Mathematics Education

Semiotics

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Edited by

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1. CONSTRUCTING KNOWLEDGE SEEN AS A SEMIOTIC ACTIVITY

> In studying the laws of signs, we are in effect studying the manifested laws of reasoning.
>
> George Boole,
> An Investigation of the Laws of Thought
> London, 1854

> The object of reasoning is to find out, from the considerations of what we already know, something else that we do not know…
>
> Charles Sanders Peirce,
> Writings of Charles S. Peirce 3.244

ABSTRACT

During the last two decades, semiotics has been attaining an important explanatory role within mathematics education. This is partly due to its wide range of applicability. In particular, the success of semiotics in mathematics education may be also a consequence of the iconicity and indexicality embedded in symbols, in general, and mathematical symbols, in particular. The introductory chapter discusses issues of signs, sign use, and communication. On the one hand, it shows how semiotics elucidates the way knowledge and experience of mathematics students can co-construct each other. On the other, it shows how students' construction of mathematical knowledge is linked to successful communication mediated by visible signs with their rule-like transformations. In this sense, the systems of signs and communication through them are closely tied when students send and receive mathematical messages.

SEMIOTICS AND COMMUNICATION

Semiotics is certainly a very old subject that has evolved over time. It is a field of study to which the best thinkers of each era have contributed. For a brief summary of its historical evolution, we refer the reader to the foreword, written by John Deely, of Doyle's 2001 book "*The Conimbricenses: Some questions on signs*". Deely, in less than four pages, presents a brief and substantive summary of the historical evolution of semiotics. He traces it from the Romans, to the Greeks, to the middle ages, to Descartes, to Locke, to Peirce, to Saussure, just to name only some.

For centuries, the notion of sign[1] has occupied the minds of great scholars like Plato, Aristotle, St. Augustine, Aquinas, Poinsot, Locke, Kant, Peirce, and Saussure (Doyle, 2001; Sebeok, 1991; Nöth, 1990). They grappled with the conceptualization of "signs", their nature, and their function in thinking and human interaction. Deely (2001) presents a historical argument to show that signs are *the universal instruments of experience and thought* and that they have, at once, ontological and epistemological functions. Signs were believed to have intrinsic meanings independent of the interpreter. In a certain sense, the meanings of the signs were considered to be objectively interpreted without taking into account the subjectivity of the interpreting Person. In the broadest sense, signs were seen as mediating perceptible entities that prompted thought, that facilitated the expression of thought, and that embodied idiosyncratic and conventional thought.

The history and the theory of semiotics co-evolved, for centuries, synchronically and diachronically (Deely, 1990; Doyle, 2001). Deely (2001) also argues that the history of semiotics is, first and foremost, the history of the achievement of a semiotic consciousness that systematically emerged from the influence and repercussion that signs had in every sphere of knowledge and experience. In a basic sense, theories of semiotics explains how knowledge and experience co-construct each other; how knowledge and experience depend on signs and actions on signs (i.e., semiosis); how signs indicate and symbolize their Objects; and how signs are interpreted.

The history and evolution of semiotics also goes hand-in-hand with the history and development of communication (Uexküll, 1957; Sebeok, 1991; Maturana & Varela, 1992). Thus, both communication and socio-cultural systems of signs are entwined when organisms, of the same species, send and receive messages. Uexküll (1957) argues that the *human semiotic reality* is different from that of any other organism. This reality, he contends, goes beyond the *adjacent environment* that an observer might objectively *see*; it involves a *subjective environment* as each Person internally might perceive it and construct it from his own perspective. In this semiotic reality, each Person's process of meaning-making follows a dynamic interweaving path. This path is not only guided by consensually constructed meanings already considered objective, but it is also guided by subjectively constructed meanings.

As a result, communicating and meaning-making are always intertwined and mediated by perceptible signs that *senders* use or produce and that *receivers* interpret through alternating-sequences of message-exchange. The meanings intended to be carried out by signs are co-constructed by senders and receivers through their own processes of interpretation. Undoubtedly, the meaning-making process undergoes cyclic and recursive transformations instead of being a linear and straightforward process. To have a glimpse at the complexity of this process, let us ponder over the description given by Ogden and Richards (1923). They consider that the meaning intended by the sender is that to which the *signifier* [*sign-vehicle*] *actually refers*; that to which the sender *ought to be referring;* and that to which he himself *believes to be referring*. On the other hand, the meaning constructed by the receiver is that to which *he actually refers*; that to which he himself *believes to be referring;* and that to which *he* himself *believes the sender to be referring*.

In general, humans construct their own meaning about themselves and about the world in the midst of communicating with others (Mead, 1962; Wertsch, 1985; Vygotsky, 1987; Lanigan, 1988). Likewise, we could say that students construct and co-construct their mathematical meanings in the midst of mathematical communications and that, if they are motivated and well-directed, such meanings will progressively become refined to approximate the objective meanings that are the focus of the mathematical exchange.

PEIRCE'S CATEGORIES AND HIS TRIADIC "SIGN"[2]

One thing is to understand that the Peircean SIGN has three components (Object, representamen/sign-vehicle, interpretant) instead of only the two components (signified, signifier) implicitly or explicitly considered before him and even after him. Another is to understand the implications that the third component of the SIGN—the interpretant—has in meaning-making. To understand meaning-making is also to understand the active role of the *interpreting Person* in the re-construction of the *real Object* of a SIGN from the cues and hints carried out by sign-vehicles which indicate only certain aspects of the *real Object*. To understand meaning-making is also to understand the interplay between objectivity (i.e., the construction of the non-subjective *real Object* of a SIGN) and subjectivity (i.e., the construction of subjective *dynamic objects* in the mind of the interpreting Person that, when integrated and unified, will eventually approximate the *real Object* of that SIGN). In other words, to understand meaning-making is to understand the role of sign-vehicles and interpretants in the conceptualization of the "real" Object of a SIGN. Briefly put, to understand meaning-making is to understand the Peircean triadic SIGN.

Peirce uses the term "real" to differentiate between true and untrue cognitions: "Cognitions whose objects are *real* and those whose objects are *unreal*, between an *ens* relative to private inward determinations, to the negations belonging to idiosyncrasy, and an *ens* such as *would stand in the long run*. The *real*, then, is that which, sooner or later, information and reasoning would finally result in, and which is therefore *independent* of the vagaries of me and you" (CP 5.311, quoted in Fisch, 1986, p. 187, emphasis added). It is under this perspective of "real" in the sense of collective and consensual that Peirce differentiates between the *real*, the *immediate*, and the *dynamic* object of the SIGN.

The *real Object* of the SIGN does not change when it is encoded into different sign-vehicles and interpreted by different people. The *dynamic object* is the changing object subjectively constructed in the mind of the interpreter as a result of ongoing interpretations of different yet interrelated sign-vehicles. The *immediate object* is constituted by those aspects of the real Object materialized in a sign-vehicle. Therefore, one-and-only-one sign-vehicle does not have the capacity to represent, at the same time, all the aspects of the *real Object*. Due to the representational limitations of sign-vehicles, several of them may be needed to represent as many aspects as possible of the *real Object* of the SIGN.

How do the three components of the Peircean SIGN co-construct each other? Figure 1 presents a diagram of the synergistic dyadic and irreducible relationships among them. This figure will be explained in detail later. In this section we try to understand the relations between the components of the SIGN and the three Peircean categories.

Peirce's semiotics is founded on his three connected categories, which can be differentiated from each other, and which cannot be reduced to one another. Peirce argued that there are three and only three categories: "He claims that he has look long and hard to disprove his doctrine of three categories but that he has never found anything to contradict it, and he extends to everyone the invitation to do the same" (de Waal, 2013, p. 44). The existence of these three categories has been called Peirce's theorem.

Firstness, secondness, and thirdness are the three categories. He considers these categories to be both ontological and phenomenological; the former deals with the nature of being and the latter with the phenomenon of conscious experience.

Firstness denotes the character of being a first, "the mode of being of that which is such as it is, positively and without reference to anything else" (CP 8.328). Phenomenologically, firstness is a condition of unmediated unreflexive access. Firsts are experience without reaction, cause without effect. It is a first level of meaning derived from bodily and sensory processes.

Peirce also argues that one cannot represent the idea of first without immediately introducing the idea of something else; that is a second. He explains that to realize a first, even if only in thought, some second must be used. Secondness is "the mode of being of that which is such as it is, with respect to a second but regardless of any third" (CP 8.328). Phenomenologically, secondness is a condition of mediated but not yet reflexive access. Seconds are experience and the reaction it causes together with the effect it provokes; but not yet a reflection on the reaction or the effect.

Peirce similarly explains that logically seconds do not involve thirds as part of their conceptions; to realize a second, even if only in thought, some third must be involved. Thirdness is "the mode of being of that which is such as it is, in bringing a second and a third into *relation* to each other" (CP 8.328, italics added). That is, thirdness is mediation between firstness and secondness. Phenomenologically, thirdness is a condition of both mediated and reflexive access. Thirds are experience and reaction together with the reflection upon that reaction. They are cause, effect, and the extension of that effect in the form of habit or convention or law. According to Peirce, we can abstract a first from a second and a third; however, we cannot abstract a first only from a second because a second will not be there without a first to which it is a second. Nonetheless, we can abstract a second from a third (de Waal, 2013).

To gain a deeper insight about these categories, we present the Chart 1 which was taken from http//en.wikipidea.org/wiki/Charles_Peirce. This chart presents firstness, secondness, and thirdness as a synthesis of two ground breaking papers from Peirce: "On a new list of categories," (1867) and "How to make our ideas clear" (1878). It compares the three categories according to criteria specified in the top row. These criteria shed light on their phenomenological and ontological aspects. We can make sense of their phenomenological dependence by reading each column from top to bottom and *vice versa*. In contrast, each row structures a description of each category that shows their ontological nature. Chart 2, synthesizes, in terms of the categories, the nature of the sign-vehicle (first row), the nature of the relation between the sign-vehicle and the Object (second row), and the relation between the sign-vehicle and the interpretant (third row). Together Charts 1 and Chart 2 lend themselves not only to understand the complexity of sign-vehicles but also to gain insight into the explanatory potential of the Peircean triadic SIGN in matters that relate to the teaching-learning of mathematics.

In Chart 2 we observe that each of the phenomenological categories (rows) allows for the emergence of the ontological categories of firstness, secondness, and thirdness. For example, the phenomenological category of firstness co-constructs the ontological categories of firstness, secondness and thirdness. Phenomenologically, the nature of the sign-vehicle is a firstness—a first level of meaning derived from bodily and sensory processes. Ontologically, however, it could be a *qualising*—a firstness or a condition of unmediated, unreflexive access. It could be a *sinsign*—a secondness or the level of cause and the effect it provokes, but not yet a reflection on the effect. It could also be a *legising*—a thirdness or the level of reflection on the cause and its effect as well as the possible formation of a habit, or a convention, or a law.

Likewise, the phenomenological secondness of the sign-vehicle (the relation between the sign-vehicle and the object) has the potential to trigger ontological forms of firstness (icon), secondness (index), and thirdness (symbol). A mathematical example may shed some light on how mathematical notations, used in different mathematical situations, could change their character, in the mind of the interpreter, from iconic, to indexical, to symbolic. Let's consider the command "add four and five." Children are induced to use the notation 4+5=____. That is, children are led to use two symbols "+" and "=", which are seen by them as icons to abbreviate the given command (ontological firstness). When they add, 4+5 = 9, they transform the initial abbreviation into the act of adding (ontological secondness). The performance of several of these additions indicating "the answer or result" has the potential to trigger, later on, the generalization "the addition of two numbers is another number." This can be symbolized as $a+b=c$ where a, b, *and* c are any numbers (ontological thirdness). Put it briefly, the phenomenological secondness of sign-vehicles has the potential to trigger nested ontological forms of firstness, secondness, and thirdness.

It is important to notice that the triad (icon, index, symbol) is not a separate and autonomous species of sign-vehicles as if it were dogs, cats, and mice, as Fisch (1986) puts it. Rather this triad is nested so that more complex sign-vehicles contains

PEIRCEAN CATEGORIES					
Name:	***Typical characterization:***	***As universe of experience:***	***As quantity:***	***Technical definition:***	***Valence*:***
FIRSTNESS	*Quality of feeling*	*Ideas, chance, possibility*	*Vagueness, "some"*	*Reference to a ground (a ground is pure abstractness of quality)*	*Essentially monadic (the quale, in the sense of the thing with the quality)*
SECONDNESS	*Reaction, resistance*	*Brute facts, actuality*	*Singularity, discreteness*	*Reference to a correlate (by its relate)*	*Essentially dyadic (the relate and the correlate)*
THIRDNESS	*Representation*	*Habits, laws, necessity*	*Generality, continuity*	*Reference to an interpretant*	*Essentially triadic (Object, sing-vehicle, interpretant)*

Chart 1. Peircean categories: firstness, secondness, and thirdness

** Valence: The capacity of one thing to react with or affect another in some special way, as by attraction or by the facilitation of a function or activity*

Ontological Categories

Phenomenological Categories

		FIRSTNESS *(qualia)*	***SECONDNESS*** *(relation)*	***THIRDNESS*** *(generalization)*
FIRSTNESS *(feeling/perception)*	**A *sign-vehicle* could be seen as a(n)**	"mere quality" ***Qualisign*** *(a firstness)*	"actual existence" ***Sinsign*** *(a secondness)*	"general law" ***Legisign*** *(a thirdness)*
SECONDNESS *(reaction)*	**A *sign-vehicle* relates to its *Object* in having some**	"*similarity* to some quality of the *Object*" ***Icon*** *(facts of firstness)*	"*existential relation* with the *Object*" ***Index*** *(facts of secondness)*	"*relation* to the *interpretant*" ***Symbol*** *(facts of thirdness)*
THIRDNESS *(reflection)*	**A sign-vehicle is represented by its *interpretant* as a "sign" of**	"possibility" ***term/rheme*** *(sign of firstness)*	"fact" ***proposition/dicisign*** *(sign of secondness)*	"reason" ***Argument*** *(sign of thirdness)*

Chart 2. The sign-vehicle framed within the categories of firstness, secondness, and thirdness

and involves specimens of simpler sign-vehicles. Symbols typically involve indices which, in turn, involve icons. Conversely, icons are incomplete indices which are, again, incomplete symbols. A nonmathematical example may shade some light on this issue. The American flag can momentarily function as an icon, an index, or a symbol. It functions as an *icon* when attention is focused both on the number of Strips (7 read and 6 white) to represent the 13 founding colonies, and on the number of Stars (50 white stars on a blue rectangle) to represent the 50 states in the union. It functions as an *index* when, in times of war, is pointed at the fighting target. It also functions as a *symbol* when it is used to honor the troops, to represent the country, or to prosecute cases on flag burning. In summary, the phenomenological secondness (which will not exist without a phenomenological firstness) of the sign-vehicle has the potential to trigger ontological forms of firstness, secondness, and thirdness.

The same can be said of the phenomenological thirdness of the sign-vehicle which involves elements of awareness and reflection. The triad (term/rheme, proposition/decising, argument) indicates ontological forms of firstness, secondness, and thirdness. Propositions are seconds that cannot come into existence without terms. Arguments cannot come into existence without terms and propositions; that is, propositions are intertwined to construct convincing or persuasive arguments. Conversely, terms are incomplete propositions which are, again, incomplete arguments.

By the same token, observing the columns of Chart 2 we see again that each of the ontological categories of firstness, secondness, and thirdness (columns) contains within itself nested forms of phenomenological categories of firstness, secondness, and thirdness. For example, the ontological firstness of a quale (real or imagined) necessarily provokes some type of iconic representation to visualise it and, in turn, a term/rheme is created to make it an object of discourse.

The challenge in the classroom is to infer at what level (firstness, secondeness, or thirdness) students interpret mathematical sign-vehicles and what is the effect of their constructed meanings in their mathematical reasoning and habits of thinking. For example, it is common knowledge that students experience difficulties both in translating mathematical word-problems into mathematical expressions and in translating geometric propositions, given in natural language, into geometric diagrams and symbolism. This indicates that students may experience and interpret these meanings at different phenomenological and ontological levels of firstness, secondness, and thirdness.

DYADIC RELATIONS AMONG THE COMPONENTS OF THE “SIGN”

Focusing on the diagram in Figure 1, we proceed with our exploration of the three dyadic relations between the components of the SIGN. We used the vertices of two joint triangles to position the three components (each pair of joined vertices stands for a component of the SIGN). We explore the dyadic relations between the three components in the clockwise and in the counter-clockwise directions. In

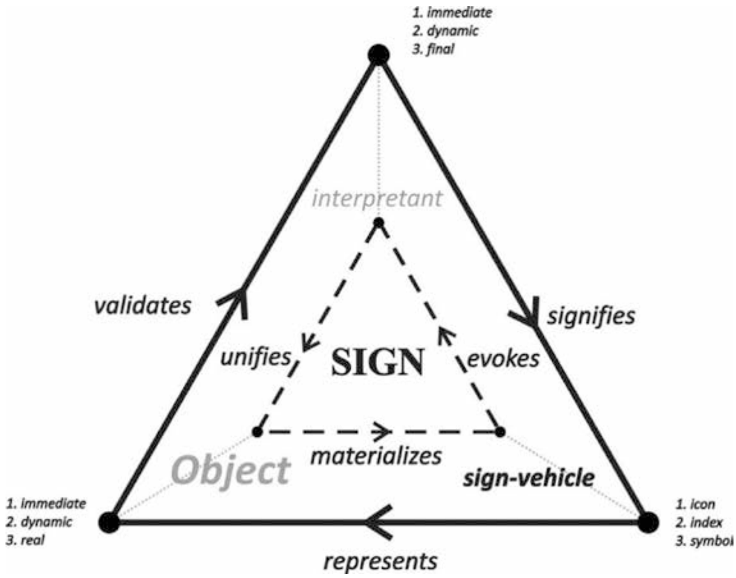

Figure 1. The sign-vehicle mediates between the object and the interpretant

this figure, the grey color of the word *Object* indicates the abstract nature of this component, and the grey color of the word *interpretant* indicates its interpretive and changing nature.

