | 10 | 1/6 | |
| B | | 40 | 1/6 | |
| C | 41 | 50 | 2/3 | 2/3 |
| A | | 9 | 0 | |
| B | | 40 | 0 | |
| C | 42 | 51 | 1 | 1 |
| Synthesis | | | | |
| Number of shares (C receives from A) | | | Resulting increment of power | |
| | | 1 | + 16.7 % | |
| from | | 2 to 40 | + 33.3 % | |
| | | 41 | + 66.7 % | |
| from | | 42 to 51 | +100.0 % | |

**TABLE 18.1**: Exchange of shares between two players (according to Shapley-Shubik index).

varying the $n$-dimensional vectors $b$ whose components take only 0 and 1 values, with condition $b_i = b_j = 0$ and $t$ is the total sum of weights. Both summations are moreover subjected to the requirement:

$$0 \le \sum_{h=1}^{n} b_h w_h < H$$

where $H$ is the minimum between $q$ and $(t - q)$.

It is worth noting that the formula $d_S = t - q + 1 - \sum_{h=1}^{n} b_h w_h$ is suitable for computing the position of the buying player when he is crucial for all the winning coalitions (with integer exchanges of shares).

In the previous example, $t = 100$, $q = 51$, $t - q = 49$, $i = 1$ and $j = 3$. The only binary vectors to be considered are $(0, 0, 0)$ and $(0, 1, 0)$. From those formulas the following values are obtained: 10, 11, 50 and 51, which generate the sequences $(51, 50, 49, 10, 9)$ for $A$ and $(9, 10, 11, 50, 51)$ for $C$.

### 18.5.3 Trade of Shares between One Player and Ocean of Players

Suppose a company has three major shareholders $A$, $B$ and $C$ and an "ocean" of minor shareholders who are not able to form a coalition to control the company. Let the initial breakdown of shares between the major shareholders be $(20, 15, 4)$; see Table 18.2. What would happen if the third shareholder starts to buy shares on the market from minor shareholders (the ocean) to increase his power index in the company?

If $C$ purchased one share from the ocean, the share distribution would become $(20, 15, 5)$ and the power factors (according to Shapley-Shubik) would be $(2/3, 1/6, 1/6)$, as the majority shareholding would go from 19.5 to 20. If $C$ purchased two shares from the ocean, the share distribution would become $(20, 15, 6)$ and the power indices would be $(1/3, 1/3, 1/3)$. This power distribution would remain the same even if $C$ were to purchase 30 shares; the situation would only change if $C$ bought 31 shares. In this case, the share distribution would become $(20, 15, 35)$ and the power factors $(1/6, 1/6, 2/3)$. With the purchase of one more share, $C$ would acquire the absolute majority and his power factor would be 100 percent.

It was proved that the Shapley-Shubik index of the raider ($i$-th player) to form coalitions is a monotonic step function of the number of shares purchased from minor shareholders (see [32]). The critical stocks $d_S$ are generated using the following formula (where $q$, $t$, and $w_h$ have the conditions indicated in the previous section):

$$d_S = -\frac{M}{tb_i - q} + b_i,$$

where

$$M = \sum_{\substack{h=1 \\ n \neq i}}^{n} (tb_h - q) w_h$$

with condition $M \geq 0$ for $b_i = 0$; $M \leq 0$ for $b_i = 1$.

In this example, the third player is involved ($i = 3$). Initially $t = 39$, $q = 19.5$, $w = (20, 15, 4)$ and these change as $w_3$ increases. The formula, with the necessary roundings, generates the critical points 5, 6, 35, and 36.

Note that the model proposed here differs from classical oceanic games (see, for example, [70], [59] and [56]) as it supposes that all the power is held by major shareholders. It is, therefore, more suitable for incomplete information in imperfect markets, where the minor shareholders are obviously excluded from the board of directors and where the means and the information the raider has renders the power of the ocean, which is not able to form a coalition, completely ineffective. (This model describes also this type of situation, because the $i$-th player could be a syndicate of shareholders.)

| Player | No. of shares bought by $C$ | Resulting distribution of shares | Resulting majority $(A+B+C)/2$ | Resulting power distribution |
|---|---|---|---|---|
| $A$ | | 20 | | 1 |
| $B$ | | 15 | | 0 |
| $C$ | 0 | 4 | 19.5 | 0 |
| $A$ | | 20 | | 2/3 |
| $B$ | | 15 | | 1/6 |
| $C$ | 1 | 5 | 20 | 1/6 |
| $A$ | | 20 | | 1/3 |
| $B$ | | 15 | | 1/3 |
| $C$ | 2 | 6 | 20.5 | 1/3 |
| $A$ | | 20 | | 1/3 |
| $B$ | | 15 | | 1/3 |
| $C$ | 30 | 34 | 34.5 | 1/3 |
| $A$ | | 20 | | 1/6 |
| $B$ | | 15 | | 1/6 |
| $C$ | 31 | 35 | 35 | 2/3 |
| $A$ | | 9 | | 0 |
| $B$ | | 40 | | 0 |
| $C$ | 32 | 36 | 35.5 | 1 |
| | | | | |
| Synthesis | | | | |
| | 1 share | bought: | +16.7 % | |
| from | 2 to 30 shares | bought: | +33.3 % | |
| with | 31 shares | bought: | +66.7 % | |
| with | More than 31 shares | bought: | +100.0 % | |

**TABLE 18.2**: Trade of shares between one player and ocean.

### 18.5.4 Remarks on Prices

The takeover can be carried out with the agreement of the current control group, which is interested in gaining a new shareholder as a result of company politics, the development outlook, etc. or against the control group. In the latter instance, the raider should expect an increase in the share offer price by increasing of the requested quantity. Such increase is artificial with respect to the real share value, shares being only part of the company in question, since it is only an added value that the raider wishes to pay so as to gain control, with the deriving benefits. An increase in the value of the company might ensue (for example, by an improved handling of politics), or there might be damage, for example, through choosing inferior suppliers, managers and political policies, although linked in other ways to the raider, or by using confidential information to pursue different goals. The new controller will then indirectly affect the share values, albeit such an influence remains more or less distanced from a temporary increase in the quotation connected to the takeover.

Another advantage for the raider is to sell the entire share package to the current control group, naturally at an increased price: This fact will decrease the share quotation, for which the small shareholders will pay the consequences. For further remarks, see Buzzacchi and Mosconi [17] and Corielli, Nicodano and Rindi [20].

During the acquisition phase, the first shares are normally bought on the market of small shareholders through silent operations, so as to avoid alarming the control group. Following possible agreements with some of the large shareholders, a takeover bid is presented with a fixed price and a pledge to buy only if a predetermined quantity is achieved.

An *a priori* evaluation of the price payable for the operation is very important for the raider. It is based on objective information (available share quantities on the market, closeness to the majority quote, the economic power of the current control group, eventual undercutting of the share, and so on) and subjective considerations (the power and cohesion of the current control group, possible collateral benefits that favor destabilizing agreements, and so on).

It has to be noted that the trend of price versus demand in the perfect market should coincide with the trend of the power position defined by a suitable index. On the other hand, in most standard cases, where small shareholders have no possibility of control, the two curves do not coincide: The raider is playing on a precise evaluation of this difference. The model presented in the previous clause can also be used to describe the effects of the formation of a syndicate of small shareholders who wish to defend their position.

### 18.5.5 Steadiness of Control

Another particular problem concerns the steadiness of the control position reached. It is not sufficient to acquire the minimum number of shares to achieve a position of power in which such power may be exercised. The current controllers may buy in turn, at an increased price, sufficient additional shares to drive the new shareholder from a position of power. It is thus necessary to buy a further "security amount", $\Delta s$ in relation to each discontinuity point $s$. How is such a quantity determined? It is clear that a purchase so as to reach the absolute majority quote could be enough to defend against counteractions, but it is also clear that the cost of such an operation might nullify the advantages.

A method is shown based on the considerations that follow [35]. When the current controllers try to buy shares on the market to regain their lost position, they could have difficulty finding them and be obliged to pay an ever-increasing price for such shares. The raider himself could offer to sell them the necessary shares at a price covering the surcharge he paid to buy the shares. With a reliable forecasting model of the quotation $p(s)$ versus the number of exchanged shares starting from the initial price $p_o$, the investor could calculate the unknown quantity $\Delta s$ by equating the sum of the inflated price paid for each share (from 0 to $s+\Delta s$) to the sum of the following inflated prices requested for each share (from $s+\Delta s$ to $s+2\Delta s$). In a model in the continuum space, the unknown $\Delta s$ is obtained from the equation:

$$\int_0^{s+\Delta s} p\left(s\right) ds - p_o\left[s+\Delta s\right] = \int_{s+\Delta s}^{s+2\Delta s} p\left(s\right) ds - p_o \Delta s$$

Being $P(s)$ the integral function of $p(s)$ in the considered interval, the problem is to find the minimum $\Delta s$ which is solution of the equation

$$P(s+2\Delta s) - 2P(s+\Delta s) = -p_o s - P(0).$$

### 18.5.6 Indirect Control

A particularly interesting problem involves those cases where an investor has a shareholding in a certain company, which, in turn, holds shares in another company, and so on. In situations of this kind, it may be useful to calculate power in the whole system.

Let a shareholder hold 20% of the shares of a company whose remaining shares are divided equally (40% and 40%) between two other shareholders. Let this company own 51% of the shares of another company which owns a quarter of the shares of a third company, whose remaining shares are divided equally among three other shareholders. What is the power of the first shareholder with regard to this latter company?

It could be said that the shareholder has a third of the power in the first company, which has total control in the second one; thus he has a third of the power in the second one. This latter company has a quarter of the power in the last one, so thus the shareholder has $(1/3) \cdot (1) \cdot (1/4) = 1/12$ of the power in this final company. It seems logical to assign, to each shareholder, an indirect control power equal to the product of the Shapley-Shubik indices. However, there are counter-examples that show how this method of proceeding in calculations can arrive at a total of company shares different to 100%. Therefore, another method must be found.

This problem has been tackled in the paper by Gambarelli and Owen [38], by transforming the set of inter-connected games into just one game, using *multi-linear extensions* introduced by Owen ([61] and [63]). The advantage of this method is that the power index considered to be the most suitable in describing the situation in question can then be applied to the unified game.

An algorithm for the automatic computation of such situations was elaborated by Denti and Prati [23] and then extended [24]. Denti and Prati's algorithm was implemented in a computer program presented in papers by Kolodziej and Stach [51] and Stach [76].

For further studies on the subject, see [16], [68], [21], [14] and [76].

Karos and Peters [50] developed a theory for computing power indices for indirect control in general cases giving a unique solution when dealing with invariant mutual control structures. In a mutual control structure agents exercise control over each other and a mutual control structure is invariant if it incorporates all indirect control relations.

Mercik and Lobos [57] introduced a power index of implicit power as a measurement of power in reciprocal ownership structures. This index, which they called the Implicit Power Index, is a modification of the Johnston index [49].

For work of a more directly applicational nature, we also recommend [7] and [21].

### 18.5.7 Global Index of De-Stability

Let's consider a set of companies which could be subjected to takeover. Is it possible to state which one is more vulnerable or to give a numerical index indicating the stability of each company? The answer was given in [35]. Let's see how to proceed. Let $n$ be the number of big investors having shares of at least one of such companies, whereas all the other shares belong to the ocean of small shareholders.

Let $A$ be the matrix of which the generic element $a_{hk}$ represents the share quantity of the $h$-th shareholder ($1 \leq h \leq n$) or of the $h$-th company ($n+1 \leq h \leq n+m$) in the $k$-th company. Let $B$ be the matrix of which the generic element $b_{hk}$ represents the Shapley-Shubik index of the $h$-th shareholder ($1 \leq h \leq n$) in the $k$-th company (being the power distributed only among the big shareholders, excluding the other companies).

Let $C$ be the matrix of which the generic element $c_{hk}$ represents the effective power (Shapley-Shubik index) of the representatives of the $h$-th shareholder in the board of directors of the $k$-th company. The generic element $d_{hk}$ of the matrix $D = C - B$ represents the difference between the theoretic and effective power; the higher values of $D$ represent a greater dissatisfaction of the $h$-th shareholder for the situation of the $k$-th company. To calculate the above defined indices, the presence of special friendship among big shareholders should be taken into account (the generalization given by Owen should then be used [62]). Let $d_k$ represent the maximum value of the $k$-th column of matrix $D$. Such value represents the maximum dissatisfaction in the considered company (the $k$-th one) and contributes to the formation of the *de-stability index* proposed by Gambarelli [35]. Other data necessary to define such index are as follows, with reference to each company (for the sake of simplicity, the index $k$ is omitted):

$w_r$ number of shares owned by the "raider"
$w_c$ number of shares owned by the control group ($0 \leq w_r < w_c$)
$q$ the majority quote
$p_z$ a former reference quotation

$p_o$ a current quotation
$s$ the power (political and economic power) of the current control group; this parameter gives indications of the relevant reaction capacity ($0 \leq s \leq 1$).

The above cited values contribute to the formation of the following preliminary indices, each taking values from 0 (=maximum stability) to 1 (=minimum stability) of the company:

| | |
|---|---|
| $c = w_r/w_c$ | ratio between the number of shares for the raider and the control group |
| $m = (t - w_r - w_c)/t$ | availability of residual shares on the market |
| $v = (q - w_c)/q$ | the vicinity to the absolute majority quota by the controlling shareholders |
| $f = \max(0, (p_z - p_o)/p_z)$ | the drop in the current quotation $p_o$ with respect to the reference quotation $p_z$ |

Thus, the global index $i$ is given by:

$$i = d^{a_1} \cdot s^{a_2} \cdot c^{a_3} \cdot m^{a_4} \cdot v^{a_5} \cdot f^{a_6},$$

where $a_1, \ldots, a_6$ are positive exogenous parameters, which can be estimated using statistical methods on historical series of past takeovers.

It has to be noted that the resulting index is still limited from 0 to 1 and it is worth 0 for minimal and 1 for maximum de-stability.

### 18.5.8 Portfolio Theory

Certain developments of the above results also involve the Theory of Portfolio Selection. It is known that traditional portfolio models imply a diversification of investments to minimize risk: The classical models of Portfolio Selection advise the saver to diversify his share portfolio in such a way as to efficiently reduce risk (the problem is solved by a multi-tasking optimization by maximizing the expected return and by minimizing the risk (see [77]). This, however, is in conflict with the relevant amount of a single stock that needs to be acquired to carry out hostile take-over bids. The connection between takeover and portfolio theories was initially dealt with by Amihud and Barnea [2] and by Batteau [6], who found a hindrance in determining the control function: This function was determined at the beginning of the 1980s by Gambarelli ([31] and [32]). A method of linking these two theories has been proposed by means of a control propensity index that can be linked to the risk aversion index (see [31] and [41]).

To summarize the optimal composition of a portfolio is determined by taking not only the expected return and variance of the classical investments into account, but also the investments with ordinary shares to be used for control.

One of the difficulties in this generalization is that, while in classical models a fixed price for buying the shares was assumed, in the new model the price is aleatory.

The method is the following:

- Identify an "index of inclination to the control" for the investor that can be connected to his risk aversion as it is used in the classical models;
- Share the capital into two classes of investment by using the new index (one for classical and one for control);
- Identify the company (or, if small, the companies with respect to the available capital) in the takeover and to identify more suitable power quotes for each company;
- Eliminate, from the companies used for classical investments, those already chosen for the takeover and those with a strong correlation with these last ones;
- Undertake the purchase of shares for the takeover silently;
- Finalize the operation.

To apply the model, the algorithms described below are used. For further application of game theory to portfolio, see [5] and [41].

To conclude this section, attention should be paid to recent work by Crama and Leruth [21], in which they show how techniques such as power indices are more suitable than cut-off methods in describing power-sharing among shareholders.

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## 18.6 Political Applications

In this section, instead, we shall look at simulations and predictions.

### 18.6.1 Introduction

We shall not go into the question of electoral systems, since we believe that the Shapley-Shubik index is best suited to both predictive and normative systems. For electoral systems, we would refer the reader to other works, such as those by Holler and Nurmi [47], Gambarelli [37], Gambarelli and Palestini [40], and Gambarelli and Stach [42].

We should start by saying that, in many cases, the application of power indices to political environments necessitates correction, inasmuch as not all coalitions theoretically possible occur in practice: For example, a coalition between the extreme Right and the extreme Left. Such corrections usually involve recourse to Owen's results [62] for the Shapley-Shubik index and successive developments (see, for instance, [64]).

### 18.6.2 Simulations

The formulas given in Section 18.5.2 may also be used to predict the changes in power relationships that follow from a shift by a subset of the electorate from one party to another.

We note that the formulas in Section 18.5.3 may be applied to political-electorate problems, since they are able to describe variations in the index subsequent to the introduction of electoral laws that permit an extension of the vote to new categories of electors (for example, immigrants, the disabled, prison inmates, young people, etc.) who are presumed to be oriented towards a particular party.

Further political applications use the results given in Section 18.5.6 relative to indirect control, inasmuch as parties are made up of various tendencies, within which there may be further diversification. Thus, Gambarelli and Owen's results [38], and successive results, may be applied to the quantification of power for each of these sub-categories within the party as a whole.

Another type of application concerns the limits imposed on small parties in electoral systems (see [33]) Here, too, we can go back to the considerations, advanced at the beginning of the sixth section, regarding affinity; it is, in any case, possible to undertake simulations of variations in the power indices for large parties when such limits are altered. For example, Table 18.3 shows how, for the Lega Lombarda following a change in the limit from 1 to 2%, there was a decrease, rather than increase, in its power (here measured using the Banzhaf-Coleman index).

Further studies concern the number of seats to be assigned to nations making up the European Parliament; a simulation based on a linear combination of population and GDP was undertaken by Bertini, Gambarelli, and Stach ([11] and [12]), based on the Banzhaf-Coleman index. However, analogous studies could be undertaken using the Shapley-Shubik index.

### 18.6.3 Predictions

Bearing in mind the considerations made at the start of Section 18.6, predictions of power relationships in a political environment cannot be made without reference to affinity and ideological distance between the various parties. Furthermore, many parliaments have two chambers, and the fact that a particular party might have two different power indices in the two chambers may create confusion concerning the overall power of that party (see, for instance, [80]). A global approach to the problem was taken by Gambarelli and Uristani [43], in which a model was elaborated (using relevant software) based on the Banzhaf index, which formulates predictions on the power of

| Party | Power in the case of a limit of | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 92% | 0% | 1% | 2% | 3% | 4-5% | 6-8% | 9% |
| DC | 33.1 | 42.2 | 43.2 | 43.6 | 46.2 | 46.2 | 50.0 | 100 |
| PDS | 18.1 | 14.1 | 14.6 | 14.5 | 15.4 | 15.4 | 16.7 | 0 |
| PSI | 14.9 | 13.9 | 14.3 | 14.5 | 15.4 | 15.4 | 16.7 | 0 |
| Lega L. | 8.5 | 8.6 | 9.4 | 8.2 | 11.5 | 7.7 | 16.7 | - |
| Rif. Com. | 5.8 | 4.9 | 4.8 | 5.5 | 3.8 | 7.7 | - | - |
| MSI | 5.3 | 4.5 | 4.4 | 5.5 | 3.8 | 7.7 | - | - |
| PRI | 3.9 | 3.1 | 3.0 | 2.7 | 3.8 | - | - | - |
| PLI | 2.2 | 1.9 | 1.7 | 1.8 | - | - | - | - |
| Verdi | 2.1 | 1.8 | 1.7 | 1.8 | - | - | - | - |
| PSDI | 2.0 | 1.7 | 1.7 | 1.8 | - | - | - | - |
| Rete | 1.6 | 1.3 | 1.2 | - | - | - | - | - |
| Pannella | 0.7 | 0.6 | - | - | - | - | - | - |
| SVP | 0.6 | 0.5 | - | - | - | - | - | - |
| Autonom. | 0.3 | 0.3 | - | - | - | - | - | - |
| Altri | 0.7 | 0.7 | - | - | - | - | - | - |
| TOTALS | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

**TABLE 18.3**: Power in Italy in case of limits on small parties (Camera + Senato, 1992).

the various parties in multi-chamber parliaments, taking affinity and hostility into account. This model was applied to all European nations with a multi-chamber parliament and, more generally, to the European Union. This model could also be developed with reference to the Shapley-Shubik index, which we consider to be more suited to problems concerning prediction.

## 18.7 Conclusions

Throughout the discussion we have undertaken thus far, we have indicated certain open problems that would be worth pursuing further. With this work, we have supplied an overview of studies regarding the application of the Shapley-Shubik index to finance and politics.

Possible further developments could concern what follows.

Regarding financial applications:

– As far as indirect control, some work is still needed in order to improve the algorithms from a computational point of view in order to reduce the calculation time.

– As far as the share shift between shareholders, some financial institutions have begun using the techniques shown in the quoted papers (though obviously without divulging the related results). Therefore, a comparison between the theoretical models and their applications (where possible) remains an open problem.

– Furthermore, the above-mentioned formulae concern the exchange of shares between two shareholders or among one shareholder and an ocean of small shareholders who cannot control the firm. Some more work regarding small shareholders who can control a firm should be developed to compare these models.

– Other open problems of the results discussed above concern the theory of Portfolio Selection. It is known that traditional portfolio models imply a diversification of investments to minimize risk. This diversification contrasts with the concentration of shares necessary for takeovers. A method of linking these two theories has been proposed by means of a control propensity index that can be linked to the risk aversion index. However, possible developments of such a theory for practical applications remain open.

– As far as the simulations, further studies could be done with respect to the composition of the European Parliament after Brexit, with seat apportionment that not only takes population size into consideration but also gross domestic products. A simulation based on a linear combination of population and GDP was undertaken by Bertini, Gambarelli, and Stach based on the Banzhaf-Coleman index. However, analogous studies could be undertaken using the Shapley-Shubik index.

Regarding political applications:

– Another problem concerns the calculation of power indices in cases of indirect control. The same models regarding finance can be applied to politics when a political party has various currents and sub-currents. Also in this case, a more efficient algorithm should be studied in order to reduce the computation time.

– A more specifically prediction-related issue concerns bicameral parliaments, where different affinities among the parties hold. The global model introduced by Gambarelli and Uristani could be improved using other methods of computation for cases of likes and dislikes (such as probabilistic indices, for instance).

– As far as the apportionment, a theorem of solution existence has been found and an algorithm has been created. Methods to reduce the related computation time should be welcomed.

For a more general discussion of the open problems, we refer to the papers by Fragnelli and Gambarelli [29] and Bertini, Gambarelli, and Stach [13].

## 18.8 Acknowledgments

This work is sponsored by research grants from the University of Bergamo by the Group GNAMPA of INDAM and by the statutory funds (no. 11/11.200.322) of the AGH University Science and Technology.

We thank the anonymous reviewer for the valuable comments and suggestions.

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# Chapter 19

## The Shapley Value in the Queueing Problem

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### 19.1 Introduction

We consider a situation when a group of agents must be served in a facility which can serve only one agent at a time.[1] Agents incur waiting costs. We assume that each agent's unit waiting cost is constant over time, but agents differ in their unit waiting costs. The *queueing problem* is concerned with finding the order in which to serve agents and the (positive or negative) monetary transfers they should receive. Furthermore, we assume that each agent's utility is equal to the amount of her monetary transfer minus her waiting cost. This queueing problem has been analyzed extensively from various perspectives: Incentive viewpoint (Dolan 1978; Suijs 1996; Mitra 2001, 2002; Mitra and Mutuswami 2011), normative viewpoint (Maniquet 2003; Chun 2006a, b; Klijn and Sánchez 2006), and strategic viewpoint (Ju, Chun and van den Brink 2014). Recently, there have been many studies to combine the incentive

[1] It is not difficult to find queueing situations in real life: Long queues at the supermarket; a waiting list in the hospital; landing slots at some U.S. airports. For other example of queueing situations, see Maniquet (2003), Kayi and Ramaekers (2010), and Mukherjee (2013).

and the normative viewpoints (Chun, Mitra and Mutuswami 2014a, b, 2015; Chun and Yengin 2017; Kayi and Raemaekers 2010).[2]

In this chapter, we give a survey on the literature which tries to solve the queueing problem by applying the Shapley value, the best-known solution developed in the cooperative game theory. To do so, we need to transform the queueing problem into transferable utility coalitional form games, or TU games, by defining the worth of a coalition. We note that the queueing problem can be divided into two subproblems depending on the existence of an initial queue.

For queueing problems without an initial queue, Maniquet (2003) takes an optimistic approach and defines the worth of a coalition to be the minimum waiting cost incurred by its members under the optimistic assumption that they are served before the non-coalitional members. By applying the Shapley value to the optimistic queueing game, he obtains the minimal transfer rule which selects an efficient queue and transfers to each agent half of her unit waiting cost multiplied by the number of her predecessors minus half of the sum of the unit waiting costs of her followers.

On the other hand, Chun (2006a) takes a pessimistic approach and defines the worth of each coalition to be the minimum waiting cost incurred by its members under the pessimistic assumption that they are served after the non-coalitional members. By applying the Shapley value to the pessimistic queueing game, he obtains the maximal transfer rule which selects an efficient queue and transfers to each agent half of the sum of the unit waiting costs of her predecessors minus half of her unit waiting cost multiplied by the number of her followers.

For some queueing situations, an initial queue may play an important role. In such cases, it is most likely that agents are served on the first-come, first-served basis. Although this rule is easy to implement, it may not be efficient when the waiting is costly for agents. Curiel et al. (1989) proposes to solve the queueing problem with an initial queue by transforming into TU games in which the worth of a coalition is defined to be the minimum waiting cost of the coalition after efficiently reordering their positions in the queue by themselves.[3] By applying the Shapley value to the initial queueing game, they obtain the connected equal splitting rule, which selects an efficient queue and allocates the cost savings obtained after reordering the positions between any two agents equally among themselves and all agents initially located between them.

Taken together, in the queueing problem, it is important how to define the worth of a coalition. Depending on the definition, the resulting allocation rule becomes very different even though the same Shapley value is applied.

This chapter is organized as follows. Section 19.2 contains some preliminaries on the queueing problem and the queueing game. Section 19.3 introduces

[2] See Chun (2016) for a survey of the literature.

[3] Strictly speaking, Curiel et al. (1989) studies how to allocate the maximal cost savings to agents.

the optimistic queueing game and investigates the properties of the minimal transfer rule (the Shapley value of the optimistic queueing game). Section 19.4 introduces the pessimistic queueing game and investigates the properties of the maximal transfer rule (the Shapley value of the pessimistic queueing game). Section 19.5 introduces the queueing problem with an initial queue and investigates the properties of the connected equal splitting rule (the Shapley value of the initial queueing game). Conclusions follow in Section 19.6.

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## 19.2 The Queueing Problem

Let $N = \{1, \dots n\}$, $n \geq 2$, be a set of agents. Each agent must be served in a facility which can handle only one agent at a time. Each agent needs the same amount of service time which is normalized to 1. Each agent $i \in N$ is characterized by her *unit waiting cost* $\theta_i \in \mathbb{R}_+$. Let $\theta = (\theta_i)_{i \in N}$ be the vector of unit waiting costs.[4]

A *queueing problem* (without an initial queue) is defined as a list $q = (N, \theta)$. Let $\mathcal{Q}^N$ be the class of all queueing problems for $N$. For each queueing problem $q = (N, \theta) \in \mathcal{Q}^N$, we assign to each agent $i \in N$ a position $\sigma_i \in \mathbb{N}_{++}$ in a queue and a (positive or negative) monetary transfer $t_i \in \mathbb{R}$. An *allocation* for $q \in \mathcal{Q}^N$ is a pair $(\sigma, t)$ where $\sigma = (\sigma_i)_{i \in N}$ denotes the vector of queue positions and $t = (t_i)_{i \in N}$ the vector of transfers. An allocation is *feasible* if all agents are assigned different positions and the sum of transfers is not positive. Thus, the set of feasible allocations $Z(q)$ consists of all pairs $(\sigma, t)$ such that for all $i$, $j \in N$, $i \neq j$ implies $\sigma_i \neq \sigma_j$ and $\sum_{i \in N} t_i \leq 0$. The agent who is served first incurs no waiting cost. If agent $i \in N$ is served in the $\sigma_i^{th}$ position, her waiting cost is $(\sigma_i - 1)\theta_i$. Each agent $i \in N$ has a quasi-linear utility function: Her utility from the bundle $(\sigma_i, t_i)$ is given by $u(\sigma_i, t_i; \theta_i) = -(\sigma_i - 1)\theta_i + t_i$.

For each $q = (N, \theta) \in \mathcal{Q}^N$, an allocation $(\sigma, t) \in Z(q)$ is *queue-efficient* if it minimizes the aggregate waiting cost, that is, for each $(\sigma', t') \in Z(q)$, $\sum_{i \in N}(\sigma_i - 1)\theta_i \leq \sum_{i \in N}(\sigma'_i - 1)\theta_i$. The efficient queue is unique if each agent has a different unit waiting cost. If there are agents with the same unit waiting costs, then the efficient queue is not necessarily unique. The agents with the same unit waiting costs have to be served consecutively but in any order. For each $q \in \mathcal{Q}^N$, let $E(q)$ be the set of all efficient queues. An allocation $(\sigma, t) \in Z(q)$ is *budget balanced* if $\sum_{i \in N} t_i = 0$. An allocation rule, or simply a *rule,* is a mapping $\varphi$ which associates with each queueing problem $q = (N, \theta) \in \mathcal{Q}^N$ a non-empty subset $\varphi(q)$ of feasible allocations. The pair $\varphi_i(q) = (\sigma_i, t_i)$ represents the position of agent $i$ in the queue and her monetary transfer in $q$.

[4]For any set $A$, $|A|$ denotes the cardinality of $A$.

For each $q = (N, \theta) \in \mathcal{Q}^N$, each $(\sigma, t) \in Z(q)$, and each $i \in N$, let $P_i(\sigma)$ be the set of predecessors of agent $i$ and $F_i(\sigma)$ the set of followers of agent $i$.

Now we introduce two rules for the queueing problem. First, suppose that there are two agents, agent 1 and agent 2, such that $\sigma_1 < \sigma_2$. If agent 1 changes her position with agent 2, then her waiting cost increases by $\theta_1$. She is indifferent whether she pays $\frac{\theta_1}{2}$ at $\sigma_1$ or receives $\frac{\theta_1}{2}$ at $\sigma_2$. On the other hand, if agent 2 changes her position with agent 1, then her waiting cost decreases by $\theta_2$. She is indifferent whether she pays $\frac{\theta_2}{2}$ at $\sigma_1$ or receives $\frac{\theta_2}{2}$ at $\sigma_2$. Therefore, we can expect that the actual transfer will be determined between $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$. Our two rules select an efficient queue and transfer either the minimum or the maximum of the two bounds in the 2-agent problem.

The minimal transfer rule (Maniquet 2003) selects an efficient queue and transfers to each agent half of her unit waiting cost multiplied by the number of her predecessors minus half of the sum of the unit waiting costs of her followers.

**Minimal transfer rule, $\varphi^M$:** For each $q \in \mathcal{Q}^N$,

$$\begin{aligned}\varphi^M(q) \quad = \quad & \{(\sigma^M, t^M) \in Z(q) \mid \sigma^M \in E(q) \ \text{ and} \\ & \forall i \in N, \ \ t_i^M = (\sigma_i^M - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma^M)} \frac{\theta_j}{2}\}.\end{aligned}$$

On the other hand, the maximal transfer rule (Chun 2006a) selects an efficient queue and transfers to each agent half of the sum of the unit waiting costs of her predecessors minus half of her unit waiting cost multiplied by the number of her followers.

**Maximal transfer rule, $\varphi^C$:** For each $q \in \mathcal{Q}^N$,

$$\begin{aligned}\varphi^C(q) \quad = \quad & \{(\sigma^C, t^C) \in Z(q) \mid \sigma^C \in E(q) \ \text{ and} \\ & \forall i \in N, \ \ t_i^C = \sum_{j \in P_i(\sigma^C)} \frac{\theta_j}{2} - (|N| - \sigma_i^C)\frac{\theta_i}{2}\}.\end{aligned}$$

The minimal and the maximal transfer rules are *essentially single-valued* in the sense that each agent ends up with the same utility at any allocation the rule selects.

Next, we formally describe cooperative games with transferable utility, or simply TU games. Let $N = \{1, \dots n\}$ be the set of *players*. A subset $S \subseteq N$ is a *coalition*. A TU game is a real-valued function $v$ defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The number $v(S)$ is the *worth* of coalition $S$. Let $\Gamma^N$ be the class of TU games with player set $N$. A *solution* is a function $\phi$

which associates with each $v \in \Gamma^N$ a vector $\phi(v) = (\phi_i(v))_{i\in N}$. The number $\phi_i(v)$ represents the utility of player $i$ in $v$.

The Shapley value (Shapley 1953) is the best-known solution for TU games. It assigns to each player a payoff equal to a weighted average of her marginal contributions to all possible coalitions with weights being determined by the size of each coalition.

**Shapley value,** $SV$**:** For each $v \in \Gamma^N$ and each $i \in N$,

$$SV_i(v) = \sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!|N\backslash S|!}{|N|!}\{v(S) - v(S\backslash\{i\})\}.$$

For each $v \in \Gamma^N$ and each $S \subseteq N$, let $\Delta_v(S)$ the dividend defined as follows: If $|S| = 1$, $\Delta_v(S) = v(S)$, and if $|S| > 1$, $\Delta_v(S) = v(S) - \sum_{T\subset S, T\neq S} \Delta_v(T)$. For each coalition $T \subseteq N$, let $u_T$ be the unanimity game defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. Then, each TU game can be expressed as a linear combination of unanimity games with coefficients being determined by the dividend of each coalition, that is, for each $v \in \Gamma^N$, $v = \sum_{T\subseteq N} \Delta_v(T)u_T$. Also, the Shapley value is alternatively calculated by using the dividend formula: For each $v \in \Gamma^N$ and each $i \in N$,

$$SV_i(v) = \sum_{S\subseteq N,\ i\in S} \frac{\Delta_v(S)}{|S|}. \tag{19.1}$$

For each $v \in \Gamma^N$, let $X(v) = \{x \in \mathbb{R}^N | \sum_{i\in N} x_i = v(N)\}$ be the set of efficient allocations for $v$ and $I(v) = \{x \in \mathbb{R}^N | \sum_{i\in N} x_i = v(N)$ and for each $i \in N$, $x_i \geq v(\{i\})\}$ be the set of *imputations* for $v$. For each $x \in X(v)$ and each $S \subseteq N$, let $e_S(v, x) = v(S) - \sum_{i\in S} x_i$ and $e(v, x) = (e_S(v, x))_{S\subseteq N} \in \mathbb{R}^{2^N}$ be the *excess vector*. The $S$-coordinate of $e(v, x)$, $e_S(v, x)$, measures the amount that the worth of coalition $S$ exceeds its payoff at $x$. For each $y \in \mathbb{R}^{2^N}$, let $\widetilde{y} \in \mathbb{R}^{2^N}$ be the vector obtained by rearranging the coordinates of $y$ in the non-increasing order. For each $y$, $z \in \mathbb{R}^{2^N}$, *$y$ is lexicographically smaller than $z$* if either (i) $\widetilde{y}_1 < \widetilde{z}_1$ or (ii) there exists $\ell > 1$ such that $\widetilde{y}_\ell < \widetilde{z}_\ell$ and for each $k < \ell$, $\widetilde{y}_k = \widetilde{z}_k$.

Now we introduce two well-known solutions of TU games. The nucleolus (Schmeidler 1969) minimizes the excesses of all coalitions in the lexicographic order on the set of imputations. On the other hand, the prenucleolus minimizes the excesses of all coalitions in the lexicographic order on the set of efficient allocations.

**Nucleolus,** $\boldsymbol{NU}$**:** For each $v \in \Gamma^N$ such that $I(v) \neq \emptyset$,

$$NU(v) \equiv \left\{ x \in I(v) \;\middle|\; \begin{array}{l} \text{for each } x' \in I(v)\backslash\{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,x') \end{array} \right\}$$

**Prenucleolus, *PN*:** For each $v \in \Gamma^N$,

$$PN(v) \equiv \left\{ x \in X(v) \;\middle|\; \begin{array}{l} \text{for each } x' \in X(v)\backslash\{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,x') \end{array} \right\}$$

For each $v \in \Gamma^N$, the *core* is the set of imputations at which no excess is greater than zero, that is, $Core(v) = \{x \in I(v) |$ for each $S \subset N$, $\sum_{i \in S} x_i \geq v(S)\}$. A TU game is *convex* if for each $S$, $T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. It is well known that a convex game has a non-empty core. Moreover, each of the Shapley value and the nucleolus selects an allocation in the core.

## 19.3 The Shapley Value in the Optimistic Queueing Game

Maniquet (2003) proposes to solve the queueing problem by applying the Shapley value. To do so, queueing problems should be transformed into TU games by appropriately defining the worth of a coalition. Maniquet (2003) introduces the optimistic queueing game $v_O$ obtained under the optimistic assumption that the coalitional members are served before the non-coalitional members. The worth of each coalition is defined as the minimum waiting cost incurred by its members when they are served before the non-coalitional members.[5] Formally, for each $S \subseteq N$,

$$v_O(S) = -\sum_{i \in S} (\sigma_i^* - 1)\theta_i$$

where $\sigma^* \in E(S, (\theta_i)_{i \in S})$. By applying the Shapley value to this game, Maniquet (2003) shows that the resulting rule coincides with the minimal transfer rule.

**Theorem 19.1** (Maniquet 2003). The Shapley value applied to the optimistic queueing game yields the minimal transfer rule.

*Proof.* For each $q = (N, \theta) \in \mathcal{Q}^N$, we calculate the dividend $\Delta_{v_O}$ of the optimistic queueing game. In fact,

1. $|S| = 1$: We assume without loss of generality that $S = \{i\}$. Then, $\Delta_{v_O}(S) = v_O(S) = 0$.

2. $|S| = 2$: We assume without loss of generality that $S = \{i, j\}$ and $\theta_i \geq \theta_j$. Then, $\Delta_{v_O}(S) = v_O(S) - \Delta_{v_O}(\{i\}) - \Delta_{v_O}(\{j\}) = -\theta_j$.

[5] Note that the cost is measured in the negative amount and the cost savings in the positive amount.

3. $|S| = 3$: We assume without loss of generality that $S = \{i, j, k\}$ and $\theta_i \geq \theta_j \geq \theta_k$. Then, $\Delta_{v_O}(S) = v_O(S) - \Delta_{v_O}(\{i,j\}) - \Delta_{v_O}(\{j,k\}) - \Delta_{v_O}(\{i,k\}) - \Delta_{v_O}(\{i\}) - \Delta_{v_O}(\{j\}) - \Delta_{v_O}(\{k\}) = -\theta_j - 2\theta_k + \theta_j + \theta_k + \theta_k = 0$.

4. $S \subseteq N$ such that $|S| > 3$: We assume without loss of generality that $S = \{1, 2, \ldots, s\}$ and $\theta_i \geq \theta_j$ for each $i \leq j$. Now, as induction hypothesis, suppose that $\Delta_{v_O}(S') = 0$ for each $3 \leq |S'| < |S|$. Then,

$$\begin{aligned}
\Delta_{v_O}(S) &= v_O(S) - \sum_{T \subset S, |T|=1,2} \Delta_{v_O}(T) \\
&= -\sum_{h=1}^{s} (\sigma_h - 1)\theta_h - \sum_{h=1}^{s} (-(\sigma_h - 1))\theta_h \\
&= 0.
\end{aligned}$$

Altogether, for each $q \in \mathcal{Q}^N$,

$$\Delta_{v_O}(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ -\min_{i \in S} \theta_i & \text{if } |S| = 2, \\ 0 & \text{if } |S| \geq 3. \end{cases} \tag{19.2}$$

For each $v \in \Gamma^N$ and each $i \in N$, we calculate the Shapley value allocation by using the dividend formula. By substituting (19.2) into the dividend formula (19.1), we obtain

$$SV_i(v_O) = -(\sigma_i - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma)} \frac{\theta_j}{2},$$

where $\sigma \in E(q)$. By using $t_i = u(\sigma_i, t_i; \theta_i) + (\sigma_i - 1)\theta_i$,

$$\begin{aligned}
t_i &= -(\sigma_i - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma)} \frac{\theta_j}{2} + (\sigma_i - 1)\theta_i \\
&= (\sigma_i - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma)} \frac{\theta_j}{2},
\end{aligned}$$

the desired conclusion. ∎

**Remark 19.1** We apply the prenucleolus to the optimistic queueing game and identify the resulting rule. First, note that the optimistic queueing game satisfies the following two conditions:

(i) for each $i \in N$, $v_O(\{i\}) = 0$,

(ii) for each $S \subseteq N$ such that $|S| \geq 2$, $v_O(S) = \sum_{T \subseteq S, |T|=2} v_O(T)$.

As shown in Kar, Mitra, and Mutuswami (2009), these two conditions are sufficient to guarantee the coincidence of the Shapley value and the prenucleolus. Therefore, these two solutions make the same recommendation in the optimistic queueing game. On the other hand, it is not difficult to show that the core of the optimistic queueing game is empty.

**Remark 19.2** Also, as shown in Chun and Hokari (2007), the serial cost sharing rule (Moulin and Shenker 1992) coincides with the minimal transfer rule. To simplify the argument, let $N = \{1, \ldots, n\}$ and $\theta \in \mathcal{Q}^N$ be such that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. Then, $Sh_n(v_O) = -\frac{n-1}{2}\theta_n$, $Sh_{n-1}(v_O) = -\frac{n-2}{2}\theta_{n-1} - \frac{1}{2}\theta_n$, and so on. Now we calculate the payoff assigned by the serial cost sharing rule. First, suppose that all agents have the same unit waiting cost of $\theta_n$. Then, the total waiting cost $-\{1 + \cdots + (n-1)\}\theta_n$ is divided equally among all agents, and each agent receives $-\frac{n-1}{2}\theta_n$. Now suppose that agent $n$ leaves and the remaining agents have the same unit waiting cost of $\theta_{n-1}$. The total waiting cost decreases by the amount of $-\{1 + \cdots + (n-2)\}(\theta_{n-1} - \theta_n)$, which should be shared equally among the remaining $(n-1)$ agents, and each remaining agent receives $-\frac{n-2}{2}(\theta_{n-1} - \theta_n)$. Since the original assignment to each agent is equal to $-\frac{n-1}{2}\theta_n$, her final assignment is $-\frac{n-2}{2}\theta_{n-1} - \frac{1}{2}\theta_n$. And so on. Therefore, the serial cost sharing and the minimal transfer rules make the same recommendation in the queueing problem.[6]

**Remark 19.3** Maniquet (2003) presents two axiomatic characterizations of the minimal transfer rule by imposing either (1) *efficiency* (a rule should choose allocations that are queue-efficient and budget balanced), *Pareto indifference* (if an allocation is chosen by a rule, then all other allocations which assign the same utilities to each agent should be chosen by the rule), *equal treatment of equals* (two agents with the same waiting cost should end up with the same utilities), and *independence of preceding costs* (an increase in an agent's waiting cost should not affect her followers) or (2) *Pareto indifference,* the *identical preferences lower bound* (each agent should be at least as well off as she would be, under *efficiency* and *equal treatment of equals,* if all other agents had the same preferences as her), *negative cost monotonicity* (an increase in an agent's waiting cost should cause all other agents to weakly lose), and *last-agent equal responsibility* (upon the departure of the agent served last, the queue should not be affected and the transfers to all other agents should be affected by the same amount).

Now we describe that the coincidence between the minimal transfer rule and the Shapley value in the optimistic queueing game still holds for two generalizations of the queueing problem.

[6] Moulin (2007) makes a similar observation for the scheduling problem in which agents have the same unit waiting cost, but differ in the amount of service time.

**Remark 19.4 Sequencing problem.** A sequencing problem generalizes a queueing problem by allowing each agent to have a different amount of service time. Thus, each agent in the sequencing problem is characterized by two parameters, the unit waiting cost and the amount of service time. Let $\alpha = (\alpha_i)_{i\in N} \in \mathbb{R}^N_+$ be the vector of service times. Then, a sequencing problem is defined as a list $q^S = (N, \theta, \alpha)$ where $N$ is the set of agents, $\theta$ the vector of unit waiting costs, and $\alpha$ the vector of service times. An allocation $(\sigma, t)$ is *feasible* if all agents are assigned different positions and the sum of transfers is not positive. Let $Z(q^S)$ be the set of all feasible allocations. An *efficient* queue is obtained if agents are served in the non-increasing order with respect to $\frac{\theta_i}{\alpha_i}$ (Smith 1956). For each $q^S = (N, \theta, \alpha)$, let $E(q^S)$ be the set of efficient queues. The minimal transfer rule $\varphi^M$ is generalized as: For each sequencing problem $q^S = (N, \theta, \alpha)$,

$$\begin{aligned}\varphi^M(q^S) \quad = \quad & \{(\sigma^M, t^M) \in Z(q^S) |\ \sigma^M \in E(q^S) \text{ and} \\ & \forall i \in N,\ \ t_i^M = \sum_{j\in P_i(\sigma^M)} \frac{\alpha_j\theta_i}{2} - \sum_{j\in F_i(\sigma^M)} \frac{\alpha_i\theta_j}{2}\}.\end{aligned}$$

As discussed in Chun (2011), this rule is obtained by applying the Shapley value to the optimistic sequencing game where the worth of each coalition is defined in the optimistic way, that is, for each $S \subseteq N$, $v_O(S) = -\sum_{i\in S}\sum_{j\in P_i(\sigma^S)} \alpha_j\theta_i$ where $\sigma^S \in E(S, (\theta_k)_{k\in S}, (\alpha_k)_{k\in S})$.

**Remark 19.5 Two-server queueing problem.** In the queueing problem (with one server), we assume that there is only one server so that only one agent can be served at one time. Here we generalize the queueing problem (with one server) by assuming the facility has two servers so that at most two agents can be served at the same time. The queueing problem with two servers (or two-server queueing problem) is defined as a list $\tilde{q} = (N, \theta, 2)$ where $N$ is the set of agents, $\theta \in \mathbb{R}^N_+$ is the vector of unit waiting costs, and 2 denotes the number of servers. An *allocation* for $\tilde{q}$ is a pair $z = (g, t)$, where for each $i \in N$, $g_i$ is the position assigned to agent $i$ and $t_i$ is her monetary transfer. An allocation is *feasible* if at most two agents are assigned to each position and the sum of monetary transfers is not positive. Let $Z(\tilde{q})$ be the set of all feasible allocations for $\tilde{q}$. For each $\tilde{q} = (N, \theta, 2)$, a feasible allocation $z = (g, t) \in Z(\tilde{q})$ is *queue-efficient* if it minimizes the total waiting costs. Let $E(\tilde{q})$ be the set of all queue-efficient allocations for $\tilde{q}$. A *rule* associates with each problem $\tilde{q}$ a non-empty subset $\varphi(\tilde{q})$ of feasible allocations. The pair $\varphi_i(\tilde{q}) = (g_i, t_i)$ represents the position of agent $i$ in the queue and her monetary transfer.

To simplify our analysis, we assume that agents are indexed in the non-increasing order of their unit waiting costs; the agent indexed 1 has the largest unit waiting cost, the agent indexed 2 has the second largest unit waiting cost, and so on. This indexing is uniquely defined except for agents with the same unit waiting cost. Those agents have to be indexed consecutively. For each

$\tilde{q} = (N, \theta, 2)$, let $D(\tilde{q})$ be the set of all possible indices. For each $d \in D(\tilde{q})$ and each $i \in N$, the efficient queue $g$ is defined as

$$g_i = \lceil \tfrac{d_i}{2} \rceil = \begin{cases} \frac{d_i}{2} & \text{if } d_i \text{ is even,} \\ \frac{d_i+1}{2} & \text{if } d_i \text{ is odd.} \end{cases}$$

For each $\tilde{q} = (N, \theta, 2)$, each $d \in D(\tilde{q})$, and each $i \in N$, let $P_i(d)$ be the set of agents with smaller indices than agent $i$ and $F_i(d)$ the set of agents with larger indices than agent $i$.

Now the minimal transfer rule is defined as: For each $\tilde{q} = (N, \theta, 2)$,

$$\begin{aligned} \varphi^M(\tilde{q}) = \quad & \{(g^M, t^M) \in Z(\tilde{q}) \mid \forall d \in D(\tilde{q}) \text{ and } \forall i \in N,\ g_i^M = \lceil \tfrac{d_i}{2} \rceil \text{ and} \\ & t_i^M = \frac{\sum_{g_j^M < g_i^M} g_j^M \cdot 2}{d_i} \cdot \theta_i - \sum_{k \in F_i(d)} \left(\frac{1}{d_k - 1} \cdot \frac{\sum_{g_j^M < g_k^M} g_j^M \cdot 2}{d_k} \cdot \theta_k\right)\}. \end{aligned}$$

As shown in Chun and Heo (2008), this rule is obtained by applying the Shapley value to the optimistic two-server queueing game where the worth of each coalition is defined in the optimistic way. For each $S \subseteq N$, its worth $v_O(S) = -\sum_{i \in S}(g_i^S - 1)\theta_i$ where $g^S \in E(\tilde{q}^S)$ and $\tilde{q}^S = (S, (\theta_i)_{i \in S}, 2)$.

## 19.4 The Shapley Value in the Pessimistic Queueing Game

On the other hand, Chun (2006a) shows that an alternative definition of the worth of a coalition results in a very different rule even if the same Shapley value is applied. He introduces the pessimistic queueing game $v_P$ obtained under the pessimistic assumption that the coalitional members are served after the non-coalitional members. The worth of each coalition is defined as the minimum waiting cost incurred by its members when they are served after the non-coalitional members. Formally, for each $S \subseteq N$,

$$v_P(S) = -\sum_{i \in S}(|N| - |S| + \sigma_i^* - 1)\theta_i$$

where $\sigma^* \in E(S, (\theta_i)_{i \in S})$. By applying the Shapley value to this game, Chun (2006a) shows that the resulting rule coincides with the maximal transfer rule.

**Theorem 19.2** (Chun 2006a). The Shapley value applied to the pessimistic queueing game yields the maximal transfer rule.

*Proof.* For each $q = (N, \theta) \in \mathcal{Q}^N$, we calculate the dividend $\Delta_{v_P}$ of the pessimistic queueing game. In fact,

1. $|S| = 1$: We assume without loss of generality that $S = \{i\}$. Then, $\Delta_{v_P}(S) = v_P(S) = -(|N| - 1)\theta_i$.

2. $|S| = 2$: We assume without loss of generality that $S = \{i, j\}$ and $\theta_i \geq \theta_j$. Then, $\Delta_{v_P}(S) = v_P(S) - \Delta_{v_P}(\{i\}) - \Delta_{v_P}(\{j\}) = -(|N| - 2)\theta_i - (|N| - 1)\theta_j + (|N| - 1)\theta_i + (|N| - 1)\theta_j = \theta_i$.

3. $|S| = 3$: We assume without loss of generality that $S = \{i, j, k\}$ and $\theta_i \geq \theta_j \geq \theta_k$. Then, $\Delta_{v_P}(S) = v_P(S) - \Delta_{v_P}(\{i, j\}) - \Delta_{v_P}(\{j, k\}) - \Delta_{v_P}(\{i, k\}) - \Delta_{v_P}(\{i\}) - \Delta_{v_P}(\{j\}) - \Delta_{v_P}(\{k\}) = -(|N| - 3)\theta_i - (|N| - 2)\theta_j - (|N| - 1)\theta_k - \theta_i - \theta_j - \theta_i + (|N| - 1)\theta_i + (|N| - 1)\theta_j + (|N| - 1)\theta_k = 0$.

4. $S \subseteq N$ be such that $|S| > 3$: We assume without loss of generality that $S = \{1, 2, \ldots, s\}$ and $\theta_i \geq \theta_j$ for each $i \leq j$. Now, as induction hypothesis, suppose that $\Delta_{v_P}(S') = 0$ for all $3 \leq |S'| < |S|$. Then,

$$\begin{aligned} & \Delta_{v_P}(S) \\ = \; & v_P(S) - \sum_{T \subset S, |T| = 1,2} \Delta_{v_P}(T) \\ = \; & -\sum_{h=1}^{s} (|N| - |S| + \sigma_h - 1)\theta_h - \sum_{h=1}^{s} (|S| - \sigma_h)\theta_h + \sum_{h=1}^{s} (|N| - 1)\theta_h \\ = \; & 0. \end{aligned}$$

Altogether, for each $q \in \mathcal{Q}^N$,

$$\Delta_{v_P}(S) = \begin{cases} -(|N| - 1)\theta_i & \text{if } |S| = 1, \\ \max_{i \in S} \theta_i & \text{if } |S| = 2, \\ 0 & \text{if } |S| \geq 3. \end{cases} \tag{19.3}$$

For each $v \in \Gamma^N$ and each $i \in N$, we calculate the Shapley value allocation by using the dividend formula. By substituting (19.3) into the dividend formula (19.1), we obtain

$$SV_i(v_P) = -(|N| - 1)\theta_i + \left\{ \sum_{j \in P_i(\sigma)} \frac{\theta_j}{2} + (|N| - \sigma_i)\frac{\theta_i}{2} \right\},$$

where $\sigma \in E(q)$. By using $t_i = u(\sigma_i, t_i; \theta_i) + (\sigma_i - 1)\theta_i$ for each $i \in N$,

$$\begin{aligned} t_i &= -(|N|-1)\theta_i + \left\{ \sum_{j \in P_i(\sigma)} \frac{\theta_j}{2} + (|N|-\sigma_i)\frac{\theta_i}{2} \right\} + (\sigma_i - 1)\theta_i \\ &= -(|N|-\sigma_i)\theta_i + \left\{ \sum_{j \in P_i(\sigma)} \frac{\theta_j}{2} + (|N|-\sigma_i)\frac{\theta_i}{2} \right\} \\ &= \sum_{j \in P_i(\sigma)} \frac{\theta_j}{2} - (|N|-\sigma_i)\frac{\theta_i}{2}, \end{aligned}$$

the desired conclusion. ■

**Remark 19.6** Now we apply the nucleolus to the pessimistic queueing game and identify the resulting rule. Surprisingly, we end up with the same rule: The Shapley value and the nucleolus coincide for the pessimistic queueing game (Chun and Hokari 2007). To show this, we introduce an *auxiliary pessimistic queueing game* $\tilde{v}_P$, in which the worth of coalition $S$ is obtained by adding $\sum_{i \in S}(n-1)\theta_i$ to $v_P(S)$, that is, for all $S \subseteq N$, $\tilde{v}_P(S) = v_P(S) + \sum_{i \in S}(n-1)\theta_i$. Note that $\tilde{v}_P$ satisfies the following conditions:

(i) for each $i \in N$, $\tilde{v}_P(\{i\}) = 0$,

(ii) for each $S \subseteq N$ such that $|S| \geq 2$, $\tilde{v}_P(S) = \sum_{T \subseteq S, |T|=2} \tilde{v}_P(T)$ and $\tilde{v}_P(S) \geq 0$.

As shown in Deng and Papadimitriou (1994) and van den Nouweland et al. (1996), these two conditions are sufficient to guarantee the coincidence of the Shapley value and the nucleolus. Finally, the coincidence for the pessimistic queueing game follows from the fact that both the Shapley value and the nucleolus satisfy *zero-independence,* which requires that adding a constant to the worth of coalitions containing player $i$ should affect her payoff by the constant.[7]

**Remark 19.7** Also, as shown in Chun and Hokari (2007), the decreasing serial cost sharing rule (de Frutos 1998) coincides with the maximal transfer rule. To simplify the argument, let $N = \{1, \ldots, n\}$ and $\theta \in \mathcal{Q}^N$ be such that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. Then, $Sh_1(v_P) = -\frac{n-1}{2}\theta_1$, $Sh_2(v_P) = -\frac{1}{2}\theta_1 - \frac{n-2}{2}\theta_2$, and so on. Now we calculate the payoff assigned by the decreasing serial cost sharing rule. First, suppose that all agents have the same unit waiting cost of $\theta_1$. Then, the total waiting cost $-\{1 + \cdots + (n-1)\}\theta_1$ is divided equally among all agents and each agent receives $-\frac{n-1}{2}\theta_1$. Now suppose that agent 1 leaves and the remaining agents have the same unit waiting cost of $\theta_2$. The total waiting cost increases by $\{1 + \cdots + (n-2)\}(\theta_1 - \theta_2)$, which should be

[7] Chun and Hokari (2007) show that the $\tau$-value (Tijs 1987) also coincides with the Shapley value in the pessimistic queueing game.

shared equally among the remaining $(n-1)$ agents, and each remaining agent receives $\frac{n-2}{2}(\theta_1 - \theta_2)$. Since the original assignment to each agent is equal to $-\frac{n-1}{2}\theta_1$, her final assignment is $-\frac{1}{2}\theta_1 - \frac{n-2}{2}\theta_2$. And so on. Therefore, the decreasing serial cost sharing and the maximal transfer rules make the same recommendation in the queueing problem.

**Remark 19.8** Chun (2006a) presents two axiomatic characterizations of the maximal transfer rule by imposing either (1) *efficiency, Pareto indifference, equal treatment of equals,* and *independence of following costs* (a decrease in an agent's waiting cost should not affect her predecessors) or (2) *Pareto indifference,* the *identical preferences lower bound, positive cost monotonicity* (an increase in an agent's waiting cost should cause all other agents to weakly gain), and *first-agent equal responsibility* (upon the departure of the agent served first, the queue should not be affected and the transfers to all other agents should be affected by the same amount).

**Remark 19.9** A bankruptcy problem is concerned with finding a reasonable compromise when the amount to divide is not sufficient to cover all claims. Let $N$ be the set of agents, $c = (c_i)_{i\in N} \in \mathbb{R}^N_+$ a claims vector, and $E \in \mathbb{R}_+$ an amount to divide. The bankruptcy problem is defined as a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. Once again, we can take two parallel perspectives on the worth of a coalition. For each coalition $S \subseteq N$, if the coalitional members have priority over the non-coalitional members, then its worth $v_O$ is defined by setting, for each coalition $S \subseteq N$, $v_O(S) = \min\{\sum_{i\in S} c_i, E\}$. On the other hand, if the non-coalitional members have priority over the coalitional members, then its worth $v_P$ is defined by setting, for each coalition $S \subseteq N$, $v_P(S) = \max\{E - \sum_{i\in N\setminus S} c_i, 0\}$. Since these two formulations are dual to each other, they give the same allocation when the Shapley value is applied.[8] However, this is not the case for the queueing problem.

Now we describe that the coincidence between the maximal transfer rule and the Shapley value in the pessimistic queueing game still holds for two generalizations of the queueing problem.

**Remark 19.10 Sequencing problem.** As in the minimal transfer rule, the maximal transfer rule $\varphi^C$ is generalized to sequencing problems as follows: For each sequencing problem $q^S = (N, \theta, \alpha)$,

$$\begin{aligned} \varphi^C(q^S) \quad = \quad & \{(\sigma^C, t^C) \in Z(q^S) |\, \sigma^C \in E(q^S) \text{ and} \\ & \forall i \in N,\ \ t_i^C = \sum_{j\in P_i(\sigma^C)} \frac{\alpha_i\theta_j}{2} - \sum_{j\in F_i(\sigma^C)} \frac{\alpha_j\theta_i}{2}\}. \end{aligned}$$

[8]See Aumann and Maschler (1985), Driessen (1998), and Thomson (2003) for details.

As discussed in Chun (2011), this rule is also obtained by applying the Shapley value to the pessimistic sequencing game where the worth of each coalition is defined in the pessimistic way, that is, for each $S \subseteq N$,

$$v_P(S) = -\sum_{i \in S} \Big( \sum_{j \in N \setminus S} \alpha_j + \sum_{j \in P_i(\sigma^S)} \alpha_j \Big) \theta_i$$

where $\sigma^S \in E(S, (\theta_k)_{k \in S}, (\alpha_k)_{k \in S})$.

**Remark 19.11 Two-server queueing problem.** Now we generalize the maximal transfer rule to two-server queueing problems. For each $\tilde{q}$, each $d \in D(\tilde{q})$, and each $i \in N$, let $m_i$ be the contribution from agent $i$, defined as

$$m_i = \begin{cases} \frac{\sum_{g_j > g_i} (g_j - g_i) \cdot 2}{|N| - d_i + 1} \theta_i & \text{if } |N| \text{ is even,} \\ \frac{\sum_{g_j > g_i, g_j < \lceil \frac{|N|}{2} \rceil} (g_j - g_i) \cdot 2}{|N| - d_i + 1} \theta_i + \frac{\lceil \frac{|N|}{2} \rceil - g_i}{|N| - d_i + 1} \theta_i & \text{if } |N| \text{ is odd.} \end{cases}$$

The maximal transfer rule is extended as: For each $\tilde{q} = (N, \theta, 2)$,

$$\varphi^C(\tilde{q}) = \{(g^C, t^C) \in Z(\tilde{q}) \mid \forall d \in D(\tilde{q}) \text{ and } \forall i \in N,\ g_i^C = \lceil \tfrac{d_i}{2} \rceil \text{ and } t_i^C = \textstyle\sum_{j \in P_i(d)} \frac{m_j}{|N| - d_j} - m_i\}.$$

As shown in Chun and Heo (2008), this rule is also obtained by applying the Shapley value to the pessimistic two-server queueing game where the worth of each coalition is defined in the pessimistic way. Let $S \subseteq N$. To define the worth of a coalition, we need to consider the cardinality of $N \setminus S$. If $|N \setminus S|$ is even, then agents in $S$ are served from the $(\frac{|N| - |S|}{2} + 1)$th position. If $|N \setminus S|$ is odd, the last position for $N \setminus S$ is composed of one agent from $N \setminus S$ and one agent from $S$. Therefore, the waiting cost of $i \in S$, $C_i(S)$, is calculated as follows. For each $i \in S$,

$$C_i(S) = \begin{cases} \{\frac{|N| - |S|}{2} + (g_i^S - 1)\} \theta_i & \text{if } |N| - |S| \text{ is even,} \\ \{\frac{|N| - |S| - 1}{2} + (g_i^S - 1)\} \theta_i & \text{if } |N| - |S| \text{ is odd and } d_i^S \text{ is odd,} \\ (\frac{|N| - |S| - 1}{2} + g_i^S) \theta_i & \text{if } |N| - |S| \text{ is odd and } d_i^S \text{ is even,} \end{cases}$$

where $g^S \in E(\tilde{q}^S)$ and $d^S \in D(\tilde{q}^S)$. The worth of coalition $S$, $v_P(S)$, is defined as

$$v_P(S) = -\sum_{i \in S} C_i(S).$$

## 19.5 The Shapley Value in the Queueing Game with an Initial Order

Up to now, we assume that there is no initial queue of the agents. However, an initial queue is given in many queueing situations. In this section, we discuss how the Shapley value is applied to the queueing problem with an initial queue.[9]

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of agents and $\theta = (\theta_i)_{i \in N} \in \mathbb{R}^N$ be the vector of unit waiting costs. Furthermore, we assume that there is an initial queue $\sigma^0$ which determines the order of agents when there is no reordering. A *queueing problem with an initial queue,* or *an initial queueing problem,* is defined as a list $q^0 = (N, \theta, \sigma^0)$. Let $\mathcal{Q}_0^N$ be the class of all initial queueing problems for $N$.

For each $q^0 = (N, \theta, \sigma^0) \in \mathcal{Q}_0^N$, we assign to each agent $i \in N$ a reordered position $\sigma_i \in \mathbb{N}_{++}$ in a queue and a monetary transfer $t_i \in \mathbb{R}$. An *allocation* for $q^0 \in \mathcal{Q}_0^N$ is a pair $(\sigma, t)$ where $\sigma = (\sigma_i)_{i \in N}$ denotes the vector of reordered queue positions and $t = (t_i)_{i \in N}$ the vector of transfers. An allocation is *feasible* if all agents are assigned different positions and the sum of transfers is not positive. Let $Z(q^0)$ be the set of all feasible allocations for $q^0$. The agent who is served first incurs no waiting cost. If agent $i \in N$ is served in the $\sigma_i^{th}$ position, her waiting cost is $(\sigma_i - 1)\theta_i$. Each agent $i \in N$ has a quasi-linear utility function: Her utility from the bundle $(\sigma_i, t_i)$ is given by $u(\sigma_i, t_i; \theta_i) = -(\sigma_i - 1)\theta_i + t_i$.

*Efficiency* requires to reorder agents in the non-increasing order of their unit waiting costs. To reorder agents in an efficient queue, we change the positions of two agents one by one. For each $q^0 = (N, \theta, \sigma^0) \in \mathcal{Q}_0^N$, let agents $i$ and $j$ be two neighbors such that $\sigma_j^0 = \sigma_i^0 + 1$. If $\theta_j > \theta_i$, the total waiting cost decreases by $(\theta_j - \theta_i)$ by reordering their positions. On the other hand, if $\theta_i > \theta_j$, they cannot gain by reordering their positions. Let $g_{ij} = \max\{\theta_j - \theta_i, 0\}$ be the cost savings for two agents $i$ and $j$. We assume that two agents with the same unit waiting cost does not change their positions.

Now we introduce a rule for the initial queueing problem which selects an efficient queue and allocates the cost savings obtained after reordering the positions between any two agents equally among themselves and all agents initially located between them.

**Connected equal splitting rule, $\varphi^{CE}$:** For each $q^0 \in \mathcal{Q}_0^N$,

$$\begin{aligned}\varphi^{CE}(q^0) \quad &= \quad \{(\sigma^{CE}, t^{CE}) \in Z(q^0) |\ \sigma^{CE} \in E(q^0) \text{ and} \\ &\qquad \forall i \in N,\ \ t_i^{CE} = (\sigma_i^{CE} - \sigma_i^0)\theta_i + \sum_{\sigma_k^0 \leq \sigma_i^0 \leq \sigma_j^0} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1}\}.\end{aligned}$$

[9]The strategic aspect of this queueing problem has been studied by Gershkov and Schweinzer (2010) and Chun, Mitra and Mutuswami (2017).

To transform initial queueing problems into TU games, Curiel et al. (1989) defines the worth of a coalition to be the maximal cost savings that the coalitional members can ensure among themselves by reordering their positions in the queue. However, in our queueing game with an initial queue (or the initial queueing game), the worth of each coalition, denoted by $v_I$, is defined to be the minimum waiting cost of the coalition after efficiently reordering their positions in the queue by themselves. Equivalently, it can be calculated as the waiting cost of the coalition in the initial queue plus the maximal cost savings of the coalition that can be ensured by reordering the positions in the queue.[10] Note that in the reordering process, the coalitional members may not jump ahead of non-coalitional members, that is, two agents in the coalition may not reorder their positions in the queue if a non-coalitional member occupies a position between them. A coalition $S \subseteq N$ is connected if for each $i$, $j \in S$ and each $k \in N$, $\sigma_i^0 < \sigma_k^0 < \sigma_j^0$ implies $k \in S$. If coalition $T$ is connected, then efficiency requires that agents should be served in the non-increasing order of their unit waiting costs and the maximal cost savings of the coalition $T$, $v^{CS}(T)$ is calculated as: $v^{CS}(T) = \sum_{i \in T} \sum_{k \in P_i(\sigma^0) \cap T} g_{ki}$ where $g_{ki} = \max\{(\theta_i - \theta_k), 0\}$. Let $S \subset N$ be a non-connected coalition. A coalition $T \subset S$ is a component of $S$ if $T$ is connected and for each $i \in S \backslash T$, $T \cup \{i\}$ is not connected. The components of $S$ form a partition of $S$ denoted by $S/\sigma^0$. Coalition $S$ can achieve its maximal cost savings when the members of each component are rearranged in the non-increasing order with respect to their unit waiting costs. The cost savings of $S$ is the sum of the cost savings of all components, that is, $v^{CS}(S) = \sum_{T \in S/\sigma^0} v^{CS}(T)$. The worth of coalition $S$ is the waiting cost in the initial queue plus its maximal cost savings, that is, $v_I(S) = -\sum_{i \in S}(\sigma_i^0 - 1)\theta_i + \sum_{T \in S/\sigma^0} v^{CS}(T)$.

By applying the Shapley value to the initial queueing game, Curiel et al. (1989) shows that the Shapley value allocates the cost savings obtained by reordering the positions between any two agents equally among themselves and the agents initially located between them. In the proof, we use the well-known fact established by Shapley (1953) that the Shapley value is the unique solution satisfying *efficiency, symmetry, dummy,* and *additivity. Efficiency* requires that the sum of utilities assigned to agents should be equal to the the worth of the grand coalition. *Symmetry* requires that if two players are symmetric in game $v$, then they should end up with the same utility. *Dummy* requires that if a player contributes nothing to all coalitions, then she should end up with 0 utility. Finally, *additivity* requires that a solution should be an additive function of games.

**Theorem 19.3** (Curiel et al. 1989). The Shapley value applied to the initial queueing game yields the connected equal splitting rule.

[10] Once again, we note that the cost is measured in the negative amount and the cost savings in the positive amount.

*Proof.* For each $q^0 \in \mathcal{Q}_0^N$ and each $i$, $j \in N$ such that $\sigma_j^0 > \sigma_i^0$, let

$$v_{ij}^{CS}(S) = \begin{cases} g_{ij} & \text{if } \{k \in N | \sigma_i^0 \leq \sigma_k^0 \leq \sigma_j^0\} \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

For connected coalition $S$, $v_{ij}^{CS}(S) = g_{ij}$ if and only if $\{i, j\} \subseteq S$. For non-connected coalition $S$, $v_{ij}^{CS}(S) = \sum_{T \in S/\sigma^0} v_{ij}^{CS}(T)$.

Therefore, for connected coalition $S$, $v^{CS}(S) = \sum_{j \in S} \sum_{i \in P_j(\sigma^0) \cap S} g_{ij} = \sum_{\sigma_i^0 < \sigma_j^0} v_{ij}^{CS}(S)$, and for non-connected coalition $S$, $v^{CS}(S) = \sum_{T \in S/\sigma^0} v^{CS}(T) = \sum_{\sigma_i^0 < \sigma_j^0} \sum_{T \in S/\sigma^0} v_{ij}^{CS}(T) = \sum_{\sigma_i^0 < \sigma_j^0} v_{ij}^{CS}(S)$. Altogether,

$$v^{CS}(S) = \sum_{\sigma_i^0 < \sigma_j^0} v_{ij}^{CS}(S).$$

Since the Shapley value satisfies *efficiency, symmetry,* and *dummy,* for each $i \in N$,

$$SV_i(v_{kj}^{CS}) = \begin{cases} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1} & \text{if } i \in \{\ell \in N | \sigma_k^0 \leq \sigma_\ell^0 \leq \sigma_j^0\} \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value satisfies *additivity,* for each $i \in N$,

$$SV_i(v^{CS}) = \sum_{\sigma_k^0 < \sigma_j^0} SV_i(v_{kj}^{CS}) = \sum_{\sigma_k^0 \leq \sigma_i^0 \leq \sigma_j^0} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1}.$$

Since the Shapley value satisfies *zero-independence,*

$$SV_i(v_I) = -(\sigma_i^0 - 1)\theta_i + SV_i(v^{CS}) = -(\sigma_i^0 - 1)\theta_i + \sum_{\sigma_k^0 \leq \sigma_i^0 \leq \sigma_j^0} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1}.$$

For each $i \in N$, by using $t_i = u(\sigma_i, t_i; \theta_i) + (\sigma_i - 1)\theta_i$,

$$\begin{aligned} t_i^{CE} &= SV_i(v_I) + (\sigma_i^{CE} - 1)\theta_i \\ &= -(\sigma_i^0 - 1)\theta_i + \sum_{\sigma_k^0 \leq \sigma_i^0 \leq \sigma_j^0} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1} + (\sigma_i^{CE} - 1)\theta_i \\ &= (\sigma_i^{CE} - \sigma_i^0)\theta_i + \sum_{\sigma_k^0 \leq \sigma_i^0 \leq \sigma_j^0} \frac{g_{kj}}{\sigma_j^0 - \sigma_k^0 + 1}, \end{aligned}$$

the desired conclusion. ■

**Remark 19.12** The queueing problem with an initial queue can be generalized to the sequencing problem with an initial queue by allowing each agent to have a different amount of service time. For each $i \in N$, let $\alpha_i$ be the service time needed by agent $i$. As studied in Curiel et al. (1989), the initial queueing game can be extended to the sequencing game with an initial queue by assuming that any two agents in a coalition can reorder their positions as long as all agents between them are included in the coalition. Now, the cost savings for any two neighboring agents $i$ and $j$ such that $\sigma_j = \sigma_i + 1$ are defined to be $g_{ij} = \max\{\alpha_i\theta_j - \alpha_j\theta_i, 0\}$.

**Remark 19.13** There is another rule widely discussed in the initial queueing problem. This rule selects an efficient queue and allocates the cost savings obtained by reordering the positions of any two agents equally between them.

**Pairwise equal splitting rule,** $\varphi^{PE}$**:** For each $q^0 \in \mathcal{Q}_0^N$,

$$\begin{aligned}\varphi^{PE}(q^0) \quad = \quad & \{(\sigma^{PE}, t^{PE}) \in Z(q^0) |\ \sigma^{PE} \in E(q^0) \text{ and} \\ & \forall i \in N,\ t_i^{PE} = (\sigma_i^{PE} - \sigma_i^0)\theta_i + \frac{1}{2}\sum_{j \in P_i(\sigma^0)} g_{ji} + \frac{1}{2}\sum_{j \in F_i(\sigma^0)} g_{ij}\}.\end{aligned}$$

It is an interesting open question to investigate whether this rule can be obtained by applying a solution of TU games after appropriately defining the worth of a coalition.

## 19.6 Conclusions

In this chapter, we give a survey on the literature which tries to solve the queueing problem by applying the Shapley value. Depending on the definition of the worth of a coalition, we end up with very different rules. Therefore, in the queueing problem, it is very important how to define the worth of a coalition. Also, similar results can be established for two generalizations of the queueing problem, the sequencing problem and the queueing problems with two servers.

In fact, the Shapley value has been successfully applied to other allocation problems such as minimum cost spanning tree problems (Bergantiños and Vidal-Puga 2007), traveling salesman problem (Yengin 2012), etc. However, there are still many interesting allocation problems in which the implications of applying the Shapley value have not been studied. It would be an interesting open question to investigate what recommendation the Shapley value makes for these allocation problems.

A slot allocation problem can be such an example. In this problem, agents differ in their unit waiting cost and the most preferred slot position (or the *peak*). Each agent's utility from her assignment is equal to the amount of monetary transfer minus her unit waiting cost multiplied by the distance from the peak to her assigned slot. It generalizes the queueing problem by allowing each agent to have a different peak. Note that for the queueing problem, each agent has the same peak at the first slot. See for details Chun and Park (2017). An ordinal version of this problem has been studied by Hougaard, Moreno-Ternero and Østerdal (2014) and a related problem of assigning landing slots to airlines by Schummer and Vohra (2013) and Schummer and Abizada (2017).

## 19.7 Acknowledgments

I am grateful to Eunju Lee, Kiyong Yun, and a referee for their comments. This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2016S1A3A2924944).

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# Chapter 20

## Sometimes the Computation of the Shapley Value Is Simple

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### 20.1 Introduction

One of the essential foundation requirements of game theory is the assumption that players are able to evaluate the utility they expect to obtain as the result of an interaction situation. As Lloyd S. Shapley wrote in the introduction of his seminal paper:

"So long as the theory is unable to assign values to the games typically found in application, only relatively simple situations -where games do not depend on other games- will be susceptible to analysis and solution." [31, p.1]

The impressive amount of papers concerning the Shapley value seems to well reflect the success of the method to convert the information contained in a cooperative game into a personal attribution that players may use as a "prospect" in the game. A relevant part of this success is due to several important contributions from the literature providing alternative axiomatic characterizations of the Shapley value [29, 36], some reformulations and generalizations [19, 20, 26], and its application over quite different domains [23, 37].

On the other hand, in order to guarantee the effective evaluation of a cooperative game in practice, players are also faced with a concrete challenge related to the difficulty of the calculation of a "sensible" value [10]. Luckily, for many classes of games studied in the literature of cooperative games, the computation of the Shapley value becomes surprisingly easy. The objective of this survey is to present some examples of this type, where the exact Shapley value of a game can be obtained avoiding the complex calculation of a weighted average of all the players' marginal contributions, as suggested by the original formula introduced in [31]. Looking at these examples, we argue that the linearity of the Shapley value plays an important role to establish a clever computational procedure for its calculus. The majority of the algorithms considered in this chapter rely on the decomposition of a given characteristic function as a sum of games where the marginal contribution of each player over all possible coalitions is driven by properties tailored to specific applications. In many cases, these properties restore axioms used in the literature for the alternative axiomatic characterizations of the Shapley value.

The chapter is organized as follows. In Section 20.3 we introduce some cooperative interaction situations where the worth of coalitions depends on the position of its members along a line. In particular, we focus on cost allocation problems where the cost of a coalition of airplanes depends on the construction and the use of an airport landing strip (Section 20.3.1); then we consider situations where countries release pollutants into a river and face the problem of how to allocate the cost of cleaning it (Section 20.3.2); in Section 20.3.3, we deal with methods for the attribution of an indivisible good and keeping into account the possibility of collusion among the players.

In Section 20.4, we discuss classes of games where the computation of the Shapley value is based on a clever decomposition of the characteristic function. To be more specific, we present some results for sequencing games, where the players, aimed at performing jobs on a machine, may cooperate and save costs rearranging their ordering in a queue (Section 20.4.1); then, we illustrate two quite different applications of the Shapley value: The first one, addresses the problem of allocating the cost of maintenance of a shared infrastructure (e.g., a railway network; see Section 20.4.2), whereas the objective of the second application is to evaluate the importance of genes for the onset of a disease based on the information provided by an experimental dataset (Section 20.4.3); finally, in Section 20.4.4, we discuss some results from the use of the Shapley value for a facility location problem over an area controlled by a centralized authority. Section 20.5 concludes.