In the counter-clockwise direction (represented by the interior triangle), the sign-vehicle materializes certain aspects of the real Object. Peirce calls these aspects of the Object represented by or materialized in the sign-vehicle the immediate object. The sign-vehicle evokes an interpretant in the mind of the Person who perceives it and who is willing to make some kind of sense. This interpretant gives rise to an object, in the mind of that Person, which has some degree of likeness to the real Object of the SIGN. Peirce calls this object a dynamic object. This dynamic object is continually modified in the mind of the interpreting Person when prior sign-vehicles are reinterpreted or when new sign-vehicles, representing the same Object, are interpreted. This is to say that the dynamic object is the result of the Person's process of integration and unification of subjective immediate and dynamic interpretants (these kinds of interpretants will be described toward the end of this section). Put it differently, the sequence of dynamic objects is the result of the Person's ongoing process of conceptualization. This counter-clockwise direction starts with the real Object of the SIGN, leading the process of conceptualization, and ends up with a re-construction of the real Object, which is somewhat similar to it. This is, in fact, the Person's conceptualization of the real Object of the SIGN. Therefore we can conclude that the interpretant plays a fundamental role in the process of conceptualization as the following quote from Peirce clearly indicates.

> Cognition is a consciousness of a sign [SIGN], and a triple consciousness: of the sign [sign-vehicle], of the real Object cognized, and of the meaning or interpretation of the sign [interpretant], which the cognition *connects* with that Object. (Peirce, CP 5.373, italics and brackets added)

In the clockwise direction (represented by the exterior triangle), the *real Object* gives validity to the Person's engendered interpretants (immediate, dynamical, and final) and the sequences of *dynamic objects* constructed during the Person's process of conceptualization. Dynamic objects are encoded into conventional or idiosyncratic sign-vehicles, which role is to represent a conceptualization of the *real Object* of the SIGN. This direction also starts with the *real Object* of the SIGN and ends up with a construction of an object that approximates it. This approximation is the Person's inferential conceptualization of the *real Object*.

Needless to say that the interpretation and re-interpretation of sign-vehicles to infer the *real Object* of the SIGN is a continual transformational process which does not happen at random or as a result of one-and-only-one act of interpretation. Sebeok (1991) argues that the Person's memory is, in effect, a reservoir of interpretants that keeps, to some extent, *connecting* the inferences of the interpreting Person to the *real Object* of the SIGN.

It is important to note that implicitly embedded in the triadic SIGN is the two-fold function of the sign-vehicle. On the one hand, the sign-vehicle is functioning as such *from within* the cognitive powers of the interpreting Person. On the other hand, the sign-vehicle is functioning as such because of the influence of cognitive powers *from without* the interpreting Person (Fisch, 1986). It is also important to keep in mind that a symbolic sign-vehicle, independently of the intention of the sender, may be interpreted by the receiver either as an icon, or as an index, or as a symbol. This is so because a symbol is a thirdness and it has embedded a secondness (index) which, in turn, has embedded a firstness (icon).

Peirce unfolds the interpretant to take into account a wide range of possible interpretations brought about by what is represented, implicitly or explicitly, in the sign-vehicle. The *immediate interpretant* is "a vague possible determination of consciousness" (R 339: 287) or "the immediate pertinent possible effect in its unanalyzed primitive entirety" (R 339: 289).

As with any possibility, an immediate interpretant can be actualized when a new interpretant is concretely formed in a specific act of interpretation. This actualization is what Peirce calls a *dynamic interpretant* even when they are merely a first hunch, a wild guess, or the product of wishful thinking. The dynamic interpretant is "the actual effect produced upon a given interpreter on a given occasion in a given stage of his consideration of the sign [sign-vehicle]" (R 339: 288). A sequence of *dynamic interpretants* produces a sequence of *dynamic objects* that progressively change.

The *final interpretant* (which Peirce also calls normal or genuine interpretant) is defined as "the effect of the sign [sign-vehicle] produced upon any mind upon which circumstances should permit it to work out its full effect" (SS, 111), or "the one interpretative result to which every Interpreter is destined to come if the sign [sign-vehicle] is sufficiently considered" (id.). Peirce argues that the final interpretant is an ideal, in that it embraces, "all that the sign [sign-vehicle] could reveal concerning the Object" (R 339: 276).

In summary, the immediate interpretant (a firstness) is an abstraction consisting in a possibility. The dynamic interpretant (a secondness) is a single actual cognitive event producing a dynamic object. The final interpretant (a thirdness) is that refined dynamic interpretant that tend to produce "good" approximations of the *real Object* of the SIGN. These approximations will continue as a result of unlimited semiosis.

THE INDEXICALITY OF SIGN-VEHICLES

Peirce's notion of icon is as old as Plato in the sense that the signifier imitates the signified; the notion of symbol was also conceptualized to a certain degree. However, several semioticians recognize as unique Peirce's contribution to the indexical aspects of sign-vehicles and the embededness of iconicity in indexicality and indexicality in symbolicity. Wells (1967) argues that Peirce's notion of index is novel and fruitful in the sense that *indication* in the form of pointing, ostention, and deixis is not only irreducible but also indispensable.

Using quotes from Peirce's writings, Sebeok (1991) summarizes, the importance of the indexicality in sign-vehicles: indexicality hinges on association by contiguity (CP 3.419, 1892), not as iconicity does, by likeness; nor does it rest, in the manner of a symbol, on intellectual operations; "indexes, whose relation to their objects consist in a correspondence in fact,…direct the attention to their object by blind compulsion" (CP 1.558, 1867).

Interpretants evoked by the indexicality of sign-vehicles generate a succession of dynamic objects relating cause-to-effect (a cause produces an effect) and *vice versa*. In other words, the indexical character of sign-vehicles moves onwards the Person's process of *objectification* and therefore the process of conceptualization (abstraction and generalization). By objectification, we mean the reconstruction of the *real Object* of the SIGN from the clues and prods of the sign-vehicles. In brief, objectification is considered to be the Person's *inferential conceptualization* of the *real Object* of the SIGN from the hints, clues, traces, and conventions carried out by sign-vehicles.

Why is it that all sign-vehicles have indexical character? An *iconic sign-vehicle* represents the *real Object* because it has a non-arbitrary and direct simulative connection to that referent. That is, icons directly relate to the *real Object* of the SIGN. An *indexical sign-vehicle* represents the *real Object* of the SIGN because it has an existential connection to that referent. This is to say that the indexicality of sign-vehicle *indicates* that something "exists" somewhere in time and space. The indexical sign-vehicle, in addition to being affected by the *real Object*, is also affected by the senses and memory of the Person whom it serves as a sign-vehicle (Sebeok, 1991). That is, indexes, as it were, allows the Person to invert the prior cause-to-effect relation to construct the effect-to-cause relation (a perceived effect indicates a cause) to infer, with a degree of certainty, the cause that produced the particular effect. In other words, based on actualities, indexes allow a Person to make inferences about the *real Object* of the SIGN.

When indexical sign-vehicles become, in the mind of the interpreting Person, the genesis of symbolic sign-vehicles, then they can also be thought of as *objects* easier to "manipulate" and work with without going back to the singularity and actuality of indexical sign-vehicles. One must keep in mind that symbolic sign-vehicles *indicate* their *real Objects* by *mental operations* and *cultural conventions*. Again, as pointed out before, icons, indices, and symbols are not three mutually exclusive kinds of sign-vehicles. In fact, they are sequentially embedded in the sense that there are no indexes deprived of iconicity nor symbols deprived of indexicality. In the classroom, this nestedness or embeddedness provides a basis for strategizing teaching sequences which may facilitate students' understanding of mathematical concepts.

For example, the graphs of polynomials allow us to visually see that the great majority of them are not one-to-one functions, and therefore the great majority of them cannot have inverse functions with respect to composition. This visualisation, based on indexicality, can also lead to some kind of generalization with respect to polynomials functions and their inverses with respect to composition and with respect to multiplication. The graphs of polynomials also allow us to see that all polynomials have multiplicative inverse functions, which at the zeros of the polynomial, will have infinite values (i.e., undefined values). This is equivalent to say that the multiplicative inverses of polynomials (which are in fact rational functions) will present undefined functional values, which were absent in the graphs of polynomial functions.

This notion of indexicality embedded in the symbolicity of mathematical sign-vehicles is, in a sense, described by Tall, Gray, Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, and Yusof (2001) when they argue that "*symbols* can *act as pivots*, switching from a focus on *processes to compute* or manipulate, to *a concept* that may be thought about as a manipulable entity" (p. 5, italics added).

Indexicality is not only embedded in mathematical symbols, diagrams, and diagrammatic reasoning, but it also characterizes the inferential process of deduction. As Peirce put it,

> An Obsistent Argument, or *Deduction,* is an argument representing facts in the Premise, such that when we come to represent them in a Diagram we find ourselves compelled to represent the fact stated in the Conclusion; so that the Conclusion is drawn to recognize that, quite independently it be recognized or not, the facts stated in the premises are such as could not be if the fact stated in the conclusion were not there; that is to say, the Conclusion is drawn in acknowledgement that the facts stated in the premises constitute an Index of

that fact which it is thus compelled to acknowledge…. (CP 2.96, 1902; quoted in Sebeok 1991, p. 129)

As a logician, Peirce undertook as his endeavor the classification of inferential thinking. He distinguishes three irreducible types of inferences: deduction, induction, and abduction. More about inferential thinking, diagrams, and diagrammatic reasoning will be found in some of the chapters in this anthology.

THE PEIRCEAN TRIADIC SIGN AND MATHEMATICS EDUCATION

Let us consider the metaphor of the German semiotician Petter Schmitter, which was adopted by Nöth (1990) in the preface of his book "*Handbook of Semiotics*," to describe the field of semiotics as a "country of different topographies." This metaphor provides a fitting portrayal of the conceptual richness of different semiotic perspectives and their influence in different fields of knowledge. It is well known that the Peircean triadic SIGN has been influential in the development of different areas of the arts and sciences. In mathematics education various semiotic theories have allowed the proposal of richer and deeper explanations for the complexity of the teaching-learning activity. Among them, Peirce's semiotics has won acceptance during the last 20 years.

Peirce's addition of the *interpretant* as a new component of his SIGN, his re-conceptualizations of the signified as the *real Object* of the SIGN, his specification of the signifier as the sign-vehicle (of the Object) with all of its diverse expressions, and the framing of these three components within his three categories, altogether, constitute a revolutionary shift in the historical evolution of semiotics. The introduction of the *interpretant* as the effect that the sign-vehicle provokes in the mind of the interpreting Person is Peirce's unique acknowledgement of the irrevocable right that each Person has to be actively involved in his own meaning-making processes.

In other words, this is his acknowledgement that the construction and reconstruction of the *real Object* of the triadic SIGN is an evolutionary inferential process in the mind of the interpreting Person. This process is inherently linked to the hints and clues about the *real Object* which are carried out by different sign-vehicles and which, in turn, trigger the formation of interpretants and the construction and refinement of dynamic objects. As said before, the interpretant is intrinsically interrelated to the sign-vehicle and to the *real Object* of the SIGN. In fact, Peirce considers that cognition is a triple consciousness that begins with perception: consciousness of the sign-vehicle, consciousness of the *real Object*, and consciousness of the meaning or interpretation of the sign-vehicle which the cognition connects with that *real Object* (Fisch, 1986). Thus, perception and consciousness are seen as evolutionary complex processes so as to scaffold simpler cognitive semiotic processes already functioning in human beings (Stjernfelt, 2014).

This evolutionary cognitive process mediated by sign-vehicles has been observed and acknowledged in the teaching-learning of mathematics. In one way or another, the Peircean semiotic perspective has been recognized in several edited books and Special Issues (Hitt, 2002; Anderson, Sáenz-Ludlow, Zellweger, & Cifarelli, 2003; Hoffman, Lenhard, & Seeger, 2005; Sáenz-Ludlow & Presmeg, 2006; Radford & D'Amore, 2006; Radford, Schubring, & Seeger 2008) as well as in numerous research papers published in well-known journals. The chapters in this anthology, explicitly or implicitly, also consider various Peircean semiotic notions.

CLASSROOM MATHEMATICAL COMMUNICATION

Within human semiotic reality, communication is essentially message exchange that depends, among other things, on: socio-cultural contexts; the content of the message; the language used to convey the message (syntax, grammar and semantics, active and passive lexicon); the means of human interaction (voice-intonation, diagrams and graphs, writing and inscriptions); and the visual means of their delivery (gestures, pointing/deixis, gazing, posture, and the like). All these variables add to the complexity of human communication (Halliday, 1978; Austin & Howson, 1979; Habermas, 1984; Bruner, 1986; Vygotsky, 1987). Even further, communication is also influenced by the behavioral dispositions and expectations of the participants and their intersubjective relations of power (Bourdieu, 1991).

Taking the Peircean semiotic perspective, the construction of meaning straddles the puzzling and yet equilibrating semiotic realities of those who interact. Meaning-making is taken to be a constructing activity mediated by socio-cultural systems of sign-vehicles. The intended goal of those who interact is to achieve some kind of consensus in back-and-forth message exchanges to construct the *real Object* that sign-vehicles endeavor to represent and materialize. On the one hand, sign-vehicles can only represent certain aspects of the *real Object* but not all of its aspects at the same time. On the other, sign-vehicles cannot be conflated with the *Objects* they represent. Given that sign-vehicles are by nature *pars pro toto,* they have their own inherent representational constrains. This is to say that to conceptualize a *real Object* several sign-vehicles are necessary to indicate as many as possible of its aspects so that it can be inferred.

Like other sign-vehicles, mathematical sign-vehicles (e.g., mathematical diagrams, notations, mathematical linguistic expressions) can only indicate some aspects of a mathematical *real Object* but not all of its aspects at the same time—they foreground some of its aspects and background others. Consequently, meaning-making of mathematical *real Objects* (i.e., concepts) can be seen as an inferential, recursive process mediated by a diversity of mathematical sign-vehicles. This process is inferential in the sense that was argued before. It is recursive in the sense that dynamic objects, constructed at a particular interpreting moment, are modified and refined in subsequent acts of interpretation. This mathematical activity mediated by mathematical sign-vehicles constitutes the semiotic activity (i.e., mathematical semiosis) of the classroom participants—activity which is continually transformed through the interpretive collaboration and elaboration of teacher and students.

The exchange of mathematical messages in the classroom has been and continues to be a challenge. This challenge has at least three causes. One is the distinctive role that writing plays in the emergence of mathematical thinking and whether or not students are willing to use it as a tool for learning. Another is the students' levels of interpretation of mathematical sign-vehicles. Still another is how students connect and keep track of their own constructed meanings.

With respect to the role of writing, Rotman (2000) argues that in order to communicate mathematically one essentially writes. He contends that writing plays not only a *descriptive* but also a *creative* role in mathematical practices. He asserts that those *things* that are described (thoughts, signifieds, and notions) and the means by which they are described (*scribbles*) co-construct each other in a synergistic manner. Mathematicians, as producers of mathematics, he argues, think their scribbles and scribble their thinking. By the same token, it could also be argued that learners of mathematics, when describing mathematical concepts or solving mathematical problems, build up their understanding by scribbling their thinking and thinking their scribbles in order to internalize and appropriate the institutionalized mathematical knowledge intended in the curriculum.

With respect to the second and third causes, one's own mathematical meanings, actual and potential, can be expanded only when we are able to integrate the meanings constructed at a given stage of conceptualization. The construction of mathematical meanings appears to be similar to the construction linguistic meanings. Linguistic systems have only a finite lexicon, but their semantics, grammar, and syntax account for an unlimited series of acceptable combinations of linguistic meanings (Gay, 1980; Rossi-Landi, 1980; Deacon, 1997). By the same token, mathematical systems have only a finite number of axioms, definitions, and concepts that, when combined, account for a large number of mathematical meanings.

DIAGRAMS AND VISUALISATION IN MATHEMATICAL THINKING

The history of visualisation within mathematics education is a long one. This fact can be seen in a series of papers published since the beginning of the 1980's. Recall the earlier texts of Presmeg (cf. 1986, 1994, 1997), Skemp (1987), Pim (1995), or Eisenberg's widely recognized paper "On understanding the reluctance to visualize" (1994). More recent examples on visualisation can be found in Arcavi (2003), Giaquinto (2007), or David and Tomaz (2012). Some of these papers focus on the practical aspects of teaching school mathematics while others are aligned with educational psychology or more sophisticated theoretical concepts.[3] Regardless of their focus, nearly all these papers emphasize how the mathematician's success owes a considerable amount to visualisation (Heintz & Huber, 2001). On the other hand, the history of mathematics shows visualisation to have been cut back and even avoided to a certain extent. In the time of Leonhard Euler the visual was also used as a means for proving or establishing the existence of a Mathematical Object, whereas the mathematicians of the 19th and 20th century reduced the use of visualisation for gaining new ideas when solving problems. Heuristics was the task of visualisation. Maybe this was one reason why dealing with visualisation became an important topic for researchers in mathematics education.

However, for some twenty years we have seen a growing interest in the use of images within cultural science. It was Thomas Mitchel's dictum that the linguistic turn is followed now by a "pictorial turn" (1994) or Gottfried Boehm (1994) "iconic turn". Their concentration on visualisation in cultural sciences is based on their interest in the field of visual arts and it is still increasing (Bachmann-Medick, 2009). But more interesting for our view on visualisation are developments within science which have introduced very sophisticated methods for constructing new images. For example, medical imaging allows us to see what formerly was invisible. Other examples could be modern telescopes, which allow us to see nearly infinite distant objects, or microscopes, which bring the infinitely small to our eyes. With the help of these machines such tiny structures become visible and with this kind of visibility they became a part of the scientific debate. As long as these structures were not visible we could only speculate about them, now we can debate about them and about their existence. We can say that their ontological status has changed. In this regard images became a major factor within epistemology.

Such new developments, which can only be hinted at here, caused substantial endeavor within cultural science into investigating the use of images from many different perspectives. Mitchell (1987), Arnheim (1969) or Hessler and Mersch (2009) are examples. The introduction to "Logik des Bildlichen" (Hessler & Mersch, 2009), which we can translate as "The Logic of the Pictorial", focusses on the meaning of visual thinking. In this book, they formulate several relevant questions on visualisation which could/should be answered by a science of images. Among these questions we read: epistemology and images, the order of demonstrating or how to make thinking visible.