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## 20.2 Preliminaries

A *Transferable Utility* (TU-)*game* (or, simply, a *game*) is a pair $(N, v)$, where $N = \{1, \cdots, n\}$ is a finite set of $n$ *players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. If the set $N$ of players is fixed, in the following we identify a game $(N, v)$ with its characteristic function $v$ and for each subset (*coalition*) $S \subseteq N$, we shall denote by $s$ its cardinality $|S|$. It is well known that a game $(N, v)$ may be expressed as a linear combination of its unanimity games defined for each coalition:

$$v = \sum_{S \subseteq N, S \neq \emptyset} \lambda_S(v) u_S,$$

where the coefficients $(\lambda_S(v))_{S \in 2^N \setminus \{\emptyset\}}$ are called *unanimity coefficients* or *dividends* of the game $(N, v)$ and the *unanimity game* $(N, u_S)$ for coalition $S \subseteq N$ is the game described by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for the unanimity game $(N, u_S)$ the players in $S$ are symmetric, while the players in $N \setminus S$ are null players, so the Shapley value[1] $\phi_i(u_S)$ of the unanimity game $u_S$ is:

$$\phi_i(u_S) = \begin{cases} \dfrac{1}{s} & \text{if } i \in S \\ 0 & \text{otherwise,} \end{cases}$$

for each $i \in N$. Exploiting the additivity of the Shapley value, we can write the Shapley value $\phi_i(v)$ of game $v$ as follows:

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{\lambda_S(v)}{s}, \tag{20.1}$$

for each $i \in N$. For further details, we address to [15].

[1] See the definition of the Shapley value and the discussion of its main properties in the introductory chapter of the book.

## 20.3 Games on a Linear Resource

In this section, we review situations where players utilize a resource that can be modelled as a line. The utility of each player depends on his/her position on the line and on who else is using the same resource.

### 20.3.1 Managing Airport Runways

The airport problem is one of the first instances where "a simple and exact expression for the Shapley value" was made available by Littlechild and Owen [17] twenty years after Shapley's proposal for a value. The authors addressed the problem of determining airport landing fees, with particular focus on the capital costs caused by the construction and the use of the airport's runway. The main assumption is that the use of such runway depends on the features of each plane, the "largest" planes needing "longest" runway. Referring to Littlechild and Thompson (see [18]), they consider a set $N$ formed by $n$ planes of $m$ different types. The cost of a runway suitable for type $i$, $i = 1, \ldots, m$, is $c_i$, where they assume, without loss of generality, that

$$0 = c_0 < c_1 < c_2 < \cdots < c_{m-1} < c_m. \tag{20.2}$$

Let $N_i$ denote the set of $n_i$ type $i$ planes, for $i = 1, \ldots, n$. For any nonempty subset $S$ of planes, the cost of a runway that is adequate to receive all the landings of the planes in $S$ is given by

$$v^A(S) = \max\{c_i : N_i \cap S \neq \varnothing\}.$$

This, together with the usual assumption on the empty coalition, namely $v^A(\varnothing) = 0$, defines the characteristic function of the airport game (see [18]).

To compute the Shapley value, the following quantities are needed

$$R_k = \bigcup_{i=k}^{m} N_i \qquad \text{and} \qquad r_k = \sum_{i=k}^{m} n_i.$$

Littlechild and Owen show that, for any $j \in N_i$, $i = 1, \ldots, m$,

$$\phi_j(v^A) = \sum_{k=1}^{i} \frac{c_k - c_{k-1}}{r_k}. \tag{20.3}$$

Suppose that in an existing airport, new planes that require a larger runway are introduced. It seems reasonable to fairly divide the costs for the expansions exclusively among the new airplanes that will actually use the new stretch of runway. The Shapley value for the airport game, defined by (20.3), guarantees this property. Actually, due to its common sense appeal, the rule had been proposed earlier by the economists Baker and Thompson (see [3, 33]). For this reason, the allocation rule is also known as the Baker-Thompson ($BT$) rule.

The game and its Shapley value can be formulated without a prior division of the planes into types. If there are $n$ planes, each with runway cost $c_i$, $i = 1, \dots, n$, and

$$0 = c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_{n-1} \leq c_n, \tag{20.4}$$

then, for any $S \subset N$, we have

$$v^A(S) = \max\{c_j | j \in S\} = c_{\bar{j}(S)},$$

where $\bar{j}(S) = \max\{j | j \in S\}$. Moreover, for each airplane $j$, $j = 1, \dots, n$

$$\phi_j(v^A) = \sum_{k=1}^{j} \frac{c_k - c_{k-1}}{n - k + 1}. \tag{20.5}$$

The same game can also be defined in terms of the increments $\alpha_k = c_k - c_{k-1} \geq 0$, $k = 1, \dots, n$. In this setting, for any $S \subset N$,

$$v^A(S) = \sum_{i=1}^{\bar{j}(S)} \alpha_i \qquad \text{and} \qquad \phi_j(v^A) = \sum_{k=1}^{j} \frac{\alpha_k}{n - k + 1} \qquad j = 1, \dots, n.$$

Airport games are known to be concave[2]. Therefore, the Shapley value lies in the game core. Fragnelli et al. [12], Norde et al. [28] and Fragnelli and Meca [13] dropped the monotonicity assumption (20.2), only to require their nonnegativity, and obtained the *generalized* and the *extended airport games*; these new games are not concave anymore, but the formulas for the Shapley value (20.3) and (20.5) remain valid.

### 20.3.2 Cleaning Rivers

The settings of the airport game can be adapted to other situations where players are arranged along a line. Consider, for instance, rivers that carry goods and "bads" across regions and countries. Ni and Wang [27] consider a model in which countries release pollutants into the river via their factories.

More in detail, suppose that a river flows through a set $N = \{1, 2, \dots, n\}$ of countries. Such countries are labelled according to the flow, country 1 being the region where the river has its source and country $n$ receiving the river's mouth. Assume also that each country $i$ releases pollutants into the river, that require cost $\alpha_i \geq 0$ to be cleaned. A problem then arises on how to share the total costs $\sum_{i \in N} \alpha_i$ for cleaning the whole river.

When the pollutants released by a certain country remain in that area and do not flow downstream, then the cost for cleaning the area in charge

[2] A game $(N, v)$ is concave if $v(S) + v(T) \geq v(S \cup T) + v(S \cap T), S, T \subseteq N$.

of a coalition $S \subset N$ of countries is $v^{LR}(S) = \sum_{i \in S} \alpha_i$. This is the Local Responsibility (LR) game. Clearly, $\phi_j(v^{LR}) = \alpha_j$, and each country bears the costs for its own pollution.

In a more realistic setting, the pollutants flow, and each country should cover the cleaning costs for its own area, as well as the downstream areas. In such a scenario, it is reasonable to assume that a coalition $S \subset N$ of countries should be responsible for the cleaning of the pollution released between the most upstream country of the coalition and the river's mouth. Therefore

$$v^{DR}(S) = \sum_{i=\underline{j}(S)}^{n} \alpha_i, \tag{20.6}$$

where $\underline{j}(S) = \min\{j | j \in S\}$. This is the *Downstream Responsibility* (DR) game in [27]. An even more realistic model is given by Alcalde-Unzu et al. [2]. In that model a transfer rate for the flow of pollutants from a country to its downstream neighbor is assumed, though no mention of the Shapley value is to be found.

Van den Brink and van der Laan [35] point out that the DR game bears a very close similarity with the airport game. In fact, if we consider the downstream costs from country $j$, $j = 1, \ldots, n$, $c_j = \sum_{k=j}^{n} \alpha_k$, we have

$$c_1 \geq c_2 \geq \cdots \geq c_n \qquad \text{and} \tag{20.7}$$

$$v^{DR}(S) = \max\{c_j | j \in S\} = c_{\underline{j}(S)} \qquad \text{for } S \subset N. \tag{20.8}$$

The resemblance with the airport game is striking, since the only difference lies in the monotonicity order of the costs, which are decreasing instead of increasing. For this reason, van den Brink and van der Laan [35] call this a *Dual Airport game*. Simply rearranging the players in the reverse order, or the recursion to generalized airport games, yields

$$\phi_j(v^{DR}) = \sum_{k=j}^{n} \frac{c_k - c_{k-1}}{k} = \sum_{k=j}^{n} \frac{\alpha_k}{k} \tag{20.9}$$

with the usual proviso $c_0 = 0$.

### 20.3.3 Auctions and Markets

The mathematical framework employed to describe the use of a linear resource by a finite set of agents works surprisingly well in economics to model players' competition over the attribution of an indivisible good.

Van den Brink and van der Laan [35] point out that the DR game shares the same Shapley value with a game defined earlier by Graham et al. [14] to describe collusion in an auction setting. We refer also to van den Brink [34] for a concise description of this situation. Consider a set of bidders $N = \{1, 2, \ldots, n\}$ participating to a second-price sealed-bid auction (though the conclusions straightforwardly apply to an English open auction as well) for the attribution of a single indivisible item. Suppose that bidders are labelled according to their valuation $V_i$, $i \in N$, of the good, so that

$$V_1 \geq V_2 \geq \cdots \geq V_n. \tag{20.10}$$

These values are private information, but if bidders collude, or, in the auction jargon, *form a ring*, they will exchange information about their own valuations with the other colluders, in order to take advantage of those outside the colluding group. If bidders split into two colluding groups: $S$ and $S^c = N \setminus S$, respectively, and if the exchange of information is complete within each colluding group, the dominant strategy for each bidder in this group is to bid $v^*(S) = \max_{i \in S} V_i$, $v^*(S^c) = \max_{j \in S^c} V_j$, respectively. In this context, the worth of coalition $S$ is given by the auction (AU) game, defined as:

$$v^{AU}(S) = \max\{v^*(S) - v^*(S^c), 0\} = V_1 - \max_{j \in S^c} V_j = V_1 - V_{\underline{\mathrm{j}}(S^c)}, \tag{20.11}$$

where the middle equation follows from the fact that coalition $S$ obtains a positive value only when it includes player 1. The value will therefore be given by the difference between the evaluation of player 1 and the highest evaluation within the complementary coalition.

Though not immediately apparent, it can be shown, using the definition of the Shapley Value, that this game has the same Shapley value of the DR game with $c_j = V_j$ for every $j \in N$, and therefore,

$$\phi_j(v^{AU}) = \sum_{k=j}^{n} \frac{V_k - V_{k-1}}{k}$$

with the usual proviso $V_0 = 0$.

A somewhat similar analysis of collusive behavior in a bidding context was provided by Briata et al. [4]. Here, the agents in $N$ have equal rights over the indivisible good. The good is then "sold" to the highest bidder, and this amount is distributed among all players according to a procedure defined by Knaster [16]. More in detail, suppose (again w.l.o.g.) that players have non-increasing valuations (20.10). If each agent bids an amount $b_i$, $i \in N$, the item is sold to the highest bidder (in case of ties, it can be given to any of them according to some rule specified in advance). Each agent $i \in N$ receives the fair share $b_i/n$ plus an equal share of the surplus, i.e., $\left(\max_{i \in N} b_i - \frac{\sum_{i \in N} b_i}{n}\right)\Big/ n$. In case an agent does not know the valuations of the others, and the agent is risk averse, a safe strategy, i.e., a strategy that maximizes the payoff of the agent, avoiding the risk of a negative payoff, is to equalize the bid to the valuation, i.e., $b_i = V_i$, $i \in N$. Conversely, if a subset $S \subset N$ of agents engage in a collusive behavior, they may reveal their valuations within the group and may sign a binding agreement on how to share their gain at the expense of those agents outside the group. The highest gain that the coalition $S$ obtains with certainty is obtained when all agents in $S$ bid $b_S = \max_{j\in S} V_j$ and their joint safe gain is given by

$$v^G(S) = \frac{n-|S|}{n^2} \sum_{i\in S} (b_S - V_i). \tag{20.12}$$

This is the gain game defined in [4]. It turns out that the Shapley value for this game is given by

$$\phi_i(v^G) = \sum_{j=1}^{n-1} \psi_{ij}(V_j - V_{j+1}) \,, \tag{20.13}$$

where, for each $j \in N \setminus \{n\}$

$$\psi_{ij} = \begin{cases} (n-j)\; c(n,j) & \text{if } i \le j \\ -j\; c(n,j) & \text{if } i > j \end{cases} \tag{20.14}$$

$$\text{and} \qquad c(n,j) = \frac{2n - 3j - j^2}{2n(j+1)(j+2)} \,.$$

Another economic application is given by Fragnelli and Meca [13] who consider a von Neumann-Morgenstern market game. Here, a seller, denoted by $n+1$, values $w_{n+1}$ the item to be sold to a set of $n$ potential buyers, each with reserve value $w_i$, $i = 1, \ldots, n$. Assuming weak monotonicity, i.e., $w_1 \ge w_2 \ge w_3 \ge \cdots \ge w_{n+1}$, this situation can be associated to a cooperative game with player set $N' = \{1, 2, \ldots, n+1\}$ and characteristic function

$$v^m(S) = \begin{cases} 0 & \text{if } n+1 \notin S \\ \max\{w_i | i \in S\} & \text{if } n+1 \in S. \end{cases}$$

This game can be decomposed as the difference of an airport game with cost vector $(w_{n+1}, w_n, \ldots, w_1)$ and an extended airport game with cost vector $(w_n, w_{n-1}, \ldots, w_1, w_{n+1})$. Consequently, the Shapley value is

$$\phi(v^m) = \begin{cases} \sum_{j=1}^{n} \frac{w_j - w_{j+1}}{j(j+1)} & \text{if } i = 1, \ldots, n \\ w_1 - \sum_{j=1}^{n} \frac{w_j - w_{j+1}}{j+1} & \text{if } i = n+1. \end{cases}$$

## 20.4 Decomposition

In the following, we present some classes of games for which the Shapley value can be easily computed after a suitable decomposition of the game. In particular, we consider the sequencing games [9], the maintenance cost games [12], the microarray games [24], and the coverage games [11].

### 20.4.1 Sequencing Games

This class of games was introduced by Curiel, Pederzoli and Tijs in 1989 [9]. They consider a queue in front of a window delivering a service and associate to each agent in the queue a job that requires a given time to be performed; the agents suffer different cost for each unit of time spent in the queue, including the time for performing their own jobs, so that it may be profitable for the agents rearranging their ordering in the queue in such a way to reduce the total cost. The situation may be represented by a 4-tuple $(N, \sigma_0, t, \alpha)$ where $N = \{1, 2, ..., n\}$ is the set of agents, $\sigma_0$ is the initial ordering, $t = (t_1, t_2, ..., t_n)$ is the vector of the time required by the jobs of the agents and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is the vector of the cost per unit of time suffered by the agents.

Given an ordering $\sigma$, the time spent in the system by agent $i \in N$ is the sum of the execution times of the jobs preceding her/his one according to $\sigma$, i.e., $\sum_{j \in P(\sigma,i) \cup \{i\}} t_j$ where $P(\sigma, i)$ denotes the set of agents preceding $i$ in the ordering $\sigma$; consequently, the cost for agent $i \in N$ is $\alpha_i \sum_{j \in P(\sigma,i) \cup \{i\}} t_j$ and the total cost of the ordering $\sigma$ is $C_\sigma = \sum_{i \in N} \alpha_i \sum_{j \in P(\sigma,i) \cup \{i\}} t_j$.

In 1956, Smith [30] proved that the optimal order $\sigma^*$ may be obtained simply reordering the agents by weakly decreasing urgency indices, where the urgency of agent $i \in N$ is $u_i = \frac{\alpha_i}{t_i}$, i.e., the ratio among the unitary cost and the execution time.

Curiel, Pederzoli and Tijs [9] used a TU-game for allocating among the agents the total saving $C_{\sigma_0} - C_{\sigma^*}$ in such a way that all agents are satisfied with the optimal order $\sigma^*$. They suppose that the set of players coincides with the set of agents $N$ and that two agents in a coalition $S \subseteq N$ may switch their positions in the current order only if all the agents in between them are in the coalition.

The gain of the switch among two adjacent agents $j$ and $i$, $j \in P(\sigma_0, i)$ is $g_{ji} = max\{\alpha_i t_j - \alpha_j t_i, 0\}$, so the worth of a connected coalition $T$ is the sum of all the gains of the possible switches, i.e.:

$$v(T) = \sum_{i \in T} \sum_{j \in P(\sigma_0,i)} g_{ji}.$$

For a coalition $S \subseteq N$, the worth is the sum of the worths of the connected subcoalitions:

$$v(S) = \sum_{T \in S \setminus \sigma_0} v(T),$$

where $S \setminus \sigma_0$ denotes the set of connected subcoalition of $S$ induced by the ordering $\sigma_0$.

A sequencing game $(N, v)$ may be decomposed according to the unanimity games defined on the connected coalitions $[j, i] = P(\sigma_0, i) \cup \{i\} \setminus P(\sigma_0, j)$ with $j \in P(\sigma_0, i)$; in other words, $[j, i]$ is the set of all the players in between $j$

and $i$, included $j$ and $i$. In this case, the dividend associated to the unanimity game $(N, u_{[j,i]})$ is $g_{ji}$.

In view of this, the Shapley value of a sequencing game $(N, v)$ is:

$$\phi_i(v) = \sum_{[h,k]\subseteq N: i\in[h,k]} \frac{g_{hk}}{|[h,k]|}, \quad i \in N.$$

Note that it is possible to compute the Shapley value without computing the characteristic function, but only the gain of the switch for each pair of agents.

### 20.4.2 Maintenance Cost Games

This class of games was introduced by Fragnelli, Garcia-Jurado, Norde, Patrone and Tijs in 2000 [12]. The problem they consider is related to an infrastructure used by several agents with different needs that correspond to a more expensive infrastructure. The users may be grouped according to the level of requirements and each coalition requires an infrastructure suitable for satisfying the needs of the most demanding agent, in a situation similar to the already mentioned airport games. The difference is that in this case both fixed and variable costs are involved. More precisely, the fixed costs are independent from the number of users of the infrastructure in a given period of time, while the variable costs depend on the number of users in the same period of time. The two kinds of costs lead to the definition of two classes of games. The first class, related to the fixed costs, called *building cost games*, is equivalent to the class of the airport games, and the Shapley value may be computed according to the formula in [17]. The second class of games, that considers the variable costs, is called *maintenance cost games* and the Shapley value may be computed with a formula that requires a suitable decomposition of the game. In the following, we devote our attention to this second class of games.

Suppose that there are $k$ disjoint groups of players $g_1, \ldots, g_k$ with $n_1, \ldots, n_k$ players, respectively; clearly, the infrastructure is at level $k$, in order to allow the usage of all the players. Suppose that an agent in the group $j \leq k$ uses the infrastructure, then the cost of restoring the infrastructure may be decomposed as the cost $\alpha_{jj}$ of renovating the infrastructure at level $j$, plus the cost $\alpha_{jh}, h = j+1, ..., k$ of renovating the infrastructure from the level $h-1$ to level $h$. Consequently, the total cost of restoring the infrastructure at level $k$ after the usage of an agent of level $j$ is $A_{jk} = \alpha_{jj} + \alpha_{j,j+1} + ... + \alpha_{jk}$.

In view of the previous reasoning, we can give the formal definition of the maintenance cost game as in [12].

**Definition 20.1** *Suppose we are given $k$ groups of players $g_1, \ldots, g_k$ with $n_1, \ldots, n_k$ players respectively and $k(k+1)/2$ non-negative numbers $\{\alpha_{ij}\}_{i,j\in\{1,\ldots,k\},j\geq i}$. The* maintenance cost game *corresponding to $g_1, \ldots, g_k$*

*and* $\{\alpha_{ij}\}_{i,j\in\{1,\ldots,k\},j\geq i}$ *is the cooperative (cost) game* $(N,c)$ *with* $N=\cup_{i=1}^{k} g_i$ *and cost function* $c$ *defined by*

$$c(S)=\sum_{i=1}^{j(S)}|S\cap g_i|A_{ij(S)},\quad S\subseteq N,$$

*where* $j(S)=\max\{j\,:\,S\cap g_j\neq\varnothing\}$ *and* $A_{ij}=\alpha_{ii}+...+\alpha_{ij}$ *for all* $i,j\in\{1,...,k\}$ *with* $j\geq i$*.*

A maintenance cost game $(N,c)$ may be decomposed as follows.

$$\begin{aligned} c(S) &= \sum_{i=1}^{j(S)}|S\cap g_i|A_{ij(S)}= \\ &= \sum_{i=1}^{j(S)}|S\cap g_i|(\alpha_{ii}+...+\alpha_{ij(S)})=\sum_{i=1}^{k}\sum_{j=i}^{k}\alpha_{ij}c^{ij}(S),\quad S\subseteq N, \end{aligned}$$

where

$$c^{ij}(S)=\begin{cases} |S\cap g_i| & \text{if } j\leq j(S) \\ 0 & \text{if } j>j(S) \end{cases}$$

for all $i,j\in\{1,...,k\}$ with $j\geq i$.

Referring to Theorem 3.1 in [12], the Shapley value for a maintenance cost game for player $i\in g_{j(i)}$ may be computed as:

$$\begin{aligned} \phi_i(c) &= \alpha_{j(i)j(i)}+\sum_{m=j(i)+1}^{k}\alpha_{j(i)m}\frac{n_m+...+n_k}{n_m+...+n_k+1} \\ &\quad +\sum_{m=2}^{j(i)}\sum_{l=1}^{m-1}\alpha_{lm}\frac{n_l}{(n_m+...+n_k)(n_m+...+n_k+1)}. \end{aligned}$$

In order to check this, notice that for every $l\in\{1,...,k\}$:

$$\phi_i(c^{ll})=\begin{cases} 1 & \text{if } i\in g_l \\ 0 & \text{otherwise.} \end{cases} \tag{20.15}$$

Suppose now that $l<m$. In this case, only players in $g_l\cup(\cup_{r=m}^{k} g_r)$ are not null players. By symmetry, $\phi_i(c^{lm})=a$ for every $i\in g_l$ and $\phi_i(c^{lm})=b$ for every $i\in\cup_{r=m}^{k} g_r$. In order to compute $a$ take $i\in g_l$ and note that for every $S\subseteq N\backslash\{i\}$:

$$c^{lm}(S\cup\{i\})-c^{lm}(S)=\begin{cases} 0 & \text{if } j(S)<m \\ 1 & \text{otherwise.} \end{cases}$$

So, if the players of $N$ are ordered at random, $a$ is the probability that player $i$ has at least one predecessor in $\cup_{r=m}^{k} g_r$. Equivalently, if the players of $N$ are ordered at random, $a$ is the probability that player $i$ is not the first player of the players in $\{i\} \cup (\cup_{r=m}^{k} g_r)$. Consequently,

$$a = \frac{n_m + ... + n_k}{n_m + ... + n_k + 1}. \tag{20.16}$$

Thus, by symmetry and efficiency,

$$b = \frac{n_l - n_l a}{n_m + ... + n_k} = \frac{n_l}{(n_m + ... + n_k)(n_m + ... + n_k + 1)}. \tag{20.17}$$

Now, in view of (20.15), (20.16) and (20.17), the computation of the Shapley value is completed.

### 20.4.3 Microarray Games and Network Centrality

The possibility to decompose a game can be very helpful when the number of players is extremely large. This is the case, for instance, of *microarray games* [24], a class of games aimed at evaluating the importance of genes in regulating or provoking a biological condition of interest, and taking into account the observed relationships in all subgroups of genes.

To be more specific, let $N = \{1, \ldots, n\}$ be a set of genes whose level of expression (i.e., the process by which genetic instructions are used in the synthesis of a gene product) is evaluated over a set $M = \{1, \ldots, m\}$ of $m$ samples or experiments (e.g., cells of patients with a genetic disease). From this information one can generate a Boolean matrix $\mathbf{B} \in \{0,1\}^{N \times M}$ (see [24] for more details) where the Boolean values 0 or 1 represent two mutually exclusive *expression status*, for example the status of normal expression (coded by 0) and the status of "abnormal" expression (coded by 1) over all $n$ genes and $m$ samples (alternative expression status for genes can be considered, for instance, the property of up- or down-regulation, etc.; see [24]).

Let $j \in M$ and let $\mathbf{B}_{\cdot j}$ be the $j$-th column of matrix $\mathbf{B}$. The *support* of $\mathbf{B}_{\cdot j}$, denoted by $sp(\mathbf{B}_{\cdot j})$, is defined as the set of genes $sp(\mathbf{B}_{\cdot j}) = \{i \in N \mid \mathbf{B}_{ij} = 1\}$. On a single sample $j \in M$, in a coalition of genes $S \subseteq N$, a sufficient requirement to realize the *association* between a biological condition of $j$ and an expression status of genes is that the set of genes showing that expression status is contained in $S$, i.e., $sp(\mathbf{B}_{\cdot j}) \subseteq S$.

The microarray game corresponding to $\mathbf{B}$ is defined as the TU-game $(N, v)$, where $v : 2^N \to \mathbb{R}_+$ is such that $v(T)$ is the rate of occurrences of the coalition $T$ as a superset of supports in $\mathbf{B}$. More precisely, we define $v(T)$, for each $T \in 2^N \setminus \{\emptyset\}$ as follows:

$$v(T) = \frac{|\Theta(T)|}{m}, \tag{20.18}$$

where $\Theta(T) = \{j \in M \mid sp(\mathbf{B}_{\cdot j}) \subseteq T,\ sp(\mathbf{B}_{\cdot j}) \neq \emptyset\}$, and $|\Theta(T)|$ is the cardinality of $\Theta(T)$ (by convention, $v(\emptyset) = 0$). In other words, $v(T)$ represents the frequency of associations realized in $T \subseteq N$ between the expression status of its genes and the biological condition of samples in $M$. A microarray game $v$ can be easily reformulated as follows [24]:

$$v = \sum_{S \subseteq N : S \neq \emptyset} \frac{\bar{\lambda}_S}{m} u_S, \tag{20.19}$$

where $\bar{\lambda}_S = \big|\{j \in M | sp(\mathbf{B}_{\cdot j}) = S\}\big|$ is the number of occurrences of the coalition $S$ as support in the Boolean expression matrix $\mathbf{B}$.

Using a property-driven approach, in [24], the authors suggest the use of the Shapley value as a measure of the relevance of genes in inducing the biological condition of interest, and keeping into account their "coalitional" interaction in the biological system. Note that if a Boolean matrix $\mathbf{B} \in \{0,1\}^{N \times M}$ is given, the computation of the Shapley value $\phi(v)$ of the corresponding microarray game $v \in \mathcal{M}^N$, in virtue of Equation (20.19), is very easy, regardless of the number of genes involved (in typical experiments, in the order of tens of thousands). To be more specific, we have that

$$\phi_i(v) = \frac{1}{m} \sum_{S \subseteq N : i \in S} \frac{\bar{\lambda}_S}{|S|} \tag{20.20}$$

for each $i \in N$ (see [24] for more details).

Another interesting field related to the analysis of genetic data deals with *co-expression networks*, which are increasingly used to explore the system-level functionality of proteins and genes[3]. In [22] the authors introduce a method based on TU-games to evaluate the *centrality* of genes in *co-expression networks*. Following the approach introduced in [22], an *association game* $(N, a)$ is first defined, where $N$ is the set of genes under study (for instance, analysed by means of a gene expression dataset) and the characteristic function $a$ assigns a worth $a(S)$ to each coalition of genes $S \subseteq N$ representing the overall magnitude of the "interaction" between the genes in $S$ and a given set of *key-genes*.

Precisely, suppose we have a finite set $K$ (with $N \cap K = \emptyset$) of key-genes and let $I \subseteq \{\{i,k\} | i \in N, k \in K\}$ be the set of *interactions* (reported by previous studies) between genes in $N$ and the key-genes in $K$. Given a set of genes $S \subseteq N$, the number of key-genes which interact with genes in $S$ can be considered as a measure of the likelihood that genes in $S$ are also involved in the regulation of the biological condition of interest. The map $a : 2^N \to \mathbb{N}$ assigning to each coalition $S \in 2^N \setminus \{\emptyset\}$ the number $a(S)$ of key-genes in $K$ which only interact (in $I$) with genes in $S$ (again, by convention, $a(\emptyset) = 0$) is the *association game* corresponding to $(N, K, I)$. In [22], the authors also introduce a second game, where gene interaction is restricted to the connections specified by an associated undirected graph $\langle N, E \rangle$. The set

[3] Roughly speaking, a co-expression network is a network where the nodes correspond to the genes, and a link between two genes is established if they are simultaneously expressed in a dataset (see, for instance, [32] for a detailed discussion on co-expression networks).

of edges $E$ indicates interaction ties between pairs of genes in $N$, i.e., a set $\{i, j\} \subseteq N$ is an element of $E$ if and only if $i$ and $j$ interact (for instance, they are significantly co-expressed). Following the approach in [25], in [22] the structure of network $\langle N, E\rangle$ is used to define the *graph-restricted game* $(N, w^a_E)$ such that

$$w^a_E(S) = \sum_{T \in C_{E_S}} a(T) \tag{20.21}$$

for each $S \in 2^N \setminus \{\emptyset\}$, where $C_{E_S}$ is the set of all the connected components in $\langle S, E_S\rangle$. The difference of the Shapley values computed on the two games $(N, a)$ and $(N, w^a_E)$ is considered in [22] as a *centrality measure* of genes in the network $\langle N, E\rangle$. Specifically, the $\gamma$-*centrality* $\gamma(a, E)$ is defined by

$$\gamma_i(a, E) = \phi_i(w^a_E) - \phi_i(a) \tag{20.22}$$

for each $i \in N$, where $\phi(a)$ is the Shapley value of the association game $a$ and $\phi(w^a)$ is the Shapley value of the corresponding graph-restricted game $w^a_E$ (i.e., the Myerson value [25]). So, genes with strictly positive $\gamma$-centrality represent those genes with a positive differential power between the graph-restricted game $w^a_E$ and the association game $a$.

It is easy to show that the characteristic function $a$ of an association game can be written as a sum of unanimity games according to the following relation,

$$a = \sum_{k \in K, N_k \neq \emptyset} u_{N_k}, \tag{20.23}$$

where $N_k = \{i \in N | \{i, k\} \in I\}$ denotes the set of genes in $N$ which have a strong interaction with a key-gene $k \in K$. A natural decomposition of a graph-restricted game based on the reformulation of the association game can be also provided (see [22] for further details), but it requires to consider all minimal components containing $N_k$, for each key-gene $k \in K$, and all of their combinations (see equation (6) in [22]). However, as the number of minimal components in a graph can be very large (especially for graphs generated from realistic datasets with thousands of genes) this option is computationally unaffordable. Therefore, in [22], the authors limit the decomposition of the graph-restricted game $w^a_E$ to the "smallest" minimal components connecting the "most associated genes" (i.e., genes that directly interact with key-genes), which are those minimal components where the most associated genes are connected to each other by a shortest path. Alternatively, in order to estimate the Shapley value of a graph-restricted game with many genes, one could adopt the strategy of sampling orderings of players and calculate an unbiased estimator of each gene's contribution as an average marginal contribution over all sampled orderings (see, for instance, the recent papers [5, 6] for the discussion of a sampling procedure to estimate the Shapley value of games with a large number of players).

Another approach to assess the centrality of genes in a co-expression network $\langle N, E\rangle$ has been presented in [7] and further studied in [8]. Given a parameter vector $k \in \mathcal{R}^N$, which specifies the *a priori* importance or *weight* of each gene in $N$, a game $(N, v_E^k)$ is defined in [7] associating to each coalition of genes $S \subseteq N$ a value $v_E^k(S)$ representing the overall magnitude of the interaction between the genes in $S$. More precisely, in [7], the set of *neighbors* of a node $i$ in a graph $\langle N, E\rangle$ is defined as the set $N_i(E) = \{j \in N : \{i, j\} \in E\}$ and the *degree* of $i$ as the number $d_i(E) = |N_i(E)|$ of neighbors of $i$ in graph $\langle N, E\rangle$. With a slight abuse of notation, we denote by $N_S(E)$ the set of neighbors of nodes in $S \in 2^N \setminus \{\emptyset\}$ in the graph $\langle N, E\rangle$, i.e., $N_S(E) = \{j \in N : \exists i \in S \text{ s.t. } j \in N_i(E)\}$. The characteristic function $v_E^k : 2^N \to \mathbb{N}$ assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$ the value

$$v_E^k(S) = \sum_{j \in S \cup N_S(E)} k_j, \tag{20.24}$$

that is the sum of the weights associated to the genes in $S$ and to the ones that are directly connected in $\langle N, E\rangle$ to some genes in $S$ (again, by convention, $v_E^k(\emptyset) = 0$). Now, for any non-empty coalition $S \subseteq N$, consider the *dual unanimity game* $(N, u_S^*)$ such that

$$u_S^*(T) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $T \in 2^N \setminus \{\emptyset\}$. In [7], the authors have shown that an equivalent way to formulate game $v_E^k$ is as a weighted sum of dual unanimity games as follows:

$$v_E^k = \sum_{i \in N} k_i u^*_{N_i(E) \cup \{i\}}. \tag{20.25}$$

Exploiting the properties of the Shapley value, it is easy to show that

$$\phi_i(k_j u^*_{N_j(E)\cup\{j\}}) = \begin{cases} \dfrac{k_i}{d_j(E)+1} & \text{if } i \in N_j(E) \cup \{j\} \\ 0 & \text{otherwise} \end{cases}$$

and, consequently, the Shapley value of a game $(N, v_E^k)$ can be computed according to the following simple relation:

$$\phi_i(v_E^k) = \sum_{j \in (N_i(E) \cup \{i\})} \frac{k_j}{d_j(E)+1} \tag{20.26}$$

for each $i \in N$. According to relation (20.26), the Shapley value of a node increases when it is connected to many nodes with a low degree. So, according to the Shapley value, a relevant gene $i \in N$ is directly connected to many other genes having few or no possibilities to “interact” with genes different from $i$. We refer to [1] for some related game theoretical notions of centrality and to the survey [21] for other applications of the Shapley value to biology.

### 20.4.4 Coverage Games

This class of games was introduced by Fragnelli, Gagliardo and Gastaldi in 2017 [11]. They faced the problem of locating units in the area controlled by an emergency service, the so-called *Emergency Units Location Problem* (*EULP*) and introduced a new class of games to deal with it.

The different candidate locations for hosting an emergency unit may be considered as interacting agents. In fact, the choice of deploying an emergency unit cannot take into account only the characteristics of a candidate location, i.e., the extension of the area that can be covered within the maximum time allowed (or with the maximum utility), the probability of a call in that area, etc., but should also consider where the other emergency units are located. In other words, an important role is played by the *marginal contribution* of an ambulance to each possible set of units located in the other candidate zones of the area, i.e., what a further candidate location may add to the service when an ambulance is located there. The average marginal contribution may be considered as a measure of the relevance of the candidate locations. The problem under analysis is a *centralized decision situation* in which the emergency management decides where to locate vehicles. Cooperative games have been widely used to deal with situations in which interacting agents realize that they may improve their payoffs by cooperating; in the EULP case, cooperation leads to supplying the best possible service to a set of users.

An *EULP* may be represented by a 4-tuple $(M, N, C, w)$ where $M = \{1, \ldots, m\}$ is the set of zones of the area, $N = \{1, \ldots, n\} \subseteq M$ is the subset of zones which are candidate locations for an emergency unit, $C = (c_{ij}) \in \mathbb{R}^{n \times m}$ is the *coverage matrix*, s.t. $c_{ij} = 1$ if an emergency unit located in $i$ covers zone $j$ and $c_{ij} = 0$ otherwise and $w \in \mathbb{R}^m$ the vector of the demands of the zones of the area.

The aim is ranking the $n$ candidate locations in the area described by the previous parameters in order to satisfy the demand in the best possible way; the utility of the service may assume only two values depending on whether the intervention is carried out in the maximal time allowed or not. In this case, the utility is zero.

Given an EULP, a new class of TU-games, the *coverage games*, is introduced.

**Definition 20.2 (Coverage games)** *The* coverage game *is the TU-game $(N, v)$ defined by*

$$v(S) = \sum_{j \in A_S} w_j \quad \forall S \subseteq N \ ,$$

*where $A_S = \{j \in M \mid \exists\ i \in S \ s.t.\ c_{ij} = 1\}$, i.e., the set of zones which are covered by at least one emergency unit located in $S$, when each zone in $S$ hosts one emergency unit.*

The value of a coalition $S$ in the coverage game is the sum of the demands of the zones that are covered locating one emergency unit in each location of $S$.

The definition of the coverage game related to an EULP does not consider the number of available units; ranking by relevance the possible locations makes the approach adaptable to a variable number of units to activate. Finally, the authors do not consider that the ambulances may not satisfy the whole demand, for instance due to the high number of calls or to their time distribution (in an efficiency-oriented case).

A good solution to the EULP is the Shapley value of the coverage games, as it is rooted in the concept of marginality: It is important to take into account not only the demand of a zone or the aggregated demand that a candidate location can cover, but mainly the contribution that an ambulance located there can add to the other locations. As it was already said, the aim is ranking by relevance the candidate locations accounting their marginal contributions; in view of this, the Shapley value represents a very good solution. Then, the available ambulances are deployed according to the ordering of relevance of the candidate locations.

The Shapley value has good fairness properties with respect to the problem. Two suitable fairness criteria are the *coverage indifference* and the *demand indifference*; these properties are related more to the problem than to the game, allowing to improve the fairness of the solution of the location problem.

Before introducing the two properties of coverage indifference and demand indifference, it is necessary to define a sub-class of coverage games, the *$j$-th zone sub-games*, in which uniquely zone $j$ has a positive demand, i.e., the demands of all the zones but $j$ are put down to zero.

**Definition 20.3 ($j$-th zone sub-game)** *Let $v$ be a coverage game. Given $j \in M$, the $j$*-th zone sub-game *of $v$ is the coverage game $v^j$ defined, for each $S \subseteq N$, by*

$$v^j(S) = \begin{cases} w_j & \text{if } j \in A_S \\ 0 & \text{otherwise.} \end{cases}$$

The coverage game $v$ is the sum of all its zone sub-games:

$$v(S) = \sum_{j \in M} v^j(S), \quad S \subseteq N.$$

**Definition 20.4** *Given a coverage game $v$ and its $j$-th zone sub-games $v^j, j \in M$, a solution $\psi$ satisfies* coverage indifference (CI) *if*

$$\psi_i(v^j) = \psi_l(v^j)$$

*for $j \in A_{\{i\}} \cap A_{\{l\}}, i, l \in N$ and satisfies* demand indifference (DI) *if*

$$\psi_i(v) = \sum_{j \in M} \psi_i(v^j).$$

It is easy to observe that, for each $i \in N,\ j \in M$,

$$\psi_i(v^j) = \begin{cases} \dfrac{w_j}{\sum_{l=1}^{n} c_{lj}} & \text{if } j \in A_{\{i\}} \\ 0 & \text{otherwise.} \end{cases} \qquad (20.27)$$

The coverage indifference looks at the situation from the point of view of the users and requires to give the same importance to the units that cover a zone allowing for equally sharing the demand of the zone among them; in a sense, each of those units has the same probability to satisfy a call coming from the considered zone. The demand indifference looks at the situation from the point of view of the emergency service provider and gives the same importance to each demand, wherever it comes from and whatever the required intervention is.

According to Proposition 2 in [11], the Shapley value of a coverage game is given by

$$\phi_i(v) = \sum_{j \in A_{\{i\}}} \frac{w_j}{\sum_{l \in N} c_{lj}}. \tag{20.28}$$

The Shapley value of a coverage game may be computed via a $n \times m$ matrix $D$, called the *division matrix*, where, for each $i \in N,\ j \in M$,

$$d_{ij} = \begin{cases} \dfrac{w_j}{\sum_{l \in N} c_{lj}} & \text{if } c_{ij} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By (20.28), the Shapley value of the coverage game for player $i \in N$ can be obtained simply summing up the values in the $i$-th row of $D$. The computational complexity of this algorithm is then polynomial in $n$ and $m$; moreover, it is not necessary to compute the characteristic function of the coverage game.

## 20.5 Conclusions

In this survey, we presented some classes of games for which the calculation of the Shapley value has a very low computational complexity. As shown by the richness of the models and the diversity of the applications discussed in this chapter, the specification of an easy formula for the Shapley value seems to be not only an essential ingredient for its successful utilization in practice, but also an important source of inspiration for researchers aimed at understanding the intrinsic nature of the most famous solution for TU-games. Our goal was to foster the dissemination of the different methodologies employed in literature to achieve easy algorithms and procedures for the computation of the Shapley value, and to provide incentives to the scholars for extending these results to other classes of games.

## 20.6 Acknowledgments

We are grateful to two anonymous referees for some useful remarks.

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# Chapter 21

## Analysing ISIS Zerkani Network Using the Shapley Value

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### 21.1 Introduction

We are witnessing a movement from large-scale mass casualty attacks (WTC 9/11 style) to multiple coordinated swarming attacks (Mumbai 2008), to lone wolf attacks on the far end of this spectrum (Berlin 2016, Nice 2016, London 2017). It appears that a mix of such attacks, empowered by a society that is digitalizing at a fast pace, is becoming more and more likely in the future. It is of the utmost importance that key members of networked organisations that conduct aforementioned attacks are identified in time by intelligence and law enforcement agencies. However, since the resources to such agencies are limited, they have to be allocated optimally. For instance there are only so many observation teams that can be deployed at any given time for a certain number of targets or there is only a certain amount of agents that can be infiltrated into a specific organisation to track a certain individual. The use of quantitative methodology during the intelligence cycle yields a more finely attuned preference ranking of targets in the social network which in turn helps to better piece together a complete picture of whom to follow, investigate and surveil and henceforth optimally allocate limited intelligence and law enforcement resources.

One approach taken in literature to identify key members in a network is using social network analysis (cf. [23]). Actually, such a quantitative approach in analysing terrorist networks is applied in [14] by using standard network measures such as degree centrality, betweenness centrality and closeness centrality. The drawback of these measures is that they only take the structure of the network into account. Other researchers have also used the standard social network centrality approach in analysing criminal and terror networks see [4], [8], [13], [16], [19] and [21]. However, in all these contributions, valuable information concerning the type and character of (communication) links between two members and also characteristics of the individuals are rarely taken into account. More recent developments, however, conceptualise not only the structure of a social network, but also take the heterogeneity of links and nodes into account [17] introduce the use of cooperative game theory to explicitly incorporate the additional information that is available to the intelligence analyst. Indeed, an appropriate cooperative game takes properties of the individuals in the network as well as properties of the relationships between individuals into account. As such, it provides tailor-made solutions since it aids in better modelling the context under consideration. Among others, they applied their model to Al Qaeda's 9/11 attack and established a ranking of the hijackers of the four planes in the attack by calculating the Shapley value (cf. [20]) of the defined cooperative game. In [12] a sensitivity analysis is applied to Al Qaeda's 9/11 attack concerning the robustness of the ranking obtained using the Shapley value. Moreover, they modified the cooperative game introduced in [17] that takes better into account the operational strength of disconnected subnetworks. A drawback of modelling social network centrality by use of cooperative game theory is found in the computational complexity of the Shapley value which in general increases exponentially with the number of players. It can be shown that computing the Shapley value is an NP-complete problem (cf. [6], [7]). Real-life situations often concern a large number of players, think for instance of large social networks. Because of computational complexity, finding the exact solution of the Shapley value is not possible, hence approximations are needed. For example, the extended WTC 9/11 network consists of 69 members (cf. [15]) which is too large to analyze using the Shapley value in order to obtain a ranking of the players in a reasonable amount of time. Heuristics to approximate the Shapley value can be found in (cf. [9], [1]). However, they only focus on simple games, i.e., games in which the coalitional values either are 0 or 1. The heuristic presented by

[5], which uses the average of a random subset of marginal vectors, can handle any cooperative game. This latter heuristic is improved in [3] and applied to the complete network of Al Qaeda's 9/11 attack consisting of 69 players.

In this chapter we combine the approach taken in [17], [3] and [12] to analyse the Zerkani network involved in the final phase of the radicalisation process leading to the November 2015 mass shootings and suicide attack in Paris and the March 2016 coordinated suicide bombings in Brussels. More precisely, we use from these chapters the centrality measure, the sensitivity analysis and the approximation of the Shapley value. However, a difference is in the definition of the cooperative games in this chapter that takes into account the individual power of a terrorist in a network, in contrary to the games considered in the other three chapters. The rankings induced by the combination of these methodologies identify Abaaoud, Abdeslam and Zerkani as key members of the network. This result is consistent with the vast qualitative research concerning this network (cf. [22], [2]). Further, we compare our results to the rankings established in [11], who used graph theoretic centrality measures degree and centrality to establish a ranking of the individuals in the Zerkani network. Finally, we note that the game theoretical analysis of the Zerkani network in this chapter contains two games that are different from the games used in the analysis of the Al Qaeda 9/11 network and the Jemaah Islamiyah network in [17], [3] and [12], respectively.

This chapter is organised as follows. Section 21.2 discusses the Shapley value and the structured random sampling method to approximate the Shapley value. The network analysis using cooperative game theory is explained in Section 21.3. The analyses of the Zerkani network are presented in Section 21.4 and Section 21.5 concludes the chapter.

## 21.2 The Shapley Value

This section recalls the definition of the Shapley value to describe the heuristic that is used in this chapter to approximate the Shapley value. This section is strongly based on Section 2 and 3 of [3].

Let $(N, v)$ be a cooperative game where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $v$ a map that assigns a value $v(S)$ to each possible coalition $S \subseteq N$. By definition $v(\emptyset) = 0$. The Shapley value is defined as

$$\varphi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_v^\sigma(i). \tag{21.1}$$

where $m_v^\sigma(i)$ is defined as $m_v^\sigma(i) = v(\{\sigma_1, \ldots, \sigma_k\}) - v(\{\sigma_1, \ldots, \sigma_{k-1}\})$. with $\Pi$ all possible orderings of the players in the grand coalition $N$.

Although the concept of marginal contributions is intuitively clear, computing the Shapley value via marginal contributions is time-expensive since all $n!$ possible orderings of the players need to be considered. Although for several special classes (e.g., airport games (cf. [18])) a time-efficient closed formula to compute the Shapley value exists, in many (real-world) situations the computation in reasonable time is not feasible if the number of players increases substantially. Therefore, there is a need for heuristics that provide a good approximation of the Shapley value.

Now we recall the heuristic called structured random sampling method introduced in [3]. The idea of this heuristic is to ensure that each player attains each position in the ordering the same number of times. As a consequence, the marginal contribution of a player to a coalition of the same size is calculated the same number of times. The intuition is that this leads to a better estimate because the calculation of the marginals with respect to coalitions of a certain size is equally distributed. To realize this, the randomly selected orderings are tweaked by swapping players to their preferred positions in the orderings. The marginal contributions of the players in these new orderings are then used to approximate the Shapley value.

The swapping method is illustrated for a 4-person game in Table 21.1. To ensure that each player attains each position in the ordering the same number of times the sample size $r$ must be a multiple of the number of players of the game. In this case, $r = 8$ orderings are randomly selected and divided into $n = 4$ groups of size $t = 2$. Observe that the number of groups is always equal to the number of players in the game. The size of a group indicates the number of times a player attains the same position in the ordering. In the two orderings in the first group, player 1 is swapped with the player at the first position. In the two orderings in the second group, player 1 is swapped with the player at the second position, etc. These new orderings are used to compute the marginal contributions of player 1 and they in turn are averaged to approximate the Shapley value of player 1. This procedure is then repeated for players 2 to $n$, each time using the original randomly selected $r$ orderings as a starting point for the swapping method. In Table 21.1, player 1 attains each position in the ordering exactly $t = 2$ times. The remaining positions in the orderings however remain random. The same holds for players 2 to $n$ when the swapping method is applied to construct the orderings for these players.

The procedure to approximate the Shapley value of an arbitrary game using the structured random sampling method is as follows.

**Procedure structured random sampling** ([3])
Input: $n$-person cooperative game $(N, v)$. (Hence, $n$ is fixed and determines the number of groups.)

1. Select a subset $\Pi_r$ of $r$ orderings from all $n!$ possible orderings, i.e., $\Pi_r \subset \Pi$, with $r = t \cdot n$ and $t \in \mathbb{N}$. (Hence, the subset must be a multiple of $n$.)

| Group | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | | Group | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | . | . | . | 1 | $\rightarrow$ | 1 | 1 | . | . | $\sigma_1$ |
| 1 | . | 1 | . | . | $\rightarrow$ | 1 | 1 | $\sigma_1$ | . | . |
| 2 | 1 | . | . | . | $\rightarrow$ | 2 | $\sigma_2$ | 1 | . | . |
| 2 | . | 1 | . | . | $\rightarrow$ | 2 | . | 1 | . | . |
| 3 | . | . | . | 1 | $\rightarrow$ | 3 | . | . | 1 | $\sigma_3$ |
| 3 | . | 1 | . | . | $\rightarrow$ | 3 | . | $\sigma_3$ | 1 | . |
| 4 | 1 | . | . | . | $\rightarrow$ | 4 | $\sigma_4$ | . | . | 1 |
| 4 | . | 1 | . | . | $\rightarrow$ | 4 | . | $\sigma_4$ | . | 1 |

**TABLE 21.1**: Swapping player 1 to his preferred positions in the orderings.

2. Divide the subset $\Pi_r$ in $n$ groups of size $t$. (This ensures that each player can attain each position in the ordering the same number of times.)

3. For each player $i$:

   (a) Swap player $i$ with the player at position $j$ for each of the $t$ orderings in group $j$, where $j \in \{1, \ldots, n\}$, resulting in a set $\Pi'_r$ of $r$ new orderings. (This ensures that each player will attain each position in the ordering the same number of times.)

   (b) Compute the marginal contributions $m_v^\sigma(i)$ of player $i$ for all new orderings $\sigma \in \Pi'_r$.

   (c) Approximate the Shapley value of player $i$ by averaging the marginal contributions obtained at step 3$b$, i.e., $\hat{\varphi}_i(v) = \frac{1}{r} \sum_{\sigma \in \Pi'_r} m_v^\sigma(i)$.

The following example illustrates how the structured random sampling procedure could be applied.

**Example 21.1 (*Structured random sampling*)** Consider the 3-person game $(N, v)$ with $v$ as in Table 21.2.

| $S$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|---|
| $v(S)$ | 0 | 6 | 18 | 0 | 30 | 42 | 24 | 60 |

**TABLE 21.2**: An example of a 3-person game.

Assume that we sample $r = 3$ from all $3! = 6$ orderings in $\Pi$, see the second column of Table 21.3. Since we have a three-player game, i.e., $n = 3$, we divide this subset into 3 groups. Since the size of the subset is chosen to be 3, i.e., $r = 3$, we have that the size of each group equals one, i.e., $t = 1$. Now consider player 1. Swapping this player with the player at the first, second and third position results in the new orderings depicted in the third column of

Table 21.3. The fourth column in this table depicts the corresponding marginal contributions $m_v^\sigma(1)$ of player 1 in the new orderings. More precisely, if $\sigma = (1,2,3)$ then $m_v^\sigma(1) = v(\{1\}) - v(\emptyset) = 6 - 0 = 6$, if $\sigma = (3,1,2)$ then $m_v^\sigma(1) = v(\{1,3\}) - v(\{3\}) = 42 - 0 = 42$, and if $\sigma = (3,2,1)$ then $m_v^\sigma(1) = v(\{1,2,3\}) - v(\{2,3\}) = 60 - 24 = 36$. Averaging these marginal contributions yields an approximation of the Shapley value for player 1, i.e., $\hat{\varphi}_1(v) = (6+42+36)/3 = 28$.

| Group | Ordering | Swap 1 | $m_v^\sigma(1)$ | Swap 2 | $m_v^\sigma(2)$ | Swap 3 | $m_v^\sigma(3)$ |
|---|---|---|---|---|---|---|---|
| 1 | $(1,2,3)$ | $(1,2,3)$ | 6 | $(2,1,3)$ | 18 | $(3,2,1)$ | 0 |
| 2 | $(1,3,2)$ | $(3,1,2)$ | 42 | $(1,2,3)$ | 24 | $(1,3,2)$ | 36 |
| 3 | $(3,1,2)$ | $(3,2,1)$ | 36 | $(3,1,2)$ | 18 | $(2,1,3)$ | 30 |

**TABLE 21.3**: The marginal contributions of randomly selected orderings of the 3-person game in Table 21.2.

Starting again from the original subset $\Pi_r$ in the second column and swapping player 2 with the player at the first, second and third position results in a new subset of orderings for player 2, see the fifth column of Table 21.3. The sixth column depicts the corresponding marginal contributions, resulting in $\hat{\varphi}_2(v) = 20$. Repeating this swapping method for the third player results in the orderings and marginal contributions depicted in the seventh and eighth column of Table 21.3, which in turn lead to $\hat{\varphi}_3(v) = 22$. Hence, $\hat{\varphi}(v) = (28, 20, 22)$.

Remark that structured random sampling does not result in an efficient allocation. This lack of efficiency is due to the fact that structured random sampling only considers marginal contributions of a single player for each sampled ordering. In spite of this drawback, the structured random sampling method outperforms the random sampling method introduced in [5] as shown in [3].

## 21.3 A New Game Theoretic Centrality Measure

In this section, we follow the idea of [17] to create two classes of games that take into account both the structure of the network and the relational and the individual strength of the members of the network. The first game is a slight modification of the game introduced in [12], the second game is a new type of game.

A network can mathematically be represented by a graph $G = (N, E)$, where the node set $N$ represents the set of members of the network and the set of links $E$ consists of all relationships that exist between these members.

The existence of a relationship between member $i$ and $j$ is denoted by $ij \in E$. For a coalition $S \subseteq N$, the subnetwork $G_S$ consists of the members of $S$ and its links in $E$, i.e., $G_S = (S, E_S)$ where $E_S = \{ij \in E | i, j \in S\}$. The strength of individuals in a network $G = (N, E)$ is represented by a set of weights on players set $N$, i.e., $\mathcal{I} = \{w_i\}_{i \in N}$ with $w_i \geq 0$, and the relational strength between members of the network is represented by a set of weights on the the edges $E$, i.e., $\mathcal{R} = \{k_{ij}\}_{ij \in E}$ with $k_{ij} \geq 0$. A coalition $S \subseteq N$ is called a connected coalition if the network $G_S$ is connected, otherwise $S$ is called disconnected.

In [12] was introduced *monotonic weighted connectivity game (mwconn)* $(N, v^{\text{mwconn}})$ with respect to network $G = (N, E)$ based on $\mathcal{I}$ and $\mathcal{R}$ in the following way. For a connected coalition $S$ we define

$$v^{\text{mwconn}}(S) = f(S, \mathcal{I}, \mathcal{R}) \tag{21.2}$$

where $f$ is a context specific and tailor-made non-negative function depending on $S$, $\mathcal{I}$ and $\mathcal{R}$ which measures the effectiveness of coalitions in the network. It can be chosen to best reflect the situation and information at hand. For a coalition $S$ that is disconnected we define

$$v^{\text{mwconn}}(S) = \max_{T \in \Sigma_S} v^{\text{mwconn}}(T), \tag{21.3}$$

where $\Sigma_S$ is the set of maximal connected components in $G_S$. Observe, that the value of each disconnected coalition is based on the most effective connected component of this coalition. Moreover, in contrary to [17] and [12] individuals can have a positive value.

An *additive weighted connectivity game (awconn)* $(N, v^{\text{awconn}})$ with respect to network $G = (N, E)$ based on $\mathcal{I}$ and $\mathcal{R}$ is for a connected coalition $S$ defined by (21.2), identically to the corresponding monotonic weighted connectivity game $(N, v^{\text{mwconn}})$. The difference between these two games is the definition of the disconnected coalitions. For a coalition $S$ that is disconnected in an additive weighted connectivity game we define

$$v^{\text{awconn}}(S) = \sum_{T \in \Sigma_S} v^{\text{mwconn}}(T). \tag{21.4}$$

Hence, here all maximal connected subsets of $S$ are effective.

**Example 21.2 (*an mwconn and awconn game on a network*)** In this example we define $f(S, \mathcal{I}, \mathcal{R})$ by

$$f(S, \mathcal{I}, \mathcal{R}) = \begin{cases} \left(\sum_{i \in S} w_i\right) \cdot \max_{ij \in E_S} k_{ij} & \text{if } |S| > 1, \\ w_S & \text{if } |S| = 1. \end{cases} \tag{21.5}$$

which takes both the individual and the relational strength into account. Now, we will illustrate the mwconn and awconn game corresponding to the network displayed in Figure 21.1. Subsequently, we present the rankings based on the Shapley value applied to the mwconn and awconn game.

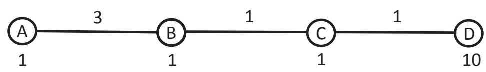


**FIGURE 21.1**: A network with information on individual and relational strengths.

We assume that individual information is available only for member $D$. He was involved in a previous attack and has financial means to support a potential new attack. Counterterrorism analysts will take this observation into account when assigning weights to the members. Here we assume that member $D$ is assigned a weight of 10 and all other members are assigned a weight of 1. This defines $\mathcal{I}$. Additionally, suppose it is observed that members $A$ and $B$ communicate more frequently than the other members of the network. Here we assume that link $AB$ is assigned a weight of 3 and all other links are assigned a weight of 1. This defines $\mathcal{R}$.

Using (21.5) the values $v^{\text{mwconn}}(S)$ as defined in (21.2) and (21.3) can be computed for each coalition $S$. Table 21.4 provides these values for each of the 16 coalitions in $N = \{A, B, C, D\}$. Simultaneously, we present the values of $v^{\text{awconn}}(S)$ in Table 21.4 using (21.2) and (21.4).

| Coalition $S$ | $\{A\}$ | $\{B\}$ | $\{C\}$ | $\{D\}$ | $\{A,B\}$ |
|---|---|---|---|---|---|
| $v^{\text{mwconn}}(S)$ | 1 | 1 | 1 | 10 | 6 |
| $v^{\text{awconn}}(S)$ | 1 | 1 | 1 | 10 | 6 |
| Coalition $S$ | $\{A,C\}$ | $\{A,D\}$ | $\{B,C\}$ | $\{B,D\}$ | $\{C,D\}$ |
| $v^{\text{mwconn}}(S)$ | 1 | 10 | 2 | 10 | 11 |
| $v^{\text{awconn}}(S)$ | 2 | 11 | 2 | 11 | 11 |
| Coalition $S$ | $\{A,B,C\}$ | $\{A,B,D\}$ | $\{A,C,D\}$ | $\{B,C,D\}$ | $\{A,B,C,D\}$ |
| $v^{\text{mwconn}}(S)$ | 9 | 10 | 11 | 12 | 39 |
| $v^{\text{awconn}}(S)$ | 9 | 16 | 12 | 12 | 39 |

**TABLE 21.4**: The values for $v^{\text{mwconn}}(S)$ and $v^{\text{awconn}}(S)$ in the example in Figure 21.1.

The outcome of the Shapley value computed for the monotonic weighted connectivity game and the additive weighted connectivity game is presented in Table 21.5.

| Member | $\phi(v^{\text{mwconn}})$ | $\phi(v^{\text{awconn}})$ |
|---|---|---|
| $A$ | 8 | $8\frac{2}{3}$ |
| $B$ | $8\frac{1}{2}$ | $8\frac{2}{3}$ |
| $C$ | $8\frac{1}{6}$ | $6\frac{2}{3}$ |
| $D$ | $14\frac{1}{3}$ | 15 |

**TABLE 21.5**: The Shapley value of $v^{\text{mwconn}}(S)$ and $v^{\text{awconn}}(S)$ for the example in Figure 21.1.

The corresponding ranking using the Shapley value with respect to the mwconn game is $R_{mwconn} = (D, B, C, A)$ and the ranking using the Shapley value with respect to the awconn game is $R_{awconn} = (D, B, A, C)$. Both rankings seem almost identical, but member $A$ was ranked last in $R_{mwconn}$ and is in $R_{awconn}$ ex aequo with B and ranked higher than $C$. On the other hand, member $B$ was in both rankings the second most important member.

## 21.4 Zerkani Network Analysis

It is known that the Islamic State (IS) built a vast network in Europe to prepare for the attacks in Paris, which also partly mobilised a network to attack in Brussels a few months later. Strategically ISIS security apparatus is said to be divided into four agencies, of which the one responsible for conducting terror attacks outside of Syria is subdivided into regions, called theaters. The theater commander responsible for Europe is believed to be Salim Benghalem who worked together with Abdelhamid Abaaoud in selecting fighters to send to war zones or fight in Europe. The main figures responsible for the tactical operations of the Paris and Brussels attacks were Abaaoud and jihadist recruiter Zerkani. The Zerkani network facilitated IS for operations in Europe, providing personnel, training, planning, attack and escape and evasion. In that sense, the Zerkani network can be seen as the main operational network responsible for the November 2015 mass shootings and suicide attack in Paris and the March 2016 coordinated suicide bombing in Brussels. Khalid Zerkani is currently imprisoned on terrorism related charges ([2]) and Abaaoud was killed during a police raid in the aftermath of the Paris attacks. The network considered in this chapter consists of 47 members obtained from [10][1].

In this section we analyse rankings of the individuals in the Zerkani network, displayed in Figure 21.2, by approximating the Shapley value for the mwconn and the awconn game. Observe that we have to approximate the Shapley value in this setting since the Zerkani network contains 47 members.

[1] We acknowledge Valens Global for providing the dataset.

We compar d perform a sensitivity iced in [12].

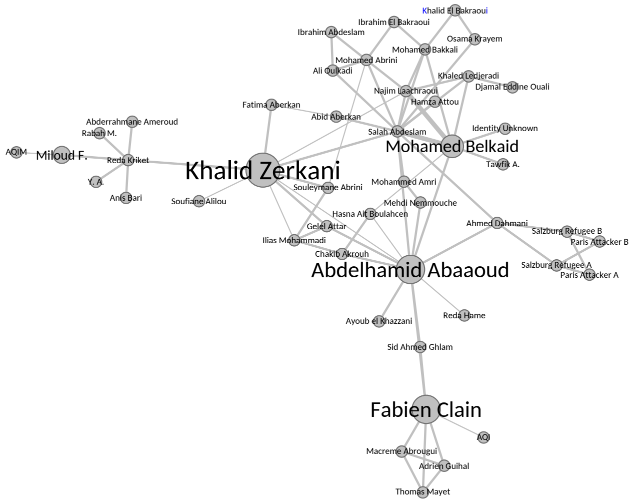


**FIGURE 21.2**: Zerkani network. *Based on data of 'Valens Global'.*

In our analysis we rank all 47 individuals but will focus on the top 10 of the ranking only per example. This is because scarce resources limit intelligence and law enforcement agencies in their operations, i.e., not everybody can be kept under surveillance all the time.

Table 21.6 provides the top-10 rankings of the Zerkani network using the game theoretical approach discussed in Section 21.4 and the graph theoretical measures degree and betweenness, respectively. The rankings obtained from the two game theoretical measures are displayed in the columns $RankingR_{mwconn}$ and $RankingR_{awconn}$, respectively. The columns $RankingR_{degree}$ and $RankingR_{betweenness}$ correspond to the rankings obtained by the graph theoretical measures degree and betweenness, respectively. Recall that the degree of a member in a network is equal to its direct neighbors in the network and the betweenness indicates the importance of a member to connect different members in a network. Formal definitions of these two measures can be found in [23]. We include these two graphs theoretical measures because they are used in [11]. [2] This results in a ranking $R_{mwconn}$, obtained from the approximated Shapley value corresponding to the mwconn game, a ranking $R_{awconn}$, obtained from the approximated Shapley value corresponding to the awconn game, $R_{degree}$ and $R_{betweenness}$, respectively.

[2] [11] analysed the Islamic State network in Europe, which contains 119 members. The Zerkani network is contained in this network and consists of 47 members.

Note we use the function $f(S, \mathcal{I}, \mathcal{R})$ defined by (21.5) as shown in Example 21.2 to define the mwconn and awconn game. The specific weights assigned to the individuals and links can be found in the appendix.

| | Ranking $R_{mwconn}$ | Ranking $R_{awconn}$ |
|---|---|---|
| 1 | Mohamed Belkaid | Mohamed Belkaid |
| 2 | Abdelhamid Abaaoud | Abdelhamid Abaaoud |
| 3 | Khalid Zerkani | Khalid Zerkani |
| 4 | Salah Abdeslam | Salah Abdeslam |
| 5 | Fabien Clain | Fabien Clain |
| 6 | Najim Laachraoui | Najim Laachraoui |
| 7 | Reda Kriket | Reda Kriket |
| 8 | Ahmed Dahmani | Ahmed Dahmani |
| 9 | Mohamed Abrini | Miloud F. |
| 10 | Khaled Ledjeradi | Khaled Ledjeradi |
| | $RankingR_{degree}$ | $RankingR_{betweenness}$ |
| 1 | Abdelhamid Abaaoud | Abdelhamid Abaaoud |
| 2 | Salah Abdeslam | Salah Abdeslam |
| 3 | Khalid El Bakraoui | Reda Kriket |
| 4 | Najim Laachraoui | Khalid El Bakraoui |
| 5 | Mohamed Abrini | Khalid Zerkani |
| 6 | Osama Krayem | Najim Laachraoui |
| 7 | Ibrahim El Bakraoui | Osama Krayem |
| 8 | Mohamed Bakkali | Hasna Ait Boulahcen |
| 9 | Mohamed Belkaid | Souleymane Abrini |
| 10 | Khalid Zerkani | Mohamed Belkaid |

**TABLE 21.6**: The top 10 rankings of the Zerkani network of Figure 21.2 according to the approximated Shapley value of mwconn and awconn game, degree and betweenness.

The two game theoretical rankings differ only in one member in the top 10, whereas the difference between the game theoretical ranking $R_{mwconn}$ (or $R_{awconn}$) and the graph theoretical ranking $R_{degree}$ (or $R_{betweenness}$) is on four positions. The terrorist Mohamed Belkaid is ranked top in the game theoretical rankings. This a little bit surprising. Indeed, he was close to Salah Abdeslam and had a history in fighting in Syria, but his involvement in the Paris and Brussels attacks is not proven. On the other hand, we see that he is also ranked in the top 10 of the two graph theoretical measures which indicates that in some way due to the information at hand, he seems to be important. Number two, three and four ranked in the game theoretical measures, and also ranked in the top 10 of the graph theoretical measures, are important members. Abdelhamid Abaaoud, ranked first in the graph theoretical measures, is considered as the mastermind of the Paris attack, but there are some doubts about his role (cf. [22]). At least, he was strongly involved in the coordination of the Paris attack during the execution of the attack. Naming the network to Zerkani reflects already his influence. His most prominent role was recruiting new members for his network and soldiers for the war in Syria. He was not actually present or coordinating the attacks of Paris and Brussels, but his earlier radicalisation process influenced strongly the people involved in the attacks. Salah Abdeslam was an important liaison in the attack of Paris. After the attack, he was the most wanted person in Europe. The first member that is in the top 10 of the game theoretical measure but not present in the graph theoretical top 10 is Fabien Clain. He is one of the planners of the Paris attack. He explored and pointed the locations of the different hits in Paris, but according to [22] there are also indications that he was the mastermind of this attack instead of Abdelhamid Abaaoud. Note that Fabien Clain is not in the top 10 of the graph theoretical measures. Najim Laachraoui and Reda Kriket, number 6 and 7 in the game theoretic ranking and also present in the graph theoretical top 10 ranking, are both important. The first is involved in the Brussels attack, but more important, he had the skills to construct bombs. The latter is an important recruiter and money man in the network. The persons ranked 8, 9 and 10 are different from the graph theoretical measures. Mohamed Dahmani is considered an explorer for the locations of the Paris attack. Mohamed Abrini is known as "the man with the hat" in the Brussels attack that was present at the airport. It turns out that he is also an important liaison in the network and that he was also involved in the Paris attack. Khaled Ledjeradi was the leader of an organisation that made false documents for the members of the Zerkani network, which enabled them to travel Europe using aliases. The members that are in the top 10 of the graph theoretical measures are almost all known as suicide bombers. So, we can conclude that 6 important members are present in all rankings. Other important people that seem to be leaders or have important skills are present in the game theoretical rankings, whereas the suicide bombers are more visible in the graph theoretical rankings. On one hand, the game theoretical rankings provide the members that are more difficult to replace due to their position or skills, on the other hand the graph theoretical rankings provide the members that cause high casualties. But the latter group is easier to replace. We want to emphasize that these observations should not lead to a discussion which ranking is the best. In fact, any ranking and the differences between different types of rankings can help intelligence to focus more on some members. It also depends on the moment. If the attack will not be operational in the near future, the leader and skill type of people are more interesting to monitor. If there is a great threat that an attack will be executed in the very near future, the suicide bombers have higher priority.

Since the difference between $R_{mwconn}$ and $R_{awconn}$ with respect to the Zerkani network is marginal, the final part of this section is devoted to the sensitivity of the $R_{mwconn}$ ranking with respect to the Zerkani network. The sensitivity analysis is done by several simulations in which the weights on nodes (i.e., members of network), weights on links (i.e., strength of communication), adding/removing links (non-presence of communication) are varied. Before we will do these simulations, we need to be able to compare different rankings. This difference will be measured using the $\rho$ measure as introduced in [12]. The following Example 21.3 illustrates this measure.

**Example 21.3 (*the $\rho$ measure to compare rankings*)** Consider the situation whe ... twork have been discov ... 4 (random) links have b ... sconnected.

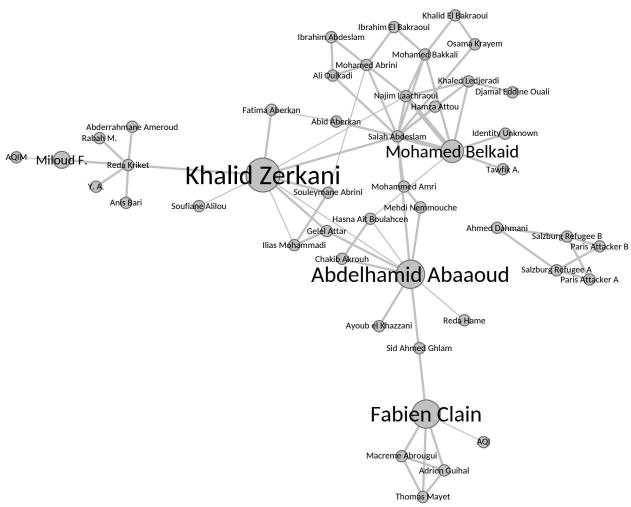


**FIGURE 21.3**: Zerkani network of Figure 21.2 with 4 (random) links removed.

Ranking $R_1$ in Table 21.7 presents the top-10 ranking of the terrorists corresponding to the network in Figure 21.3, obtained by calculating the approximated Shapley value of the mwconn game. The computation of $\rho$, which encapsulates the amount of difference between rankings, is based on values assigned to the ranking positions. These depend on the total number of members in the network, in this example there are 47 members, and the number of top ranked players that are of interest. In this chapter we focus on the 10 highest ranked players. Each position in the top 10, i.e., position $i$, with $i = 1, \ldots 10$, is assigned the value $\frac{10-i+1}{10}$ and each member outside the top

| | Ranking $R_1$ |
|---|---|
| 1 | Mohamed Belkaid |
| 2 | Abdelhamid Abaaoud |
| 3 | Khalid Zerkani |
| 4 | Salah Abdeslam |
| 5 | Fabien Clain |
| 6 | Najim Laachraoui |
| 7 | Reda Kriket |
| 8 | Mohamed Abrini |
| 9 | Khaled Ledjeradi |
| 10 | Hasna Ait Boulahcen |

**TABLE 21.7**: The rankings of the Zerkani network of 21.3 according to the Shapley value applied to the mwconn and awconn game.

10, i.e., position $i$, with $i = 11, ...47$, is assigned the value $\frac{i-10}{37}$. To calculate $\rho$ for this setting, the aforementioned value of each player in the top 10 of $R_{\text{mwconn}}$ that is no longer present in the top 10 of the new ranking $R_{m_1}$ is summed and added to the aforementioned value of each player in $R_{mwconn}$ not present in its top 10 that enters the top 10 in $R_{m_1}$, totalling $\rho$. Observe that the values assigned to the positions are chosen in such a way that highly ranked members that leave the top 10 in $R_{\text{mwconn}}$ and lowly ranked members that enter the top 10 in $R_1$ result in a large value of $\rho$.

The maximal value of $\rho$ in this case is obtained when the top-10 in $R_{mwconn}$ is replaced by the terrorists in the bottom-10 of $R_1$, resulting in the maximal value $(1 + \frac{9}{10} + ... + \frac{1}{10}) + (\frac{28}{37} + \frac{29}{37} + ... + \frac{36}{37} + 1) \approx 14.28$. In general, the value of $\rho$ will be relatively low when two rankings do not differ too much. In fact, if the top-10 only shifts their position internally, then the value of $\rho = 0$. Note that $\rho(R_{mwconn}, R_1) = \frac{3}{10} + \frac{5}{37} \approx 0.4351$ since Ahmed Dahmani at position 8 in the original ranking leaves the top-10 in $R_1$ and Hasna Ait Boulahcen at position 15 in the original ranking enters the top-10 in $R_1$.

In our sensitivity analysis, we will first focus on network structure and vary the number of links present in the network. We will not only investigate scenarios in which a percentage of the links is removed from the network, but also scenarios in which a percentage of the links is added to the network. Hence, we consider the situation when an intelligence agency has gathered information about the structure of some network but does not know for sure whether all lines of communication are discovered or play a role in the network. Second, we will focus on individual and relational strength and investigate scenarios in which different weights on individuals and links are used. In practice, field experts have to decide on the exact heights of the weights assigned to individuals and links. The magnitude of such weights should reflect individual characteristics (e.g., financial means, skills to create an explosive) or the importance of a specific type of communication (e.g., email communication, exchanging explosive materials). Obviously, it is difficult to quantify these numbers accurately. Therefore, we also want to check what impact minor changes in the assigned weights have on the ranking of the members of the network.

With respect to network structure, we ran 500 simulations in which up to 4 links were randomly added or deleted. We computed the resulting values of $\rho$. With respect to individual strength, we ran 500 simulations in which each of 4 randomly selected individuals received an additional random weight equal to 1, 2, 3 or 4. To investigate the effect of relational strength, we selected four random links and each selected link received an additional weight of 1, 2, 3 or 4, running 500 simulations. The results of this sensitivity analysis are depicted in Figure 21.4. Note that (another) simulation shows that the expected value of $\rho$ is approximately 8.37 when randomly ranking the 47 members and not using network structure, individual strength and relational strength.

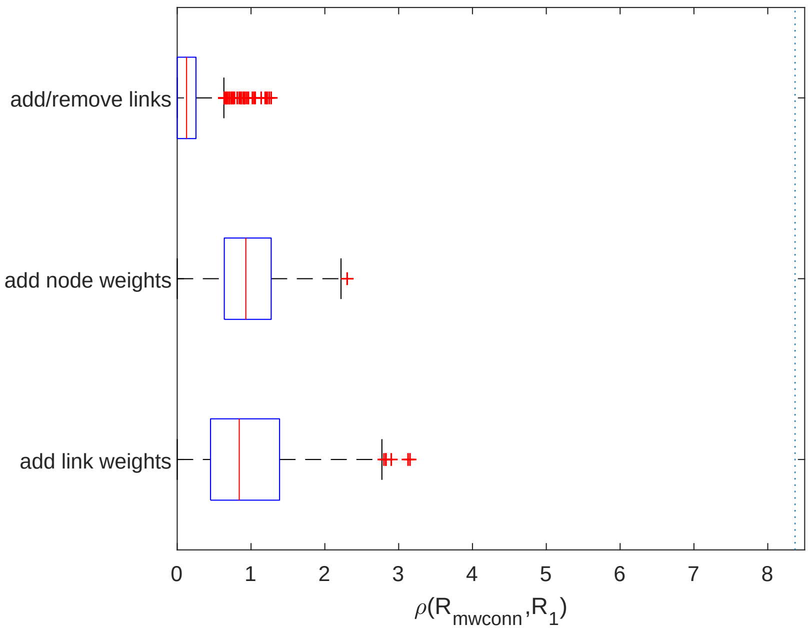


**FIGURE 21.4**: Boxplots of $\rho$-values for the sensitivity analysis of Zerkani network on network structure, individual strength and relational strength.

From Figure 21.4 it follows that the ranking resulting from the monotonic weighted connectivity game is robust to small changes in the network structure and the weighing of links and individuals. In all cases, the value of $\rho$ is significantly less than the value $\rho = 8.37$ obtained by a random ranking.

Note that $\rho \geq \frac{1}{10} + \frac{1}{37} \approx 0.13$ in case one individual is replaced in the original top-10 by an individual not yet present in the top-10. Two individuals leaving the original top-10 will yield a value of $\rho$ of at least $\frac{1}{10} + \frac{2}{10} + \frac{1}{37} + \frac{2}{37} \approx 0.38$. For three individuals, this value is at least 0.76.

When varying the number of links present in the network, the top-10 of the original ranking is seen to remain virtually unchanged. In 50% of the cases the ranking remains unchanged and in another 30% of the cases only one individual leaves the top-10. Even at the (relatively few) outliers, at most four individuals leave the top-10. When adding extra weights to nodes in the network, the top-10 differs slightly from the original ranking. In 35% of the cases, at most three individuals in the top-10 are replaced. In most other cases, at most four individuals are replaced. Even the (relatively few) outliers are seen to differ significantly from the value of $\rho$ when using a random ranking. The original ranking is seen to be more sensitive to adding extra weights to links. Although at most one individual in the top-10 is replaced in 50% of the cases, and at most two individuals are replaced in another 30% of the cases, the remaining 20% of the cases show more variability in the value of $\rho$. Still, all significantly outperform a random ranking.

We can conclude that $R_{\text{mwconn}}$ provides a ranking for the Zerkani network that is globally robust against link changes as well as weight changes.

## 21.5 Conclusions

In this chapter we analysed the Zerkani network that was complicit in the terror attacks in Paris and Brussels. Two types of cooperative games are used that take both network structure and individual characteristics of the members of the network into account. Subsequently, the Shapley value as a measure of social centrality is approximated to provide rankings of the members of the network. We compared these rankings to traditional network theoretic measures. Finally a robustness analysis representing the incomplete information available to intelligence agencies is conducted by simulations.

We conclude that the ranking obtained by use of the game theoretic Shapley value ranked those individuals high which seemed of great importance to the preparation of the attack. This in contrast to standard network theoretic measures that ranked field attackers (such as shooters and suicide bombers) higher. It turned out that the rankings of both different cooperative games were almost equal, from which we conclude that the ranking obtained by the Shapley value is quite robust.

It will be clear that the methodology behind $R_{mwconn}$ can be applied to other datasets as well, actually to each dataset representing a social network. These datasets need thus not be restricted to terrorist networks but could also be applied to, e.g., criminal networks. Moreover, other variations in our choices of $f(S, I, R)$ should be studied to better fit specific settings.