When we consider these short deliberations, then we can recognize two positions. We have a long tradition of visualisation within mathematics education together with an interest in certain theoretical backgrounds. At the same time, there are several recent developments within cultural science concerning visualisation. Hence there is a need to find some means of transmission to bring ideas and research questions from cultural science to mathematics education. A theory-based example of such a means of transmission could emphasize the iconic and indexical aspect of mathematical sign-vehicles (mathematical representations) rather than emphasizing only their symbolic aspects. That is, the teaching and learning of mathematics is a semiotic activity. The semiotics of Charles S. Peirce has been very helpful in the development of other sciences like medicine, chemistry, crystallography, cinematography, theater, literature, linguistics, architecture and the visual arts, just to mention some.

Diagrammatic thinking, one of most important notions in the Peircean theory, has become a tool for investigating mathematical activities (Dörfler, 2005; Hoffmann et al., 2005).

INTEGRATING THE SUMMARIES OF THE CHAPTERS

The above introductory deliberations describe the theoretical background of the chapters in this volume. Let's now take an overview on each of the chapters. Based on the Peircean semiotic theory, the considerations in these texts focus at least on three characteristics of the teaching and learning of mathematics. In particular, this includes questions of argumentation and communication discussed in the first two chapters. Closely related with these questions there are certain issues of visualisation which can be seen as an attempt to teach and learn mathematics with the use of visual signs. The chapters in the third part of this anthology apply results from the Peircean theory to describe activities when students are asked to solve problems.

Communication

Kadunz's chapter "Geometry a means of argumentation" is a historical and theoretical paper. It discusses the relation between the development of geometry and argumentation in ancient Greece as well as the role of argumentation in the emergence of the Greek democracy. Based on these historical co-evolving developments, Kadunz puts forward three organizing semiotic principles for the teaching-learning of geometry. These principles take into account the nature of geometric signs (sign-vehicles) and their role in the development of geometric argumentation. This chapter presents avant-garde notions on the development of geometric thinking and the teaching and learning of geometry.

Adalira Sáenz-Ludlow and Shea Zellweger introduce in "Classroom mathematical activity when it is seen as an inter-intra double semiotic process of interpretation: a Peircian perspective" the theoretical background of this anthology. It is their goal to introduce the semiotic of Ch. S. Peirce as a viable instrument to describe the teaching and the learning of mathematics where these two activities can be seen as a double semiotic process of interpretation. The authors explain that the formation of students' mathematical concepts is guided by the teacher but is also determined by a process of inter-intra interpretation of the learning student. To describe this formation, the authors take a thorough look at Peirce's triadic sign to describe its use within mathematics education. They argue that the developmental construction of mathematical signs in the classroom—mathematical semiosis—is grounded among others things in the presence of a certain semiotic reality. This reality can be found in classroom practices but also in systems of communication as well.

Visualisation

In her chapter "Visualisation for different mathematical purposes" Caroline Yoon adds a special view on the use of visualisation when learning mathematics. While visualisation is often suggested as a heuristic tool, she concentrates on visualisation as an instrument to support students when they try to generalize mathematics or when they communicate mathematical ideas. To achieve this goal, Yoon presents a case study where visualisation is used in calculus teaching. In this study the author demonstrates that it is the nature of the task proposed which connects different semiotic activities with different types of successful visualisations when doing mathematics.

Tessa Miskell and Caroline Yoon offer in their text "Visualising cubic reasoning with semiotic resources and modelling cycles" a semiotic view, on students' activity, of how to model mathematically. They show, that the mere presence of visible diagrams or physical manipulatives cannot guarantee students successful reasoning. In three case studies they exhibit the effectiveness of visible semiotic tools on the way how students can use these tools. In this respect, a semiotic tool is a successful instrument to enable students to visualise their modelling activities and to support them to test and examine their mathematical approaches.

Kadunz's chapter "Diagrams as a means for learning" is an example of diagrammatic reasoning. The chapter analyzes the cooperative and elaborative mathematical activity of two school students who are presented with a novel task. Their activity illustrates how students produce and use sign-vehicles (inscriptions and diagrams) to guide their creative activity, to communicate their thinking, and to solve the mathematical situation that was presented to them. Kadunz explains the students' problem solving activity using important epistemological notions of the Peircean semiotic theory (e.g., collateral knowledge, and theorematic and corollarial deductions).

Perry et al. analyze an episode of classroom interaction in a geometry class of pre-service teachers. This interaction is analyzed using a model of classroom communication based on students' constructions of dynamic objects when they interpret geometric sign-vehicles and the use of theoretical elements they have already constructed. In this classroom, theoretical elements are included as the need arises and students are expected to conform to them, to solve geometric tasks, and to explore geometric situations conducive to the establishment of old and new geometric propositions. The data for this empirical study comes from a longitudinal program for the teaching-learning of geometry for teachers and from a longitudinal teaching-experiment.

In her chapter "Abduction in proving: A deconstruction of the three classical proofs of the proposition 'The angles in any triangle add 180'" Adalira Sáenz-Ludlow successfully uses Peirces' semiotics to clarify some relevant questions from the heart of mathematics. To prove a theorem has always been one of the main activities of mathematicians. The use of Peirce's notion of abduction together with Kant's notion of intellectual intuitions and perceptual judgments enable Sáenz-Ludlow to analyze the three classical proofs of the above mentioned theorem. The investigation of these proofs and the use of abduction to describe their constructions presents semiotics, again, as an instrument to characterize the learning of mathematics.

PROBLEM SOLVING

Christof Schreiber explains another semiotic view on problem solving in "Semiotic analysis of collective problem-solving processes using digital media". Backed by Perice's triadic sign relations, Schreiber illustrates a chat session of students when solving a mathematical problem by inscriptions in written or in graphical form only.

To describe student's activities when solving problems, Victor V. Cifarelli concentrates in his text "The importance of abductive reasoning in mathematical problem solving" on abduction as a particular kind of reasoning. Ch. S. Peirce argued that the well-known reasoning activities induction and deduction are not enough to describe the finding of new ideas when solving problems. Peirce suspected that mainly by abduction learning individuals generate hypotheses to explain surprising facts. Using the results of a case study, Cifarelli suggests that, in the sense of Peirce, teachers should focus on students' mathematical thinking and learning to support them to investigate their own interpretations of mathematics.

NOTES

1 In general, the word *sign* is used, somentimes, to refer to the *object* itself and, other times, to the mode of representation of the *object*. The reader is then left with the task of interpreting either meaning from the context in which the word *sign* is used.
Saussure conceptualizes *sign* as the dyadic entity (signified, signifier): the signified refers to the *object* or referent of the *sign* and the signifier refers to the representation of the *object*.
Peirce conceptualizes *sign* as the triadic entity (Object, representamen/sign-vehicle, interpretant): the *Object* refers to the signified, the representamen or sign-vehicle refers to the signifier, and the interpretant refers to the effect of the sign-vehicle in the mind of the Person interpreting the *sign*.

2 The word SIGN, in upper case letters only, is reserved to refer to the Peircean triadic sign (Object, sign-vehicle, interpretant). See Sáenz-Ludlow and Zellweger in this volume.

3 For example, Jerome Bruner and his view on the use of images, Jean Piaget and his learning theory, and George Lakoff and Mark Johnson and their views on metaphors.

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COMMUNICATION

2

GERT KADUNZ

2. GEOMETRY, A MEANS OF ARGUMENTATION

ABSTRACT

Arguing and proving are essential elements in mathematics. To learn these skills from a mathematical point of view, elementary geometry is often used as a paradigmatic part of mathematics to demonstrate and to learn how to find an argument or how to construct a proof. This text presents several reasons why elementary geometry can be seen as a fruitful part of mathematics to learn these abilities. The chapter starts with a view on the use of geometry of ancient Greeks. Within this first part some sociological reasons are presented to which objective geometry was used in their political life. The second part of this text offers a semiotically based view on geometry. There three further reasons, all based on and motivated from the use of geometry signs, are provided. All reasons help us to understand why geometry is intimately connected to learn how to argue and how to prove.

INTRODUCTION

Educational aims currently under discussion[1] as well as publications in the didactics of mathematics consider argumentation and reasoning to be essential elements for the teaching of mathematics. Argumentation and reasoning are expected from students as early as the first level of secondary school. A significant portion of school mathematics at this level is devoted to elementary Euclidian geometry. Publications in the didactics of mathematics cite school geometry as an essential subject to develop students' processes of argumentation and proving (e.g., Graumann, 1996; Kadunz & Sträßer, 2009).

The text presented here pursues, from a semiotic position, the question of why geometry is suitable as a means of learning how to argue and how to demonstrate the validity of something. To do so, attention has to be given, in a semiotic sense, to the specific forms of construction and usage of geometry signs. From this perspective, three essential aspects are to be considered because, in the long term, they will contribute to the development of skilful argumentation. These three aspects will be supported by different theories throughout the chapter.

- The use and creation of geometric signs is closely connected to the geometric relations connected with these signs.

- In general, geometry signs rarely support algorithms as we know them from algebra or analysis. This non-support is partly substituted by the promotion of other relieving activities.
- The particular plane of a geometric configuration (which differentiates it from the two-dimensionality of e.g., an algebraic equation) results in the fact that the completion of this geometric configuration is made more difficult if only for no other reason than the particular position/situation of the geometry signs with respect to each other. The configuration hides its genesis and this has particular consequences.

The above three points outline the mathematical and didactical direction of this text. One consequence of this view of geometric signs is the almost obligatory use of argumentation or reasoning for the construction, interpretation, and reconstruction of geometric configurations. In addition, it is also important to take into account the fact that argumentation in geometry needs not only verbal but also written linguistic signs.

Denise Schmandt-Besserat (1997), in her many works as archaeologist, pursues, among other things, the question of the emergence of writing. Through the use of archaeological finds and her respective interpretations, she succeeds in seeing the origin of writing not as duplication of oral language but as a consequence of the economic and military needs of the people of Mesopotamia. Thus she demonstrates that the first writing was of a numerical nature. Is it possible to find examples of similar needs—as it were everyday needs—in order to find a *new* interpretation not only *for* the learning of school geometry but also *for* how to start it? I will argue that this question can be answered on the positive. This new start relates to the beginnings of geometric argumentation in ancient Greece. Using the three aspects of the didactic program outlined above, the origin of geometric proof and the transformation of geometry from utilitarian to scientific discipline in ancient Greece (6th cent – 4th cent B.C.) can be more readily understood.

Thus the structure of my text is predetermined. First, I will turn to the geometry of the Ancient Greeks at the time of the origin of its new use. Then, I will discuss the three essential aspects, listed above, about the nature of geometry and the development of geometric argumentation.

A PARTICULAR VIEW OF THE GEOMETRY OF (ANCIENT) GREEKS

When we consider the geometry of the Greeks, the utilitarian aspect of geometry for solving everyday problems was not emphasized even though it was the main focus in Egypt and Mesopotamia. Instead, the purpose of the Greeks was the formulation theorems and the forms of geometric reasoning. From a socio-historical point of view, which reason(s) could be given for this change of emphasis? The formulation of such reasons sheds light on the success of geometry as a guiding science in the ancient Greek culture. From a semiotic position, which respects the role of the signs in geometry, possible reasons will present themselves as conclusions as to why geometry, as an argumentative science, played an influential part in the development of Greek democracy.

Thesis: The Greeks developed their use of geometry in order to organize their democratic social order. Geometry, or more precisely, argumentation through geometric reasoning, was seen as a paradigmatic example of consequential speech. Characteristic elements in the use of the geometric signs support this.

When one reads relevant publications on the history of (Greek) mathematics (Szabo, 1969, 1994; Becker, 1975; Scriba & Schreiber, 2005), they concentrate, essentially, on a geometrically accurate presentation of the development of geometry in ancient Greek while historical citations are rather short. Mostly, philosophers' statements are inserted only at the beginning of relevant parts of an exposition to embed them in similar but finalized forms of thinking, to which geometry then also belonged. Questions as to why the Greeks' approach to geometry and its new use changed permanently, in the precise period between the 8th and the 6th centuries BC, are not posed explicitly. So the hope remains that texts which concentrate on the development of Greek thinking viewed from a position of cultural sciences will produce more results. However, at this moment, there is nothing to be found about this issue.

The collection "Early Greek Thinking" (cf. Rechenauer, 2005) contains an article entitled "On the Origin of the Written Records about Thales' Geometry" (Dührstein, 2005). It is true that specific geometric problems which are traditionally (ibid., pp. 89–90) attributed to Thales of Milet are presented. However, questions about the motives for this new view of geometry are not posed. What are discussed are questions on the existence of Thales, as well as connections between Thalesian geometry and, for example, Greek astronomy.

A similar picture is given in the book "The Knowledge of the Greek" (cf. Brunschwig, 2000) through the text "The Proof and the Idea of Science" (Lloyd, 2000). Here, geometry is discussed as a source of a specific way of thinking (ibid., pp. 240–241), but the focus of the argumentation is laid on the philosophy of Parmenides. The section "Mathematics" (cf. Knorr, 2000), in the same volume, describes the phases of Greek mathematics more as a report than an interpretation. Questions as to reasons for the emergence of these phases are not posed.

The literature shows a different side to itself when the explanations of the origin of Greek thinking consider social and political dimensions simultaneously. As an example, I refer to Jean-Pierre Vernant (1982). In the section "The Intellectual Universe of the Polis" (ibid., pp. 44–48) of his comments, Vernant reports on the origin of the processes of negotiation for the making of decisions in public space. The advancement of citizens of a polis together with their enrolment into military service was tied to the right to take part in the decision-making processes. "Within the polis the status of a soldier is at one with that of a citizen; who has a place in the military structure of a city also has it in the political organization. Thus the changes in weaponry, which occurred in the middle of the 7th century, …, create a new type of warrior, convey a new social status upon him and let his personality appear in a completely different light" (ibid., p. 58). What is hinted at here, in just a few words, correlates with the above reference to Schmandt-Besserat and her view on the emergence of mathematics from particular needs. Obviously essential changes had taken place in the world of Greek life which entailed a serious change in social behaviour. In particular, the joint participation in decision-making processes demanded previously unpractised behaviour: through discussion and objection and essentially without regard for rank (cf. Vernant, p. 61) decisions should be taken together. How had such a development come about?

In "The Birth of Science" Andre Pichot (1995) developed a plausible picture of those times. Let us follow his explanations (cf. Pichot, pp. 243–246). The ancient Greek settlement area at the time of the 8th century essentially comprised the Greek mainland, the islands in the Aegean, the coastal areas of Asia Minor, southern Italy and Sicily. The period starting from about 1200 BC to the 800BC saw times of intense migration as a result of which the density of the population decreased. Individual cities formed city states, creating small and very small kingdoms that kept to themselves. This situation lasted until the 8th century BC from when on the Greeks again actively returned to the use of writing, navigation boomed, and ceramic and metalwork flourished. It is the time of Homer and Hesiod. How did these city states organize themselves? In the ancient city states power and wealth lay in the hands of the aristocracy. In the course of time this power was distributed among the well-to-do. What was one of the reasons for this division of power? As so often is the case in the history of mankind it is the above-mentioned military factor which drove the change. The number of the people living in these cities was so small that people who did not belong to the aristocracy also had to be recruited for military missions. That also meant that the large number of weapons present through the emergence of iron manufacturing demanded a well-organized army, which required that those who were well off also joined the military. Such a partaking in military duties resulted in the person's eligibility to share in power. That power was executed by civil servants, with a council to aid them. Sovereignty laid with the people's assembly which reflected the power relationships within the military. In the year 594 BC, Solon opened up the Athens assembly to all citizen classes. Numerous reports about the nature and terror of several Greek tyrants show that such attempts at democracy were often only short-lived. Despite this we can record with Pichot that public life was reorganized anew, that constitutional laws were enacted, which also regulated and restrained power.

In consequence, these city states were confronted with the problem that the people in the assembly had, among other things, to learn the activity of amicably reaching agreement. The verb used to denote argumentation was deikmüni (show, demonstrate). It has been preserved in its Latin translation as "demonstrate". It is to be found in Plato in the shape of "apodeixis", meaning "rational argumentation", a description which clearly referred to something. The description of mathematical argumentation, as used by Aristotle in his Analytica priora, was explained in the form of syllogisms. The development of the meaning of deikmüni was, according to Lucio Russo (2005) in “The Forgotten Revolution”, tied to the development of Greek democracy (in the 5th century). A further phenomenon can be confirmed at this point of time. The Greek language began to change. Whereas the passed down recordings of Old Greek from the times before the 6th century show a language exercising itself through the relating of legends, a different function now started to gain in importance. Language became the means of arranging organizational processes. With language, argumentation takes place in the form of statement and counter-statement. From where did the Greeks draw the ability to use language in such an unfamiliar manner for them? A quote by the Roman Quintillian shall point the way to a possible explanation of the reason.

> From the preceding geometry proves the following, and from the known the unknown. Do we (speakers) not do this also in speaking? Yes, does the conclusion from the preceding sentences not consist almost exclusively of syllogisms?... For if the matter demands it (the speaker) will use syllogisms or at any rate the enthymeme which is of course a rhetorical syllogism. After all, the most powerful proofs are generally called grammatikai apodeixis (providing proof by drawing); but what does discourse/statement seek more than proof? Consequently .... there is no way being an orator is possible without geometry.

(Quintillian, Instituto Oratoria, I, x§§37–38, from Russo, p. 197) This quotation by the Roman Quintillian takes me back to my initial thesis: it was geometry which had a substantial part in the development of argumentative speech. How was the emergence and development of this combination of elementary geometry and argumentation?