## 21.6 Appendix

Here we provide the weights we have assigned to the links and members (=node) of the network. Initially each link and member receives a weights equal to one. This weight is increased using the tables below.

| Relationship | Weight on link | Extra weight start node |
|---|---|---|
| 'Associate of' | 2 | 0 |
| 'Brother of' | 1 | 0 |
| 'Commander of' | 2 | 2 |
| 'Family relationship' | 1 | 0 |
| 'Funded' | 1 | 2 |
| 'Lived with' | 2 | 0 |
| 'Nephew of' | 1 | 0 |
| 'Recruiter of' | 1 | 1 |
| 'Supporter of' | 1 | 1 |
| 'Traveled to Syria with' | 2 | 0 |
| 'Traveled with' | 2 | 0 |

**TABLE 21.8**: The weights assigned to links and starting nodes.

Applying Table 21.8 results in the following members having a weight larger than one as shown in Table 21.9.

| Node | Weight |
|---|---|
| Abdelhamid Abaaoud | 4 |
| Fabien Clain | 4 |
| Khalid Zerkani | 5 |
| Miloud F. | 2 |
| Mohamed Belkaid | 3 |

**TABLE 21.9**: The members with a weight larger than one.

## 21.7 Acknowledgments

We thank the Department of Econometrics and Operations Research of TiSEM, Tilburg University, the Department of Military Operational Art & Science, Netherlands Defense Academy and the Department of Mathematics, Fontys University of Applied Sciences for supporting this research.

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# Chapter 22

## A Fuzzy Approach to Some Shapley Value Problems in Group Decision Making

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### 22.1 Introduction

The Shapley value is one of the classical methods for deriving appropriate payoffs in cooperative games, in particular for simple games. It has very good properties for defining *a priori* values in cases where the payoffs for each possible coalition are precisely defined. Due to the nature of this collection of papers, the authors treat the concepts of cooperative games and the Shapley value as known to the reader. However, the following works should be mentioned: Felsenthal and Machover (1997) and Taylor and Zwicker (1999). When decisions are made on the basis of a vote to obtain a group decision, often the assumption of determinism is inappropriate. This article considers an adaptation of the Shapley value to situations in which the weights of the individual players in voting games are not deterministic. This means that the number of votes required to pass a decision (the quota) and the set of winning coalitions are not precisely defined. Also, the number of votes cast might be affected by a quorum. The concept of fuzzy sets and a modified version of the characteristic form of a game are utilised to appropriately modify the concept of the Shapley value to such situations. The theory is illustrated by examples including decision making in parliament and the Council of the European Union.

If we assume that parties are more concerned about getting into office than about implementing a specific policy, any coalition consisting of two or more parties can form, regardless of the parties' policy positions. The classical Shapley value (or Shapley-Shubik value, which is a particular case of the Shapley value for simple games) is appropriate when all of the possible coalitions are equally probable, the weight of each player (e.g., the number of votes available to a party) and the quota (the number of votes required to pass a motion) are deterministic numbers and power is split evenly between the players in a winning coalition.

The assumption that all of the possible coalitions are equally probable has been hotly debated from the very beginning. Owen (1977) proposed an approach based on so-called pre-coalitions, which leads to some coalitions being by definition impossible. Moreover, an individual probability can be assigned to the formation of any possible coalition. Obviously, the question arises in this situation as to what method should be used to define the Shapley value for each player (i.e., how should the payoff obtained by a coalition be split between its individual members). Many articles assume that the payoff of a winning coalition is split equally between the members. In some applications, e.g., in determining the division of profits among a set of firms which are interdependent, possibly in a complex manner, a more complicated approach is used to adapt this division to the ownership structure: See Gambarelli and Owen (1994), or more recently Bertini et al. (2016) and Stach (2017).

The main goal of this chapter is to apply fuzzy theory to cases of voting games in which the weight of each player and the quota are not deterministic numbers in order to assess the voting power of the players. In standard voting games, both the weights of players and the rules of voting are fixed. On the other hand, in parliamentary decision making, individual players may have differing levels of freedom to choose how to vote on a particular motion (not to mention the possibility of strategic behaviour). Hence, in the opinion of the authors, there is a need for a more flexible approach to defining the number of players in a voting game, their weights and the conditions for winning a vote. Another goal is to consider a form of inverse problem, where a set of desirable voting powers of the players are given (e.g., in EU council votes, it may be deemed desirable that the power of a member state is proportional to its population size). If such a set of power indexes is attainable, it might result from a large range of voting weights which, e.g., minimize the distance between the power indexes and the desired set of indexes. Hence, such voting weights may be interpreted as fuzzy, in its intuitive sense. Such an inverse problem refers to finding a set of appropriate weights given the desired voting powers to satisfy a given optimality criterion.

In particular, the authors state that the definition of the Shapley values should be adapted to the following situations:

**a)** When there exists a single dominant player who possesses the power to pass any decision. In the standard majority game, such a player has a Shapley value of 1, independently of whether this player possesses 51% or 91% of the votes.

**b)** There exist more than one criterion deciding the desirability of a coalition. It is clear that the Shapley value will generally depend on the criteria employed. Hence, there exists a question as to which of a number of criteria should be employed to calculate the Shapley value[1] or how should these criteria be aggregated.

**c)** There exists a clear difference between the relative powers of players (according to the Shapley value) and their relative weights. Such discrepancies occur quite regularly and are related to the lack of linearity of the Shapley value in simple games.

**d)** There can be minority winning coalitions[2], e.g., in practice, members of a parliamentary party may abstain or be absent. In this case, a coalition which does not have a majority of the members of parliament may be able to win a vote. The application of the classical Shapley value in such games is by definition impossible.

The aim of this article is to analyse the above cases and propose possible modifications of the Shapley value. These examples are mostly based on simple voting games, although the concepts presented here can be applied to

[1] One example of a situation in which the result of a vote is based on two criteria is the dilemma associated with voting in the Council of the European Union: Each country has one vote as a state, but since the states have widely differing populations, an additional requirement is added related to the percentage of the total EU population represented by the countries voting for a motion (this will be defined more precisely later). However, since countries have different values of GDP according to which they pay into EU funds, it would also be possible to introduce another criterion based on GDP. It should be noted that recently there have been discussions regarding the introduction of a 0-1 criterion regarding a condition for democratic legal governance. A slightly different problem occurs when the value of a coalition depends not only on the number of votes it possesses, but also on, for example, its internal stability. Such examples can be described by cooperative games in which the characteristic function is multi-valued. The authors consider the second type of game.

[2] A good example of this situation is the existence of minority governments, which despite not having a majority can last for a relatively long time in many countries including the Netherlands, Finland and Italy).

cooperative games of a more general form. This chapter is organized as follows: Section 22.2 presents some preliminary definitions and notation. Section 22.3 considers the problem of defining the Shapley value for majority games. Section 22.4 considers cases where the quota is not fixed, under the assumption that a motion is passed if and only if the number of votes for is greater than the number of votes against. In Section 22.5, we present an analysis of situations in which the value of a coalition is multi-dimensional and we propose a solution based on a synthetic value. In Section 22.6 we consider the problem of significant discrepancies between the relative powers and the relative weights of the players and propose a solution to this problem based on a measure of consistency between the relative weights and the relative powers. The final section gives a summary.

## 22.2 Preliminaries

Indeterminancy is normally modelled using either probability theory or fuzzy theory. Probability theory is normally appropriate when the distribution of variables can be estimated on the basis of historical data. On the other hand, when government coalitions are being formed immediately after an election or one is considering votes in atypical situations (e.g., when party discipline is not applied), the authors would argue that a fuzzy approach is more appropriate[3]. In this case, the behaviour of the decision makers should be assessed by experts. The goal of this paper is to present a framework for carrying out such an analysis and leaves the problem of how precisely such problems should be defined (e.g., the appropriate definition of fuzzy numbers defining the voting weights of the players) for future research.

This section presents fundamental concepts related to fuzzy sets. These concepts will be adapted to solving some of the problems described above. Concepts and notation related to the Shapley value will not be presented here, since we use standard notation, which should be well known to the readers of this book. Various concepts of fuzziness have been applied to the analysis of cooperative games (see, e.g., Branzei et al., 2008 and Mares, 2013).

### 22.2.1 Some Elements of the Theory of Fuzzy Sets

Here we present some fundamental concepts from the theory of fuzzy sets. Zadeh (1965) introduced the idea of a fuzzy set. A fuzzy set $\tilde{A}$ in a space $X$

[3]However, it should be noted that probabilistic approaches can be adopted even when we do not possess any data, either with a non-informative Bayesian prior or using a prior generated by experts, see, e.g., Driver and Alemi (1995). For a discussion of the relationship between fuzzy sets and probability measures see, e.g., Dubois and Prade (1989) and (1993).

is a set of ordered pairs: $\{(x, \mu_A(x) : x \in X)\}$, where $\mu_A : X \to [0,1]$ is the membership function of the fuzzy set.

Given two fuzzy sets $\tilde{A}$ and $\tilde{B}$, the degree of membership of the element $x$ in the set $\tilde{A} \cap \tilde{B}$ is given by:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}. \tag{22.1}$$

An interval fuzzy number $\tilde{X}$ is family of real, closed intervals $[\tilde{X}]_\lambda$, where $\lambda \in [0,1]$, such that: $\lambda_1 < \lambda_2 \Rightarrow [\tilde{X}]_{\lambda_1} \subset [\tilde{X}]_{\lambda_2}$ and $I \subseteq [0,1] \Rightarrow [\tilde{X}]_{\sup I} = \cap_{\lambda\in I}[\tilde{X}]_\lambda$. For a given $\lambda \in [0,1]$, the interval $[\tilde{X}]_\lambda$ is called the $\lambda$-level of the fuzzy number $\tilde{X}$. This level will be denoted by $[\tilde{X}]_\lambda = [\underline{x}(\lambda), \overline{x}(\lambda)]$.

The function $\mu_X(x) = \sup\{\lambda : x \in [\tilde{X}]_\lambda\}$ is called the membership function of the fuzzy number $\tilde{X}$. The value $\mu_X(x)$ can be interpreted as a measure of the possibility that the fuzzy number $\tilde{X}$ takes the (crisp) value $x$.

An interval fuzzy number $\tilde{X}$ is called L-R type, if its membership function is of the following form (Dubois and Prade, 1978):

$$\mu_X(x) = \begin{cases} L\left(\frac{\underline{m}-x}{\alpha}\right) & \text{for} \quad x < \underline{m} \\ 1 & \text{for} \quad \underline{m} \le x \le \overline{m} \\ R\left(\frac{x-\overline{m}}{\beta}\right) & \text{for} \quad x > \overline{m}, \end{cases} \tag{22.2}$$

where: $L(x)$ and $R(x)$ are continuous non-increasing functions $x; \alpha, \beta > 0$.

The functions $L(x), R(x)$ are called the shape functions of the fuzzy number. The most commonly used shape functions are: $\max\{0, 1-x^p\}$ and $\exp(-x^p)$, $x \in [0,\infty), p \ge 1$. An interval fuzzy number for which $L(x), R(x) = \max\{0, 1-x\}$ and $\underline{m} = \overline{m} = m$ is called a triangular fuzzy number and will be denoted by $(m, \alpha, \beta)$. The authors will also use truncated triangular fuzzy numbers, whose membership function is of the form $L(x) = \max\{0, 1-x\}$ for $x \ge 0$ and $L(x) = 0$ for $x < 0, R(x) = \max\{0, 1-x\}$ and $\underline{m} = \overline{m} = m$. A truncated triangular fuzzy number will be denoted by $(m, \alpha, \beta)_T$.

Let $\tilde{X}, \tilde{Y}$ be fuzzy numbers with membership functions $\mu_X(x)$ and $\mu_Y(y)$, respectively, and let $z = f(x,y)$ be a real function. According to Zadeh's extension principle, the membership function of the fuzzy number $\tilde{Z} = f(\tilde{X}, \tilde{Y})$ is of the form:

$$\mu_Z(z) = \sup_{z=f(x,y)} (\min[\mu_X(x), \mu_Y(y)]). \tag{22.3}$$

If we wish to compare two fuzzy numbers, i.e., define the measure of possibility that a realization of the fuzzy number $\tilde{X}$ is not less (or greater, respectively) than a realization of $\tilde{Y}$, then we can use the formulas proposed by Dubois and Prade (1988):

$$Pos(\tilde{X} \ge \tilde{Y}) = \sup_{x\ge y}(\min[\mu_X(x), \mu_Y(y)]) \tag{22.4}$$

$$Pos(\tilde{X} > \tilde{Y}) = \sup_x \inf_{y\ge x}(\min[\mu_X(x), 1-\mu_Y(y)]). \tag{22.5}$$

The possibilistic expected value, $E(\tilde{X})$, and possibilistic variance, $Var(\tilde{X})$, of the fuzzy number $\tilde{X}$ are given by (see Carlsson and Fullér, 2001)

$$E(\tilde{X}) = \int_0^1 \frac{1}{2}\left(\underline{x}(\lambda) + \overline{x}(\lambda)\right) d\lambda \tag{22.6}$$

$$Var(\tilde{X}) = \int_0^1 \left(\frac{\overline{x}(\lambda) - \underline{x}(\lambda)}{2}\right)^2 d\lambda. \tag{22.7}$$

## 22.3 The Shapley Value for Majority Voting Games

One of the most common applications of the Shapley value to analysing phenomena of social life is modelling group decisions using the theory of cooperative games. The classical approach assumes that there are relatively few players (parties), who are treated as voters who have differing weights. For example, Mercik and Ramsey (2015) analysed the use of the power of veto in the United Nations Security Council. Sosnowska (2014) analysed the Governing Council of the European Central Bank. Nurmi and Meskanen (1999) analysed EU institutions. It seems that one of the sources of discrepancy between the results obtained and empirical observation lies in the unrealistic assumptions regarding the number of players and their weights.

The different approaches considered here take into account various aspects of decision making bodies. These aspects can be split into three types:

**a)** Decision bodies where the "players" vote en bloc based on party affiliation and/or according to whether a player is a member of the government or of the oppostion, but there is uncertainty about the number of players from a party taking part in a vote, due to unforeseen circumstances.

**b)** Decision bodies where the members of a party can vote differently from the leader of the party.

**c)** Decision bodies in which the outcome of the formation of a coalition can be described by a multi-dimensional value, e.g., the value of a coalition of parties might be measured by both the total number of members and the coherence of the political views of the members of the coalition. This leads to the consideration of more general cooperative games than simple games.

The following example, which analyses the structure and number of seats possessed by each party in the Polish parliament, is used as a tool to introduce more realistic assumptions which take these aspects into consideration.

| Party or parliamentary club | Immediately after the election | At the end of 2017 |
|---|---|---|
| PiS | 235 | 237 |
| PO | 138 | 136 |
| Kukiz 15 | 42 | 30 |
| N | 28 | 26 |
| PSL | 16 | 15 |
| MN | 1 | 1 |
| WiS | - | 6 |
| ED | - | 4 |
| Non-affiliated | - | 5 |
| Total | 460 | 460 |
| Rae's F index | 0.6358 | 0.6383 |
| Coleman's index of collective action | 0.5 | 0.5 |

Source: http://www.sejm.gov.pl

**TABLE 22.1**: The Polish parliament in the 2015-2019 term of office. (PiS - Law and Justice, PO - Citizens' Platform, Kukiz 15, N - Modern, PSL - Polish Peasants' Party, MN - German Minority, WiS - Freedom and Solidarity, ED - European Democrats).

**Example 22.1** *Politicians moving between parties and changes in the number of parties.*

*In the everyday life of various decision bodies, not only do the weights (number of seats) of parties change, but the number of players (parties) can change. We wish to see how this affects the Shapley value. We start by analysing the present situation in the Polish Sejm (parliament), which is described in Table 22.1.*

The value of Coleman's index of collective action[4], given by $C^c = \frac{\omega}{2^n}$, where $\omega$ denotes the total number of winning coalitions and $n$ denotes the number of voters, is 0.5 immediately after the election and at the end of 2017.

Rae's index of fractionalisation[5] (at the beginning of the present Sejm and end of 2017 equal to 0.6358 and 0.6383, respectively), which summarises infor-

[4]Calculated using POWERSLAVE Mark I, available from http://powerslave.utu.fi/index.html

[5]Rae's fractionalisation index is nothing else than the complement of the Herfindahl-Hirschman concentration index (HH) (https://www.justice.gov/atr/herfindahl-hirschman-index) known in economics as a measure of the size of firms in relation to their industrial sector as a whole and an indicator of the amount of competition among them. This concentration index is calculated as follows: $HH = \sum_{i=1}^{n} s_i^2$ or the normalised HH index $HH^* = \frac{HH-1/n}{1-1/n}$ for $n > 1$. The Effective Number of Parties, $ENP = \frac{1}{\sum_{i=1}^{n} s_i^2} = 1/(1-F)$, (Laakso and Taagepera, 1979) is also used, but in our opinion it is less intuitive than Rae's F-index.

mation about the number of parties and their relative size, can be computed using the formula:

$$F = 1 - \sum_{i=1}^{n} s_i^2, \tag{22.8}$$

where $s_i$ is the proportion of parliamentary seats held by party $i$. The closer the value of the Rae index to one (its maximum value), the more fractionalized the system is. It can be seen that according to the F index, fractionalisation has only marginally increased since the beginning of this parliament's term, despite the number of parties increasing from 6 to 9. From the point of view of the Shapley value, one party (PiS) holds a majority of the seats in parliament[6]. Since passing a motion requires a standard majority of the votes given, PiS can form the government on its own. In such a case, the Shapley value for this party is equal to 1 and the measure of the ability of the parliament to implement a motion (index of collective action) does not depend on how fragmented the rest of the parliament is (see Example 22.1). In practice, this would mean that it does not matter whether the opposition parties unite their forces or act independently. The *a priori* Shapley value does not differentiate between any such division of seats in parliament.

Reality is somewhat different when we consider the dynamics of a specific decision body:

**a)** Since the last general election, 12 new members from various parties have entered parliament due to seats becoming vacant. The degree to which these members follow party lines is an important question.

**b)** The rate of attendance during votes is rather low. Consequently, the quota $q$ has varied.

**c)** A dozen or so members of parliament have changed their party affiliation.

**d)** The strength of party discipline ("following the leader") varies across parties.

It seems that the level of party discipline could be particularly important in the case of a party that has more 50% of the seats in parliament. It seems reasonable to assume that it is rational for such a party to try and increase the size of its majority (especially taking into account the possibility of losing an absolute majority due to splits or a loss of seats), which is a form of insurance against "disloyalty" amongst party members. This will be modelled using the concept of fuzzy numbers.

[6] As Taagepera (1999: 502) noted, "once a party has more than 50%, how much does it matter whether it has 53% or 57%?" Intuitively, in such a situation the fragmentation index should probably be defined as F=0. Obviously, one should also consider that there also exists a "constitutional" majority (i.e., any changes to the Polish constitution require at least 2/3 of the votes). However, Shapley values need to be completely recalculated when the majority required changes (this holds for any quota).

For simplicity, we assume that the number of seats held by each party participating in the current parliament (2015-2019) are fixed values and the parties are split into two groups: PiS and "the opposition"[7]. Since PiS has more than 50% of the seats, its *a priori* Shapley value is equal to 1. This is always the situation when one of the players has a greater weight than the rest of the players taken together. When the weights of the players are fixed, then there is no sense in analysing the Shapley-Shubik power index. The situation is different when the weights of the players can be described by fuzzy variables, e.g., it is not assumed that all the members of a particular party behave in the same way. Assume (using the data regarding the current Polish parliament) that the weights (number of seats) of PiS at the beginning of its term of office and at the end of 2017 are given by the triangular fuzzy numbers $\tilde{A}(PiS) = (235, 4, 0)$ and $\tilde{A}(PiS) = (237, 6, 0)$, respectively[8]. This means that the number of voters from PiS:

**a)** immediately after the election is between 231 and 235, such that the most likely value is 235,

**b)** at the end of 2017 is between 231 and 237, such that the most likely value is 237.

A fuzzy number whose membership function has such a shape (a right-angle triangle) corresponds to a party which is still strongly unified, but allows some departures from party discipline. In intuitive terms, such a number can be said to exhibit left-sided fuzziness, i.e., the realization of such a fuzzy number cannot be greater than (be to the right of) the most likely value. In mathematical terms, this corresponds to a membership function of the form given by Equation (22.2) which satisfies $R(x) \equiv 0$ and $L(x)$ is not identically equal to zero. Right-sided fuzziness can be defined in an analogous manner.

It is much more difficult to define a fuzzy number that might describe the weight of the opposition, since the level of cohesion within the opposition (initially 5 parties, now 8) is much lower than within a single party. For convenience, assume that the weights of the opposition immediately after the election and at the end of 2017 are described by the numbers $\tilde{A}(rest) = (225, 8, 0)$ and $\tilde{A}(rest) = (223, 12, 0)$, respectively. We define the value of the characteristic function for this simple game as follows (see Gladysz and Mercik 2018):

$$v(T) = Pos(\tilde{A}(T) \geq q\tilde{A}_0 + \epsilon), \tag{22.9}$$

where $\tilde{A}_0$ is the number of valid votes cast, $q$ defines what proportion of the votes describes a winning majority. Assume that a standard majority is required to form a winning coalition, i.e., $q = 1/2$ and $\epsilon = 1$. Hence, the quota may be estimated using the fuzzy number $q\tilde{A}_0 + \epsilon = (460, 12, 0) + 1 = (231, 6, 0)$

[7] This is a particular case of a two-party system, which, in practice, does not exist anywhere in Europe.

[8] This is a very conservative assumption that the ruling party never loses its majority. The Polish parliament has 460 seats.

| Immediately after the election | | | At the end of 2017 | | |
|---|---|---|---|---|---|
| Coalition $T$ | $\tilde{A}(T)$ | $v(T)$ | Coalition $T$ | $\tilde{A}(T)$ | $v(T)$ |
| {PiS} | (235, 4, 0) | 1 | {PiS} | (237, 6, 0) | 1 |
| {rest} | (225, 8, 0) | 0 | {rest} | (223, 12, 0) | 0 |
| {PiS, rest} | (460, 12, 0) | 1 | {PiS, rest} | (460, 18, 0) | 1 |

Source: authors' own calculations.

**TABLE 22.2**: Values of the characteristic function for the game $(\tilde{N}, \tilde{v})$: PiS vs. rest (left-sided fuzziness).

| Immediately after the election | | | At the end of 2017 | | |
|---|---|---|---|---|---|
| Coalition $T$ | $\tilde{A}(T)$ | $v(T)$ | Coalition $T$ | $\tilde{A}(T)$ | $v(T)$ |
| {PiS} | (231, 0, 4) | 1 | {PiS} | (219, 0, 18) | 1 |
| {rest} | (217, 0, 8) | 0 | {rest} | (211, 0, 12) | 0.59 |
| {PiS, rest} | (448, 0, 12) | 1 | {PiS, rest} | (430, 0, 30) | 1 |

Source: Authors' own calculations. The characteristic functions were derived on the basis of Equations (22.5) and (22.9). These are real-valued functions. The Shapley value was calculated according to the classical formula (see Shapley, 1953).

**TABLE 22.3**: Values of the characteristic function for the game $(\tilde{N}, \tilde{v})$: PiS vs. rest (right-sided fuzziness).

immediately after the election and $q\tilde{A}_0 + \epsilon = 1/2(460, 18, 0) + 1 = (231, 9, 0)$ at the end of 2017, respectively[9]. Tables 22.2 and 22.3 present the appropriate values of the characteristic function for these two coalitions immediately after the election and at the end of 2017, on the basis of the above assumptions. Table 22.2 illustrates a case where it is assumed that the largest party (which has more than 50% of the seats in parliament) can ensure the loyalty of at least 231 members in any particular vote and thus allow up to 4 (initially) or 6 (at the end of 2017) members not to follow party discipline (based on left-sided fuzziness). Note that in such a case, even if there were no fuzziness in the weight of the opposition, the power index of PiS would still be equal to one. In Table 22.3, it is assumed that the cohesion of the ruling party is decreasing in time. The right-sided fuzziness at the end of 2017 in this case corresponds to the government ensuring an absolute majority with possibility measure 1/3, i.e., 18 members do not necessarily follow party discipline. In this case, it is possible for the opposition to win a vote.

Based on the values of the characteristic functions presented in Tables 22.2 and 22.3, we obtain the following Shapley values:

[9]Empirical data from the Polish parliament indicate that there are always a number of the 460 members of parliament who do not participate in a particular vote. This means that the left-sided fuzziness of the quota should be stronger than we have assumed.

For left-sided fuzziness: Both immediately after the elections and at the end of 2017, PIS has a power index equal to 1. Hence, the opposition has power index 0 at both times.

For right-sided fuzziness: Immediately after the elections, PiS and the opposition have power indexes equal to 1 and 0, respectively. At the end of 2017, these indexes are 0.705 and 0.295, respectively.

Obviously, these results show that the government is clearly stronger than the opposition. However, it enables a quantitative assessment (not considered by Taagepera) of the effect of party discipline. When the government has a small majority, the opposition can make use of the absence or disloyalty of members of the ruling party. In everyday terms, members of the ruling party should always be present at votes and vote in line with their party leader. Other members of parliament can behave as they want, but be aware of situations where the government is weakened by absence or indiscipline, which gives them the opportunity to win a vote.

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## 22.4 Variability in the Quota $q$

In this section we consider the problem of variability in the quota $q$. Again we note that, according to the classical approach to defining the Shapley value for simple games, the quota $q$ determining whether a coalition is winning or not is defined to be a constant. This allows us to classify each of the possible coalitions $T \subseteq N$ as winning or losing. Transforming this quota into a fuzzy number radically changes the situation: Some coalitions may be both winning and losing to some degree. For example, in the vote described below by Example 22.2 (see Table 22.4) - which corresponds to a project submitted to a vote in the Polish parliament - the actual quota (the first to exceed 50% of the valid votes cast) was equal to 202. Such a value cannot be applied to an *a priori* approach to defining the Shapley value, since it could lead to situations in which there are two winning coalitions (the Polish parliament consists of 460 members, so 231 votes guarantee a majority). In this section we consider a model that takes into account that the quota $q$ may vary and as a consequence leads to an *a priori* analysis of winning coalitions that do not form a majority of the decision body as a whole. Since, in practice, minority governments exist, the authors feel that such an approach makes sense. In order to model such votes, expert opinion is required on the tendency for members of a party to vote in line with the leader of the party (often termed "party discipline") or not to take part in a vote.

**Example 22.2** *A fuzzy quota. Let us consider a vote that took place in the Polish parliament on 10/01/2018 regarding the first reading of the citizens'*

| | Government | Opposition |
|---|---|---|
| Nominal number of seats | 237 | 223 |
| Present at vote | 226 | 177 |
| Votes to reject | 166 | 36 |
| Votes to accept | 58 | 136 |
| Abstentions | 2 | 5 |
| Absent | 11 | 46 |

**TABLE 22.4**: Vote in the Polish parliament on 10/01/2018 regarding the rejection of a citizens' project for an Act on the Rights of Women and Birth Control.

| Date | 2017 |
|---|---|
| Number of votes for the government | (231,173,6) |
| Number of votes against the government | (211,175,12) |
| $Pos(\tilde{X}_{YES} > \tilde{X}_{NO})$ | 0.57 |

**TABLE 22.5**: Number of votes for parliamentary government-opposition game.

*project for an Act on the Rights of Women and Birth Control. Table 22.4 presents how the members of the government and opposition (which consisted of seven parties) voted on the project at its first reading.*

The vote itself was rather paradoxical. The project was submitted by the opposition. However, a significant proportion of the opposition voted to reject the project (36 - 16%) and a larger number of members of the opposition were absent. On the other hand, 58 (24%) of the members of the governing party voted not to reject the motion (including the leader of the government party). We will give a general analysis of the problem. Let a given legislative body $N$ be composed of $K$ disjoint subsets $N_k$ (groups, parties, parliamentary clubs, etc.), where $k = 1, 2, \ldots, K$, the $k$-th group has weight $n_k$ and $\sum_{k=1}^{K} n_k = n$. Let $k_0$, $1 \leq k_0 \leq K$, describe the number of groups forming the cabinet (for the sake of convenience, assume that it is the first $k_0$ groups in lexicographic order).

We denote by $\tilde{X}_k$, $\tilde{Y}_k$ the triangular fuzzy numbers describing the number of members of group $k$ that conform to the leader's vote (including the leader) and go against the leader's vote, respectively (other members of the group are assumed to be absent or abstain). Let $\tilde{X}_{YES}$ and $\tilde{X}_{NO}$ denote the fuzzy numbers describing the total number of votes "for" the government, and "against" the government, respectively. It follows that

$$\tilde{X}_{YES} = \sum_{k=1}^{k_0} \tilde{X}_k + \sum_{k=k_0+1}^{K} \tilde{Y}_k$$

$$\tilde{X}_{NO} = \sum_{k=k_0+1}^{K} \tilde{X}_k + \sum_{k=1}^{k_0} \tilde{Y}_k$$

A motion is passed if the number of "Yes" votes is greater that the number of "No" votes. Hence, the power index of a cabinet is defined as the following measure:

$$Pos(\tilde{X}_{YES} > \tilde{X}_{NO}).$$

By definition, this index lies in the interval $[0, 1]$ and the larger the index, the greater the power of the government.

Table 22.5 presents the fuzzy number of votes for the government and the fuzzy number of votes against the government, together with the power index of the government at the end of the year 2017 according to the current structure of the Polish parliament. The most likely value and the right spreads are the same as in Example 22.1. Moreover, it is assumed that the minimum possible number of votes for the government is 58, and for the opposition 36.

It can be observed that the power index of the government at the end of 2017 was equal to 0.89 and was slightly smaller than immediately after the election, when this power index was equal to 0.92. In other words, as time has passed, the power of the government has decreased. Moreover, such an approach enables us to derive the power of a minority government. In this case, it is necessary to derive the value of the possibility that the government obtains a larger number of votes than the opposition. One may assume that when this power index is decreasing over time, it becomes likely that the government will fall in the near future.

## 22.5 Multi-Dimensional Descriptions of the Value of a Coalition

The situation described by Shapley (1953) is pretty unambiguous: The Shapley value was designed to define the division of the profits gained on the basis of the level of a player's participation and the characteristic function was given by monetary values. However, the ease of use and universality of this approach soon led to applications in many different fields[10]. It was also assumed that

[10]For example, some of the most recent applications, out of a large group, can be found in Lobos and Mercik (2017) and Pilling et al. (2017).

various Shapley values could be obtained by changing the assumptions regarding the weights of players. For example, the Shapley-Shubik power indexes of the 16 regions of Poland (see Mercik et al., 2004) vary according to the way in which the weights of the regions are defined. For example, based on population size, the Mazowieckie region (which includes Warsaw) should have 13 out of 100 seats in the Polish senate, whereas based on GDP it should have 19 seats. This corresponds to a Shapley-Shubik power index equal to 0.1378 based on population size and 0.2144 based on GDP. Obviously, it would be favourable to the Mazowieckie region to define the number of seats according to GDP (particularly since the power index indicates that the power of the region would be large in comparison to the percentage of seats held - assuming that the standard majority voting rule is used). On the other hand, the Lubelskie region in the east would have 2 fewer seats if they were allocated on the basis of GDP rather than population size.

The decision making process of the EU council of ministers is another subject of such analysis. Many attempts have been made to assess the power of individual states based on their populations and their ability to form a winning coalition. This approach can involve a number ($k$, where $k \geq 2$) of criteria that can be based on various types of variable (categorical, ordinal or continuous), which can interact with each other in different ways. When weights and quotas are deterministic, such a voting game can be reduced to a simple voting game. This is due to the fact that the criteria for a winning coalition can be aggregated and thus the game can be defined as a standard simple voting game.

When the weights of the players are fuzzy, the aggregation of such criteria is more complex. Making an appropriate balance between the criteria in some way seems to be an appropriate approach in this case. This leads to a set of criteria being replaced by a single synthetic criterion. In the authors' opinion this should be a fuzzy criterion that takes into account the non-linearity of the Shapley-Shubik index[11].

A similar, although slightly different, problem is encountered if the value of a coalition in a general cooperative game is multi-dimensional. For example, one component of such a value might be the financial profit that a coalition can bring and the second the mutual pleasure obtained by forming such a coalition (i.e., there are two criteria for valuing a coalition). In this case, it is unclear how these criteria should be combined, e.g., what weights should the criteria be given? Example 22.3 indicates possible approaches to such problems and describes one such approach, that of defining a fuzzy criterion.

**Example 22.3** *We begin with an example describing a situation in which the value of a coalition is based on two criteria. Consider a 3-player game with*

[11]Note that in the European Union Council, decisions are undertaken on the basis of two criteria. For a given decision to be passed (after Brexit), it is necessary that 15 countries representing at least 65% of the population of the EU support it. An analysis of this voting procedure can be found in Gladysz et al. (2018).

| Coalition | Criterion 1 $v_1(\{.\})$ | Criterion 2 $v_2(\{.\})$ |
|---|---|---|
| $\{a\}$ | 1 | 12.3 |
| $\{b\}$ | 1 | 12.3 |
| $\{c\}$ | 1 | 12.5 |
| $\{a, b\}$ | 3 | 25 |
| $\{a, c\}$ | 5 | 29.7 |
| $\{b, c\}$ | 2 | 30 |
| $\{a, b, c\}$ | 10 | 40 |

**TABLE 22.6**: The characteristic function for a cooperative game based on two criteria.

| | Criterion 1 | | | Criterion 2 | | |
|---|---|---|---|---|---|---|
| Permutation | $a$ | $b$ | $c$ | $a$ | $b$ | $c$ |
| $\{a, b, c\}$ | 1 | 2 | 7 | 12.3 | 12.7 | 15 |
| $\{a, c, b\}$ | 1 | 5 | 4 | 12.3 | 10.3 | 17.4 |
| $\{b, a, c\}$ | 2 | 1 | 7 | 12.7 | 12.3 | 15 |
| $\{b, c, a\}$ | 8 | 1 | 1 | 10 | 12.3 | 17.7 |
| $\{c, a, b\}$ | 4 | 5 | 1 | 17.2 | 10.3 | 12.5 |
| $\{c, b, a\}$ | 8 | 1 | 1 | 10 | 17.5 | 12.5 |
| Shapley Value | 4 | 2.5 | 3.5 | 12.417 | 12.567 | 15.017 |
| Normalised Shapley Value | 0.40 | 0.25 | 0.35 | 0.310 | 0.314 | 0.375 |

**TABLE 22.7**: Marginal inputs of the players to the grand coalition according to two criteria.

*players $\{a, b, c\}$, in which the characteristic function is defined on the basis of two criteria (see Table 22.6). We can calculate the Shapley value based on each of the criteria individually (see Table 22.7).*

The Shapley values presented in Table 22.7 indicate that, depending on the criterion used, we obtain different relative values of the Shapley value for the players $\{a, b, c\}$. By using normalised values, we can compare different criteria. It can be seen that the power of player $\{c\}$ is little affected by the choice of criterion to be used, while player $\{b\}$ has more power under criterion 2. It is clear that the use of different criteria (weights) can lead to different normalised Shapley values. For this reason, the authors feel that there exists a need to either aggregate a set of criteria into a single synthetic criterion or to define another method for adapting a set of criteria to a particular decision-making scenario. The authors propose that this may be achieved by using a fuzzy criterion.

| Coalition | Standardised value based on criterion 1 $v_1(\{.\})$ | Standardised value based on criterion 2 $v_2(\{.\})$ | Mean value of coalition (synthetic value) | Standard deviation for coalition |
|---|---|---|---|---|
| $\{a\}$ | 0.043478 | 0.07602 | 0.059749 | 0.000529 |
| $\{b\}$ | 0.043478 | 0.07602 | 0.059749 | 0.000529 |
| $\{c\}$ | 0.043478 | 0.077256 | 0.060367 | 0.000571 |
| $\{a,b\}$ | 0.130435 | 0.154512 | 0.142473 | 0.013976 |
| $\{a,c\}$ | 0.217391 | 0.18356 | 0.200476 | 0.04018 |
| $\{b,c\}$ | 0.086957 | 0.185414 | 0.136185 | 0.016532 |
| $\{a,b,c\}$ | 0.434783 | 0.247219 | 0.341001 | 0.175795 |

**TABLE 22.8**: Standardised values of the characteristic function for Example 22.3.

In order to compare Shapley values based on a set of $k$ criteria, where $k \geq 2$, we propose that the corresponding characteristic functions (weights) be normalised according to the formula $\overline{x}_{ij} = \frac{x_{ij}}{\sum_i x_{ij}}$, where $x_{ij}$ denotes the value of the characteristic function for coalition $i$ based on the $j$-th criterion and taking the overall value of such a coalition to be the mean normalised value of that coalition over all the criteria[12], $\overline{x}_{i.} = \frac{1}{k}\sum_{j=1}^{k}\overline{x}_{ij}$. This defines the synthetic value of the $i$-th coalition based on a set of criteria. Table 22.8 presents the appropriate calculations for Example 22.3.

These operations

1) enable the comparison of coalitions according to various criteria,

2) free the calculations from “the curse of high dimensions”.

Since the evaluations of coalitions may be made according to completely different units, unless some form of normalisation is applied, then it is impossible to compare the value of coalitions according to these two evaluation criteria. Based on such a standardisation, apart from the expected value of the standardised weights of coalitions, we can also calculate the variance of the value of a coalition, $Var_{i.} = \frac{1}{k}\sum_{j=1}^{k}(\overline{x}_{ij} - \overline{x}_{i.})^2$ . Table 22.8 gives the expected values and standard deviations (the square root of the variance) for each coalition.

On the basis of these two parameters and Equations 22.6 and 22.7, we derive the characteristic function for each coalition under the assumption that each of these values is a symmetric triangular fuzzy number[13]. Table 22.9 gives the values of the characteristic function for Example 22.3.

[12]Here we use the arithmetic mean. However, it is clear that other definitions of the mean can be used depending on the context of the problem.

[13]In our analysis here we consider triangular fuzzy numbers. However, it is possible to use fuzzy numbers of any form.

| Coalition $T$ | $\tilde{v}(T)$ |
|---|---|
| $\{a\}$ | (0.06, 0.001, 0.001) |
| $\{b\}$ | (0.06, 0.001, 0.001) |
| $\{c\}$ | (0.06, 0.001, 0.001) |
| $\{a,b\}$ | (0.14, 0.024, 0.024) |
| $\{a,c\}$ | (0.20, 0.069, 0.069) |
| $\{b,c\}$ | (0.14, 0.029, 0.029) |
| $\{a,b,c\}$ | (0.34, 0.304, 0.304) |

**TABLE 22.9**: Synthetic values of the characteristic function of the game considered in Example 22.3.

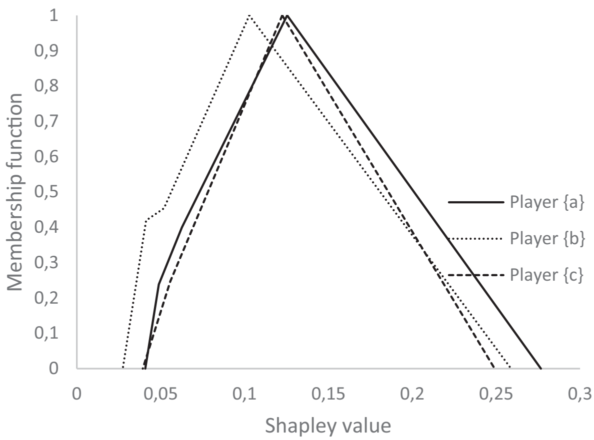


**FIGURE 22.1**: Shapley value for player $a$ - continuous line, player $b$ - dotted line, player $c$ - broken line (from Example 22.3).

Next, we can derive the marginal value of each player entering into the grand coalition $\{a, b, c\}$. It is assumed that these values are non-negative fuzzy numbers. The appropriate calculations are given in Table 22.10.

In the following step, using the Shapley value, together with Zadeh's extension principle, we determine the fuzzy Shapley values for each player in this game. Figure 22.1 illustrates the fuzzy Shapley values for the three players: $\{a\}$, $\{b\}$, $\{c\}$.

In order to compare the power indexes of the players $\{a\}$, $\{b\}$, $\{c\}$, we use the measure $Pos(\tilde{X} > \tilde{Y})$ defined by Equation 22.5. We also consider the definition of the membership function for the intersection of two fuzzy sets defined by Equation 22.1.

Pos(Shapley value of Player $\{a\}$ is the maximum Shapley value) $= 0.55$,

Pos(Shapley value of Player $\{b\}$ is the maximum Shapley value) $= 0.43$,

Pos(Shapley value of Player $\{c\}$ is the maximum Shapley value) $= 0.45$,

| | Synthetic marginal value | | |
|---|---|---|---|
| Permu-tation | a | b | c |
| $\{a, b, c\}$ | (0.06,0.001,0.001) | (0.14,0.024,0.024 )-(0.06,0.001,0.001)=$(0.08, 0.202, 0.202)_T$ | (0.34,0.304,0.304 )-(0.14,0.024,0.024)=$(0.20, 0.329, 0.329)_T$ |
| $\{a, c, b\}$ | (0.06,0.001,0.001) | (0.34,0.304,0.304)-(0.20,0.069,0.069)=$(0.20, 0.329, 0.329)_T$ | (0.20,0.069,0.069)-(0.06,0.001,0.001)=$(0.14, 0.071, 0.071)_T$ |
| $\{b, a, c\}$ | (0.14,0.024,0.024)-(0.06,0.001,0.001)=$(0.08, 0.030, 0.030)_T$ | (0.06,0.001,0.001) | (0.34,0.304,0.304)-(0.14,0.024,0.024)=$(0.20, 0.329, 0.329)_T$ |
| $\{b, c, a\}$ | (0.34,0.304,0.304)-(0.14,0.029,0.029)=$(0.20, 0.477, 0.477)_T$ | (0.06,0.001,0.001) | (0.14,0.029,0.029)-(0.06,0.001,0.001)=$(0.08, 0.030, 0.030)_T$ |
| $\{c, a, b\}$ | (0.20,0.069,0.069)-(0.06,0.001,0.001)=$(0.14, 0.071, 0.071)_T$ | (0.34,0.304,0.304 -(0.20,0.069,0.069)=$(014, 0.374, 0.374)_T$ | (0.06,0.001,0.001) |
| $\{c, b, a\}$ | (0.34,0.304,0.304)-(0.14,0.029,0.029)=$(0.20, 0.477, 0.477)_T$ | (0.14,0.029,0.029)-(0.06,0.001,0.001)=$(0.08, 0.030, 0.030)_T$ | (0.06,0.001,0.001) |

**TABLE 22.10**: Marginal values of the players entering into the grand coalition according to the synthetic criterion.

It can be seen that this measure is greatest for player $\{a\}$. The measure of the possibility that the Shapley value of player $\{a\}$ is greater than the Shapley value of the remaining players is 0.55.

## 22.6 Discrepancy between the Weight and Shapley Value of a Player

The concept of the *a priori* Shapley value is intuitive when it is applied to problems that can be precisely defined. However, in the social sciences, particularly the theory of group decision making, we often observe a huge discrepancy between the relative weights of the decision makers (players) and their Shapley values. This is particularly true in simple games that model decision making via the process of voting (see Example 22.4). In this section we consider inverse problems where desired indexes of the players are given (e.g., in the case of EU votes, it might be deemed desirable that the power of a member state is proportional to its population size) and the weights of the players in a voting procedure are to be chosen using some form of optimality criterion. Generally, the set of voting weights which minimize the distance between the desired set of power indexes and the Shapley values will not be unique. Hence, these voting weights can be thought of as fuzzy, in an intuitive sense.

**Example 22.4** *Consider the following two voting games* $(d; w) : (51; 49, 49, 2)$ *and* $(d; w) : (51; 33, 33, 34)$*, where* $d$ *denotes the number of votes required for a motion to be passed and* $w$ *denotes the vector of the weights of the players. In both games, a standard majority of the house is required in order to pass a motion, i.e., in this case* $d = 51$*. Both games involve three players (parties), but the distributions of the weights of these three players are very different. However, the Shapley value for both of these games is equal to* $(1/3, 1/3, 1/3)$*. It is relatively simple to show that in such a majority voting game whenever the weights are integers that sum to 100 and the largest weight is less than 50, then the Shapley-Shubik power index of each of the players is equal to 1/3.*

Brückner and Peters (1996) and Mercik (1999) consider the concept of the appropriate distribution of power. It is assumed that power should be proportional to voting weight. In this context, the balance of power in the game (51; 33, 33, 34) is more appropriate than the balance of power in the game (51; 49, 49, 2). Asymptotic results indicate that these discrepancies tend to be very small when there are a large number of players (see Alon and Edelman, 2010). However, in many cases the number of players (or parties) tends to be small.

To analyse the discrepancies between voting weights and voting powers, the voting weights are standardised so that they sum to one by dividing each weight by the sum of the initial weights. We state that the balance of power is correct to degree $(\epsilon^L, \epsilon^R)$ in terms of a decision body, if for each player $i \in N$, the relative weight of player $i$, $w_i$, satisfies

$$\epsilon^L \leq |w_i - \phi_i(d; w)| \leq \epsilon^R,$$

where $\phi_i(d; w)$ denotes the Shapley value (power index) in the voting game $(d; w)$.

In an ideal situation, i.e., one where the relative values of the voting weights coincide with the Shapley values, $\epsilon^L = \epsilon^R = 0$. In all other situations, we have

$$\epsilon^L = \min_{i \in N} |w_i - \phi_i(d; w)| \text{ and } \epsilon^R = \max_{i \in N} |w_i - \phi_i(d; w)|.$$

Such a fuzzy approach to the discrepancy between the voting weights and powers of various players also enables us to assess whether the assigned weights are appropriate to the power of a player. With this as a goal, we define the discrepancy between a standardised vector of voting weights $w$ and a vector of power indices $p$ according to the $L_k$ norm, as

$$c_k(w, p) = \left[ \sum_{i=1}^{n} |w_i - p_i|^k \right]^{1/k}.$$

Such measures of discrepancy can be applied to, e.g., voting in the EU Council. The weights of the individual states can be based on their population; GDP of each country can be given an equal weight. The power of each state results from a given voting procedure (see Example 22.5).

On the basis of Example 22.4, we can see that based on a given Shapley-Shubik value it is possible to define a set of fuzzy weights, i.e., there is a large set of triples of weights that give the same Shapley-Shubik value. Moreover, in practice, since there is no ideal correspondence between the relative weights of the players and the Shapley-Shubik value, we aim for a balance of power that is approximately correct.

One might consider an inverse problem, which for a given Shapley-Shubik value derives the weights that achieve a given discrepancy from a given vector of power indexes, while maximising the distance from a vector of voting weights.

Table 22.11 gives vectors that maximise the Euclidean distance from (49, 49, 2) while constraining the discrepancy, $c_2(w, p)$, to be less than or equal to some required value and ensuring a given vector of power indexes. In the first two cases, the constraint on the discrepancy becomes binding before the Shapley-Shubik value changes. In the third case, the constraint on the Shapley-Shubik value becomes binding, since increasing the weight of Party 3 means that it has the power to block any motion. When the weight of Party 3 is strictly between 49 and 51, then it has the power to block any motion, but

| $\epsilon_2$ | $w_1$ | $w_2$ | $w_3$ | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|---|---|---|
| 0.005 | 33.129 | 33.129 | 33.742 | 0.33333 | 0.33333 | 0.33333 |
| 0.05 | 31.293 | 31.292 | 37.416 | 0.33333 | 0.33333 | 0.33333 |
| 0.2 | 25.494 | 25.506 | 49.000 | 0.33333 | 0.33333 | 0.33333 |
| 0.2 | 24.500 | 24.500 | 50.99999 | 0.16667 | 0.16667 | 0.66667 |

**TABLE 22.11**: Numerical results for maximising the Euclidean distance of a weight vector from (49, 49, 2) while constraining the discrepancy between the voting weights and the given voting powers (measured according to the $L_2$ norm) to be $\leq \epsilon_2$ (see Example 22.4).

cannot form a majority on its own. In this case, the Shapley-Shubik vector is (1/6, 1/6, 2/3), since Party 3 turns a coalition into a winning coalition if and only if it is not the first to join. Increasing the voting weight of Party 3 to almost 51, while evenly decreasing the weights of the other parties will ensure such a Shapley-Shubik vector and satisfy the constraint on the discrepancy. Since a wide range of weight vectors correspond to the same Shapley-Shubik value, in this sense we can interpret the weights of parties to be fuzzy variables.

It should be noted that this index of discrepancy is to some degree dependent on the number of players in a game. For example, when the $L_1$ norm is used to calculate the discrepancy, i.e., $c_1(w,p) = \sum_{i=1}^{n} |w_i - p_i|$, it is possible to standardise this index by calculating the mean discrepancy $\overline{c}(w,p) = \frac{c_1(w,p)}{n}$ or calculate the mean relative discrepancy

$$c_r(w,p) = \sum_{i=1}^{n} \frac{|w_i - p_i|}{n w_i}.$$

Such standardised indexes enable us to compare various decision bodies and voting rules independently of the number of players and the methods of ascribing weights to the players. It should be noted that such measures of discrepancy are equal to zero if and only if the standardised vector of weights corresponds exactly to the Shapley-Shubik value (e.g., if the weights of the member states of the EU council are proportional to their population size, then for the discrepancy to be equal to zero, the power index of each state must be proportional to its population). The greater the deviances between the weights and power indexes of the players, the larger such a discrepancy measure is. Hence, we can say that larger standardised measures of discrepancy correspond to situations in which there is a greater degree of disequilibrium between the weights of the players and their influence in a decision making procedure.

**Example 22.5** *Discrepancy between the weights (according to chosen parameters) and the power of EU members in the European Union Council before and after Brexit.*

In recent times, there have been a number of articles (e.g., Mercik and Ramsey, 2017) on the effect on the Shapley-Shubik power indexes of the EU countries. Here, we consider this problem from the point of view of the fuzziness of weights, in particular analysing the discrepancy between the voting weights of individual countries and their power indexes.

This analysis is based on the rules introduced by the Lisbon treaty, in order to reflect possible changes in the membership of the EU and the populations of individual states. This is described by Koczy (2012) and http://www.consilium.europa.eu/en/council-eu/voting-system/. In order for a vote to be passed, the following two conditions must be satisfied:

**1.** 55% of the member states must be in favour (i.e., before Brexit 16 of 28 states, after Brexit 15 of 27 states). It should be noted that in special cases 72% of the states must be in favour, but this variant is not considered here.

**2.** The states voting for a proposition should represent at least 65% of the EU population, with the additional condition that to block a proposition, any coalition representing at least 35% of the population must contain at least four states.

Table 22.12 presents the values of the power indexes of the EU countries both before and after Brexit, together with their populations and GDP measured in both Euros and in terms of purchasing power. Table 22.13 presents the same data after normalisation, so that the observations of each variable sum to 100, which enables comparison.

Table 22.14 presents the standardised values of the discrepancy between the weights of countries according to various parameters and their Shapley-Shubik index according to the EU council voting procedure. It can be seen that:

**1.** The discrepancy based on the assumption that each has equal weight shows a slight increase after Brexit.

**2.** According to the other three definitions of weights (one based on population size and the others on the size of the economy), Brexit leads to a fall in the standardised discrepancy between the weights of the players and their power indexes. This analysis suggests that Brexit should improve the cohesion of the EU council and the fuzziness of the weights ascribed to the players is reduced. However, this might be a premature conclusion due to the dynamics between the member states.

**Example 22.6** *The case of the unification of Germany - a forgotten EU enlargement.*

Let us analyse now the consequences of Germany's reunification. In 1990 the population of Germany increased by 17 million. Standard regression analysis (Mercik et al., 2004) shows that taking as a starting point the distribution

| | | GDP | | Before Brexit | | After Brexit | |
|---|---|---|---|---|---|---|---|
| Country | Country per se | mill. Euro | mill. PPS | % pop. | S-S | % pop. | S-S |
| Germany | 1 | 3025900 | 2932714 | 15.96 | 14.43 | 18.29 | 17.32 |
| France | 1 | 2183631 | 2020155 | 13.06 | 11.25 | 14.97 | 13.28 |
| UK | 1 | 2568941 | 2051389 | 12.74 | 10.93 | - | - |
| Italy | 1 | 1636372 | 1663314 | 11.96 | 10.17 | 13.71 | 12.03 |
| Spain | 1 | 1081190 | 1221366 | 9.14 | 7.53 | 10.47 | 9.00 |
| Poland | 1 | 427737 | 756850 | 7.48 | 6.32 | 8.57 | 6.99 |
| Romania | 1 | 160353 | 323486 | 3.91 | 3.75 | 4.48 | 4.00 |
| Neth. | 1 | 678572 | 625463 | 3.33 | 3.28 | 2.82 | 3.52 |
| Belgium | 1 | 409407 | 378359 | 2.22 | 2.42 | 2.54 | 2.60 |
| Greece | 1 | 176023 | 220074 | 2.13 | 2.36 | 2.44 | 2.53 |
| Czech Rep. | 1 | 163948 | 258653 | 2.07 | 2.31 | 2.37 | 2.47 |
| Portugal | 1 | 179379 | 230404 | 2.04 | 2.29 | 2.34 | 2.45 |
| Hungary | 1 | 108748 | 191925 | 1.94 | 2.21 | 2.22 | 2.37 |
| Sweden | 1 | 444617 | 346946 | 1.92 | 2.20 | 2.20 | 2.35 |
| Austria | 1 | 337286 | 313841 | 1.69 | 2.03 | 1.94 | 2.17 |
| Bulgaria | 1 | 44162 | 95794 | 1.42 | 1.83 | 1.63 | 1.94 |
| Denmark | 1 | 266240 | 201726 | 1.11 | 1.60 | 1.27 | 1.68 |
| Finland | 1 | 207220 | 169621 | 1.08 | 1.58 | 1.24 | 1.66 |
| Slovakia | 1 | 78071 | 119367 | 1.07 | 1.58 | 1.23 | 1.65 |
| Ireland | 1 | 214623 | 193292 | 0.91 | 1.46 | 1.04 | 1.52 |
| Croatia | 1 | 43897 | 70441 | 0.83 | 1.40 | 0.95 | 1.45 |
| Lithuania | 1 | 37124 | 61390 | 0.57 | 1.21 | 0.65 | 1.24 |
| Slovenia | 1 | 38543 | 48948 | 0.41 | 1.10 | 0.47 | 1.11 |
| Latvia | 1 | 24378 | 36617 | 0.39 | 1.09 | 0.45 | 1.09 |
| Estonia | 1 | 20461 | 28091 | 0.26 | 1.00 | 0.30 | 0.98 |
| Cyprus | 1 | 17421 | 19711 | 0.17 | 0.93 | 0.19 | 0.91 |
| Lux. | 1 | 52113 | 44351 | 0.11 | 0.89 | 0.13 | 0.86 |
| Malta | 1 | 8797 | 10969 | 0.08 | 0.87 | 0.09 | 0.83 |

Percentages of the entire EU population are calculated based on data from 1st Jan., 2015 according to Eurostat (http://ec.europa.eu/eurostat/data/database). PPS - purchasing power standard.

**TABLE 22.12**: Chosen parameters describing EU members before and after Brexit (the countries are ordered according to decreasing population, SS - Shapley-Shubik index scaled to sum to 100).

| | GDP | | Before Brexit | | After Brexit | |
|---|---|---|---|---|---|---|
| Country | Euro | PPS | % pop. | S-S | % pop. | S-S |
| Germany | 20.68 | 20.04 | 15.96 | 14.43 | 18.29 | 17.32 |
| France | 14.92 | 13.80 | 13.06 | 11.25 | 14.97 | 13.28 |
| UK | 17.55 | 14.02 | 12.74 | 10.93 | - | - |
| Italy | 11.18 | 11.37 | 11.96 | 10.17 | 13.71 | 12.03 |
| Spain | 7.39 | 8.35 | 9.14 | 7.53 | 10.47 | 9.00 |
| Poland | 2.92 | 5.17 | 7.48 | 6.32 | 8.57 | 6.99 |
| Romania | 1.10 | 2.21 | 3.91 | 3.75 | 4.48 | 4.00 |
| Neth. | 4.64 | 4.27 | 3.33 | 3.28 | 2.82 | 3.52 |
| Belgium | 2.80 | 2.59 | 2.22 | 2.42 | 2.54 | 2.60 |
| Greece | 1.20 | 1.50 | 2.13 | 2.36 | 2.44 | 2.53 |
| Czech Rep. | 1.12 | 1.77 | 2.07 | 2.31 | 2.37 | 2.47 |
| Portugal | 1.23 | 1.57 | 2.04 | 2.29 | 2.34 | 2.45 |
| Hungary | 0.74 | 1.31 | 1.94 | 2.21 | 2.22 | 2.37 |
| Sweden | 3.04 | 2.37 | 1.92 | 2.20 | 2.20 | 2.35 |
| Austria | 2.30 | 2.14 | 1.69 | 2.03 | 1.94 | 2.17 |
| Bulgaria | 0.30 | 0.65 | 1.42 | 1.83 | 1.63 | 1.94 |
| Denmark | 1.82 | 1.38 | 1.11 | 1.60 | 1.27 | 1.68 |
| Finland | 1.42 | 1.16 | 1.08 | 1.58 | 1.24 | 1.66 |
| Slovakia | 0.53 | 0.82 | 1.07 | 1.58 | 1.23 | 1.65 |
| Ireland | 1.47 | 1.32 | 0.91 | 1.46 | 1.04 | 1.52 |
| Croatia | 0.30 | 0.48 | 0.83 | 1.40 | 0.95 | 1.45 |
| Lithuania | 0.25 | 0.42 | 0.57 | 1.21 | 0.65 | 1.24 |
| Slovenia | 0.26 | 0.33 | 0.41 | 1.10 | 0.47 | 1.11 |
| Latvia | 0.17 | 0.25 | 0.39 | 1.09 | 0.45 | 1.09 |
| Estonia | 0.14 | 0.19 | 0.26 | 1.00 | 0.30 | 0.98 |
| Cyprus | 0.12 | 0.13 | 0.17 | 0.93 | 0.19 | 0.91 |
| Lux. | 0.36 | 0.30 | 0.11 | 0.89 | 0.13 | 0.86 |
| Malta | 0.06 | 0.07 | 0.08 | 0.87 | 0.09 | 0.83 |

**TABLE 22.13**: Chosen normalised parameters (scaled to sum to 100) describing EU members before and after Brexit (S-S: Shapley-Shubik index).

| | $\overline{c}(w,p)$ before Brexit | $\overline{c}(w,p)$ after Brexit |
|---|---|---|
| GDP (Euros) | 0.014739 | 0.011900 |
| GDP (PPS) | 0.010512 | 0.008177 |
| Equal weight | 0.028119 | 0.029906 |
| % pop. | 0.007095 | 0.006071 |

**TABLE 22.14**: Standardised values of the index of discrepancy (mean difference) between voting weights and power indexes for Example 1.5 (before and after Brexit).

of weights from 1986, one can estimate the number of seats a state possesses in the EU Council of Ministers as:

$$seats = 1.17026 + 1.00513\sqrt{population},$$

where population is measured in millions. Prior to unification, the EU council had 76 seats (the distribution is presented in Table 22.15 and 54 votes were required for a motion to be passed (at least 70% of the votes). This formula gives East Germany (the DDR) 5 votes. The result of reunification could be interpreted as a pre-coalition of West Germany and the DDR where the DDR is the entering country. Table 22.15 presents three different scenarios[14]:

1. German unification without any correction of the number of seats in the EU Council of Ministers.

2. Treating the DDR as a single state that is entering the EU and thus giving it five seats in the EU Council according to its population.

3. Treating the unified Germany as a single country, which has the seats given to both West Germany and the DDR.

The values of the power index for each of the three scenarios above lead to the following conclusions regarding the measure of mean discrepancy between weights based on population and voting power:

1. Keeping the status quo (scenario 1) corresponds to a standardised measure of discrepancy equal to 0.0038.

2. Treating the DDR entering the EU as a country in itself (scenario 2) corresponds to a standardised measure of discrepancy equal to 0.0037.

[14]Probably Germany paid a price for acceptance of its reunification by not asking for a new distribution of weights. An earlier agreement between De Gaulle and Adenauer at the beginning of the 1950s guaranteed that even incorporation of DDR should not change the number of seats held by Germany. This new distribution of weights evidently changes the power of individual countries. This was the origin of the problems with the distribution of weights in each successive enlargement: Underestimation of the weight of Germany.

| Country | Seats 1986 | S-S (×100) | Seats 1990 DDR separate | S-S (×100) | Seats 1990 Unified Germany | S-S (×100) |
|---|---|---|---|---|---|---|
| (West) Germany | 10 | 13.42 | 10 | 12.92 | 15 | 21.03 |
| Italy | 10 | 13.42 | 10 | 12.92 | 10 | 12.78 |
| France | 10 | 13.42 | 10 | 12.92 | 10 | 12.78 |
| the Netherlands | 5 | 6.37 | 5 | 6.05 | 5 | 5.75 |
| Belgium | 5 | 6.37 | 5 | 6.05 | 5 | 5.75 |
| Luxembourg | 2 | 1.18 | 2 | 2.61 | 2 | 2.51 |
| UK | 10 | 13.42 | 10 | 12.92 | 10 | 12.78 |
| Denmark | 3 | 4.26 | 3 | 3.07 | 3 | 3.00 |
| Ireland | 3 | 4.26 | 3 | 3.07 | 3 | 3.00 |
| Greece | 5 | 6.37 | 5 | 6.05 | 5 | 5.75 |
| Spain | 8 | 11.13 | 8 | 9.34 | 8 | 9.13 |
| Portugal | 5 | 6.37 | 5 | 6.05 | 5 | 5.75 |
| DDR | - | - | 5 | 6.05 | - | - |
| Total no. of seats | 76 | | 81 | | 81 | |

Source: Authors' calculations.

**TABLE 22.15**: Hypothetical corrected distribution of weights for the enlargement of the EU in 1990 resulting from the unification of Germany (these weights were only applied in practice in 1986).

**3.** Giving the reunified Germany the same number of seats as the sum of the number of seats of the two component states (scenario 3) gives a measure of mean discrepancy equal to 0.0064.

It is clear that giving Germany the sum of the number of seats corresponding to its component parts would radically increase the discrepancy between population sizes and power indexes. The regression analysis indicates that the number of seats is approximately proportional to the square root of population size. Hence, a more reasonable approach in this case would have been to redefine the number of seats held by Germany according to this analysis (this would lead to an increase in the number of seats held by Germany to 11).

## 22.7 Conclusions

The Shapley value is undoubtedly one of the most effective tools for analysing group decisions. It enables us to define the expected payoff, position and power of a player according to a given procedure for group decision making. In particular, analysis of the Shapley value is the basic tool to use when the analysis is carried out *a priori*, i.e., when we may assume that the players involved can form any possible coalition and each coalition is as likely as any other. In this sense, analysis based on the classical Shapley value answers the question "What can happen?", but does not answer the question "What will happen?". This leads to many situations where our intuition and observations of the results of real decision procedures are in stark conflict with such an analysis. For example, when we treat parliamentary decisions as a game played between the government and the opposition, then the Shapley-Shubik power index of a minority government is by definition equal to zero. In practice, such governments have held power for significant periods of time. It is thus clear that in practice the power index of such a government should be greater than zero.

This article has illustrated various applications of a fuzzy approach to adapting the use of the Shapley value for group decision making. The most commonly used approach to such problems in practice is to use a voting procedure. Three distinct elements need to be defined: The weights of the players (or votes), the quota (or quotas) and the way in which votes are aggregated. In some cases, multiple criteria must be satisfied before a decision is accepted, or in a similar problem, coalitions may be valued according to different criteria. In the second case, the value of a coalition may be treated as a vector.

According to the classical approach to simple voting, the weights of player (interpreted here as parties) are deterministic. However, the model describes human behaviour and such behaviour cannot in many scenarios be foreseen. In votes, the classical quota most commonly is related to nominal values (absolute majority of votes) corresponding to a given decision body. However, we know that in practice it suffices that the number of votes for a motion simply needs to exceed the number of votes against. Hence, due to absence and abstaining, the empirically observed quota is less than the nominal quota (this is practically always the case in the Polish parliament). The empirical quota (a majority of the valid votes) is applied to decide the result of the vote. This fact should be taken into account when assessing the power of individual decision makers, which can be done using a fuzzy approach. This article has shown how this can be achieved, for example by defining the weights of the players as fuzzy numbers.

According to the classical approach, any party that holds a majority of the seats is assumed to have all the power, independently of how large the majority is. It is thus clear that the *a priori* approach to deriving the Shapley-Shubik power indexes of parties often leads to clear dissonance between one's intuition and the results of such an analysis.

Similarly, by using the *a priori* approach to calculating the Shapley-Shubik power index, we are applying a method of solving a simplified (one-dimensional) decision problem. Often the dissonance between intuition and the results obtained by using such an analysis can be explained by factors that have not been taken into account in the model. This is not related to the fact that sometimes multiple criteria need to be satisfied in order to accept a motion (e.g., in the EU council a minimum number of countries representing a minimum percentage of the EU population are required to pass a motion), but that there exist other criteria for approving a decision which are not explicit. In such cases, we need to find a healthy compromise between empirical decision processes and the rigour inherent in the definition of the Shapley value, including the method of aggregating votes (the method of voting). Again, the authors feel that such a fuzzy approach is an appropriate compromise.

At the end, it is necessary to say a couple of words about the range of applications of such an approach. This article has presented several examples to which the Shapley value has been applied and how to utilise a fuzzy approach in these situations. Obviously, these examples do not give a comprehensive range of problems in which a fuzzy approach seems appropriate. The goal of this article was to show that such an approach is appropriate in a range of situations, particularly those in which the behaviour of decision makers is unpredictable. The authors have argued that in such cases a fuzzy approach is more appropriate than the classical *a priori* approach and could serve to analyse a wide range of practical examples. The authors intend to formalise the approaches presented here in future work.

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# Chapter 23

## Shapley Values for Two-Sided Assignment Markets

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## 23.1 Introduction

The aim of this chapter is to bring together two of the main contributions of Lloyd Shapley to the theory of games. One of them is the Shapley value, which is the objective of the entire book, and as the reader already knows is a single-valued solution for coalitional games. The second one is the study of cooperative two-sided markets. In these markets there are two finite and disjoint sets of agents (men and women, students and colleges, resident doctors and hospitals, buyers and sellers) and each agent wants to be matched with one, and only one, agent on the opposite side. These markets can be of two types, depending on whether money is involved in the transaction. Gale and Shapley [6] introduce the marriage problem and the college admission problem, where each agent has a preference list on the agents on the opposite side, and defines the notion of stable matching. Then, the deferred acceptance algorithm is proposed to obtain the stable matching between men and women (or students and colleges) that is optimal for one side of the market.

Some years later, Shapley and Shubik [27] introduced the assignment game, as a model for a two-sided market with buyers on one side and sellers on the other side (take for instance a real state market) where, different from the marriage model, utility can be fully transferred by means of money. Each seller has one object on sale and each buyer wants to buy one object. The transaction of buyer $i$ with seller $j$ is valued at $a_{ij} \geq 0$ and this value is shared between the two agents by means of the price $p$ paid. Hence, the buyer's payoff is $a_{ij} - p$ and the seller's payoff is $p$. From this market situation [27] defines a coalitional game and proves the core is non-empty and, similar to the marriage model, it has a lattice structure with an optimal core allocation for each sector.

After Shapley and Shubik [27], most of the work done on the assignment game has focused on the core and other solution concepts that are a core selection. An expression of the two-optimal core allocations, each one optimal for one side of the market, is given in [5] and [12]; the extreme core allocations are analyzed in [1], [10], [18] and [22]; the fair division point, which is the midpoint of the segment determined by the two optimal core allocations is studied in [30] and [16]; algorithms for the computation of the nucleolus and an axiomatic characterization of this solution are given in [28], [13] and [11]; the stability of the core and the existence of von Neumann-Morgenstern stable sets for the assignment game are studied in [29] and [21].

However, little is known about the Shapley value for assignment games. The reason is twofold. On one side, the Shapley value of an assignment game may not lie in the core, meaning that it may be blocked by some coalition of agents that, by acting on its own, can make each of its members better off. In addition to that, the computation of the Shapley value of an assignment game requires the worth of all coalitions and this implies solving a combinatorial optimization problem for each coalition. Compared to that, the core and most of the aforementioned solutions that are a core selection can be obtained just from the valuation matrix that gathers the worth of each buyer-seller pair.

Nevertheless, the Shapley value has the advantage that each agent is rewarded according to his/her importance or influence in the market situation. Take for instance an assignment market with two buyers and one seller. Assume that the first buyer values the object of the seller higher than the second buyer does. Then an optimal matching will match the first buyer with the seller and leave the second buyer unmatched. In any core allocation, the first buyer will pay for the object a price in between his own valuation and the valuation of the second buyer, while this second buyer will get a zero payoff. As a consequence, in any solution that is a core selection, the second buyer receives nothing although his/her presence in the market has an influence on what the other two agents get. If this second buyer were not present, the price paid by the first buyer could go down to zero and remain in the core. Instead, the Shapley value will give this second buyer a positive payoff, since this buyer has a positive marginal contribution in the order in which he/she enters immediatley after the seller.

In this chapter we will first survey what is known about the Shapley value of the assignment game: Conditions on the valuation matrix that guarantee that the Shapley value remains in the core and an axiomatic characterization of this solution in this class of market games. In the second part of the chapter, we assign to each assignment game a single-valued solution that always selects a core element. This solution is the Shapley value of a related assignment game: The only assignment game with reservation values that is exact and has the same core as the initial market.

## 23.2 The Shapley and Shubik Assignment Game

Let $M$ be a finite set of $m$ buyers and $M'$ a finite set of $m'$ sellers, these sets being disjoint. Each buyer $i \in M$ is willing to buy at most one house and each seller $j \in M'$ has exactly one house on sale. Assume $h_{ij} \geq 0$ is how much buyer $i \in M$ values the house of seller $j \in M'$ and $c_j \geq 0$ is how much seller $j \in M'$ values his own house, meaning he will not sell his house for a lower price. Then, whenever $h_{ij} \geq c_j$, there is room to agree on some price $h_{ij} \geq p \geq c_j$ and the joint profit of this trade is $(h_{ij} - p) + (p - c_j)$. As a consequence, we consider the matrix $A = (a_{ij})_{(i,j) \in M \times M'}$ where $a_{ij} = \max\{h_{ij} - c_j, 0\}$ for all $i \in M, j \in M'$. We will refer to matrix $A$ as the *valuation matrix*. The *assignment market* is defined by the triple $(M, M', A)$.

A matching $\mu$ between buyers in $M$ and sellers in $M'$ is a bijection between a subset of $M$ and a subset of $M'$ such that the cardinality of $\mu(M)$ is $\min\{m, m'\}$. We write $(i,j) \in \mu$, or equivalently $\mu(i) = j$ or $\mu^{-1}(j) = i$. We say $i \in M$ is unmatched if $i \notin \mu^{-1}(M')$ and $j \in M'$ is unmatched if $j \notin \mu(M)$. Moreover, we denote by $\mathcal{M}(M, M')$ the set of matchings between $M$ and $M'$, and by $\mathcal{M}_A(M, M')$ those which are optimal with respect to a given valuation matrix $A$: $\mu \in \mathcal{M}_A(M, M')$ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$.

The first rigorous presentation of this two-sided market as a coalitional game, the *assignment game*, is in Shapley and Shubik [27]. The worth of each coalition $S \subseteq M \cup M'$ is defined by

$$w_A(S) = \max_{\mu \in \mathcal{M}(S \cap M, S \cap M')} \sum_{(i,j)\in\mu} a_{ij}.$$

An important property of this coalitional function is that players on the same side of the market appear as *substitutes* in coalitional terms, while agents on opposite sides appear as *complements* (see [25]): Given two agents $i, j$ not contained in $S \subseteq M \cup M'$,

$$w_A(S \cup \{i,j\}) - w_A(S \cup \{i\}) \leq w_A(S \cup \{j\}) - w_A(S) \text{ if } i,j \in M \text{ or } i,j \in M',$$
$$w_A(S \cup \{i,j\}) - w_A(S \cup \{i\}) \geq w_A(S \cup \{j\}) - w_A(S) \text{ if } i \in M \text{ and } j \in M'.$$

The paper of Shapley and Shubik focuses on the *core* of the game. They show that the core of the assignment game is non-empty and moreover $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is in the core $C(w_A)$ of the assignment game $(M \cup M', w_A)$ if and only if there exists a matching $\mu \in \mathcal{M}(M, M')$ such that

$$\begin{array}{l} u_i + v_j = a_{ij} \text{ for } (i,j) \in \mu, \\ u_i = 0 \text{ if } i \text{ unmatched by } \mu,\ v_j = 0 \text{ if } j \text{ unmatched by } \mu \\ u_i + v_j \geq a_{ij} \text{ for } (i,j) \in M \times M',\ u_i \geq 0 \text{ for all } i \in M,\ v_j \geq 0 \text{ for all } j \in M'. \end{array} \tag{23.1}$$

Notice that (a) any matching $\mu$ supporting a core allocation must be optimal, and moreover any optimal matching can play this role; (b) the core is determined by the valuation matrix, with no need to compute the complete characteristic function, since only individual rationality and pairwise stability, that is those core constraints related to individual coalitions and mixed-pair coalitions (the ones formed by a buyer and a seller) are relevant to describe the core of the assignment game; and (c) there are no side payments in the core other than the price a buyer pays a seller to acquire his object on sale.

The second main contribution in the paper of Shapley and Shubik [27] is the study of the structure of the core of the assignment game. If we consider, on the core elements, the partial order defined by one side of the market, for instance $(u, v) \leq_M (u', v')$ if and only if $u_i \leq u'_i$ for all $i \in M$, it turns out that the core has the structure of a *complete lattice* with respect to this order. A consequence of this lattice structure of the core is the existence of two special extreme core points. In one of them, $(\overline{u}^A, \underline{v}^A)$, each buyer maximizes her payoff in the core and each seller minimizes his, while in $(\underline{u}^A, \overline{v}^A)$ each seller maximizes his core payoff and buyers minimize theirs. What is remarkable is that all agents on the same side of the market, despite competing for the best deal, obtain their maximum core payoff in the same core element.

The maximum core payoff of an agent, be it a buyer or a seller, is her/his marginal contribution to the grand coalition (see [5] and [12]). That is, given an assignment game $(M \cup M', w_A)$,

$$\overline{u}_i^A = w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M'), \text{ for all } i \in M, \tag{23.2}$$
$$\overline{v}_j^A = w_A(M \cup M') - w_A(M \cup (M' \setminus \{j\})), \text{ for all } j \in M'. \tag{23.3}$$

Given an optimal matching $\mu$, and from the core constraints and the lattice structure of the core, we immediately obtain the minimum core payoffs of those assigned agents $(i, j) \in \mu$:

$$\underline{u}_i^A = a_{ij} - \overline{v}_j^A \text{ and } \underline{v}_j^A = a_{ij} - \overline{u}_i^A. \tag{23.4}$$

As for the remaining extreme core allocations, it is shown in [7] that every extreme core allocation of an assignment game is a *marginal worth vector*. Since assignment games use to be non-convex games, not all marginal worth vectors of an assignment game are extreme core points, only those that satisfy the core constraints. This provides a way to obtain the extreme core allocations: Compute all marginal worth vectors and select those that belong to the core. However, this procedure is very costly since you need the worth of all coalitions.

Most studies on the assignment market focus on the core and in particular on the two extreme core elements that are optimal for one side of the market. However, in several instances, it seems unfair to solve the situation by allocating the utility of an optimal matching by means of one of these two extreme points that favours one sector and damages the opposite. A first attempt to propose a not so extreme solution can be found in [30]. There, G.L. Thompson defines the *fair division point* of an assignment market $(M, M', A)$, as the midpoint between the buyers-optimal core allocation and the sellers-optimal core allocation:

$$\tau(w_A) = \frac{1}{2}(\overline{u}^A, \underline{v}^A) + \frac{1}{2}(\underline{u}^A, \overline{v}^A). \tag{23.5}$$

Since the core is a convex set, the above allocation always lies in the core. Some monotonicity properties of this solution are studied in [16].

## 23.3 The Shapley Value of the Assignment Game

We will denote by $\phi(w_A)$ the Shapley value of an assignment game $(M \cup M', w_A)$. Recall that value allocates to each agent the weigthed average of his/her marginal contributions to all possible coalitions.

When the Shapley value of an assignment game lies in the core, then it is also not an extreme point, as long as the core does not reduce to a singleton. But the problem is that the Shapley value is often outside the core in these games. To see that, notice that whenever the valuation matrix is not square, that is, there are not as many buyers as sellers, then unassigned agents receive a null payoff in any core element of the game. However, as long as an unassigned agent can make a positive profit with some agent of the opposite side, the Shapley value will give these agents a positive payoff.

Before revising the work of M. Hoffman and P. Sudölter that provides sufficient conditions that guarantee that the Shapley value of an assignment game is a core allocation, we need to recall some additional facts.

An assignment game is *exact* if for each coalition $S \subseteq M \cup M'$ there exists a core allocation $x \in C(w_A)$ such that $\sum_{k \in S} x_k = w_A(S)$.

The following two examples that are gathered in Table 23.1 show that assignment games may not be exact. The first market contains three buyers

$M = \{1,2,3\}$ and three sellers $M' = \{1',2',3'\}$, while the second market has only two agents on each side.

| | **1'** | **2'** | **3'** |
|---|---|---|---|
| **1** | 1 | 1 | 0 |
| **2** | 0 | 1 | 1 |
| **3** | 1 | 0 | 1 |

| | **1'** | **2'** |
|---|---|---|
| **1** | 3 | 3 |
| **2** | 2 | 2 |

**TABLE 23.1**: Two examples of non-exact assignment games.

The reader will easily check that the core of the first assignment game in Table 23.1 is the segment $[(1,1,1;0,0,0),(0,0,0;1,1,1)]$. Then, in any core allocation $(u,v)$, $u_1 + v_3 = 1 > 0 = a_{13}$, which shows that coalition $\{1,3'\}$ never attains its worth in a core allocation. The core of the second assignment game in Table 23.1 is also a segment and the minimum core payoff of buyer 1 is $\underline{u}_1^A = a_{11} - \overline{v}_1^A = 3 - 2 = 1 > 0$. Hence, coalition $\{1\}$ never attains its worth, $w_A(\{1\}) = 0$, in a core allocation.

Solymosi and Raghavan [29] characterizes, in terms of the valuation matrix, the exactness of an assignment game. To this end, they introduce two properties for a non-negative square matrix $A$.