To answer this question let us go back in time into the 6th century BC and visit two people with very famous names. Both may be seen as representatives of a multitude of other classical surveyors of the time. Thales of Milet, known to us mainly through the theorem named after him, lived about 625–550 in the town of Milet in Ionia (today’s Turkey). In varying sources, he is sometimes called a Phoenician, sometimes a Greek. In any case, he seems to have been a successful and clever businessman, who also undertook business journeys to Egypt. His numerous contacts probably took him to the area of Mesopotamia. What he imported from these two highly developed cultures was—besides highly marketable goods—knowledge of Egyptian and Mesopotamian geometry. To this body of knowledge may be counted, if you will believe the classical author Proclus Diadochus (3rd century AD), five theorems of Euclidian geometry (cf. Pichot, 1995, pp. 334–336):

1. The circle is bisected by its diameter.
2. If two straight lines intersect, the opposite angles formed are equal. (Scheitelwinkelsatz = theorem of opposite/vertical angles)
3. Angles at the base of isosceles triangles are equal.
4. A triangle is defined if the base and the base angles are given.
5. Any angle inscribed in a semicircle is a right angle.

It is impossible to determine conclusively whether these theorems were indeed imported to Greece by Thales or not. The use of such theorems which can be constructed with the simplest means is what is significant for my argument.

I will briefly return to the members of a people's assembly who were discussed above. What was their social standing and which means of shaping their lives were available to them? One can assume that they certainly did not suffer from material need. The society of ancient Greece was a slave society, despite all their achievements, in which the ruling Greeks were in possession of extremely profitable means of production. Slaves were the cheap machinery which had to carry out whatever work was required. This specific social situation enabled the Greek patricians to turn to more particular problems. The applicability of geometry was not among them in those times. The specification of reasons for the universal validity of a geometric situation was probably of greater significance. It will not be possible to give an answer as to who was first to pose this question. That it was Thales himself is doubted by the relevant authors (cf. e.g., Scriba, 2005). It is also of no relevance for my endeavor. Of essential importance, however, are the living conditions of people in ancient Greece, which provided them with the means, the motives, and the opportunity to practice geometry. The means were mostly the geometric theorems of the Egyptians and Mesopotamians, the motive was the necessity to practice arguing, and the opportunity arose from the social situation of those Greeks who saw themselves as belonging to the aristocracy.

As I mentioned two people earlier, I will add to Thales, the at least equally famous, Pythagoras of Samos, as a contrast. Similarly to Thales, Pythagoras had close contact with the geometry of the ancients, which he became acquainted with during his travels. His motives for practicing geometry as a form of argumentation were others: everything is a number was the motto of the sects of the Pythagoreans. Questions of metaphysics of a highly speculative nature were the main-spring for him and his followers in practicing mathematics and geometry. His/their reasons for engaging with geometric problems therefore had different causes.

I look back again at the sequence of means, motive, and opportunity. We can bring our argument relating to Greek geometry to a close and record geometry as a means of acquiring the means for presenting an argument. However, if we remain on such a historically-led level of argumentation an aftertaste remains. Why was geometry, in particular, which fulfilled these needs of the Greeks? Through the contacts of the Greeks with Asian culture, other means, like a board game, could have been used to acquire or practice arguing. As a first answer one could point to the successful usability of geometry. Although the Greeks, due to the work of their slaves, may have had little interest in increasing the efficiency of their daily routines, applications of geometry in seafaring would have left an impression. A further reason, which from my point of view cannot be underestimated, may be the social acceptance of geometry in Egypt and Mesopotamia. From ambitious people like the Greeks, both societies must have elicited their admiration for their intellectual achievements. And certainly geometry was one of those achievements. And let us not forget that by the 6th century BC, geometry and arithmetic had already had more than a thousand years of history (Scriba, 2005). Notwithstanding all of this, it should be possible to provide specific reasons/arguments for the success of the new way of thinking about geometry in Greece. To approach this question, I will concentrate on the usage of geometric signs, as formulated at the beginning of this essay. To do that, I will leave this historical reflection and enter into semiotic considerations.

FOR THE LEARNING OF GEOMETRY

How can we justify this new way of using geometry, which is obviously no longer used, both to structure the given physical environment and to organize thinking? In the introductory section, three possible features of geometric signs that could possible support these forms of structuring and organizing were given. In this section they will be complemented by further reasons. I would like to briefly repeat them here. Then I will present them in more detail. Finally, I will indicate possible arguments for their validity.

1. *The construction of geometrical concepts by means of visible signs is essentially determined by those relations; relations which define these terms.*
2. *In contrast to elementary arithmetic, algebra, and calculus, the signs of geometry do not support algorithmic transformations.*
3. *When we take a look at a geometric construction, this view does not show us the history of the construction.*

What arguments and references can be presented for these three claims? These arguments are presented in the following sections. The next section is about my first claim, the construction of geometrical concepts.

RELATIONS BETWEEN VISIBLE SIGNS AND THEIR ASSOCIATED GEOMETRY

The first part of my text was historically oriented. Now, I shall switch to the learning of mathematics and take the opportunity to remind the reader at a book worth reading on the didactics of teaching geometry. The title of this book is “Operative Genese der Geometrie”[2] (by Peter Bender & Alfred Schreiber, 1985) and it was published again in 2012 in the form of a reprint. Some sections of this book should be used as a first argument to back my first claim.

The aim of “Operative Genese” is the reception and implementation of a particular view of the development of science, concentrating on ideas from Hugo Dingler, a philospher from the beginning of the 20th century. From Dingler’s perspective, Bender and Schreiber developed their principle of operational concept formation (POCF) for the development of geometry. Although the prime focus of Bender and Schreiber’s book is spatial geometry and a large variety of its applications, it also contains further ideas how geometry can be used to structure parts of our everyday life. I would suggest that POCF is a tool for interpreting the relationship between signs and concepts that I mentioned earlier. I will not concentrate on the use of geometry as a structuring agent for our environment, as the authors intended, but will focus on the construction of concepts of plane geometry.

Bender and Schreiber's claims follow some constructivist positions in stating that the basic concepts of geometry do not develop by abstraction, that is, by disregarding features, but by seeing and implementing properties into "objects". These "objects" include the geometrical signs. This view of signs as realization of objects leads to material realization. Bender and Schreiber suggest that this implementation is ruled by norms, which can be seen as the operational basis for the production and use of geometry. To illustrate this, the authors present the construction of a cube. However, there is no need to focus on spatial geometry. The production of basic concepts of plane geometry (line, circle, polygon, perpendicular line, parallel line, etc.) can also be interpreted with the POCF.

They argue that "Geometrical concepts have to be constructed in an operative way; i.e., starting from some certain purposes, standards for the production are developed to meet those purposes. These standards, mostly homogeneity requirements, are implemented in procedures and guidelines for their implementation and are, therefore, the substantive basis of the corresponding terms" (Bender & Schreiber, p. 26).

Therefore the construction of concepts is determined by the production of corresponding signs. We only need to consider the instructions we offer to the student when she/he has to draw a circle or a perpendicular. Her/his drawing of the geometrical sign always obeys the intended specification. Furthermore, the successfully drawn sign reinforces the geometrical concept within the learners mind. When constructing concepts, Bender and Schreiber name this interplay between the construction of signs according to certain specifications and their successful use "operativity."[3]

However, a semiotic reason for my first claim has not been found. Which semiotic[4] terms can be used to describe the relationship between students' activity when drawing signs and the control of this drawing activity by obeying certain relations? Peirce's concept of index can be seen as an opportunity to describe the emergence of the sign (e.g., sign of a circle, sign of a parallel line etc.) as the current result of the activity of the learner when constructing such a sign.

In the conventional use of indexical signs, these signs are to refer to some desired goal. In this respect, indices refer to something that is present, or in terms of activities, this something is already done. Smoke refers to the fire, the weather-cock to the wind, and the signpost refers to a direction. An alternative point of view and thus an extension of the use of the term index, which takes this rather special property of geometric sign generation into account, is offered by a text presented by Sybille Krämer where she concentrates on the use of the word "tracks"[5] in (Krämer, 2007). Using her terminology, we can describe the interplay of production and control discussed above. How does Krämer proceed?

In a first approach, she sees the track as a sign or indication of something mainly unintentionally left in the past. The fingerprint of the burglar or the scents of the animal in the wild are examples of such tracks. When reading such a track the past meets the present. “Just as the simultaneity is the system of order of the index, the non-simultaneity is thus the ‘system of order’ of the track” (Krämer, 2007, p. 164). In this respect, tracks, traces or marks, are all indices that refer to something not (yet) visible. If I consider my first claim, I think that the socially regulated interpretation of a track, such as the “correct” use of a simple closed curve like a circle, becomes visible during the activity of producing the circle and when using the result to do some further activities. To use Krämer’s words, “tracks embody … the expectation …” (ibid., p. 166).

The reading of tracks and the associated expectation of a successful interpretation is not new in the cultural sciences field. The Italian historian Carlo Ginzberg identified this “epistemological method” in articles on art history or psychoanalysis (depth psychology) (cf. Krämer, 2007, pp. 168–170). These texts open up an initially hidden reality on the basis of traces usually in the form of incidental and unimportant details and construct their particular view of the unknown. Krämer asks whether it might be possible that “... we must recognize that the person following the track ... becomes the constructor of the referenced object, to which the track seems to refer in a ‘quasi natural way’?” (Ibid., p. 171). If we agree with Krämer, then the reading of a track becomes the construction of a sign rather than a reference to something. Geometrically speaking the track just develops and controls our own geometrical activity. In other words, referring to Krämer again (see Kramer, 2007, p. 178), the production of a geometric sign—at least when learning elementary geometry—becomes a constructive projection. With Krämer’s deliberations a first semiotic position is marked.

Can we now determine the sign more precisely with the help of Peircean theory? Helmut Pape (2007) presents the text “Footprints and Proper Names: Peirce’s theory” that takes up this question. Let’s follow his argument, which come to the conclusion, in a nutshell, that activities act as the mediator between the general and the particular.

Peirce’s use of the word index is a broad one. In a certain, generous fashion, he saw an indexical aspect implied in nearly every kind of sign (Pape, 2007, p. 41). A listing of Peirce’s use of “index” in Pape (ibid., pp. 41–42) illustrates this generosity. What all these examples have in common is their special link between signifier and signified. In this case, an index is characterized as a sign, “that refers to an object not because of a resemblance or analogy with it, also not because it is linked to the general characteristics exhibited by the object, but because there is a dynamic (including spatial) connection between the individual object on the one hand, and, on the other, the senses and the memory of the person, for whom it serves as a sign” (Peirce, 2–305, quoted in Pape, 2007, p. 43).

What can we imagine such a dynamic connection consists of? For Peirce, Pape writes, it is the effect or the strength of the indexed sign when used on the senses of the person. Here we find a perceptual dynamic in the shape of an exchange with the environment. As we will see later, it is this exchange that may determine the learning of the use of geometric signs. The peculiarity of the individual, visible, geometric sign is not "a certain indexical representation of an object. It's about individual things only insofar as they are tangible and demonstrable and can be detected within contexts and situations in a dynamic relational fashion. […] All the world and space relations of the index are determined only by their performative epistemic activation in the situation of their use" (Ibid., p. 46).

What importance can we attribute to the sensory perception of an indexical sign? Peirce thinks that the index is a link to "the senses and the memory of the person […] , for which it serves as a sign" (Peirce, 2–305, quoted in Pape, 2007, p. 47). An elaborate process of perceiving, understanding and communicating goes along with it. Pape clearly states four points (a, b, c, d), of which two (c and d) are of particular significance for my question.

a. Indices denote what is in real (existential) relations to each other – or what is presented as if it is in relations.
b. Indices orientate us cognitively towards a particular environment, which is relevant for the assessment of indexical facts.
c. All indices are regarded as signs, which are directed to a reality characterized by relations that guide our perception-based activities.
d. All statements about individual circumstances contain either implicit or explicit indices. By these indices, the hard facts of the world are involved in the use of language. This is done by speaking about individual situations and circumstances (p. 49).

With this quotation, I could now continue my own deliberation on geometry, since the two points (b) and (c) describe the construction and use of signs within elementary geometry. However, let's stay for a moment with Peirce in order to formulate a possible use of indices to describe the relationship of the particular and the general. Peirce presents here as an example, the discovery of supposedly human footprints in the sand by Robinson Crusoe, "which can be used in two ways as a sign. That footprint Robinson came across in the sand which has been carved into the granite of glory, was for him an indication (index) of a creature living on his island, and at the same time it awakened in him a symbol, the idea of man" (Peirce, 4531, quoted in Pape, 2007, p. 50). The footprint as a sign that a particular person left behind, will spark a symbolic use.

How can such a change be understood? As an example one can refer to the use of language. It is the index that adds the "meaning" to the user when we think of language as a symbol system. "… Because the external sign functions relate the general symbolizing language to present experience, they break up the structure of the description through a kind of performance. So the linguistic representations can be clearly related to an individual event, object, or an individual person." (Ibid., p. 53). Here, Pape looks to Wittgenstein and to his proposition that the construction of meaning of a word arises from its use. These indexical demonstrable relations play a central role in that process. If we were to speak without any indexical reference, our language would, in the extreme case, refer to any objects and ultimately be irrelevant.

"It would be an exaggeration to claim that we can never say what we are talking about. But in another sense, it is completely true. The meaning of the words normally depend on our tendency to associate qualities with each other and […] of our ability to recognize similarities. However, the experience is held together and is only recognizable due to forces that act on us" (Peirce, 3–419, quoted in Pape, 2007, p. 53). This mutual causality between the references to the particular, on the one hand, in the form of perceivable signs by the senses and, on the other hand, to the general rules about the use or production of signs are aspects of the signs in geometry. In this sense, these are indices which point to themselves and, at the same time, allow the universality to shine through their rule-based production. Thus, performance combines the general with the particular.

From these perspectives, which are also compatible with Dingler's "operativity,"[6] we can interpret the special role of geometry signs. During an intentional drawing activity sometimes we read relations into the construction, whether we are learning geometry as beginners or as experienced mathematicians. We draw, for example, one line parallel to another taking care that they are equidistant, i.e., that the intended relation of being parallel to one given line is specified. During this sign activity what the drawers sees "shows" the track of their current activity (Krämer) since it shows whether we are right or wrong. Tools may shorten this drawing phase but also distract us from the intended relations. This could—for example, when using dynamic geometry software (DGS)—in extreme cases lead to a complete separation of the activity (click with the mouse) from the intended relation (visible sign). It would only be when the DGS was used to vary a given straight line, and the behaviour of the drawn sign has to be interpreted again, then the connection between the visible sign and the associated relationship could be re-established (cf. Arzarello, 2002). Such an intended and target-controlled sign activity can be observed in complex configurations too. We add a sign to open up new perspectives on a geometric construction. We find this of elementary proofs in geometry, where the addition of signs is a successful strategy.

What does this mean for the initially formulated assertion that geometry is a tool of reasoning with? Looking at the comments on the use of "track", at least one thing seems to be certain. When we try to learn the basic concepts of geometry or to use them in the sense of geometry, we consider the defining relationships during the construction process. The visible trace of the signs of activity tells us whether we are drawing correctly. Thus, the perpendicular has to take this direction, because it is defined as such. If it does not take this direction then I cannot use it as a perpendicular. And we are always performing this kind of activity, at least as long as we are learning basic terms. The activity is performed because something is defined in a particular manner. The emergence of the visible has an immediate justification.

The signs activity is therefore controlled by the relationship, and the visible trace "reports" back to me if I am constructing correctly. Therefore, relationship and its visible sign are thus closely connected to one another.

GEOMETRY CANNOT "CALCULATE"

I come now to the second claim. If we look at Euclidean geometry then we recognize a lack of algorithms in contrast to, for example, elementary algebra. To put it simply, we cannot calculate in geometry. To back this claim, I would like to continue the idea of the interaction between visible geometric signs and their corresponding relations. My aim is to give reasons why the development of algorithms within elementary geometry makes little sense, but that, simultaneously, this apparent deficiency demands and causes a further specific property in geometry. This property consists in the use of definitions, theorems, or other known geometric constructions while we are drawing a geometric construction or proving a theorem.

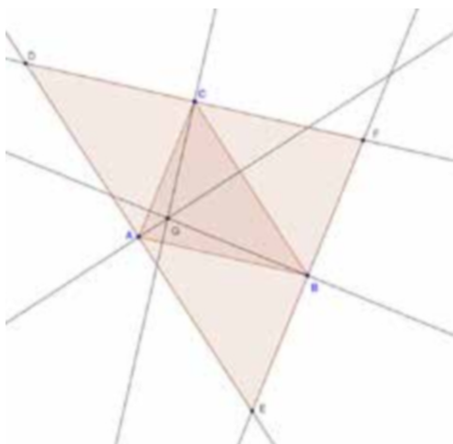

Figure 1. Orthocenter

Let us first look at elementary algebra or arithmetic, where the successful use of mathematics is rather often determined by the applications of algorithms. With the help of such algorithms we can handle parts of problems or proofs in a quasi-mechanical manner by rule based transformations. Consider, for example, an algebraic proof for Pythagoras's theorem (see Kadunz, 2000) in contrast to a proof using the signs of elementary geometry.

One reason for the success of such an algorithmic approach may lies in the two-dimensionality of the written. This two-dimensionality enables us to "walk" through a calculation line by line. Consequences for mathematics education can be found in Kadunz (2006). If we are doing geometry, then this dimensionality is increased.

The use of signs in geometry is always determined by their position with respect to each other. In this respect a construction even in plane geometry is already three-dimensional. As a consequence the reading of a finished geometric construction is often a difficult task. I will concentrate on this question within the third part of my explanations. At this point, I want to focus on the fact that within geometry algorithmic transformations are hardly feasible because of the intimate combination of geometry signs and their relations.

The only conceivable way of using a geometry signs in a new way within a drawing is to change its use intentionally. Hence the use of a sign always fulfils a specific task in a configuration. For example, let us take the proof of the orthocenter in a triangle, in which the same line can be seen as a bisector (triangle DEF) or as an altitude (triangle ABC) (cf. Figure 1). This switch is not the result of an algorithm but the consequence of a certain view of the drawing on the part of the mathematician. Ladislav Kvasz (2008) describes the impossibility of transforming by algorithms within geometry as a lack of expressiveness of the signs of elementary geometry.

What are the consequences of this apparent lack?[7] Can we gain something from this obvious lack – compared to e.g., elementary algebra? As an example let us solve the algebraic equation $(2-x)^2 = 3x+1$. After a short series of transformations based on the rules of elementary algebra, we will get the equation $x^2-7x+3 = 0$. Pupils, if practiced, recognize this expression and calculate the solution, using the formula for quadratic equations. In this sense, rule-based transformations can lead the way to configurations which remind us (sometimes) of a well-known theorem/formula.

Geometry is different. Let us consider another example. We are looking for the position of a sailing boat which can be seen from two different points on the shore joined by a given angle. A successful approach solving this task uses the application of the inscribed angle. If the pupil does not know this theorem then the likeliest solution would be the use of DGS. However, it is inconceivable that this sort of solution is proof against examination using the rules of Euclidian geometry. The very first step in the solving process requires knowledge of a geometric theorem. Such geometrical knowledge determines the path to the solution. Hardly any parts of the solution are supported by activities depending on algorithms. We always have to refer to a theorem or a definition. This is a complex activity but also presents a challenge to the pupil.

How can we describe this use of theorems and definitions? Mathematics education offers here—in addition to cognitive sciences and computer science—the notion of modules. I refer to documents about the learning of mathematics which were published from the mid-1980s onwards and more particularly to an article by Willi Dörfler (1991) with the title "The computer as a cognitive tool and cognitive medium". In this paper Dörfler reports about the use of modules to describe the learning of mathematics. Let us take a look at some of his main arguments. Cognitive psychologists report that experts achieve their performance to a considerable extent by access to highly-structured knowledge. Within this knowledge units are directly accessible and operationally usable. For instance a chess grandmaster surveys a great variety of positions on the chessboard before making a move. Similarly experienced mathematicians can easily access numerous knowledge packages in their memory, which they then apply to different problem situations. Such packets can be algorithmic processes, for example, but also knowledge of theorems and their application. One could also say, metaphorically speaking, that experiences are transformed into modules of thinking and that these modules are knowledge in a condensed form. In my view, it is remarkable that the knowledge of a proof or some kind of inner structure is not relevant to the successful use of such a theorem.

Modules, though, are not always modules as they differ in their purpose. For example, algorithmic procedures facilitate calculations, help us to reduce our effort when solving a problem. The waiter in the restaurant calculates without knowing why the algorithm is correct or the business science student calculates the inverse of a matrix without knowing, in most cases, why the algorithm works. In this respect the application of an algorithmic procedure is a form of aid. If on the other hand we look at geometry,[8] then we regularly have to use, as already mentioned, theorems or definitions from geometry. These are different forms of modules. While algorithms relieve, theorems in geometry shorten the process of finding a solution. All we have to know in this case is the interface of the theorem—what are the conditions of applicability and what is the result of its usage. If we use these modules effectively then a proof or a calculation can become very short. In a nutshell, theorems and formulas shorten whereas algorithms relieve.

The above view of geometry reveals a characteristic feature of it. When doing geometry the access to encapsulated knowledge, theorems and definitions, in the form of modules is necessary and helpful. It can be seen as a consequence of the nature of geometric signs. One could also say that the lack of algorithms in geometry forces us to use such modules. Drawing and proving in geometry can be characterized by an extended use of these modules where nearly every step within the solution has to be backed by reasons for their use. Even on a very elementary level geometry forces us to justify our activity.

THE SECRET STORY

Geometric constructions hide their history. This again can be seen as a result of the signs used in geometry. The numerous design elements even in rather elementary drawings—e.g., the circumcenter of triangle or Euler's line—obstruct the view of the development of the construction. In contrast, when we think of elementary algebra, we can easily reproduce the 'visible' activities of a solution, because algebra is written line by line, whereas the geometric signs are superimposed. Similar to the lack of algorithms in geometry another lacks becomes visible. Geometric constructions are difficult to read. Can we profit didactically from this weakness? A certain hermeneutics which enables to successfully read a construction could be the profit. The difficult task of reconstructing a finished drawing can be facilitated by a written description of the drawing activities. This description[9] follows the rules of linear writing. Thus, the history of the drawings genesis is revealed. However, the price is the use of a second sign system. We know such descriptions from (older) textbooks or even alternatives in various DGS system, which repeat the construction at the touch of a button. I conclude my remarks on this point by referring to the Irish surveyor Oliver Byrne. In his 1847 published book "The Elements of Euclid" presented geometrical constructions or theorems primarily with the help of geometrical signs. Whenever possible, Byrne avoided labels with letters. In place of these indices, he used colours or dots hatching and the like (cf. Figure 2). Even the

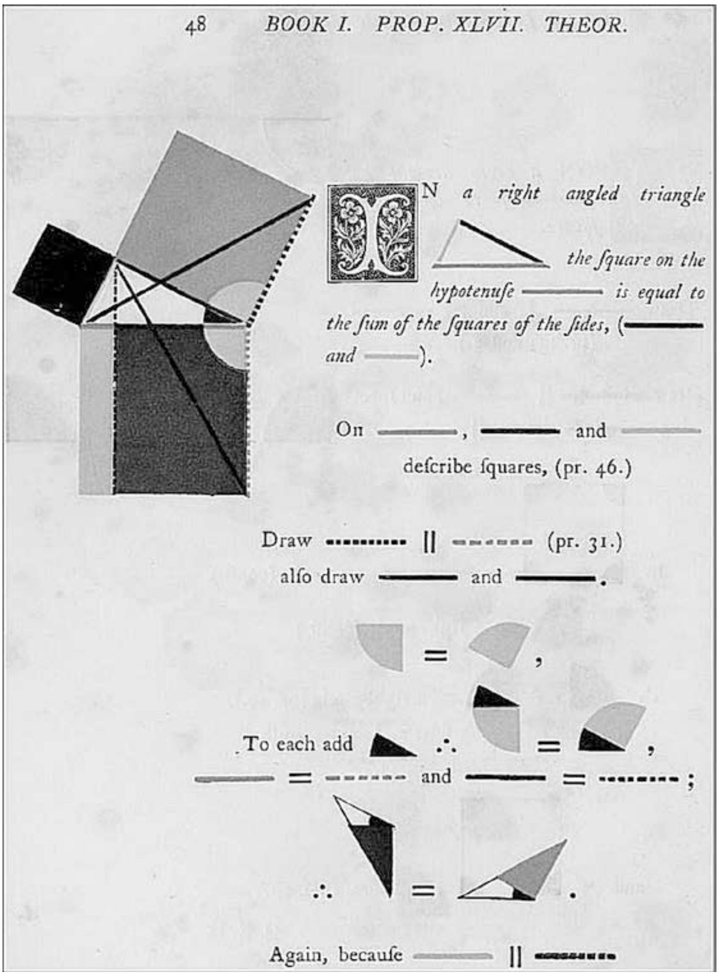

Figure 2. Pythagoras

location of parts of a structure served as an index. Thus any proof of a theorem was presented as a linear sequence of geometry signs.

Whether it is the classic written description of a construction or its repetition with the help of a DGS, the simultaneously appearing relational structure is deployed, in any case, before the observers' eyes. What was simultaneous becomes chronologically linear. The fineness of the description, the granularity, can be adapted to the learners. The interpretation or hermeneutics of a geometric design is determined by the use of signs.

SUMMARY

The considerations in part 2 presented the use of geometry signs for the learning of geometry taking into account three different but complementary aspects. Using these three perspectives reasons have been found, in addition to the historical discourse in part 1, why in ancient Greece, geometry had a special role. Geometry as a tool for reasoning and validation helped to build democratic structures. Interpreted semiotically, this is also a feature of geometry signs. At first the visible geometric sign and the corresponding geometric relations are closely linked. A semiotic interpretation of this relationship could be made through the presentation of special concept of "tracks" and especially through observing the indexical use of geometrical signs. Pupils (should) construct the signs of geometry by constant control of the visible by the corresponding geometrical relationship. The drawing activity is controlled by the geometrical relationship. These relationships are always thought along and can be used to argue the activity.

As a second point, the lack of algorithmic transformations in geometry was presented. This lack has the consequence that within geometric constructions or proofs in most cases theorems and definitions has to be used. This usage of theorems, for example, must be always justified. This is in a sharp contrast to an algorithmically oriented transformation. When working on theorems we need to give reasons why we take the next step.

As a third point the hiding of the history of a geometric construction was presented. In order to read a finished geometric construction we need to use a second sign system in addition to the geometrical one. With the help of this alternative sign system, the nonlinear set of relations is represented linearly. We can read geometry.

All these aspects of the geometry signs may have been reasons why people in ancient Greece have used geometry in order to learn how to argue. This applies in a similar manner for the learning of geometry in school.

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NOTES

[1] See the educational standards for mathematics in the final year of secondary schooling (8th grade) on the website of the Austrian Federal Institute for Educational Research (https://www.bifie.at/node/49) (24th February 2014).

[2] "The genesis of geometry from an operational point of view."

[3] "Operativitaet"in German.

[4] I focus on the semiotics of Charles S. Peirce.

[5] "Spur" in German covers many different meanings. English uses many separate words for these meanings i.e., track, tracks, traces, mark, evidence, clue etc.

[6] "Operative Genese".

[7] If we draw a construction in geometry then a way to perform transformations can be done by using software for doing geometry (DGS) and concentrating on the drag mode. Examples can be found in publication e.g., by Reinhard Hölzl (1999) or Ferdinando Arzarello (2002).

[8] Of course all parts of mathematics offer an enormous number of theorems and definitions. In this respect all we can say about the use of theorems within geometry can be said about mathematics at all. However, it is the lack of algorithms in geometry that is in the focus of my interest.

[9] "Konstruktionsgang" in German.

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ADALIRA SÁENZ-LUDLOW AND SHEA ZELLWEGER

3. CLASSROOM MATHEMATICAL ACTIVITY WHEN IT IS SEEN AS AN INTER-INTRA DOUBLE SEMIOTIC PROCESS OF INTERPRETATION

A Peircean Perspective

ABSTRACT

Semiotic reality is a fundamental part of our common reality. Where we stand in this chapter looks upon the teaching-learning of mathematics as a double semiotic process of interpretation. It takes place within the socio-mathematical semiotic reality that teachers and students inherit and jointly activate in the classroom. We argue that, during interpretation, the formation of students' mathematical conceptions and the attainment of their mathematical Concepts is constructed not only with the guidance of teachers. It also follows a progressive and corrective process of inter-intra interpretation. We emphasize that teachers' awareness of the evolving nature and refinement of their own processes of interpretation and, especially, their awareness of the interpretations that takes place in the students, is essential to maintain a collaborative and dynamic teaching-learning signifying practice. Our understanding of the Person-Object relation agrees with Vygotsky when we claim that objectification is a special case of internalization. This objectification takes place during Self-Other external activity aided by Self-Self internal activity. Taking a Peircean perspective not only puts a special emphasis on intra-placed mathematical sign-interpretant formation, but it also puts a high focus on intra-abstracting-objectification that takes place in each and every student.

INTRODUCTION

We consider the teaching-learning of mathematics to be a signifying practice, one that is framed in a complex socio-mathematical classroom that functions as an extended semiotic system. Embedded in this larger system, the discourse of teachers and students is mediated by a variety of mathematical, linguistic, and paralinguistic SIGNS. In this chapter, the word SIGN, *used only in upper case*, stands for the unified and undividable relation among the three components of the Peircean "sign".

In the classroom signifying practice, teachers and students interpret and give meaning to different kinds of socio-mathematical SIGNS. All forms of mathematical expression have intrinsic meanings and inner workings (Rotman 1988, 2000; Ernest, 2006). These expressions, significantly present in what lies ahead, also engage the subjective element of the meaning-making process of the Interpreters.

Under the lens of the Peircean triadic system of SIGNS, we look upon *classroom interpretation* as a progressive, ever changing *mental signifying process*. During this signifying process, Person *X* not only interacts with *other people* (Self-Others or Inter) but also, as we will see, when Peirce adds the third component to his more extended system of SIGNS, Person *X* also co-acts with the *Self* (Self-Self or Intra). These interpretations lead to the refining of inter-intra *cycles of objectification* that follow from intentionally constructed and highly coordinated *sign-interpretant formations.* During this meaning-making process, mathematical SIGNS are encountered in the network of socio-cultural *semiotic systems* (Wilder, 1981) and upon which mathematical semiotic systems are fully grounded.

Obvious it is that semiotic reality is significantly embedded in the natural world. Ignore it, maybe; pretend that it is not there, maybe. However, try as we will, try as we may, there is no way to make it go away. Include it we should because semiotic reality will always remain a fundamental part of our common reality. This is the same semiotic reality that teachers and students inherit and jointly activate in the classroom. Therefore, along with staying anchored to the natural world, any approach to mathematics education that does not in some way find a place for the central presence of semiotic reality is an approach that falls short, some would say far short, of its full potential.

As this chapter unfolds, it is easy to suppose that the presence of teachers and students is being ignored. This is far from being the case. In fact, there are four major layers built into the unfolding of this chapter. Whenever we start with Peirce and the topic of semiotics, this topic is so wide and so inclusive that we must start with *semiotic reality* in its far reaching and in its most general sense (G). For example, the scope of semiotic reality is so extended that we can now say that it includes the realization that people not only use SIGNS but that plants and animals also send signals (Sebeok, 1972; Deely, 1990).

(1) In the earlier part of the chapter we will focus on SIGN activity, also called semiosis, when, in general (G), each and every Person *X* makes any use of SIGNS. (2) As the chapter proceeds, we will look at semiosis as it takes place in the mathematics education community, namely, among (M)athematicians, (T)eachers, and (S)tudents. (3) Once the full scope and depth of semiotic reality is in place, the emphasis will be aimed at mathematical activity in the classroom (T and S). (4) Coming then to the primary and central goal in mathematics education, we will end by giving high focus to the *intra-abstracting-objectification* that, in some degree, takes place in each and every learner (T or S).

This chapter is divided into four sections. In the first section, when we activate the beginning part of Peirce's system, we sketch what we call a clarifying adaptation of the three main components of his triadic system of SIGNS. For us, these components are called sign-object, sign-vehicle, sign-interpretant; here also called *so*, *sv*, *si*, respectively. We also activate the beginning part just enough to call on the subcategories of each of the three components. (1) The sign-object *so* subdivides into *immediate, dynamic,* and *Real*; here called *io*, *do*, *RO*. (2) The sign-vehicle *sv* subdivides into *icon*, *index*, and *symbol*; here called *sv-icon*, *sv-index*, *sv-symbol*. (3) The sign-interpretant *si* subdivides into *intentional, effectual,* and *communicational*. After the main outline of this working frame is in place, we will look more closely at the use of only one SIGN, the use of any one SIGN in general (G).

In the second section, we introduce the use of standardized mathematical SIGNS, and we examine the central and focal role that sign-interpretant formations play in the emergence and the refinement of *mathematical conceptions*. These are the subjective formations that, in stages, will eventually approximate to the Real Object of the (M) athematicians, namely, the mathematical Concept, here called *RO(M).* We use the three components of the Peircean SIGN to unfold what happens when teachers and students progressively (a) construct their own mathematical conceptions when they *decode* standardized mathematical SIGNS and then (b) *encode* these conceptions back again *into* the given standardized mathematical SIGNS. Following from (a) and (b), teachers and students construct, re-construct, and refine their mathematical conceptions until they will be coordinated and integrated sign-objects that, at any given stage, will become their best understanding of a given *RO(M)*.

In the third section, we use Peirce's triadic SIGN to present our view of classroom interpretation. This view covers the teaching-learning of mathematics when it is seen as a *double semiotic process of interpretation*, a double process in which both teachers and students actively participate. Interpretation in the classroom is examined in terms of *inter-interpretation* and *intra-interpretation*, or what we will sometimes call *inter-intra interpretation*. Each process will be examined both as a reiteration and as a refinement of triangular cycles of objectification: (i) decoding-objectification, (ii) abstracting-objectification, and (iii) encoding-objectification.

The third section introduces an exception. This chapter is organized in terms of *inter-intra*, but in this section, intra-interpretation comes *before* inter-interpretation. It is much easier to present the separate triangles in Figure 5 *before* we introduce the two kinds of lines that interlace those triangles in Figure 6. What comes next after this section will continue in terms of *inter-intra*.

In the fourth section, we use the notion of inter-intra interpretation to call attention to a fundamental commonality that exists between Peirce and Vygotsky. It will point not only to the socio-cultural aspects of cognition but also to an important relation that exists between objectification and internalization.

PEIRCE'S TRIADIC SIGN

Historically, signs in the broadest sense were seen as mediating entities that prompt thought, that facilitate the expression of thought, and that embody original and conventional thought (Nöth, 1990). Signs themselves were believed to have intrinsic meanings, meanings that were realized when signs were translated into other signs, meanings that were *independent of the Interpreter*. Signs were thought to be *dyadic entities* constituted by signifier and signified (Saussure, 1972; Nöth, 1990; Vasco, Zellweger & Sáenz-Ludlow, 2009). Here notated as the pair (signifier, signified) or (sign-vehicle, sign-object). Note that, as indicated in Figure 1, if we start with the two components contained in the dyadic notion of sign, this leaves us with only one bidirectional relation (A), the relation between the signifier (sign-vehicle) and the signified (sign-object).

Central to the position taken by Peirce, about a century-and-a-half ago, is the key step he took when he transcended the dyadic conception of sign. He proposed that each and every sign should also have a third component, namely, what he called "interpretant" and what we will call sign-interpretant. Adding this third component extends the dyadic notion of sign to a triadic notion. We notate this triadic notion as SIGN to differentiate it from the dyadic notion of sign.

The triadic SIGN extends the dyadic sign to the part that is *intra*, to what happens *after* the mental arrival of a signifier, to what happens to the cognitive activity that takes place in the mind of an *Interpreter*. The third component, along with including the *Interpreter*, contains the world of *intra-placed* sign-interpretants. It follows that the *Interpreter*, any Person *X,* plays a double role: the role of Interpreter-Receiver who *decodes from* sign-vehicles, and the role of Interpreter-Sender who also *encodes into* standardized or idiosyncratic sign-vehicles.