An $m \times m$ non-negative matrix $A$ has:

- **a dominant diagonal** if and only if
  $a_{ii} \geq a_{ij}$ and $a_{ii} \geq a_{ji}$, for all $i,j \in \{1,2,\ldots,m\}$, and
- **a doubly dominant diagonal** if and only if
  $a_{ij} + a_{kk} \geq a_{ik} + a_{kj}$, for all $i,j,k \in \{1,2,\ldots,m\}$.

We say an assignment market (or game) is *square* when it has as many buyers as sellers. This can always be achieved by adding dummy agents on the short side of the market. Also, given an assignment market $(M,M',A)$ and an optimal matching $\mu$, agents in $M$ and $M'$ can always be ordered in such a way that the entries of the pairs in $\mu$ are on the main diagonal of matrix $A$.

**Theorem 23.1 (Solymosi and Raghavan, 2001)** *Let $(M,M',A)$ be a square assignment market with an optimal matching on the main diagonal. Then, the assignment game $(M \cup M', w_A)$ is exact if and only if $A$ has a dominant diagonal and a doubly dominant diagonal.*

When the optimal matching $\mu$ of a square assignment market is on the main diagonal, the dominant diagonal condition states that each agent is matched with a partner with whom the highest value is generated. It is straightforward to check that an assignment market has a dominant diagonal if and only if the minimum core payoff of each agent is zero. The doubly dominant condition states that whenever an optimally matched pair joins another mixed pair, the group has no incentives to interchange partners. The two definitions above do not depend on which optimal matching $\mu$ is placed on the main diagonal.

Notice that the first matrix in the previous table has a dominant diagonal but not a doubly dominant diagonal, since for instance $a_{13} + a_{22} < a_{12} + a_{23}$. Conversely, the second matrix in the table has not a dominant diagonal since $a_{22} < a_{12}$.

Given an arbitrary coalitional game, one can always define another game that is exact and has the same core as the first one. However, if we restrict to the class of assignment games, this exact game with the same core may not remain in the class. Notice that if for an assignment game there exists an agent, let us say a buyer $i \in M$, with a positive minimum core payoff, $\underline{u}_i^A > 0$, then no element $x$ in the core of an assignment game with the same core will satisfy $x_i = w_A(\{i\}) = 0$.

We may only require exactness for mixed-pair coalitions and then, for any assignment game we can guarantee existence of another assignment game with the same core and this weaker exactness property. An assignment game $(M \cup M', w_A)$ is *buyer-seller exact* if for any $(i, j) \in M \times M'$ there exists a core allocation $x = (u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$. It is shown in [17] that given any square assignment game $(M \cup M', w_A)$ there exists a unique matrix $\overline{A}$ that is buyer-seller exact and gives rise to the same core, $C(w_A) = C(w_{\overline{A}})$. By its definition, this buyer-seller exact representative $\overline{A}$ is the maximum matrix among those leading to the same core as $C(w_A)$.

Under the assumption that $A$ is square and $\mu$ is an optimal matching, for all $(i, j) \in M \times M'$, the entry in this matrix $\overline{A}$ is given by

$$\overline{a}_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} + w_A(M \cup M' \setminus \{\mu^{-1}(j), \mu(i)\}) - w_A(M \cup M'). \quad (23.6)$$

Moreover, an assignment game $(M \cup M', w_A)$ is buyer-seller exact if and only if $A$ has a doubly dominant diagonal, once an optimal matching has been placed on the main diagonal.

Given any assignment game $(M \cup M', w_A)$, the entries of the buyer-seller exact representative $\overline{A}$ can be obtained by solving a combinatorial optimization problem for each $(i, j) \in M \times M'$, but also by means of an iterative procedure that will be revised in the next section.

### 23.3.1 Balancedness Conditions

M. Hoffmann and P. Sudhölter [9] prove that, for assignment games, exactness is a sufficient condition for the Shapley value to belong to the core. As a consequence, if we check that a square valuation matrix, with an optimal matching on the main diagonal, has a dominant diagonal and a doubly dominant diagonal, then the Shapley value will satisfy all core constraints.

**Theorem 23.2 (Hoffmann and Sudhölter, 2007)** *Let $(M \cup M', w_A)$ be a square assignment game. If $(M \cup M', w_A)$ is exact, then $\phi(w_A) \in C(w_A)$.*

The exactness condition in Theorem 23.2 is not necessary for the Shapley value to be in the core of the assignment game. Take for instance the first assignment game in Table 23.1. The game is not exact, but the Shapley value is $\phi(w_A) = (0.5, 0.5, 0.5; 0.5, 0.5, 0.5)$ and belongs to the core of the game.

A remarkable subclass of assignment markets, known as *assortative assignment markets*, was introduced by G. S. Becker [2]. These markets model special bilateral assignment problems where agents on each side can be ordered by some trait, with the consequence that the mating of the likes will take place. Formally, an assignment market $(M, M', A)$ where $M = \{b_1, b_2, \ldots, b_m\}$, $M' = \{s_1, s_2, \ldots, s_{m'}\}$ and $a_{ij}$ stands for the valuation of buyer $b_i$ of the object of seller $s_j$, is assortative if it satisfies:
*(a) supermodularity*, that is, any $2 \times 2$ submarket has an optimal matching on its main diagonal,

$$a_{il} + a_{kj} \leq a_{ij} + a_{kl} \text{ for all } 1 \leq k \leq i \leq m \text{ and } 1 \leq l \leq j \leq m'; \text{and}$$

*(b) monotonicity* (non-decreasing row and column entries),

$$a_{kl} \leq a_{ij} \text{ for all } 1 \leq k \leq i \leq m \text{ and } 1 \leq l \leq j \leq m'.$$

With no loss of generality, we may assume the valuation matrix to be square, $m = m'$, simply adding null entries as first rows or columns of matrix $A$. Then, there is always an optimal matching on the main diagonal. Recently, F.J. Martínez de Albéniz et al. [14] have introduced a single-valued solution for these markets that can be computed easily from the valuation matrix and coincides with several well-known solutions for coalitional games. Moreover, they characterize when the Shapley value belongs to the core of a square assortative assignment market.

**Theorem 23.3 (Martínez de Albéniz, Rafels and Ybern, 2019)** *Let $(M, M', A)$ be a square assortative assignment market and $\phi(w_A)$ the Shapley value of the assignment game $(M \cup M', w_A)$. Then, $\phi(w_A)$ belongs to the core if and only if the valuation matrix is of the form*

$$A = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_2 & \cdots & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_m \end{pmatrix} \quad \textit{for some } 0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_m. \tag{23.7}$$

Moreover, the aforementioned authors show that for valuation matrices like those in (23.7) the Shapley value is $\phi_i(w_A) = \frac{a_{ii}}{2}$ for all $b_i \in M$ and $s_i \in M'$, that is, each agent is paid one half of the profit this agent obtains with his/her optimal partner.

Notice that, because of the monotonicity requirement, the matrices that satisfy (23.7) are the only assortative valuation matrices with a dominant diagonal. Hence, when the Shapley value of an assortative assignment game is in the core, it coincides with the fair division point. Moreover, the matrices (23.7) trivially satisfy the doubly dominant diagonal property.

**Theorem 23.4** *In the class of square assortative assignment markets, the Shapley value belongs to the core if and only if the game is exact.*

Another well-known subclass of assignment markets was introduced by E. von Böhm-Bawerk and also studied in [27]. These markets are characterized by the fact that each buyer values all the same objects on sale, that is, for all $i \in M$, $h_{ij} = h_i$ for each $j \in M'$. Then, for all $(i, j) \in M \times M'$, $a_{ij} = \max\{0, h_i - c_j\}$. These two-sided markets with homogeneous goods were introduced as a model for a horse market and are known as *Böhm-Bawerk horse markets*. It is well known, see for instance [27], that the core of these assignment games reduces to a segment.

The second example in Table 23.1 corresponds to a Böhm-Bawerk horse market and it can be checked that the Shapley value of this four-player game is $\phi(w_A) = (1.\overline{6}, 1; 1.1\overline{6}, 1.1\overline{6})$ and does not belong to the core since $u_1 + v_1 = 2.83 < 3 = a_{11}$.

The reader will easily check, that every Böhm-Bawerk horse market is an assortative assignment market. To this end, simply order the buyers in such a way that $0 \leq h_1 \leq h_2 \leq \cdots \leq h_m$ and the sellers such that $c_1 \geq c_2 \geq \cdots \geq c_{m'} \geq 0$, and check that the valuation matrix satisfies supermodularity and monotonicity.

Whenever all buyers in a Böhm-Bawerk market have the same valuation of the objects, that is, $h_i = h_{i'}$ for all $i, i' \in M$, and all sellers value their objects the same, $c_j = c_{j'}$ for all $j, j' \in M'$, then the valuation matrix is constant and we say the market is a *glove market* or a *symmetric market*.

We now introduce another subclass of Böhm-Bawerk horse markets that we name *extended square glove markets*. A square assignment market $(M, M', A)$ where $m = m'$ is an extended square glove market if there exists $r \in \{1, 2, \ldots, m\}$ and $c \geq 0$ such that $a_{ij} = c$ for all $r \leq i \leq m$ and $r \leq j \leq m$ and $a_{ij} = 0$ otherwise.

Then, as an immediate consequence of Theorem 23.3, we complete a result in [19] and characterize those Böhm-Bawerk horse markets such that their Shapley value belongs to the core.

**Theorem 23.5** *Let $(M \cup M', w_A)$ be a square Böhm-Bawerk horse market where valuations of buyers are in increasing order and valuations of sellers are in decreasing order, and $\phi(w_A)$ its Shapley value:*

$$\phi(w_A) \in C(w_A) \Leftrightarrow w_A \text{ is an extended square glove market.}$$

*Proof.* Let $(M \cup M', w_A)$ be a square Böhm-Bawerk horse market, with $m = m'$, and assume agents have been ordered such that $0 \leq h_1 \leq h_2 \leq \cdots \leq h_m$ and $c_1 \geq c_2 \geq \cdots \geq c_{m'} \geq 0$. Then define $r = \min\{k \in \{1, 2, \ldots, m\} \mid h_k - c_k > 0\}$ and notice that $a_{ij} = 0$ for all $1 \leq i < r$ and $1 \leq j < r$. Now, from Theorem 23.3, $\phi(w_A)$ is in the core if and only if the valuation matrix $A$ is of the type (23.7). This implies that all entries $a_{ij}$ with $r \leq i \leq m$ and $r \leq j \leq m$ coincide and that $a_{ij} = 0$ otherwise. ■

To summarize, for square Böhm-Bawerk horse markets and square assortative assignment markets, the dominant diagonal property (that in these cases is equivalent to exactness) characterizes the core membership of the Shapley value. As far as we know, no characterization of the core membership of the Shapley value is known.

Neither are known formulas for the Shapley value of an arbitrary assignment game in terms of its valuation matrix. However, Shapley and Shubik [26] give a formula for the particular case of a glove market. Let us assume $(M, M', A)$ is a glove market with $m \geq m'$ and $a_{ij} = 1$ for all $(i, j) \in M \times M'$. Then, the Shapley value is

$$\begin{aligned} \phi_i(w_A) &= \frac{1}{2} - \frac{m-m'}{2m} \sum_{k=0}^{m'} \frac{m!\, m'!}{(m+k)!\,(m'-k)!} \text{ for all } i \in M, \\ \phi_j(w_A) &= \frac{1}{2} + \frac{m-m'}{2m'} \sum_{k=1}^{m'} \frac{m!\, m'!}{(m-k)!\,(m'+k)!} \text{ for all } j \in M'. \end{aligned}$$

Notice that when $m = m'$, then $\phi_k(w_A) = \frac{1}{2}$ for all $k \in M \cup M'$ and the Shapley value of the square glove market coincides with the fair division point.

### 23.3.2 Axiomatic Characterization

Another approach to the study of the Shapley value in assignment games is the axiomatic approach. It is clear by now that the Shapley value is not pairwise-stable but, which properties does it satisfy? R. van den Brink and M. Pintér [4] take this approach and they first investigate whether the well-known axiomatizations of the Shapley value for coalitional games still individualize the Shapley value in the class of assignment games. Recall for instance the original axiomatization due to Shapley [24] by means of *efficiency*, *symmetry*, *null player property* and *additivity*; Young's axiomatization [31] replacing additivity and null player property by *strong monotonicity*; van den Brink's axiomatization [3] replacing additivity and symmetry by *fairness*; and Hart and Mas-Colell [8] using the potential function and a reduced game consistency.

van den Brink and Pintér find out that none of these characterizations work on the class of assignment games in the sense that there exist other solutions that satisfy these axioms. As an example of an alternative solution satisfying these axioms, they introduce the solution that is obtained as the average of the *buyers come first* (BCF) and the *sellers come first* (SCF) solutions. The BCF solution for assignment games is obtained by allocating to every seller her average marginal contribution over all permutations where all the buyers enter before any seller. Obviously, every buyer earns zero in any such marginal vector, since there are no sellers available when buyers enter the market. Reversely, the SCF solution for assignment games is obtained by allocating to every buyer his average marginal contribution over all permutations where all the sellers enter before any buyer (and thus every seller earns zero). Taking the average over these two solutions, they obtain a solution on the class of assignment games that, on this class, satisfies efficiency, additivity, symmetry, the null player property, strong monotonicity and fairness.

Given a buyer-seller market $(M, M', A)$ let the *buyers come first* situation define the TU game $(M', w_A^M)$ in which the worth of coalitions of sellers is defined by

$$w_A^M(S) = w_A(M \cup S) \text{ for all } S \subseteq M'. \tag{23.8}$$

Similarly, for the *sellers come first* situation, consider the TU-game $(M, w_A^{M'})$ in which

$$w_A^{M'}(S) = w_A(M' \cup S) \text{ for all } S \subseteq M. \tag{23.9}$$

The solution proposed in [4] is

$$\left(\frac{1}{2}Sh(w_A^{M'}), \frac{1}{2}Sh(w_A^M)\right) \in \mathbb{R}^M \times \mathbb{R}^{M'}.$$

It basically captures the idea of the Shapley value but for markets in which one sector is fully available to form coalitions with agents from the other market sector. It is easy to see that $(\frac{1}{2}Sh(w_A^{M'}), \frac{1}{2}Sh(w_A^M))$, in general, is not a core allocation of the game $(M \cup M', w_A)$.

Once seen that the known axiomatizations do not characterize the Shapley value in the class of assignment games, van den Brink and Pintér look for other properties that characterize this solution. Since assignment games can be seen as a particular case of *games with cooperation restricted by a communication graph*, as introduced in Myerson [15], the above authors base their study on Myerson's characterization of the Shapley value on that class of games.

To this end, let us denote by $\Gamma_0^{M \times M'}$ the class of assignment markets with set of buyers $M$ and set of sellers $M'$, that is $(M, M', A)$ for some $m \times m'$ valuation matrix $A$. We use this notation to remark that in this model coalitions formed by only one individual have null worth, in comparison with a related model we will introduce in the next section. Since the set of agents $M \cup M'$ is fixed, whenever it is convenient, we can identify each such market $(M, M', A)$ with its valuation matrix $A$.

A (single-valued) solution $\varphi$ on the domain $\Gamma_0^{M \times M'}$ is a function $\varphi : \Gamma_0^{M \times M'} \longrightarrow \mathbb{R}^M \times \mathbb{R}^{M'}$.

Following van den Brink and Pintér [4], a *submarket* of an assignment market $(M, M', A)$ is defined by a set of buyers and sellers such that all buyers in the set have zero valuation for the goods offered by the sellers outside the set, and all buyers outside the set have zero valuation for the goods offered by sellers inside the set.

**Definition 23.1** *Given* $\emptyset \neq S \subseteq M$ *and* $\emptyset \neq T \subseteq M'$, $(S, T)$ *is a submarket of* $(M, M', A) \in \Gamma_0^{M \times M'}$ *if* $a_{ij} = 0$ *for all* $(i, j) \in (S \times (M' \setminus T)) \cup ((M \setminus S) \times T)$.

Now, *submarket efficiency* requires that the sum of the payoffs of all agents in a submarket equal the worth of the submarket.

**Definition 23.2** *A solution $\varphi$ on the domain $\Gamma_0^{M\times M'}$ satisfies submarket efficiency if for all markets $(M, M', A)$ and all of its submarkets $(S, T)$, it holds*

$$\sum_{k\in S\cup T} \varphi_k(A) = w_A(S\cup T).$$

The second axiom will be *valuation fairness.* Valuation fairness of a solution applied to assignment markets implies that decreasing the valuation of one particular buyer for the good offered by a particular seller to zero changes the payoffs of this buyer and seller by the same amount.

**Definition 23.3** *A solution $\varphi$ on the domain $\Gamma_0^{M\times M'}$ satisfies valuation fairness if for all $(i^*, j^*) \in M \times M'$, and all $(M, M', A)$ and $(M, M', A')$ such that $a'_{ij} = a_{ij}$ for all $(i, j) \in (M \times M') \setminus \{(i^*, j^*)\}$ and $a'_{i^*j^*} = 0$, it holds*

$$\varphi_{i^*}(A) - \varphi_{i^*}(A') = \varphi_{j^*}(A) - \varphi_{j^*}(A').$$

**Theorem 23.6 (van den Brink and Pintér, 2015)** *The Shapley value is the unique solution on $\Gamma_0^{M\times M'}$ that satisfies submarket efficiency and valuation fairness.*

The Shapley value satisfies an even stronger valuation fairness property which states that changing the valuation of one particular buyer for the good offered by a particular seller in any way, changes the payoffs of this buyer and seller by the same amount.

Let us mention to conclude this section that the Shapley value also satisfies *pairwise monotonicity.* This property requires that the payoffs of buyer $i \in M$ and seller $j \in M'$ do not decrease if we only increase the valuation of buyer $i$ for the good offered by seller $j$. The proof is straightforward from the definition of the Shapley value payoff of an agent as the weighted average of all his/her marginal contributions: Increasing only valuation $a_{ij}$ does not affect the worth of coalitions not containing $i$ or $j$ and does not decrease the worth of coalitions containing both $i$ and $j$. This monotonicity property is also satisfied by other single-valued solutions of the assignment game, such as the ones that select the buyers-optimal core allocation, the sellers-optimal core allocation or the fair division point (see [16]).

## 23.4 The Shapley Value of a Related Market

In order to reconcile the notion of Shapley value with core stability in assignment markets, we may associate with each such market some related exact assignment market.

The first attempt is done in Núñez and Rafels [20]. There, to any square assignment market $(M, M', A)$, a unique exact assignment market $(M, M', A^c)$ can be associated that is defined on the same set of agents and with a core that is a translation of the core of the initial market. The new valuation matrix $A^c$ is defined by

$$a_{ij}^c = \overline{a}_{ij} - \underline{u}_i^A - \underline{v}_j^A, \text{ for all } (i,j) \in M \times M', \tag{23.10}$$

where $\overline{a}_{ij}$ are the values in the buyer-seller exact representative as defined in (23.6) and $(\underline{u}^A, \underline{v}^A)$ is the vector of minimum core payoffs of $(M, M', A)$ as defined in (23.4).

From the above definition, it is easy to check that $(M \cup M', w_{A^c})$ is an exact assignment game and

$$C(w_A) = \{(\underline{u}^A, \underline{v}^A)\} + C(w_{A^c}).$$

Hence, it is natural to consider the translation of the Shapley value of the exact game $(M \cup M', w_{A^c})$ as a single-valued core selection for the initial assignment game $(M \cup M', w_A)$.

**Definition 23.4 (Núñez and Rafels, 2009)** *Given a square assignment market* $(M, M', A)$ *in* $\Gamma_0^{M \times M'}$*, the translated Shapley value* $\phi^t(w_A)$ *is*

$$\phi^t(w_A) = (\underline{u}^A, \underline{v}^A) + \phi(w_{A^c}).$$

By its definition, it is straightforward to see that, for any square assignment market $(M, M', A)$, $\phi^t(w_A)$ belongs to the core of the original assignment market, that is, $\phi^t(w_A) \in C(w_A)$, and also to find examples that show that it differs from other well-known core solutions like the nucleolus and the fair division point.

However, to compute the translated Shapley value, we need to obtain the buyer-seller exact representative $\overline{A}$ which, according to (23.6), means solving $(m \times m') + 1$ combinatorial optimization problems.

Núñez and Solymosi [22] propose an algorithm that obtains the matrix $\overline{A}$ in polynomial time. To do so, we need to enlarge the domain of assignment markets and allow for individual reservation values. Assignment games with reservation values are introduced by Owen ([23]).

### 23.4.1 Assignment Markets with Reservation Values

Let $M$ and $M'$ be the sets of buyers and sellers, respectively. Each buyer $i \in M$ has a non-negative valuation $a_{ij} \in \mathbb{R}_+$ for the object of seller $j \in M'$, and also a reservation value $a_{i0} \geq 0$. Each seller $j \in M'$ has also a reservation value $a_{0j} \geq 0$.

By introducing a fictitious agent on each side of the market, we summarize these valuations in a matrix $A = (a_{ij})_{(i,j)\in M_0 \times M'_0}$, where $M_0$ and $M'_0$ are the sets of buyers and sellers, respectively, enlarged with the fictitious agents, and by convention $a_{00} = 0$. Then, an *assignment market with reservation values* is defined by $(M, M', A)$, where $A$ is an $(m+1)\times(m'+1)$ non-negative matrix with $a_{00} = 0$, and the set of all such markets will be denoted by $\Gamma^{M\times M'}$.

Given a non-empty subset of buyers $S \subseteq M$ and a non-empty subset of sellers $T \subseteq M'$, a matching is a partition of $S \cup T$ in mixed-pair coalitions and singletons. We denote by $\mathcal{M}(S,T)$ the set of matchings between $S$ and $T$. Given a market $(M, M', A)$, a matching $\mu$ is optimal if the addition of the values of the elements of partition $\mu$ is not less than the addition of values for any other partition $\mu' \in \mathcal{M}(M, M')$. Then, the corresponding *assignment game with reservation values* is $(M \cup M', w_A)$ where, for all $i \in M$ and $j \in M'$, $w_A(\{i,j\}) = a_{ij}$, $w_A(\{i\}) = a_{i0}$, $w_A(\{j\}) = a_{0j}$, and for all coalition $R \subseteq M \cup M'$, $w_A(R)$ is the maximum worth obtained over all possible partitions of $R$ in mixed-pair coalitions and singletons. Notice that, although its valuation matrix is $(m+1)\times(m'+1)$, the player set of the corresponding assignment game with reservation values is $M \cup M'$ and has cardinality $m + m'$.

It is straightforward to check that the core of an assignment game with reservation values $(M \cup M', w_A)$ is determined by the valuation matrix, with no need to compute the worth of all coalitions. Given any optimal matching $\mu$ of the market $(M, M', A)$ with reservation values, the core is

$$C(w_A) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} \text{for all } i \in M \text{ and } j \in M' \\ u_i + v_j \geq a_{ij} \text{ and } u_i + v_j = a_{ij} \text{ if } \{i,j\} \in \mu, \\ u_i \geq a_{i0},\ v_j \geq a_{0j}, \\ u_i = a_{i0} \text{ for all } \{i\} \in \mu, \\ v_j = a_{0j} \text{ for all } \{j\} \in \mu. \end{array} \right\}$$

Each assignment game with reservation values $(M \cup M', w_A)$ is strategically equivalent[1] to a Shapley and Shubik assignment game (that is, an assignment game with null reservation values). More precisely, it is strategically equivalent to the game $(M \cup M', w_{\tilde{A}})$ where $\tilde{A}$ is the $m \times m'$ matrix defined by $\tilde{a}_{ij} = \max\{0, a_{ij} - a_{i0} - a_{0j}\}$ for all $(i,j) \in M \times M'$. In fact, $\Gamma^{M\times M'}$ is the set of all games that are strategically equivalent to some assignment game with set of buyers $M$ and set of sellers $M'$.

[1] A game $(N, v')$ is strategically equivalent to a game $(N, v)$ if there exists a vector $d \in \mathbb{R}^N$ such that $v'(S) = \sum_{i\in S} d_i + v(S)$ for all $S \subseteq N$.

As a consequence, the core of an assignment game with reservation values inherits the properties of the core of the Shapley and Shubik assignment game: It is always non-empty and it has a lattice structure with one optimal core allocation for each side of the market.

### 23.4.2 The Shapley Value of the Exact Assignment Game with the same Core

We will now consider a Shapley and Shubik assignment game $(M \cup M', w_A)$ and obtain the unique exact game with the same core. This exact game, may not be in the class $\Gamma_0^{M\times M'}$ but will belong to the class $\Gamma^{M\times M'}$ of assignment games with reservation values. To this end, we will assume without loss of generality that there are the same number of players of both types (i.e., the dimension of the underlying valuation matrix $A$ is $m \times m$).

Notice that any Shapley and Shubik assignment game $(M \cup M', w_A)$ can be represented as an assignment game with null reservation values just defining the $(m+1) \times (m+1)$ matrix $A^0$ obtained from $A$ by adding the row and column of individual values $a_{i0} = a_{0j} = 0$ for all $i \in M$ and $j \in M'$, and $a_{00} = 0$. Notice that $(M \cup M', w_{A^0}) \in \Gamma^{M\times M'}$ and $C(w_A) = C(w_{A^0})$.

In order to obtain a unified notation, once we have introduced a fictitious row player and a fictitious column player, we identify the 'real' mixed-pair coalition $\{i, j\}$, $i \in M$, $j \in M'$ with the ordered pair $(i, j)$; we write $(i, 0)$ for single-player coalition $\{i\}$, $i \in M$, and $(0, j)$ for $\{j\}$, $j \in M'$; finally, $(0, 0)$ denotes the coalition of the two fictitious players. Since the type of the players is determined by their positions in the ordered pairs, it will be convenient to use a common set $N_0 = \{0, 1, 2, \ldots, m\}$ of indices, where $m$ is the number of agents on each side of the market.

We assume that the rows and columns of the augmented (square) valuation matrix $A^0$ are arranged such that the diagonal assignment $\{(i,i) : i \in \{1, 2, \ldots, m\}\}$ is optimal, i.e., $w_A(M \cup M') = \sum_{i=1}^m a_{ii}$.

With all the above conventions, the core of the assignment game $(M \cup M', w_{A^0})$ induced by matrix $A^0$ is

$$C(w_{A^0}) = \Big\{(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \,|\, u_i + v_i = a_{ii}\ \forall i \in N_0,\ u_i + v_j \geq a_{ij} \forall i, j \in N_0\Big\}, \tag{23.11}$$

where $u_0 = v_0 = 0$ is set by convention.

Given the assignment game $(M \cup M', w_A)$, let us consider the following algorithm.

---

**Algorithm** Cover [Núñez and Solymosi, 2017]

**Initially**, let $A^0 = A_{M_0 \times M_0}$ be a square $(m+1) \times (m+1)$ matrix with $a_{i0} = a_{0j} = a_{00} = 0$ for all $i \in M$ and $j \in M'$ and an optimal matching on the main diagonal. Set $r = 1$.

**Iteration** $r$: Compute the $(m+1)\times(m+1)$ matrix $A^r$ from matrix $A^{r-1}$ as follows:

$$a^r_{ij} := \max\left\{a^{r-1}_{ij}, \max\{a^{r-1}_{ik} + a^{r-1}_{kj} - a^{r-1}_{kk} : k \in N_0 \setminus \{i,j\}\}\right\} \text{ for all } i,j \in N_0. \tag{23.12}$$

If $A^r = A^{r-1}$, then STOP, else set $r := r+1$ and start a new iteration.

**Output**: Matrix $A^e$ where $e$ is the first $e \geq 1$ for which $A^e = A^{e-1}$.

---

It is proved in [22] that the COVER algorithm ends in at most $m$ iterations, where $m$ is the number of agents on each side of the market, and its number of elementary operations is $\mathcal{O}(m^4)$. Also, in each step of the algorithm, an optimal matching is on the main diagonal and the core of the related market with reservation values coincides with the core of the initial game: $C(w_{A^r}) = C(w_{A^0}) = C(w_A)$ for all $1 \leq r \leq e$. Moreover, the output $A^e$ of the algorithm satisfies the doubly dominant diagonal property.

We now show that the output of the algorithm COVER is the only valuation matrix that defines an exact assignment game with reservation values with the same core than the initial Shapley and Shubik assignment game.

**Theorem 23.7** *Let* $(M, M', A) \in \Gamma_0^{M\times M'}$ *be an assignment market and* $(M, M', A^e) \in \Gamma^{M\times M'}$ *the assignment market with reservation values where* $A^e$ *is the output of the algorithm* COVER. *Then, the game* $(M \cup M', w_{A^e})$ *is exact and* $C(w_{A^e}) = C(w_A)$.

*Proof.* From the properties of the algorithm COVER in [22], we know $C(w_{A^e}) = C(w_A)$. We only need to prove exactness. Let us define the valuation matrix $\tilde{C}$ with reservation values that results from $A^e$ by substracting $a^e_{i0}$ from each row $i \in M$ and $a^e_{0j}$ from each column $j \in M'$. Then, the entries of $\tilde{C}$ are

$$\begin{aligned} c_{ij} &= a^e_{ij} - a^e_{i0} - a^e_{0j}, \text{ for all } (i,j) \in M \times M', \\ c_{i0} &= a^e_{i0} - a^e_{i0} = 0, \text{ for all } i \in M, \\ c_{0j} &= a^e_{0j} - a^e_{0j} = 0, \text{ for all } j \in M'. \end{aligned} \tag{23.13}$$

Notice first that, since $A^e$ satisfies the doubly dominant diagonal property, then $\tilde{C} \geq 0$. Since its individual reservation values are null, the assignment game with reservation values $(M \cup M', w_{\tilde{C}})$ can be identified with the Shapley and Shubik assignment game $(M \cup M', w_C)$ with $C = (c_{ij})_{(i,j)\in M\times M'}$. It can be checked from (23.13) that for any $S \subseteq M$ and $T \subseteq M'$, it holds

$$w_{A^e}(S \cup T) = \sum_{i\in S} a^e_{i0} + \sum_{j \in T} a^e_{0j} + w_C(S \cup T). \tag{23.14}$$

The above equation means that the games $(M \cup M', w_{A^e})$ and $(M \cup M', w_C)$ are strategically equivalent and, as a consequence, if we define the payoff vector

$(a^e_M, a^e_{M'})$ by $(a^e_M)_i = a^e_{i0}$ for all $i \in M$ and $(a^e_{M'})_j = a^e_{0j}$ for all $j \in M'$, the cores of the two games are related just by a translation:

$$C(w_{A^e}) = \{(a^e_M, a^e_{M'})\} + C(w_C). \tag{23.15}$$

Moreover, matrix $C$ has a dominant diagonal and a doubly dominant diagonal. Indeed, this follows from the doubly dominant diagonal property of matrix $A^e$: For all $i, j \in \{1, 2, \ldots, m\}$, we have $a^e_{0j} + a^e_{ii} \geq a^e_{0i} + a^e_{ij}$ which implies

$$a^e_{0j} + (c_{ii} + a^e_{i0} + a^e_{0i}) \geq a^e_{0i} + (c_{ij} + a^e_{i0} + a^e_{0j}) \text{ and hence } c_{ii} \geq c_{ij};$$

and $c_{ij} \geq c_{ji}$ is obtained in a similar way. Also, for all $i, j, k \in \{1, 2, \ldots, m\}$,

$$a^e_{ij} + a^e_{kk} \geq a^e_{ik} + a^e_{kj} \Longrightarrow c_{ij} + c_{kk} \geq c_{ik} + c_{kj}.$$

As a consequence, by Theorem 23.1, $(M \cup M', w_C)$ is an exact assignment game, which implies $(M \cup M', w_{A^e})$ is also exact. To prove this, for any $S \subseteq M$ and $T \subseteq M'$, let $(u, v) \in C(w_C)$ be such that $u(S) + v(T) = w_C(S \cup T)$ and define the payoff vector $(u', v') \in \mathbb{R}^M \times \mathbb{R}^{M'}$ by $u'_i = a^e_{i0} + u_i$ for all $i \in M$ and $v'_j = a^e_{0j} + v_j$ for all $j \in M'$. It then follows from (23.15) and (23.14) that $(u', v') \in C(w_{A^e})$ and $u'(S) + v'(T) = w_{A^e}(S \cup T)$. ∎

A first immediate consequence of the above theorem is that $(a^e_M, a^e_{M'})$ is simply the vector $(\underline{u}^A, \underline{v}^A)$ of minimum core payoffs of the market $(M, M', A)$ and also that for all $(i, j) \in M \times M'$, $a^e_{ij}$ is the worth of the mixed-pair $(i, j)$ in the buyer-seller exact representative, $a^e_{ij} = \overline{a}_{ij}$. As a consequence, the $m \times m$ matrix $C$ defined in (23.13) is simply the matrix $A^c$ defined in (23.10). Hence, taking into account that the Shapley value is covariant with respect to strategic equivalence, we get

$$\phi(w_{A^e}) = (a^e_M, a^e_{M'}) + \phi(w_C) = (\underline{u}^A, \underline{v}^A) + \phi(w_{A^c}) = \phi^t(w_A)$$

which shows that the Shapley value of the exact assignment game with reservation values coincides with the translated Shapley value of Definition 23.4, as stated below.

**Theorem 23.8** *Let $(M, M', A)$ be a square assignment market and $(M, M', A^e)$ the only exact assignment market with reservation values such that $C(w_{A^e}) = C(w_A)$. Then,*

$$\phi(w_{A^e}) = \phi^t(w_A).$$

The first matrix in Table 23.2 is the valuation matrix $A$ of an assignment market $(M, M', A)$ where $M = \{1, 2\}$ and $M' = \{1', 2'\}$. The second matrix is the valuation matrix $A^e$ of the associated exact assignment market with reservation values that can be computed by means of the algorithm Cover.

Next we have computed three single-valued solutions for the above example: The Shapley value, the fair division point and the translated Shapley

| $A$ | **0'** | **1'** | **2'** |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **1** | 0 | 4 | 1 |
| **2** | 0 | 3 | 2 |

| $A^e$ | **0'** | **1'** | **2'** |
|---|---|---|---|
| **0** | 0 | 1 | 0 |
| **1** | 0 | 4 | 1 |
| **2** | 0 | 3 | 2 |

**TABLE 23.2**: A market $(M, M, A)$ and its related exact assignment market.

value. This example illustrates that the Shapley value of the related exact assignment market (or translated Shapley value) may differ from the fair division point.

| | |
|---|---|
| Shapley value | $\phi(w_A) = (1.58, 1.25; 2.25, 0.92) \notin C(w_A)$ |
| Fair division point | $\tau(w_A) = (1.5, 1; 2.5, 1) \in C(w_A)$ |
| Shapley value of the exact market | $\phi^t(w_A) = (1.42, 1.08; 2.58, 0.92) \in C(w_A)$ |

## 23.5 Conclusions

We have proposed in this chapter to consider, for any assignment game, the single valued solution that assigns the Shapley value of the only exact assignment game with reservation values that has the same core. From Theorem 23.8, we know that this solution will always lie in the core of the initial assignment game. Of course, when the initial assignment game is already exact, $\phi(w_{A^e})$ coincides with the Shapley value $\phi(w_A)$.

As far as we know, there is no axiomatic characterization of the Shapley value of the related exact market. Since this solution selects a core allocation, we easily deduce that it satisfies the submarket efficiency property of van den Brink and Pintér axiomatization of the Shapley value. Hence, it will not satisfy valuation fairness since we know this solution may differ from the Shapley value.

## 23.6 Acknowledgments

The authors acknowledge support from research grant ECO2017-86481-P (Agencia Estatal de Investigación (AEI) y Fondo Europeo de Desarrollo Regional (FEDER)) and 2017SGR778 (Generalitat de Catalunya).

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# Chapter 24

## *The Shapley Value in Minimum Cost Spanning Tree Problems*

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## 24.1 Introduction

The Shapley value is a very appealing solution concept that is characterized by its reliance on contributions [51] and satisfies many interesting properties in the general set of cooperative games, such as "additivity" [38] and "balanced contributions". A drawback is that the Shapley value payoff vector might not be stable in the sense of core selection: Even for games for which the core is nonempty, the Shapley value might propose allocations giving some coalitions incentives to secede.

An interesting family of balanced games in which the Shapley value has nonetheless received considerable attention is the class of minimum cost spanning tree (*mcst*) problems, which model situations where a group of agents, located at different nodes of a network, require a service provided by a source and do not care if they connect directly to the source or through other agents already connected. The cost of an edge between two nodes has to be paid when used, but cost remains invariant if more than one agent uses it to connect to the source.

As the cost of the efficient network connecting all agents to the source is easily found, we consider the problem of sharing the cost of that network among its users. There are many ways to define a TU-game from the mcst problem, depending on our assumptions on property rights and on the behavior of non-cooperating agents. The most commonly used game is the private mcst game, which limits the nodes that a coalition can use to those of its members.

Regardless of how the game is defined, a mcst problem typically induces a game with a large nonempty core and, moreover, it allows population monotonic allocations schemes [35]. However, for the private mcst game, the Shapley value does not always belong to its core. This fact has led some authors to claim that the Shapley value is not a good solution concept in mcst problems [40].

Yet, even if we are interested in the private mcst game, we can use the Shapley values of some reasonable alternative cost games that belong to the core of the private mcst game. Moreover, those solutions rely on contributions and maintain the nice properties of "additivity", "balanced contributions", among others, in the context of minimum cost spanning tree problems.

This chapter surveys the growing literature on mcst games and it is organized as follows: In Section 24.2 we present the model and some basic definitions. In Section 24.3 we define three different cooperative cost games that can be associated to a minimum cost spanning tree problem. In Section 24.4 we describe the respective Shapley values associated to the previous cooperative cost games. In Section 24.5 we review their axiomatic characterizations. In Section 24.6 we study the weighted versions of the Shapley value and also compare it with other solution concepts such as the nucleolus. In Section 24.7 we comment on some studies of the Shapley value in other problems related to mcst. Finally, in Section 24.8 we conclude.

## 24.2 Definitions

We first define general cost games, before introducing minimum cost spanning tree problems.

### 24.2.1 Cooperative Cost Games

Let $\mathcal{N} = \{1, 2, \dots\}$ be a (countably infinite) set of potential agents, and let $N = \{1, \dots, n\}$ be a generic nonempty, finite set of $\mathcal{N}$.

A *(cost sharing) game* is a pair $(N, C)$ where $C$ is a *cost function* that assigns to each nonempty *coalition* $S \subseteq N$ the cost $C(S) \in \mathbb{R}_+$ that agents in $S$ should pay in order to receive the service.

For any $S \subseteq N$, let $x(S) = \sum_{i \in S} x_i$. A *preimputation* is an allocation $x \in \mathbb{R}^N$ such that $x(N) = C(N)$. Given $S \subseteq N$ and $x \in \mathbb{R}^N$, we denote as $x_S \in \mathbb{R}^S$ the restriction of $x$ to $\mathbb{R}^S$.

We define the set of stable allocations as $Core(C)$. Formally, an allocation $x$ belongs to $Core(C)$ if it is a preimputation such that $x(S) \leq C(S)$ for all $S \subset N$.

### 24.2.2 Minimum Cost Spanning Tree Problems

We assume that the agents in $N$ need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set $Z$, define $Z^p$ as the set of all non-ordered pairs $(i, j)$ of distinct elements of $Z$. In our context, any element $(i, j)$ in $Z^p$ represents the (undirected) edge between nodes $i$ and $j$. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and $c_e$ representing the cost of edge $e$. Given $E \subset N_0^p$, its associated cost is $c(E) = \sum_{e \in E} c_e$. For simplicity, we write $c_{ij}$ instead of $c_{(i,j)}$ for all $i, j \in N_0$.