In consequence, we cannot confuse the sign-interpretant with the Interpreter. The sign-interpretant is the construction that is formed in the mind of a Person *X* who is the Interpreter. In the eye of a Constructivist, the sign-interpretant is the mental construction that is formed *after* the mental arrival of a sign-vehicle. This construction has a dynamic and evolutionary formation in the mind of the Interpreter (i.e., Person *X*). Construction that emerges in the midst of Self-Self or Self-Other interaction.

Why do we come to Peirce? Because without Peirce's third component, the *intra* that exists in semiotic reality is not made a part of the dyadic notion of sign. We cannot say it more emphatically. This comes back to Figure 1. When there is no third component, there is no formal connection in "the system of signs" to both sides of the double process of interpretation. It follows that a "dyadic system of signs" falls far short of what we need. For us, in keeping with Peirce, a good system of triadic SIGNS should reach out and incorporate not only the presence of Self-Other but also the presence of Self-Self.

We notate Peirce's triadic SIGN as the triplet (sign-object, sign-vehicle, sign-interpretant) or (*so*, *sv*, *si*). This triplet could also be expressed as (signified, signifier, sign-interpretant) which is the extension of the pair (signified, signifier). To better understand this triadic notion, we call on the lower and upper levels of the tetrahedron in Figure 1. The lower level is located at the base of the tetrahedron, there showing the three components—*so*, *sv*, *si*. The upper level is located at the peak of the tetrahedron. The peak is for the *triadic unity* of these three components, the triadic SIGN. Note that, as indicated in Figure 1, when Peirce added the intra-placed sign-interpretant as a third component, he also added two new bidirectional relations among the three components: relation (B) between the sign-object and the sign-interpretant, and relation (C) between the sign-vehicle and the sign-interpretant.

Even though we support and follow Peirce's triadic system, we acknowledge that Peirce himself uses his own terminology in such a way that it sometimes leads the reader to ambiguity and confusion. This introduces both a strong precaution and a serious risk whenever we try to quote from his writings, especially given the many decades and the many stages across which he created his system. For example, he sometimes uses the word "sign" to refer not only to his triadic SIGN itself but also to the sign-vehicle component of the triplet (sign-object, sign-vehicle, sign-interpretant). More explicitly, we encapsulate the nominal ambiguity as follows: sign = SIGN = (sign-object, sign, sign-interpretant). So when reading Peirce one must pay close attention to the contextual meaning he intends.

Avoiding this ambiguity lies behind our efforts to select vocabulary that will present a clarifying adaptation of his triadic SIGN. In our notation, we refer to the triadic SIGN as follows: SIGN = (sign-object, sign-vehicle, sign-interpretant) = (*so*, *sv*, *si*). We do this by the way we label the four vertices of the tetrahedron in Figure 1. As shown at the peak vertex of the tetrahedron, the word SIGN, *used only in upper case*, stands for the unified and undividable totality that identifies the triadic relation among the components of the triplet. This tells us that the word SIGN stands for a fundamental and defining property of Peirce's semiotic system. The other three vertices in the base of the tetrahedron, sign-object, sign-vehicle, sign-interpretant, *always expressed in lower case*, refer to the three components of the triadic SIGN. Concisely, taking off from our clarifying adaptation, we will enter Peirce's system by way of the vocabulary that goes with the four vertices of the tetrahedron in Figure 1.

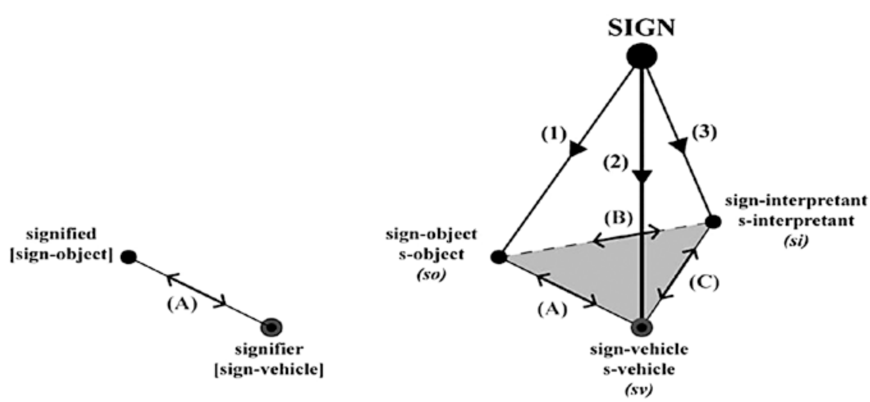


Figure 1. Dyadic and triadic conceptions of signs

Peirce defines SIGN as a *triadic relation* among its three components, a relation that determines a unified and undividable totality. He argued, on the one side, that thought can be known *between people* only by external *sign-vehicles* of some kind and, on the other side, that the only thought that Person *X* can cognize is thought that initiates the construction of *sign-interpretants*. This "one side, other side" distinction is at the heart of the inter-intra double semiotic process of interpretation. In what lies ahead, (1) "between people" will refer to the agents of Self-Others sign-interpretant formation that takes place during inter-interpretation, and (2) "a Person" will refer to an agent of Self-Self sign-interpretant formation that takes place during intra-interpretation. The same distinction will also claim center stage when we call attention to a fundamental commonality that exists between Peirce and Vygotsky.

When the intra-placed sign-interpretant is introduced as the third component, the meaning of SIGNS is located in two worlds—the world of the *intended meanings* of Senders and the world of the *interpreted meanings* of Receivers. This distinction pulls semiosis into the foreground when interpreted meanings take form, converge to, and agree with intended meanings. Such a convergence emerges mediated by SIGNS of different semiotic systems used when thinking and communicating. This tells us that the meaning of SIGNS, specifically, what the Sender *encodes into* sign-vehicles and what the Receiver *decodes from* sign-vehicles, emerges through *repeated* exchanges and *repeated* inter-intra interpretations. These exchanges and repetitions, both in the Sender and the Receiver, prompt the emergence, the construction, and the refinement of increasingly improved intra-placed sign-interpretants. What was said above does not exclude, in any way, the possibility of self-communication in which the same Person plays, alone, the roles of Sender and Receiver. This is the case of Self-Self cognitive activity.

One might think that the *sign-object* component of a SIGN is completely *encoded into* only one sign-vehicle and that it can be *decoded from* that sign-vehicle all at once. However, as we will see, three difficulties follow. (1) Just one sign-vehicle cannot completely indicate the many-sided aspects of the Real Object of a SIGN. It can only indicate at least one aspect of it. (2) Sign-interpretants prompted by a sign-vehicle and constructed at different times by Person *X* may or may not, at once, come close enough to the *intended* immediate sign-object that was *encoded into* a given sign-vehicle. (3) Sign-vehicles could function as *sv-icons*, *sv-indexes*, or *sv-symbols* depending on the contexts in which they are used and how they are interpreted in that context.

We are now ready to look more closely at only one SIGN, at any one SIGN in general (G). Peirce argues that it may be more convenient to say that, in a certain way, a sign-vehicle is determined by "a Complexus or Totality of Partial Objects" (Peirce, 1909, p. 492). He calls this Complexus or Totality of Partial Objects the Real Object of the SIGN. Here we notate it as *RO(G)*. *RO(G)* could be material, imagined, or conceptual (whether it be conventional or idiosyncratic). The adjective "Real" in Real Object does not mean that the Object necessarily has to have a material existence in the real world. The adjective "Real" expresses the compounded comprehensiveness of a multifaceted Object.

One or more selected aspects of *RO(G)* are *offered in* and *obtainable from* the explicit form of a sign-vehicle. Thus this sign-vehicle only *presents* certain selected aspects but never *all* aspects of *RO(G)* at the same time. That is, a sign-vehicle *serves RO(G)* only when it helps to make explicit and to specify some selected aspects of it. This is to say that to comprehend *all* aspects of the *RO(G)* of a given SIGN, these aspects need to be *represented by* different sign-vehicles. As a result, Peirce conceptualizes three subcategories of the sign-object component of the SIGN: the Real Object, the immediate object, and the dynamic object. We notate these objects as *RO(G)*, *io*, and *do*, respectively.

These subcategories of the sign-object of the SIGN are described in the following paragraphs. The first paragraph is for the grounding subcategory of the sign-object. It is the target object, also called the Real Object *RO(G)*. The second paragraph is for the immediate sign-object *io*. It refers to those aspects of the Real Object that the Sender encodes into a sign-vehicle. The third paragraph is for the dynamic sign-object *do*. It refers to those aspects that the Receiver decodes after the mental arrival of the sign-vehicle.

First, the Real Object is the grounding subcategory of the sign-object component of a SIGN. The goal of the Interpreter is to make the best effort to approach the target sign-object, which is the Real Object *RO(G)*. Amid the process of interpretation, the Interpreter-Receiver generates cycles of objectification that approximate the immediate sign-object encoded into a sign-vehicle. In each cycle of objectification, the Interpreter generates sequences of sign-interpretants that will become sequences of dynamic sign-objects and that will be refined to approximate the immediate sign-object. These dynamic sign-objects are also determined by collateral successions of added experience. Peirce insists that the search for better dynamic sign-objects calls for *inquiry* and *discovery.* In the long run, the Interpreter-Receiver isolates and identifies (decodes) the aspect(s) of *RO(G)* that the Interpreter-Sender has encoded into one or more sign-vehicles.

Second, the immediate object is a subcategory of the sign-object component of a SIGN. It refers only to the aspect-object that a given sign-vehicle represents. It comes into existence only after at least one aspect of the Real Object has been selected and successfully carried into, that is, *encoded into* what will become its given sign-vehicle. In effect, the immediate sign-object refers to one or more selected *aspects* intended to represent the Real Object. Peirce argues that the immediate object is the "Object *within* the Sign [sign-vehicle]" (1977, p. 83, italics added). In other words, the immediate sign-object is the object "as the Sign [sign-vehicle] itself represents it, and whose Being is thus dependent upon the

Representation of it in the Sign [sign-vehicle]" (CP 4.536). While the immediate sign-object participates in a certain generality, it also brings specificity into focus. Thus, the immediate sign-object is a representation of some aspects of the Real Object of a SIGN and it serves to stimulate further semiosis (Corrington, 1993).

Third, the dynamic object is another subcategory of the sign-object component of a SIGN. It is constructed in the mind of the Interpreter as the product of sign-interpretants. It is always constructed *after* the mental arrival of the aspect-containing sign-vehicle. It is constructed when the Receiver makes an effort to pull out, to *decode* the immediate sign-object carried by the aspect-containing sign-vehicle. As Peirce argues, the dynamic object is the "Object *outside* the Sign [sign-vehicle]" (1977, p. 83, italics added), or that object "which, from the nature of things, the Sign [sign-vehicle] *cannot* express, which it can only *indicate* and leave the Interpreter to find out by collateral experience" (CP 8.314, italics added).

In general, under the Peircean semiotic lens, the cognitive process of Person *X* can be seen as the progressive refinement of subjective dynamic sign-objects prompted by immediate sign-objects encoded in sign-vehicles. This refinement is prompted and sustained by how Person *X* interprets aspect-containing immediate sign-objects carried by sign-vehicles. Along with constructing intra-placed sign-interpretants, Person *X*'s interpretations follow from interactions that take place both with Self-Others and within the Self-Self.

In the following section we unfold this refinement as a cognitive process that starts with beginning mathematical conceptions and that converges to mathematical Concepts.

FROM MATHEMATICAL CONCEPTIONS TO THE ATTAINMENT OF MATHEMATICAL CONCEPTS

We shift now to the lower level of the tetrahedron in Figure 1 and how it functions in mathematical conceptualization. Sign-vehicles play a primary and fundamental role in the formation and refinement of mathematical conceptions. Very much in the subjective domain (intra), these conceptions are formed during mathematical semiosis, when Person *X* decodes mathematical immediate sign-objects io's from mathematical sign-vehicles sv's. These conceptions, in stages, will eventually become the formal mathematical Concept RO(M) (the Real Object of the Mathematician M). It is during this developmental semiosis that Person *X* establishes the cognitive and epistemic aspects of the Person-Object relation.

In other words, standardized mathematical sign-vehicles serve as *mediators.* Sign-vehicles come between the other two components, between mathematical immediate sign-objects and mathematical sign-interpretants—*io(sv)si*. In fact, sign-vehicles play the role of mediating cognitive tools, which in Vygotsky's terms are called psychological tools. More specifically, (1) sign vehicles, serving as psychological tools, are *determined by* the immediate sign-objects that they carry and (2) sign-vehicles will also *determine* many possible dynamic sign-objects in the mind of the Interpreter.

It is important to note that this *twofold determination* calls for two complementary mathematical acts. The first is made when Person *X* (Interpreter-Sender) encodes a selected mathematical immediate sign-object into a selected mathematical sign-vehicle—[(*io)*](*sv*). Second is made when Person *X* (Interpreter-Receiver) *decodes* this mathematical immediate sign-object *from* that sign-vehicle to obtain a sign-intepretant from which his dynamic-object is constructed—[*(io)(sv)*] (*do)*. Note that this SIGN structure not only occupies a fundamental part of mathematical semiosis but that it is also clearly made explicit in Peirce's system of SIGNS.

The primary and fundamental role of sign-vehicles becomes even more interesting. As already mentioned, three kinds of sign-vehicles are connected to the sign-objects when they are sub-classified into *sv*-icons *sv*-index, and *sv*-symbols (1) A *sv*-icon is a sign-vehicle that bears a resemblance to its sign-object, such as the drawing of a triangle when it is taken to be the representation of the class of trilateral figures. (2) A *sv*-index has a cause-effect connection to its sign-object, such as the connection of the letter "x" to an unknown quantity. (3) A *sv*-symbol is connected to the sign-object by habit, as established by consensus, such as when ">" stands for the relation "greater than." Most sign-vehicles in mathematics belong to this subcategory. Note that, also for Peirce, the word "symbol" refers only to a subcategory of the component called sign-vehicle. This three-fold sub-classification adds to the challenge of selecting clarifying and aspect-specifying sign-vehicles.

As already mentioned, a single mathematical sign-vehicle can stand for only some of the aspects of a mathematical Concept. Therefore, different but interrelated mathematical sign-vehicles should be chosen from different standardized *systems* of mathematical SIGNS to convey a given mathematical Concept. These systems are well-established and carefully connected collections of SIGNS that extend across an extremely wide range of vocabularies, notations, algorithms, tables, graphs, diagrams, metaphors, analogies, models, arguments, proofs, etc. Even though the sign-vehicles of these systems of SIGNS were at first open to idiosyncrasy and unconventionality, they have acquired, by now, a high degree of standardization.

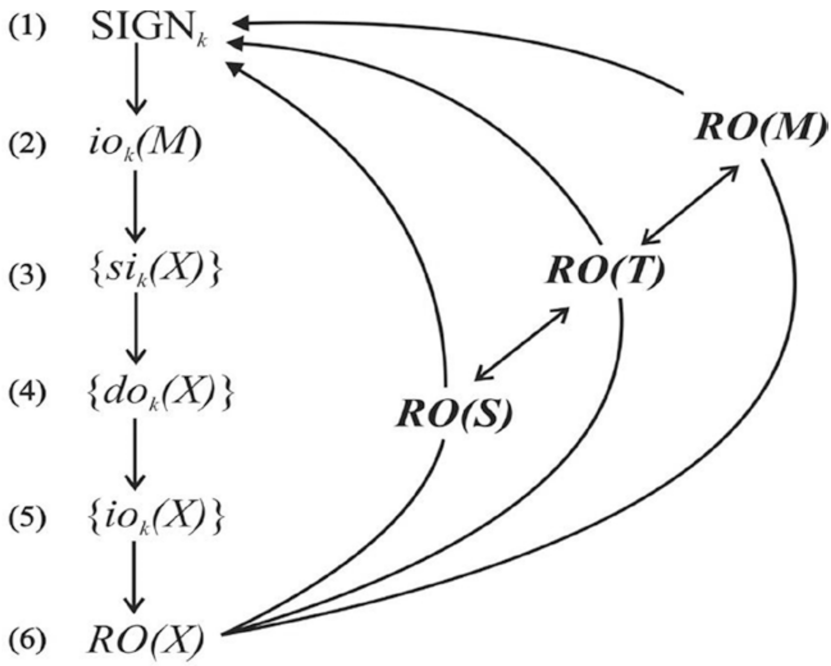
In effect, from a mathematical perspective, Person *X* interprets mathematical immediate sign-objects that have been encoded into standardized mathematical sign-vehicles. Then, he constructs and refines sign-interpretants to obtain dynamic sign-objects that will approximate the mathematical immediate sign-objects encoded by mathematicians into sign-vehicles. At each stage of this process, Person *X* makes every effort to attain the best approximation he can, at that moment, of *io(M)* and, later on, of *RO(M)*. When a sign-vehicle, carrying the *io(M)*, is interpreted by Person *X*, he generates *si(X)*'s and subsequent *do(X)*'s and *io(X)*'s, which in turn will approach the *io(M)* and, later on, will converge to *RO(M),* the mathematical Concept. This process is also improved when Person *X* calls on personal collateral observations and insights based on prior mathematical knowledge and experience.

In general, the distinctions and the complementarities between the mathematical immediate sign-object as *intended* by *M* and as *interpreted* by Person *X* have implications for the mathematical semiosis of Person *X*. This activity is not only confined to the self-reference of SIGNS. It also reaches out and includes personal, inter-personal, and social experiences. These experiences may also become relevant to an ongoing semiosis even though they may be only virtually semiotic with respect to that semiosis.

Consequently, during this mathematical semiosis, sign-vehicles that carry the intended mathematical immediate sign-object of the mathematician, *io(M)*, are the sign-vehicles that prompt Person *X* to generate sign-interpretants. Some of them can become dynamic mathematical sign-objects, *do(X)*'s, that give rise to the emergence and the refinement of personal mathematical *conceptions, io(X)*'s. These conceptions will eventually isolate and identify the intended *io(M)*.

As *RO(M)* is represented by different *io(M)*'s encoded into different but interrelated *sv*'s, the mathematical conceptions of Person *X* will emerge from personal interpretations. At each stage of the process of interpretation, the *decoded do(X)*'s will progressively constitute themselves into *io(X)*'s that, again, will progressively constitute themselves into a coherent unity *RO(X)*, which is the Real Object interpreted by Person *X* and taken by him as his approximation of *RO(M)*). Thus *RO(X)* is the result of a process of interpretation during which Person *X* makes every effort to approach *RO(M)*. This process will continue as long as Person *X* stays interested in increasing his mathematical understanding.

In what follows we will describe the mathematical semiosis of Person *X* in two levels. Here we need a soft warning. Since we are entering only the beginning part of Peirce's system, we do not climb into the layers of his more extended system that contains 10, 28, and 66 classes of SIGNS (Farias & Queiroz, 2003). When we do no more than stay within the scope of our working frame, it is still the case that describing the two levels will also serve as an example of how detailed this approach can become, when it is needed. Nevertheless, at first glance to a beginner, saying this much could easily be looked upon as climbing into a system of SIGNS that is too elaborate and overextended. Note, however, that the challenge is still open as to how these two levels would be described if Peirce's more extended system were activated.



(1) one mathematical representation of *RO(M)*

(2) mathematical immediate sign-object *encoded* by *M into* one sv_k

(3) *X*'s sequence of sign-interpretants induced by sv_k

(4) *X*'s dynamic sign-objects induced by the sequence of sign-interpretants

(5) *X*'s first mathematical conceptions

(6) *X*'s first approximation of *RO(M)*

Figure 2. First level of semiosis: Person X decodes only one standardized mathematical to attain his initial mathematical conceptions and his first approximation RO(X) of RO(M)

Figure 2 represents the first level of semiosis when Person *X* (*T* or *S*) uses only one SIGN and decodes its corresponding sign-vehicle. This first level is the simplest level of mathematical semiosis that appears when, in stages, Person *X* decodes only one standardized mathematical sign-vehicle sv_k. This mathematical semiosis takes place when *M* becomes an Interpreter-Sender who encodes selected aspects of a mathematical concept io_k (*M*) into one sign-vehicle sv_k and when Person *X* (*T* or *S*) becomes an Interpreter-Receiver who decodes it.

When Person X decodes sv_k, the $io_k(M)$ of sv_k elicits in Person X a sequence of sign-interpretant formations $\{si_k(M)\}$. The sub-index k of si indicates its association with sv_k. The recurrent sequences $\{si_k(X)\}$ generate recurrent sequences of mathematical dynamic sign-objects $\{do_k(X)\}$. These sequences represent the evolving subjective understanding of Person X. When these interrelated sequences are coordinated and integrated, they generate the sequence of interpreted mathematical immediate sign-objects $\{io_k(X)\}$, which represents the initial conceptions of Person X that comes to be an approximation of $io_k(M)$. Thus, when the sequence $\{io_k(X)\}$ is integrated and coordinated it constitutes itself into an *RO(X)* that Person X takes it to be his first approximation of *RO(M)*.

Figure 3 represents the mathematical semiosis when Person X uses a selected assortment of SIGNS and decodes their corresponding sign-vehicles. This second level is the more complex level of mathematical semiosis that appears when, in stages, Person X decodes the same RO(M) not from one but from a well chosen assortment of standardized mathematical sign-vehicles $\{sv_k\}_k$. This mathematical semiosis takes place when M becomes an Interpreter-Sender who encodes selected aspects of a mathematical concept $\{io_k(M)\}$ into different sign-vehicles $\{sv_k\}_k$ and when Person X (T or S) becomes an Interpreter-Receiver who decodes them. When Person X decodes the set $\{sv_k\}_k$, he generates a sequence of sequences $\{\{si_k^j(X)\}_j\}_k$ of intra-placed mathematical sign-interpretants associated with each element of the set $\{io_k(M)\}_k$. The super-index j of $si_k(X)$ indicates the sequence of sign-interpretants that is constructed when Person X decodes each sv_k.

This sequence of sequences then generates a second sequence of sequences $\{\{do_k^j(X)\}_j\}_k$ of mathematical dynamic sign-objects for each sv_k. Subsequently, this second sequence of sequences generates a more refined sequence of sequences of decoded mathematical immediate sign-objects $\{\{io_k^j(X)\}_j\}_k$. These more refined sequences constitute an improvement in the mathematical conceptions of Person X after every sv_k of the assortment is decoded in coordination with the others. When this latter sequence of sequences is integrated and coordinated, it converges to $\{io_k(M)\}_k$. Thus, this convergence is what, at this stage, Person X takes to be the best approximation *RO(X)* of *RO(M)*.

Especially important in this process is a consideration of the interpretation that takes place *between people*, when the mathematical sign-object (*RO*, *io*, or *do*) is in the mind of one Person (for example, *M*) and the mathematical sign-interpretant and mathematical dynamic sign-object (*si*, *do*) is in the mind of *another Person* (for example, *T*, or *S*). In other words, we need to consider not only what is determined in the mind of the Interpreter-Encoder (intentional sign-objects and intentional sign-interpretants) but, specially, what is also constructed in the mind of the Interpreter-Decoder (interpreted sign-objects and constructed sign-interpretants).

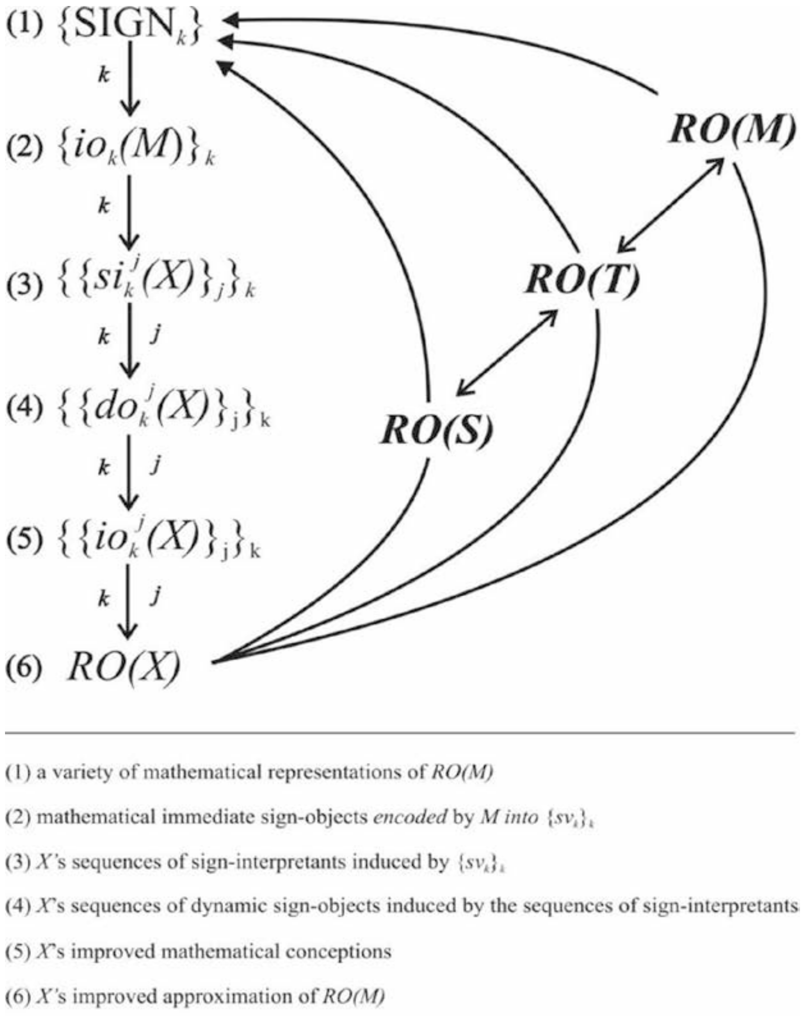


Figure 3. Second level of semiosis: Person X decodes a selected assortment of sv_k's to attain more refined mathematical conceptions $\{\{io_k^j(X)\}_j\}_k$ and, at the same time, a better approximation RO(X) of RO(M)

For communication to take place, reaching some sort of agreement (communicational sign-interpretants) is a necessary condition. In our case, standardized mathematical SIGNS will achieve their communicative function only if the agreement to be reached is whatever is expected to be *commonly understood* between Interpreter-Encoders and Interpreter-Decoders. Thus, a mathematical agreement is, in essence, the communicative invariance of mathematical SIGNS.

These are the meanings that transcend subjective interpretations, that transcend particular contexts, and that transcend any given moment in time. These are the meanings that converge to the intended meanings encoded into the second component, namely, the sign-vehicle. Even though agreement may not come in its complete totality, the classroom participants should agree, at least, on some of the essential aspects of any given mathematical Concept *RO(M)*. Aspects that are represented and carried by a set of standardized mathematical aspect-specifying sign-vehicles.

CLASSROOM MATHEMATICAL ACTIVITY

Within the Peircean semiotic approach that we have taken, we will present our view of *classroom mathematical activity* when it is seen as a *double semiotic process of interpretation*. We consider the teaching-learning of mathematics to be a complex semiotic process of interpreting standardized mathematical SIGNS, a process in which both teachers and students actively and intentionally participate.

But what is happening in the classroom is not limited to just teachers and students (T and S). Given the full presence of semiotic reality as it exists in the classroom, Person *X* could be *M*, *T*, or *S*. It is a given that *M* has gone through his own developmental stages of mathematical intra-interpretation, the stages in which *M* attains the construction of mathematical Concepts *RO(M)*. In the classroom mathematical activity, T and S also go through their own developmental stages of mathematical intra-interpretation, the stages in which T and S attain their best approximations of *RO(M)*. Even though it is obvious that *M* as a Person is almost always not physically present in the classroom, the work of *M*, namely, the selected mathematical Concepts and related sign-vehicles that point to constructions are always present. This also calls attention to the developmental stages of inter-interpretation that first go from *M* to *T, T*[*RO(M)*]. Then, ideally, they will go from *T*'s interpretation of *RO(M)* to *S, S*[T[*RO(M)*]]. Then, ideally, they will go from *S*[T[*RO(M)*]] to *T, T*[*S*[T[*RO(M)*]]]. These cycles of intra-inter interpretation among M, T, and S ground the classroom mathematical activity.

During classroom communication, sign-interpretants play an important role in the semiotic activity of both Interpreter-Sender (*M*, *T*, or *S*) and Interpreter-Receiver (*M*, *T*, or *S*). This occurs when they seek to attain some sort of consensus. Given an Interpreter-Sender with an intentional sign-interpretant in mind, what is *encoded into* a sign-vehicle is a selected mathematical immediate sign-object.

When the Interpreter-Sender *encodes* an immediate sign-object *into* one or more aspect-specifying sign-vehicles with a particular *intentional* sign-interpretant in mind, the Interpreter-Receiver is expected to *decode* it *from* the given sign-vehicles and, from this, to produce dynamic sign-interpretants and to construct dynamic sign-objects and, from them, approximations of each encoded immediate sign-object. Next, the Interpreter-Receiver becomes an Interpreter-Sender, and the cycles of semiosis will continue until some common ground—consensus or communion—is attained. In Peirce's terminology, the common ground attained by both Interpreter-Sender and Interpreter-Receiver is called a *quasimind,* a *cominterpretant,* or *a commens*.

Needless to say, intra-interpretation in the classroom coexists with inter-interpretation. They can be separated only for the purpose of analysis and description. Both occur within a semiotic reality that is not only mathematical but also social. Consequently, in the classroom, the cycles of objectification of Person *X*, both intra and inter, generate each other synergistically.

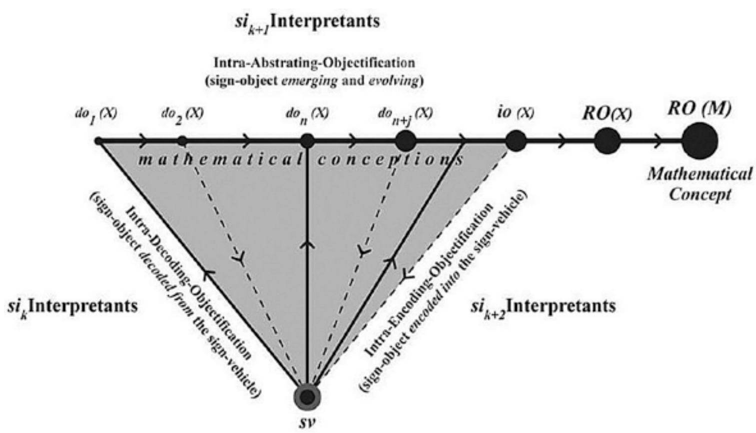


Figure 4. Triangular cycles of objectification of Person X that take place during intra-interpretation: (intra-decoding-objectification), (intra-abstracting-objectification), (intra-encoding-objectification)

Intra-Interpretation

We consider intra-interpretation to be a triangular cyclic process of objectification. Figure 4 shows the three components of the cycle: *intra-decoding-objectification*, *intra-abstracting-objectification*, and *intra-encoding-objectification*. During this process, Person *X* decodes a given standardized *sv* and constructs *si(X)*'s and *do(X)*'s to produce *io(X)*'s that are then encoded back into the same *sv or* related *sv*. Each cycle produces more refined dynamic sign-objects *do(X)*'s that are better approximations of the immediate sign-object *io(M)* initially *encoded into* a given sign-vehicle. These cycles continue, consciously or unconsciously, until Person *X* is satisfied with the construction of an *io(X)* and an *RO(X)*. Both of them will eventually converge to the mathematicians' *io(M)* and *RO(M)*.

Figure 5 shows the triangular cycles of objectification of the classroom participants—*M*, *T*, S_i and S_{i+1}—when each goes through their own triangular cycles. At this point, we also elect to describe briefly what happens in the semiotic reality of mathematicians. Mathematicians begin when they create their own mathematical conceptions by means of intra-abstracting-objectification or when they *decode* existing mathematical sign-objects from standardized mathematical sign-vehicles by means of intra-decoding-objectification. In this way, mathematicians construct their own *do*(*M*)'s and refine them so that these dynamic sign-objects cohere with the logic of broader mathematical systems. This is done through repeated intra-abstracting-objectifications, which eventually lead to the construction of new and better *RO*(*M*)'s. Finally, the mathematicians select certain aspects, *io(M)*'s, that identify, specify, and represent their *RO(M)*'s, which they then *encode into* idiosyncratic or conventional sign-vehicles that are communicated to others.

After the work of the mathematicians has been carried into the classroom, teachers together with the students always start with standardized mathematical *sv*'s. They *decode* them to generate their own mathematical dynamic sign-objects, here expressed as *do*(*T*), *do*(S_i), and *do*(S_{i+1}). In the long run, these dynamic sign-objects give rise to the formation of their mathematical conceptions. Usually, the first mathematical conceptions that are constructed by the students could be very different from what mathematicians *intended* when they *encoded* their *io(M)*'s into standardized *sv*'s for their *RO(M)*'s.

When *T*, S_i, and S_{i+1} make an effort to construct their own mathematical conceptions, they will continue to modify and refine their *interpreted io*(*T*), *io*(S_i), and *io*(S_{i+1}) so that they will converge to the *intended io(M)*. Eventually, they will construct what they consider to be their own "mathematical sign-objects" seen as their best understanding of *RO(M)* or mathematical Concept.

All of this is brought into focus by means of *triangular cycles of intra-objectification*. It tells us that the refinement, the coordination, and the integration of a sequence of *do*'s seek to isolate and make explicit the *io(M)* carried by a given standardized *sv*. Selecting different *io(M)'s* and encoding them into different *sv*'s tend to specify more general aspects of *RO(M)*.

When the classroom participants produce their own triangular cycles of intra-objectification and, consequently, their own cycles of signification, they also produce more abstract levels of intra-interpretation. Nevertheless, for us, *intra-interpretation* is nothing more than a mathematical personal process that will also be influenced by the collaborative interaction among the classroom participants. In effect, intra-interpretation keeps pace in parallel with *inter-interpretation*, which is the focus of the next section.

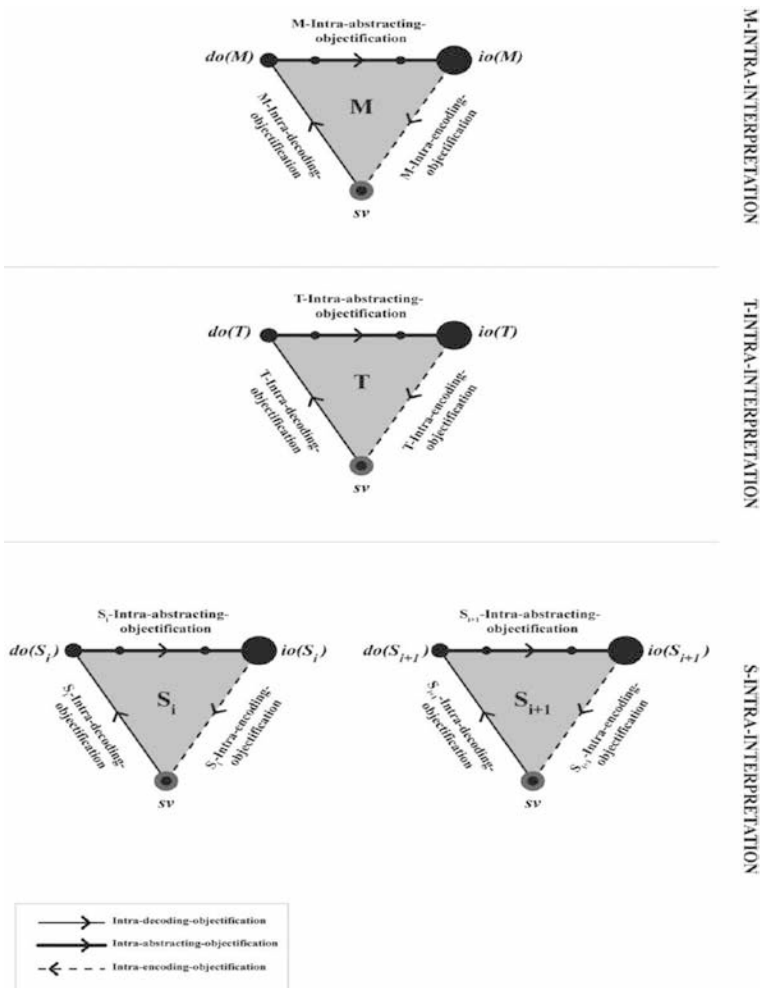


Figure 5. Intra-interpretation: Triangular cycles of intra-objectification of person X (M, T, or S) in the mathematics classroom

Inter-Interpretation

Continuing with the same format, we consider inter-interpretation to be a triangular cyclic process of *objectification*, a process aided by the presence and collaboration of others. Again, the three steps are *inter-decoding-objectification*, *intra-abstracting-objectification,* and *inter-encoding-objectification*. As before, the straight-edged triangles in Figure 6 show that the classroom participants—*M*, *T*, S_i, and S_{i+1}—when each activates their own cycles of intra-interpretation. In keeping with parallel pacing, now the curve-edged triangles of inter-interpretation connect with the straight-edged triangles of intra-interpretation. Semiotic reality is such that both sets of triangles not only coexist. They also interact synergistically. For us, the hyphen in "inter-intra" is a well placed visual sign-vehicle that stands for this synergy.