Since $c$ assigns a cost to all edges $e$, we often abuse language and call $c$ a *cost matrix*. Let $\Gamma$ be the set of all cost matrices. A mcst problem is a pair $(N_0, c)$. When there is no ambiguity, we identify a mcst problem $(N_0, c)$ by its cost matrix $c$.

Given $l \in N_0$, a *cycle* $p_{ll}$ is a set of $K \geq 3$ edges $(i_{k-1}, i_k)$, with $k \in \{1, \dots, K\}$ and such that $i_0 = i_K = l$ and $i_1, \dots, i_{K-1}$ distinct and different than $l$. Given $l, m \in N_0$, a *path* $\psi_{lm}$ between $l$ and $m$ is a set of $K$ edges $(i_{k-1}, i_k)$, with $k \in \{1, \dots, K\}$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $\Psi_{lm}(N_0)$ be the set of all paths between nodes $l$ and $m$.

A *spanning tree* is a non-oriented graph without cycles that connects all elements of $N_0$. A spanning tree $t$ is identified by the set of its edges.

We call *mcst* a spanning tree that has a minimal cost. It can be obtained using a greedy algorithm, for example Prim [36], Kruskal [29], or Borůvka's [19] algorithms.

## 24.3 Associated Cooperative Cost Games

Having established how to connect efficiently all agents to the source, we now examine how to share the cost of such connections. To derive from a mcst problem a cooperative game that represents the cost for each coalition to act alone, we need to determine the rules of the game, i.e., what exactly a coalition is allowed to do when it is alone. In the context of mcst problems, we consider three possibilities:

**The private mcst game** The cost assigned to coalition $S \subset N$ is computed by assuming that nodes in $S$ should connect without using nodes $N \setminus S$, i.e., nodes outside $S$ are unavailable.

**The public mcst game** The cost assigned to coalition $S \subset N$ is computed by assuming that agents in $S$ may use edges in $N \setminus S$, paying the costs of the edges they use.

**The optimistic mcst game** The cost assigned to coalition $S \subset N$ is computed by assuming that nodes in $N \setminus S$ are already connected to the source, and hence agents in $S$ just need to connect either to a node in $N \setminus S$ or to the source.

We analyze each of these possibilities one by one.

### 24.3.1 The Private mcst Game

The most common assumption in the literature is that a coalition only has access to the nodes of its members to connect to the source. In this approach, we assume that agents have property rights over their respective nodes, forcing a coalition to only use the nodes of its members. We thus call the resulting game the *private mcst game*.

Formally, let $c_S$ be the restriction of the cost vector $c$ to the coalition $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the mcst of the problem $(S_0, c_S)$. We say that $C$ is the *stand-alone cost* for the private mcst game associated with $c$.

### 24.3.2 The Public mcst Game

An alternative approach is to suppose that there are no property rights on nodes: A coalition $S$ can use the nodes of its neighbours in $N \setminus S$ to connect to the source if they desire so. We call the resulting game the *public mcst game*. It was first explicitly considered in [18], and also examined and contrasted with the private game in [44, 49].

We thus obtain the following characteristic cost function. For all $S \subseteq N$, we have

$$C^{Pub}(S, c) = \min_{T \subseteq N \setminus S} C(S \cup T, c).$$

It is thus obvious that for all $S \subset N$, $C^{Pub}(S, c) \leq C(S, c)$ and $C^{Pub}(N, c) = C(N, c)$.

### 24.3.3 The Optimistic mcst Game

We finally consider the case where agents are the last to move: Others have connected to the source, and they only need to add themselves to the tree.[1] A coalition $S$ can either connect to the source or to any node in $N \setminus S$. We call the resulting game the *optimistic mcst game*. It was first used in the context of mcst problems[2] by Bergantiños and Vidal-Puga [12].

Formally, let $c^{+S}$ be the cost matrix on $S_0$ defined as $c^{+S}_{0i} = \min_{j \in N_0 \setminus S} c_{ij}$ and $c^{+S}_{ij} = c_{ij}$ otherwise, for all $i, j \in S$.

We thus obtain the following cost function. For all $S \subseteq N$, we have

$$C^{+}(S, c) = C\left(S, c^{+S}\right).$$

By contrast, the two other approaches are pessimistic, as the stand-alone cost of a coalition is computed under the assumption that the other agents are not connected.

It is obvious that for all $S \subset N$, $C^{+}(S, c) \leq C^{Pub}(S, c) \leq C(S, c)$ and $C^{+}(N, c) = C^{Pub}(N, c) = C(N, c)$.

### 24.3.4 Example

Consider the mcst problem described in Figure 24.1, for which $N = \{1, 2, 3\}$ and the cost of each edge is indicated on it.

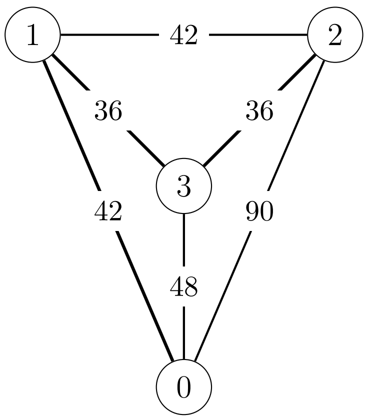


**FIGURE 24.1**: Minimum cost spanning tree problem.

[1] See [6] for a real-life example of this situation.
[2] Maniquet [32] considered the same idea in the context of queueing problems.

| $S$ | $C(S,c)$ | $C^{Pub}(S,c)$ | $C^{+}(S,c)$ |
|---|---|---|---|
| $\{1\}$ | 42 | 42 | 36 |
| $\{2\}$ | 90 | 84 | 36 |
| $\{3\}$ | 48 | 48 | 36 |
| $\{1,2\}$ | 84 | 84 | 72 |
| $\{1,3\}$ | 78 | 78 | 72 |
| $\{2,3\}$ | 84 | 84 | 72 |
| $\{1,2,3\}$ | 114 | 114 | 114 |

There is a single mcst in this game: $t^* = \{(0,1),(1,3),(2,3)\}$. The functions $C$ and $C^{Pub}$ differ only for agent 2: In the private game she can only connect to the source by using edge $(0,2)$, at a cost of 90. In the public game, she can connect at a lower cost of 84 by using trees $\{(0,1),(1,2)\}$ or $\{(0,3),(2,3)\}$. The optimistic game is quite different, and only the grand coalition has the same cost as in other approaches. For example, agent 2 can now free-ride on the edges established by other agents and can connect by adding edge $(2,3)$ at a cost of 36.

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## 24.4 The Shapley Value

In what follows, a *solution* is a function that assigns to each mcst problem $(N_0, c)$ a preimputation $y \in \mathbb{R}^N$. As in most cost sharing problems, the Shapley value is a natural candidate to share the cost in a mcst problem. We study three ways to do so, depending on the cooperative game associated to the mcst problem.

### 24.4.1 The Kar Solution

The Shapley value of the private mcst game is known in the literature as the Kar solution, following the axiomatic analysis of the method in [28]. Formally, we define the ***Kar solution*** as $y^K(c) = Sh(C(N,c))$. Ando [1] shows that computing the Kar solution is #P-hard even if the edges are restricted to costs of 0 or 1.

The Shapley value of the public mcst game has not received much more attention. We define it here as $y^{K^{Pub}}(c) = Sh(C^{Pub}(N,c))$.

Whether we are applying it to the private or public version of the game, there is one major problem: Even though the cores of the private and public mcst problems are always non empty, the Shapley values might not be in them, a fact observed as early as in [17].

In our running example, we obtain $y^K(c) = (28, 55, 31)$ and $y^{K^{Pub}}(c) = (29, 53, 32)$. Notice that whether we are in the private or public game, the stand-alone cost for coalition $\{2, 3\}$ is 84. The (private) Kar solution assigns them a joint cost of 86, while the public version assigns them 85. Therefore, both are unstable.

Following this observation, researchers have proposed solutions that are in the core. We focus in this paper on the two solutions that are based on the Shapley value, the folk and the cycle-complete solutions. Other stable solutions are based on the network-building algorithms, and include the Bird solution [17], the Dutta-Kar solution [22] and the obligation rules [5, 7, 8, 31, 42].

### 24.4.2 The Folk Solution

The folk solution can be obtained by applying the Shapley value to two different situations.

The first one is by transforming the cost matrix into an *irreducible cost matrix*, which is such that no edge cost can be reduced without reducing the cost of the grand coalition to connect to the source [17]. From any cost matrix $c$, we can define the *irreducible cost matrix* $c^*$ as follows:

$$c^*_{ij} = \min_{\psi_{ij} \in \Psi_{ij}(N_0)} \max_{e \in \psi_{ij}} c_e \text{ for all } i, j \in N_0.$$

Interestingly, it is not difficult to verify that for any irreducible matrix $c$, $C(\cdot, c) = C^{Pub}(\cdot, c)$. That is, once we have transformed the cost matrix into its irreducible cost matrix, the property rights on the nodes are irrelevant. One can thus argue that using the irreducible matrix will yield solutions that are closer in spirit to the public approach than the private one [45]. Following [11], the ***folk solution*** is defined as $y^f(c) = Sh(C^*(N, c))$, where $C^*(N, c)$ is the stand-alone game induced by the irreducible cost matrix $c^*$.

In our running example, we obtain the following irreducible matrix: Agent 3 is linked to the source with path $\{(0, 1), (1, 3)\}$ for which the most expensive edge costs 42, so $c^*_{03} = 42$. Agent 2 is linked to the source with path $\{(0, 1), (1, 3), (2, 3)\}$ for which the most expensive edge costs 42, so $c^*_{02} = 42$. Agents 1 and 2 are linked to each other with the path $\{(1, 3), (2, 3)\}$ for which the most expensive edge costs 36, so $c^*_{12} = 36$. The costs of other edges stay unchanged. This results in $C^*(S, c) = 42$ if $|S| = 1$, $C^*(S, c) = 78$ if $|S| = 2$ and $C^*(N, c) = 114$. Thus, $y^f(c) = (38, 38, 38)$ due to the symmetry of the agents. See Figure 24.2.

Bird [17] first studied $Core(C^*(N, c))$, and called it the *irreducible core*. Since $C^*$ is a concave cost game (Proposition 3.3c in [11]), its Shapley value belongs to the irreducible core. Finally, since $Core(C^*(N, c)) \subseteq Core(C^{Pub}(N, c)) \subseteq Core(C(N, c))$, we have the following result.

**Theorem 24.1 (Bergantiños and Vidal-Puga [11])** *For all $c \in \Gamma$, $y^f(c)$ is in $Core(C(N, c))$.*

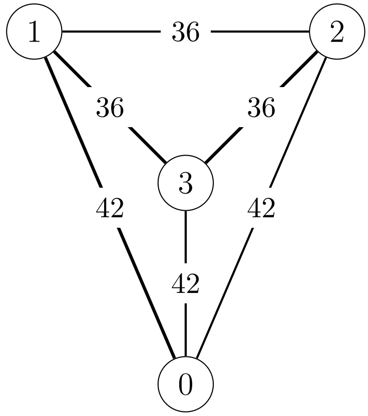


**FIGURE 24.2**: Irreducible matrix associated to Example 24.1.

The folk solution is thus remarkably stable: It is always in the core, regardless of how we define the core.

The second definition of the folk rule using the Shapley value is through the optimistic version of the cost game. Bergantiños and Vidal-Puga [11] show that the folk rule is the Shapley value of the optimistic game, i.e., $y^f = Sh(C^+(N,c))$. This is due to the fact that the private mcst game associated with the irreducible cost vector is dual to the optimistic cost game, i.e., $C^*(S,c) + C^+(N \setminus S, c)$ is independent of $S$ (Theorem 1 in [12]).

Other definitions of the folk solution are possible. Bergantiños and Vidal-Puga [11] show that $y^f$ can also be defined using Prim's algorithm [36] on the irreducible matrix. Feltkamp et al. [24], using another network-building algorithm due to Kruskal [29], were the first ones to define the folk rule with the name *Equal Remaining Obligations* (ERO) rule, renamed as $P$-value in [20]. Equivalence between ERO and folk rules was first pointed out in [10, 13]. Bergantiños and Vidal-Puga [16] use yet another network-building algorithm, due to Borůvka [19].

Given the different definitions and names, $y^f$ has been dubbed the *folk solution* by Bogomolnaia and Moulin [18]. We also use their term throughout the chapter.

### 24.4.3 The Cycle-Complete Solution

As seen in the previous subsection, the folk solution proposes a stable allocation, but one in which we have introduced a lot of symmetry. While in the running example all agents pay the same amount, this is not a general result. It does, however, introduce a lot of symmetry by keeping only the information contained in the mcst. The idea of the cycle-complete solution, proposed in

[43], is to try to keep more information from the original matrix while still proposing a stable allocation.

The method used is conceptually close to the one used to generate the folk solution, with changes made to the cost matrix, before taking the Shapley value of the corresponding (private) mcst game. Instead of looking at paths between pairs of edges, we look at cycles: For edge $(i,j)$, we look at cycles that go through node $i$ and node $j$. If there is one such cycle such that its most expensive edge is cheaper than a direct connection through edge $(i,j)$, we assign this cost to edge $(i,j)$.

From any cost matrix $c$, we formally define the cycle-complete cost matrix $c^{**}$ as follows:

$$\begin{aligned} c_{ij}^{**} &= \max_{k\in N\setminus\{i,j\}}\left(c^{N\setminus\{k\}}\right)_{ij}^{*} \text{ for all } i,j\in N \\ c_{0i}^{**} &= \max_{k\in N\setminus\{i\}}\left(c^{N\setminus\{k\}}\right)_{0i}^{*} \text{ for all } i\in N \end{aligned}$$

where $\left(c^{N\setminus\{k\}}\right)^{*}$ indicates the matrix that we first restricted to agents in $N\setminus\{k\}$ before transforming into an irreducible matrix.

The ***cycle-complete solution*** is defined as $y^{cc}(c)=Sh(C^{**}(N,c))$, where $C^{**}(N,c)$ is the stand-alone game induced by $c^{**}$.

In our running example, the only change we make to the original cost matrix to obtain the cycle-complete matrix is to edge $(0,2)$. We can build cycle $\{(0,1),(1,2),(2,3),(0,3)\}$ that contains both 0 and 2 and for which the most expensive edge costs 48. Thus, $c_{02}^{**}=48$.

In the private mcst game, the only change is to the stand-alone cost of agent 2, which goes from 90 to 48. Thus, $y^{cc}(c)=(35,41,38)$.

Trudeau [43] shows that $C^{**}(N,c)$ is a concave game, and thus its Shapley value is in $Core(C^{**}(N,c))$. Since $Core(C^{**}(N,c))\subseteq Core(C(N,c))$, the cycle-complete solution is in the core.[3]

**Theorem 24.2 (Trudeau [43])** *For all $c\in\Gamma$, $y^{cc}(c)$ is in $Core(C(N,c))$.*

In general, $y^{cc}$ is not in $Core(C^{Pub}(N,c))$. Notice that the two approaches are incompatible, as the cycle-complete approach is about bargaining for the use of an outside edge, which the public game supposes is available for free.

[3] Trudeau and Vidal-Puga [49] show that if $c$ is such that all edge costs are 0 or 1, then $Core(C^{*}(N,c))=Core(C^{Pub}(N,c))$ and $Core(C^{**}(N,c))=Core(C(N,c))$.

## 24.5 Axiomatic Analysis

In this section we focus on the axiomatic characterization of the three methods defined in the previous section.[4] This means that we find properties, or axioms, that are satisfied by the solution and such that no other can satisfy them simultaneously. Throughout this section, $y(c)$ is a generic solution.

The first property requires that a solution propose a core allocation.

**Core selection** Let $c \in \Gamma$. Then, $y(c) \in Core(C(N,c))$.

We now define three new properties. One of them is stronger and the other two are weaker than core selection.

The stronger version of core selection requires that no agent be worse off when new agents join the society. Formally,

**Population mononicity** Let $c \in \Gamma$ and $S \subset N$. Then, for all $i \in S$, $y_i(c) \geq y_i(c_S)$.[5]

A weaker version of core selection is due to Branzei et al. [20]. It requires undominance in only some coalitions:

**Upper bounded contribution** Let $c \in \Gamma$ and $P \subset N_0$ such that, for all $i, j \in P$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, $y(P \cap N) \leq \min_{i \in P \cap N} c_{0i}$.

Obviously, we are interested in axioms that are related with mcst problems. Among the numerous characterizations of the Shapley value in the general transferable utility game case, the *balanced contributions property* proposed by Myerson [34] is particularly interesting, since it is inspired by a property of edge deletion previously defined in [33].

In the mcst problem context, we say that a solution satisfies equal treatment if variation in the cost of an edge affects equally both adjacent nodes. Formally,

**Equal treatment** Let $c, c' \in \Gamma$ be such that $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \setminus \{i, j\}$. Then, $y_i(c) - y_i(c') = y_j(c) - y_j(c')$.

Equal treatment is, clearly, a fairness axiom. A weaker version that applies only when the change in the cost of the edge does not affect the total cost of the mcst problem is proposed by Trudeau [46]:

**Weak equal treatment** Let $c, c' \in \Gamma$ be such that $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \setminus \{i, j\}$ and $C(N, c) = C(N, c')$. Then, $y_i(c) - y_i(c') = y_j(c) - y_j(c')$.

[4] A similar exercise was done in [44].

[5] We use $y(c_S)$ to designate the cost allocation to the mcst game involving only agents in $S$.

For the next axiom, we need some additional notation. An edge $(i, j)$ between agents $i, j \in N$ is *relevant* if $c_{ij} \leq \max\{c_{0i}, c_{0j}\}$. An edge is *strictly relevant* if $c_{ij} < \max\{c_{0i}, c_{0j}\}$, *irrelevant* if it is not relevant, and *weakly irrelevant* if it is not strictly relevant.

Let $\overline{\Gamma}$ be the set of elementary cost matrices with no irrelevant edges.

Notice that an irrelevant edge will never belong to an optimal tree. A path $\psi_{ij}$ is an *irrelevant path* if it contains a weakly irrelevant edge. If all paths between $i$ and $j$ are irrelevant, then (one of) the efficient way(s) of connecting $\{i, j\}$ to the source is to connect them both directly to it. In other words, agent $i$ does not help agent $j$ connect to the source in a cheaper manner, and vice versa.

We say that an allocation satisfies *group independence* if we can partition agents in groups such that members of different groups only have irrelevant paths between them. Then, they never have any gain to cooperate with each other, even when considering the connection problem of subgroups of $N$. Formally,

**Group independence** Let $c \in \Gamma$ be such that there exists a partition $\mathcal{P}$ of $N$ such that for all $i \in S$ and $j \in T$, and all $S, T$ distinct in $\mathcal{P}$, we have that all $\psi_{ij} \in \Psi_{ij}(N_0)$ are irrelevant paths. Then, for all $i \in S \in \mathcal{P}$, $y_i(c) = y_i(c_S)$.

The next axiom is a stronger version of group independence, since it only requires the partition of $N$ (which does not need to be unique) to be able to connect to the source independently.

**Separability** Let $c \in \Gamma$ be such that there exists a partition $\mathcal{P}$ of $N$ such that $C(N, c) = \sum_{S \in \mathcal{P}} C(S, c)$. Then, for all $i \in S \in \mathcal{P}$, $y_i(c) = y_i(c_S)$.

Another version of group independence is when we build a partition of the set of followers of some node. Take, for example, the mcst depicted in Figure 24.3. Both nodes 2 and 3 always connect to the source through node 1. They form two different branches. When these branches obtain no benefits by connecting with other agents and the costs inside them are not lower than the costs on the path from the source to the linking node, then we should be able to remove one of the branches in order to compute the allocation of the others.

**Branch cutting** Let $c \in \Gamma$, $S \subset N$ and $k \in N_0 \setminus S$. If:

- all the nodes in $S$ are followers of node $k$,
- for all $i \in S$, $j \in N \setminus S$, $j \neq k$, $c_{ij}$ is a weakly irrelevant edge, and
- for all $i, j \in S \cup \{k\}$, $c_{ij} \geq c_e$ for all $e$ in a path from node $k$ to the source in any optimal tree,

then

$$y_i(c) = \begin{cases} y_i(c'_S) & \text{if } i \in S \\ y_i(c_{N \setminus S}) & \text{if } i \in N \setminus S, i \neq k. \end{cases}$$

where $c'_{0i} = c_{ki}$ and $c'_{ij} = c_{ij}$ for all $i, j \in S$.

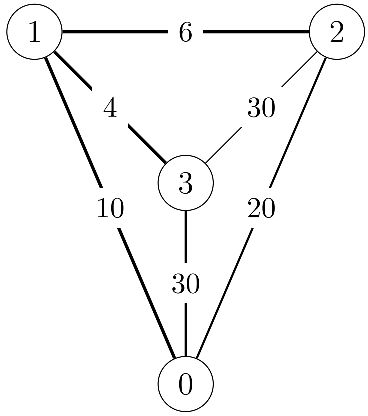


**FIGURE 24.3**: Nodes 2 and 3 constitute two branches of node 1.

Notice that branch cutting does not say anything about the cost share of node $k$ (in case $k \neq 0$). But, in this case, $y_k(c)$ can be deduced from budget balance once the other cost shares are known.

**Theorem 24.3 (Kar [28])** *The Kar rule is the unique solution which satisfies equal treatment, and group independence.*

Another relevant property of the Shapley value, in the context of cooperative game theory, is additivity. A natural definition of additivity in the context of mcst problems is to assume that $y(c + c') = y(c) + y(c')$, where $c + c'$ is defined in the natural way, i.e., $(c + c')_{ij} = c_{ij} + c'_{ij}$ for all $i, j \in N_0$.

However, no solution can satisfy this version of additivity in general. To see why, consider $N = \{1, 2\}$, $c_{12} = c'_{12} = 0$, $c_{01} = c'_{02} = 0$, and $c_{02} = c'_{01} = 1$. Then, $y_1(c) + y_2(c) = y_1(c') + y_2(c') = 0$, whereas $y_1(c + c') + y_2(c + c') = 1$.

The difficulty with this example is that there exists no tree that is optimal in all three problems $c$, $c'$, and $c + c'$. We can define a weaker version of this property by requiring additivity only between mcst problems that share at least an optimal tree $t$ such that, if we order the edges of $t$ in non-decreasing cost, then we can obtain the same order in both problems.

**Restricted additivity** Let $c, c' \in \Gamma$ be such that there exists a common optimal tree $t^* \in \mathcal{T}^*(c) \cap \mathcal{T}^*(c')$ and an order $\pi$ of the edges in $t^*$ such that $c_{\pi_1} \leq c_{\pi_2} \leq \cdots \leq c_{\pi_n}$ and $c'_{\pi_1} \leq c'_{\pi_2} \leq \cdots \leq c'_{\pi_n}$. Then, $y(c + c') = y(c) + y(c')$.

A sufficient condition for such an optimal tree to exist is that both problems share a common ordering of the edges according to their cost.

**Piece-wise linearity** Let $c, c' \in \Gamma$ be such that there exists an ordering $e_1, e_2, \ldots$ of the edges such that $c_{e_1} \leq c_{e_2} \leq \ldots$ and $c'_{e_1} \leq c'_{e_2} \leq \ldots$. Then, for all $\alpha, \beta > 0$, $y(\alpha c + \beta c') = \alpha y(c) + \beta y(c')$.

The next property is due to Trudeau [46] and uses the fact that if we do not have irrelevant edges, then there always exists a mcst in which a single agent is connected to the source. We can then divide the problem, first sharing the cost of the unique connection to the source, before sharing the cost to connect the remaining agents to that source-connected agent.

**Problem separation** Let $\bar{c} \in \Gamma$ be such that $\hat{c}_{ij} = 0$ for all $i, j \in N$. Let $\tilde{c}, \dot{c} \in \Gamma$ be such that $\tilde{c}_{0i} = \dot{c}_{0i} = \max_{i \in N} \hat{c}_{i0}$ and $\dot{c}_{ij} = 0$ for all $i, j \in N$. Then, if $\hat{c} + \tilde{c} - \dot{c} \in \overline{\Gamma}$, $y(\hat{c} + \tilde{c} - \dot{c}) = y(\hat{c}) + y(\tilde{c}) - y(\dot{c})$.

In the preceding property, the problem of connecting a single agent to the source is represented by $\hat{c}$, while the problem of connecting the remaining agents to the source-connected agent is in $\tilde{c}$. Since we added a large source-connection cost in that second problem, it is removed by subtracting $\dot{c}$.

Trudeau [46] also proposes a weaker version of problem separation that applies only to problems for which there is no edge used in a mcst that is more expensive than the cheapest edge connecting an agent to the source.

**Weak problem separation** Let $\hat{c} \in \Gamma$ be such that $\hat{c}_{ij} = 0$ for all $i, j \in N$. Let $\tilde{c}, \dot{c} \in \Gamma$ be such that $\tilde{c}_{0i} = \dot{c}_{0i} = \max_{i \in N} \hat{c}_{0i}$ and $\dot{c}_{ij} = 0$ for all $i, j \in N$. Then, if $\hat{c} + \tilde{c} - \dot{c} \in \overline{\Gamma}$ and $c_e \leq \min_{i \in N} c_{0i}$ for all edge $e$ in an optimal tree, $y(\hat{c} + \tilde{c} - \dot{c}) = y(\hat{c}) + y(\tilde{c}) - y(\dot{c})$.

The remaining properties are self-explanatory.

We require that agents that play the same role pay the same amount. Formally[6],

**Symmetry** Let $c \in \Gamma$ and $i, j \in N$ such that $c_{ik} = c_{jk}$ for all $k \in N_0 \setminus \{i, j\}$. Then, $y_i(c) = y_j(c)$.

The next property says the allocation does not depend on irrelevant edges.

**Independence of irrelevant edges** Let $c \in \Gamma$ and let $\overline{c} \in \Gamma$ be defined as $\overline{c}_{ij} = \min\{c_{ij}, \max\{c_{i0}, c_{j0}\}\}$ and $\overline{c}_{i0} = c_{i0}$ for all $i, j \in N$. Then, $y(c) = y(\overline{c})$.

**Theorem 24.4 (Trudeau [46])** *The Kar rule is the only solution that satisfies weak equal treatment, group independence, piece-wise linearity, problem separation, symmetry, and independence of irrelevant edges.*

The Kar rule also satisfies other nice properties, such as *cost monotonicity* [22], which states that an increase of the cost of an edge cannot benefit any adjacent agent. Formally,

[6] Some theorems below use, in the original article, the stronger property of anonymity, which requires that the allocation not depend on the name of the agents. All of them hold with symmetry.

**Cost monotonicity** Let $c, c' \in \Gamma$ be such that, for some $i \in N$ and $j \in N_0$, $c_{kl} = c'_{kl}$ for all $k, l \in N_0 \setminus \{i, j\}$, and $c_{ij} < c'_{ij}$. Then, $y_i(c) \leq y_i(c')$.

The property incentives efficiency, as it prevents nodes from benefitting by increasing their connection costs. In case sabotage of non-adjacent connection costs is possible, a stronger version of cost monotonicity is desirable. *Solidarity* states that an increase of the cost of an edge does not benefit *any* agent (and not only its adjacent ones). Formally,

**Solidarity** Let $c, c' \in \Gamma$ be such that $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$. Then, for all $i \in N$, $y_i(c) \leq y_i(c')$.

Despite the other nice properties, the Kar rule does satisfy neither solidarity nor, as shown in [22] and in Subsection 24.4.1, stability.

The next theorem links the Kar rule and the folk rule:

**Theorem 24.5 (Trudeau [46])** *A solution satisfies weak equal treatment, group independence, piece-wise linearity, weak problem separation, symmetry, and independence of irrelevant edges if and only if it is a convex combination of the Kar rule and the folk rule, i.e., there exists* $\alpha \in \mathbb{R}$ *such that* $y = \alpha y^K + (1 - \alpha) y^f$.

There exist several characterizations of the folk rule using the restricted additivity or piece-wise linearity:

**Theorem 24.6 (Branzei et al. [20])** *The folk rule is the only solution that satisfies upper bounded contribution, piece-wise linearity, and symmetry.*

Clearly, we can replace upper bounded contribution by core selection in this characterization.

**Theorem 24.7 (Bergantiños and Vidal-Puga [14])** *The folk rule is the only solution that satisfies separability, restricted additivity, and symmetry.*

**Theorem 24.8 (Bergantiños et al. [8])** *The folk rule is the only solution that satisfies core selection, restricted additivity, symmetry, and solidarity.*

The next axiom states that if all the nodes are close to each other and are at the same distance to the source, then any increase in the cost to the source should be shared equally among the agents. An example is depicted in Figure 24.4. All agents are equally far away from the source. So, an optimal tree should connect any of them to the source and then the others connect to the source through this one. The property of equal share of extra costs states that the cost allocation should be the same as before, and the extra cost (6 in our example) is shared equally among the agents, i.e., $y_i(c') = y_i(c) + 2$ for all $i$, where $c$ is the cost matrix on the left and $c'$ is the one on the right.

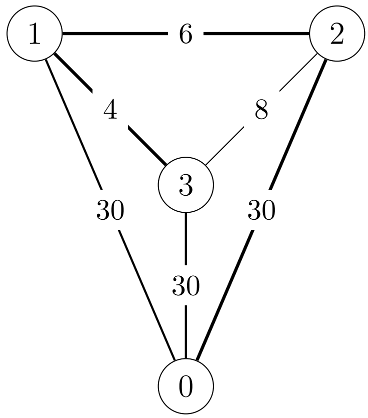


**FIGURE 24.4**: Example of equal share of extra costs.

**Equal share of extra costs** Let $c, c' \in \Gamma$ and $x_0, x_0' \in \mathbb{R}$ be such that $c_{0i} = x_0 > x_0' = c'_{0i} \geq c_{ij} = c'_{ij}$ for all $i, j \in N$. Then, for all $i \in N$, $y_i(c) = y_i(c') + \frac{x_0 - x_0'}{n}$.

The next property is a weaker version of equal share of extra costs:

**Equal share of cost reduction** Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, for all $j \in N$, $y_j(c') = y_j(c) - \frac{x}{n}$.

An opposite viewpoint is that the cost reduction should be assigned solely to the agent that is closer to the source and can connect freely to all others.

**Full share of cost reduction** Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, $y_i(c') = y_i(c) - x$ and $y_j(c') = y_j(c)$ for all $j \in N \setminus \{i\}$.

A compromise viewpoint from both previous properties is to make sure the fraction of the savings going to the cheapest source connector is the same in all such situations:

**Constant share of cost reduction** Let $c, c' \in \Gamma$, $i \in N$ and $x \in [0, c_{0i}]$ such that $c_{0i} \leq c_{0j}$, $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ otherwise and, for all $j \in N \setminus \{i\}$, there exists a path $\psi \in \Psi_{ij}(N_0)$ such that $c_e = 0$ for all $e \in \psi$. Then, there exists $\alpha \in \mathbb{R}$ such that $y_i(c') = y_i(c) - \frac{x}{n}(1 + (n-1)\alpha)$ and $y_j(c') = y_j(c) - \frac{x}{n}(1 - \alpha)$ for all $j \in N \setminus \{i\}$.

The following property, as equal share of extra costs, applies to problems where all the agents are equally far away from the source. It states that the cost sharing should be done in the same order as we find an optimal tree, i.e., by picking up one agent randomly, and connecting her to the source.

**Decomposition** Let $c \in \Gamma$ and $x_0 \in \mathbb{R}$ be such that $c_{0i} = x_0 \geq c_{ij}$ for all $i, j \in N$. Then, for all $i \in N$,

$$y_i(c) = \sum_{j \in N \setminus \{i\}} \frac{y_i(c^j)}{n} + y_i(\hat{c})$$

where $c^j_{0k} = c_{jk}$ and $c^j_{kl} = c_{kl}$ for all $k, l \in N \setminus \{j\}$, and $\hat{c}_{0j} = x_0$ and $c^j_{kl} = 0$ for all $k, l \in N$.

The following property also applies to problems where all the agents are far away from the source, but without requiring them to be equally far away nor any other similarity. It says that, in this case, agents should share the extra cost in the same way in both problems.

**Constant share of extra costs** Let $c, c' \in \Gamma$ such that $c_{0i} = \max_{j,k \in N_0} c_{jk}$ and $c'_{0i} = \max_{j,k \in N_0} c'_{jk}$ for all $i \in N$ and $x1$ be the cost matrix defined as $x1_{0i} = x$ and $x1_{ij} = 0$ for all $i, j \in N$, for some positive real number $x$.

Then, $y(c + x1) - y(c) = y(c' + x1) - y(c')$.

**Theorem 24.9 (Bergantiños and Vidal-Puga [11])** *The folk rule is the only solution that satisfies population monotonicity, solidarity, and equal share of extra costs.*

**Theorem 24.10 (Bergantiños and Kar [5])** *The folk rule is the only solution that satisfies population monotonicity, symmetry, solidarity, and constant share of extra costs.*

**Theorem 24.11 (Trudeau [45])** *The folk rule is the only solution that satisfies core selection, piece-wise linearity, branch cutting, decomposition, and equal share of cost reduction.*

The next theorem links the folk rule and the cycle-complete rule:

**Theorem 24.12 (Trudeau [45])** *A solution satisfies core selection, piece-wise linearity, branch cutting, decomposition, and constant share of cost reduction if and only if it is a linear combination of the folk rule and the cycle-complete rule, i.e., there exists* $\alpha \in [0, 1]$ *such that* $y = \alpha y^f + (1 - \alpha) y^{cc}$.

A strengthening of constant share of cost reduction to give the savings to the agent with the cheap cost to the source yields a characterization of the cycle-complete solution.

**Theorem 24.13 (Trudeau [45])** *The cycle-complete rule is the only solution that satisfies core selection, piece-wise linearity, branch cutting, decomposition, and full share of cost reduction.*

## 24.6 Correspondences with Other Concepts

We discuss how other solution concepts have been used in the mcst literature, and the links that have been found with the Shapley value.

### 24.6.1 Weighted Shapley Values

The weighted versions of the Shapley value [27, 37] have also played a relevant role in mcst problems. Moreover, [17, 21] also use the term weighted Shapley value to refer to restricted orders in the contribution vectors so that an optimal tree is constructed via Prim's algorithm following that order. Bird [17] proves that this solution belongs to the irreducible core.

In what follows, we use the definition of weighted Shapley values first suggested by Shapley [37] and studied by Kalai and Samet [27].

Bergantiños and Lorenzo-Freire [9, 10] study the weighted Shapley values of the optimistic game introduced in [12] and prove that they are obligation rules. Moreover, they characterize these rules using population monotonicity, solidarity, and weighted version of equal share of extra cost where the extra cost is divided proportionally to the weights of the agents.

Trudeau [47] obtains a family of weighted Shapley values when studying an extension of mcst problems in which some agents do not need to be connected to the source.

Gómez-Rúa and Vidal-Puga [25] study mcst situations in which agents can merge in advance, paying their internal costs. They show that this situation can lead to inefficiencies and prove that the weighted Shapley value of the irreducible cost vector, with weights given by the size of the nodes, is immune to this manipulation. It also inherits most of the nice properties of the folk rule, such as population monotonicity, core selection, solidarity, and piece-wise linearity.

### 24.6.2 The Core and the Nucleolus

The *excess* of a coalition $S$ in a TU game $(N, v)$ with respect to a preimputation $x$ is defined as $e(S, x, C) = C(S) - \sum_{i \in N} x_i$. The vector $\theta(x)$ is constructed by rearranging the $2^n$ excesses in (weakly) increasing order. If $x, y \in \mathbb{R}^N$ are two allocations, then $\theta(x) >_L \theta(y)$ means that $\theta(x)$ is lexicographically larger than $\theta(y)$. As usual, we write $\theta(x) \geq_L \theta(y)$ to indicate that either $\theta(x) >_L \theta(y)$ or $x = y$.

The *nucleolus* of the game $C$ is the set

$$Nu(C) = \{x \in X : \theta(x) \geq_L \theta(y) \forall y \in X\}$$

where $X = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = C(N), x_i \geq v(\{i\}) \forall i \in N\right\}$ is the set of *imputations* (individually rational preimputations). When $X \neq \emptyset$, as it is the case for the TU games we study here, it is well known that $Nu(C)$ is a singleton, whose unique element we denote, with some abuse of notation, also as $Nu(C)$.

Let $\Pi$ be the set of permutations of $N$. For all $\pi \in \Pi$, let $y^\pi \in Core(C)$ be the allocation that lexicographically maximizes the allocations with respect to the order given by the permutation. The permutation-weighted average of extreme points of the core is the average of these allocations: $\gamma(C) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^\pi(C)$. If the game is concave, $\gamma$ is the Shapley value. It is also closely related to the selective value [50] and the Alexia value [41], the permutation-weighted average of leximals.

Consider the subset of mcst problems known as elementary[7] mcst problems: For any $i, j \in N_0$, $c_{ij} \in \{0, 1\}$. Let $\Gamma^e$ be the set of elementary cost matrices.

**Theorem 24.14 (Trudeau and Vidal-Puga [48, 49])** *For all $c \in \Gamma^e$, we have $y^{cc}(c) = Nu(C(N, c)) = \gamma(C(N, c))$.*

**Theorem 24.15 (Trudeau and Vidal-Puga [48, 49])** *For all $c \in \Gamma^e$, we have $y^f(c) = Nu(C^{Pub}(N, c)) = \gamma(C^{Pub}(N, c))$.*

## 24.7 The Shapley Value in Other Related Problems

Some of the different versions of the Shapley value have also been studied in different subclasses and extensions of mcst problems, but it is still a very unexplored field of research. In particular, the following literature focuses on the extensions of the folk solution in the private game case.

Dutta and Mishra [23] and Bahel and Trudeau [2] extend the folk rule to minimal cost arborescence problems, where the cost vector describing the cost of connecting each pair of nodes is not necessarily symmetric. An extension of the cycle-complete solution is offered in the latter.

Bergantiños and Gómez-Rúa [3, 4] extend the folk rule to mcst problems with groups, where agents are grouped by a partition, such that the nodes inside each member of the partition (a group) are more related to each other than to any node in another group.

[7] These cost games are named *information graph games* by Kuipers [30].

Subiza et al. [39] study the folk rule in simple mcst problems, where only two different costs are possible.

Finally, from a non-cooperative point of view, the folk rule appears as subgame perfect Nash equilibrium cost allocation in several mechanisms applied to mcst problems [15, 26].

## 24.8 Conclusions

In this chapter we have reviewed the literature on the minimum cost spanning tree problems. This literature is unique in that most of the allocation methods considered are Shapley values. There are (at least) five different ways to define a game based on a mcst problem, before taking the Shapley value. The games vary depending on how we set the ground rules: Who has access to which nodes, what cost we consider for each edge, etc. The solutions vary depending on whether we care about core stability, if we allow coalitions to use the nodes of their neighbours, and if we take an optimistic or pessimistic view of the game. The corresponding axiomatic analysis reflects the choices made in how we define the game.

## 24.9 Acknowledgments

Christian Trudeau acknowledges financial support by the Social Sciences and Humanities Research Council of Canada [grant number 435-2014-0140]. Juan Vidal-Puga acknowledges financial support by the Spanish Ministerio de Economía y Competitividad (ECO2014-52616-R), Ministerio de Economía, Industria y Competitividad (ECO2017-82241-R), and Xunta de Galicia (GRC 2015/014). The authors thank William Thomson for helpful comments.

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