More specifically, a diagram that lays out the inter-intra connections and that indicates this synergy can be seen by following the two kinds of arrows in Figure 6. Note especially that, and this is a high focal point in our analysis, both sets of triangles have a single common side, the side with the thick horizontal edge, the side in the middle of each cycle of intra-interpretation, namely, *intra-abstracting-objectification.* Later we will look again at this thick horizontal edge. Then we will point to critical moments in the construction of sign-interpretants, construction that takes place in the socio-semiotic reality of each and every student.

In Figure 6, the teacher's inter-decoding objectification is indicated by the solid curved segment starting at the mathematicians' *sv* and ending at the upper left vertex of the teacher's triangle, *do(T)*. This decoding objectification links mathematicians *M* and teachers *T*. It is the first step in the teachers' process of inter-interpretation. The inter-decoding-objectification of the teacher is followed by own process of intra-interpretation. More specifically, T-intra-abstracting-objectification is shown by the side with the thick horizontal edge of the teacher's triangle. This objectification sustains the transformation of *do*(*T*)'s into *io*(*T*)'s. Also shown in Figures 2, 3, and 4, when the interpreted *io(X)*'s (from standardized mathematical *sv's*) are collectively coordinated and integrated, they will move more closely to approach the intended *io(M)*.

It is important to note that the T-intra-interpretation of *sv*'s is the starting point of the interaction between *T* and *S*'s. Not only are *sv*'s sub-classified into *sv*-icon, *sv*-index, and *sv*-symbol. Not only do *sv*'s play a major role when they serve as mediators that stand between sign-objects and sign-interpretants. But also the inter-action of the teacher shines significantly when skills are expressed during those moments when the same *sv*'s are first presented to the students.

When *T* conveys mathematical meanings to the students, he seeks to encode interpreted *io*(*T*)'s to match the meanings carried by standard mathematical *sv*'s. The teacher's mathematical *sv*'s are, in turn, decoded by the students—S_i- and S_{i+1}-inter-decoding-objectification—who then engage in constructing their own *do*(S_i)'s and *do*(S_{i+1})'s.

Continuing with Figure 6, the students' inter-decoding-objectifications are indicated by: (1) the solid curved segments starting at the teacher's *sv* and ending at the upper left vertices of the students' triangles, *do*(S_i) and *do*(S_{i+1}); and (2) the solid curved segments starting at the *sv*'s of the students and also ending at the upper left vertices of the students' triangles, *do*(S_i) and *do*(S_{i+1}).

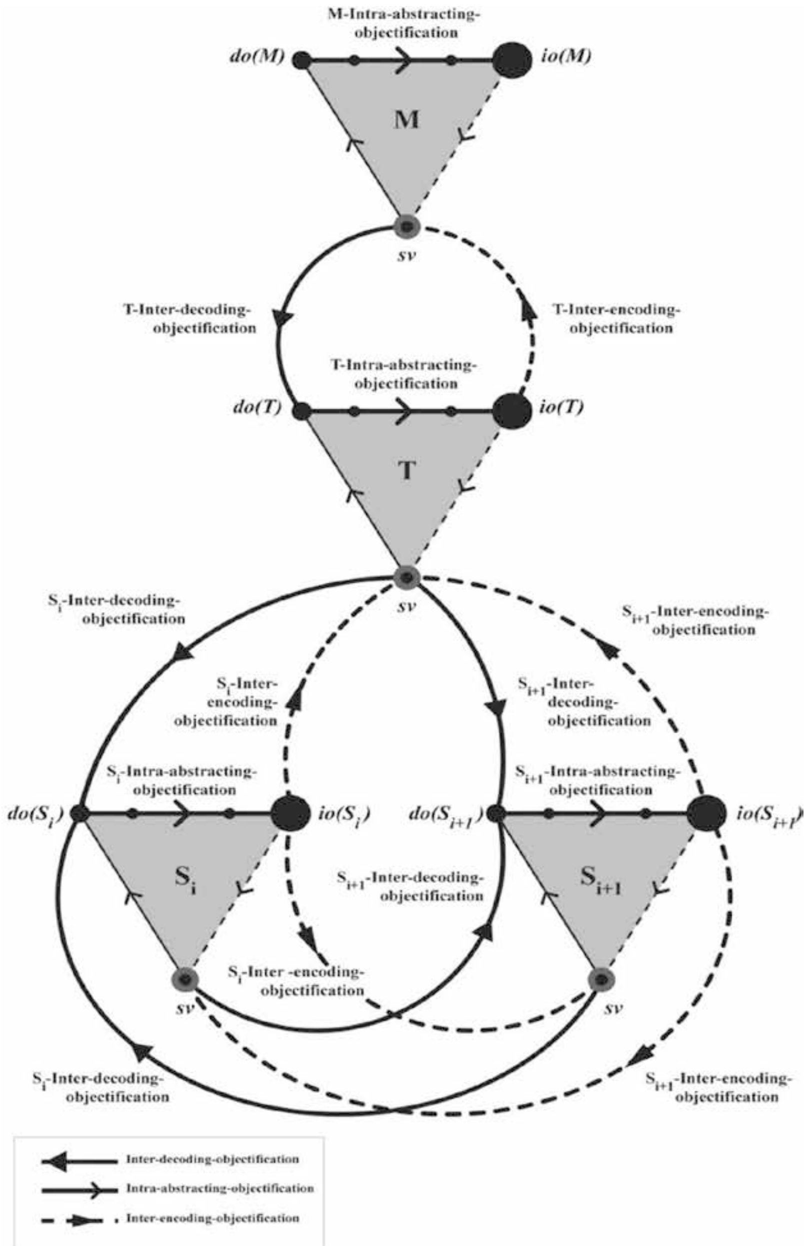


Figure 6. Inter-intra interpretation: Triangular cycles of inter-objectification and intra-objectification of Person X (M, T, or S) in the mathematics classroom

These inter-decoding-objectifications lead to S_i- and S_{i+1}-intra-abstracting-objectifications that transform $do(S_i)$ and $do(S_{i+1})$ into $io(S_i)$ and $io(S_{i+1})$. What follows is the students' inter-encoding objectifications indicated by the dashed curved segments that start at $io(S_i)$ and $io(S_{i+1})$ and end at the *sv* of either the teacher's triangle or the triangle of the other student.

Here we need a forceful alert. Look again at Figure 6 and the thick horizontal edges of the students' triangles. It is during this highly specialized mental activity, mathematically specific, that each and every student engages in intra-abstracting-objectification. Again, in inter-interpretation as in intra-interpretation, we give central attention to these high focal mental moments. Especially sensitive to the semiotic presence of the intra-placed sign-interpretants contained in Peirce's third component, it is the students' intra-abstracting-objectification that not only anchors their cycles of Self-Others *inter-interpretation* but also anchors their cycles of Self-Self *intra-interpretation*.

Consequently, in light of the double semiotic process of inter-intra-interpretation and cast within the limits of resourcefulness and ingenuity, the central and primary goal of teachers is to facilitate and sustain an ongoing *intra-abstracting-objectification* in each and every student.

PEDAGOGICAL IMPLICATIONS OF INTER-INTRA INTERPRETATION

From a Peircean perspective, we have analyzed classroom mathematical activity as a *double semiotic process of interpretation*, a process that is both inter and intra, one that, grounded in the use of standardized mathematical SIGNS, is situated not only in Self-Others but also in Self-Self.

Consequently, this view of interpretation accounts not only for the teacher's semiotic process of interpretation and not only for the students' semiotic process of interpretation. Clearly at another focal spot in our analysis, it also accounts for the teacher's interpretation of the students' interpretation. Being aware of these three parallel semiotic activities would improve not only standard teaching practice. It would also improve the learning conditions that are available to the students.

Giving special attention to the teachers' interpretation of the students' interpretation of mathematical sign-vehicles should encourage and motivate the creation, the organization, and the re-organization of instructional sequences. Such sequences ought to help students refine their inferred mathematical dynamic sign-objects and to approximate both the immediate sign-objects encoded in mathematical sign-vehicles and the Real Objects of mathematical SIGNS. The Real Objects of mathematical SIGNS—Concepts—are abstract objects apprehended by the mind through the mediation of sign-vehicles.

These sequences of mathematical objects, immediate objects encoded into and carried by sign-vehicles and dynamic objects elicited by these sign-vehicles, will allow students to experience their own learning of mathematics as an ongoing process of construction, refinement, and approximation. This is the subjective process of intra-abstracting-objectification aided by inter- and intra-decoding-objectification and by inter- and intra-encoding-objectification. These objectifications are dependent not only on the *inter-actions* among teachers and students but also on the *intra-actions* of the students within their Selves. This intentional, reciprocal, and self-reciprocal engagement of teachers and students in the interpretation of standardized mathematical sign-vehicles will not only regulate the teaching practice of teachers, but it will also regulate the learning practice of students.

VYGOTSKY, PEIRCE, INTERNALIZATION, OBJECTIFICATION, AND THE PERSON-OBJECT RELATION

As seen in sections 2 and 3, the developmental stages of mathematical inter-intra interpretation are grounded in the ongoing effort of Person *X* (T and S) to decode mathematical Concepts *RO(M)* from standardized mathematical sign-vehicles. This decoding is a deconstructive-constructive act given that sign-vehicles represent some but not all aspects of *RO(M)*. In these sections, the process of *objectification* is described in terms of *triangular cycles* that are synergistically *inter* and *intra.* These cycles are the fundamental components of the *double semiotic process of inter-intra interpretation*.

The ongoing process of interpretation comes into existence after the mental arrival of mathematical sign-vehicles *sv*. These sign-vehicles have been deliberately selected, first by *M* and then by *T*, to carry aspect-specifying mathematical immediate sign-objects *io(sv)*. These immediate sign-objects carried by sign-vehicles bring about the construction of mathematical dynamic sign-objects [*io(sv)*]*do*. These dynamic sign-objects lead to the construction of approximations of mathematical Real Objects [[*io(sv)*]*do*]*RO(T)* and [[*io(sv)*]*do*]*RO(S)* that each time will approach more closely the mathematical Concept *RO(M)*. All of this, along the way, encourages and calls forth the critical mental moments, namely, the moments that not only prompt the formation of good mathematical sign-interpretants *si* in each and every classroom participant but also prompt good contact with the socio-semiotic reality of the mathematics classroom.

The synergy between the *inter planes* and the *intra planes* of cognitive and semiotic development is not a new notion. Vygotsky clearly argued that the dialectic between the intramental and the intermental planes produces a constant evolutionary development not only in word meaning and problem-solving strategies but also in sign (i.e., sign-vehicle) use.

> We have found that sign operations appear as the result of a complex and prolonged process subject to all the basic laws of psychological evolution. *This means that sign-using activity in children is neither simply invented nor passed down from adults*; rather it arises from something that is originally not a sign operation and becomes one only after a series of qualitative transformations. (Vygotsky, 1978, pp. 45–46; italics in the original)

The above quote indicates that Vygotsky's notion of *internalization* is cast within a frame of a widely conceived semiotic reality that is socially rooted, historically developed, and based on sequential qualitative transformations. He argues that a *transformation* of an *interpersonal* process into an *intrapersonal* one is the result of a long series of developmental events. This transformation is essentially an *operation* that initially represents an *external activity* and then is *reconstructed* and begins to occur *internally*. In this process, an *interpersonal process* is *transformed* into an *intrapersonal one*.

Vygotsky (1986) also defines *internal* activity (*intra*) in terms of semiotically mediated *external* social activity (*inter*). For him, this is the key to understand what happens during the emergence and the refinement of conceptions (*intra*). According to Vygotsky, "everything internal [intra] in higher forms *was* external [inter], that is, for others it *was* what it now *is* for oneself" (Wertsch, 1985, p. 62, italics added).

The Vygotskian notion of internalization of external activity serves as an umbrella for the particular case of internalization of mathematical Concepts. These notions can also be seen, through the Peircen lens, in the double semiotic process of inter-intra interpretation. As expected, this calls for a social setting that puts inter-interpretation first in time because it makes possible what will emerge later in intra-interpretation. Inter-interpretation and intra-interpretation of standardized mathematical sign-vehicles can happen only when there is a synergistic coexistence of external activity (Self with Others) with internal activity (Self with Self) directed toward the construction, the reconstruction, and the approximation of mathematical sign-objects (immediate, dynamic, and Real) built-in mathematical SIGNS.

We can safely say that there is a fundamental commonality that exists between Peirce and Vygotsky: the notion of *internalization*. When we consider *interpretation* to be a *double semiotic process*, and in keeping with Peirce's intra-placed sign-interpretant, we can infer that *inter* and *intra* processes of interpretation are semiotically mediated, intimately interrelated, and essential to internalization. This tells us that Vygotsky's view of *internalization* is essentially not different from what we have said about *triangular cycles of objectification* based on Peirce's sub-classification of the sign-object component of the SIGN. Along with constructing intra-placed sign-interpretants, these objectifications follow from interactions that take place both with Self and Others (inter-mental) and with Self and Self (intra-mental).

Therefore it can be said that in the teaching-learning of mathematics *objectification*, within a Peircean perspective, is a special case of *internalization,* within a Vygotskian perspective. In other words, objectification is the internalization of mathematical Concepts when Person *X* attains, from first efforts to latter refinements, approximations of the Real Objects of standardized mathematical SIGNS.

Consequently, the Person-Object relation is established, from beginning to end, in the inter-intra interpretation that takes place after the mental arrival of standardized mathematical sign-vehicles. In other words, the Person-Object relation is established in the midst of the inter-intra interpretation that prompts an evolutionary cognitive development. Such development can be seen through sequential refinements, a progressive developmental transformation of subjectively interpreted dynamic sign-objects. These refinements are prompted and sustained when Person *X* interprets the aspect-containing immediate sign-objects encoded into and carried by mathematical sign-vehicles, and when Person *X* generates sign-interpretants and dynamic sign-objects that approximate not only to the intended immediate sign-objects but also to the Real Objects of mathematical SIGNS—mathematical Concepts.

CONCLUSION

Along the way, we have collected information *about* SIGNS, as we went from SIGNS in General, to SIGNS in the mathematics education community (M, T, S), and then to SIGNS in the classroom (T, S). All of this information was carried forward, as we came to the high focal mental activity that takes place in the mathematical thinking of T and S—intra-abstracting-objectification.

It is well recognized that a highly specialized and an extremely precise use of sign-vehicles is the life-blood of mathematics. As we see it, we need some vocabulary, some carefully chosen words, also called sign-vehicles, which will help us discuss both the nature of SIGNS and how they are used in mathematics. Why? Because the treatment of mathematical sign-vehicles in school mathematics is often limited to giving directions that only tell us how to exercise the proper use of mathematical sign-vehicles without taking into account the students' interpretation. Rarely, at a level above, at a meta-level, are we told anything *about* semiotic reality and *about* the nature of mathematical sign-vehicles.

To meet this need is why we come to semiotics, to Peirce, and to the tetrahedron in Figure 1, now seen as a base from which to construct a grammar at a meta-level, namely, a grammar that presents a more exact way of talking *about* SIGNS in general and *about* mathematical SIGNS in particular.

Mathematical semiosis in the classroom can be seen not only as a double semiotic process of *inter-intra interpretation*. Fundamental to its existence and standing strong within a Peircean perspective, it is also grounded in the clear presence of *semiotic reality.* This semiotic reality appears when systems of mathematical SIGNS are introduced, thereby giving rise to the emergence and refinement of mathematical conceptions, mathematical Concepts, and habits of mathematical thinking. This semiotic reality also appears in systems of classroom practice and in systems of communication that are social and cultural. These three systems are all manifestations of the functioning of living, open, dynamic systems.

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VISUALISATION

CAROLINE YOON

4. VISUALISATION FOR DIFFERENT MATHEMATICAL PURPOSES

ABSTRACT

Visualisation is often suggested as a useful heuristic for generating new ideas when one is stuck on a problem. Yet generating ideas is just one aspect of mathematical activity. Visualisation can also help students generalise mathematical discoveries and communicate mathematical ideas. This chapter describes how the nature of a person's visualisation can change depending on the purpose for which it is used, and how this in turn influences the mathematical structure one perceives. A pair of participants used gestures to visualise maxima and minima in an antiderivative problem. Their visualisation techniques changed to encourage more local analysis when they began generalising a rule for discovering maxima and minima. Then, they developed a simple graphical visualisation tool to communicate their rule to layperson clients. The study highlights the need for students and teachers to be aware of the different mathematical purposes for which visualisation can be used, and the kinds of semiotic activity that can facilitate each case.

INTRODUCTION

Visualisation is often considered to be a generative activity that helps us develop new insights. Students solving mathematics problems are encouraged to visualise during the initial stages of their problem solving activity by using diagrams, graphs and gestures to discover mathematical patterns and relationships (Tall, 2004). When it comes to generalising their discoveries and communicating their findings, however, students are usually encouraged to use more formal kinds of semiotic systems such as written language, mathematical symbol notation and algebra. Yet visualisation can play a productive role in these latter stages of mathematical problem solving, just as it does in the earlier stages.

This chapter describes a case study of a pair of participants, who use visualisation for these three purposes: to discover, generalise and communicate a rule for determining whether an x-axis intercept on a graph of a function corresponds to a maximum or a minimum on its antiderivative graph. As their purposes for engaging in visualisation change, so does the nature of their semiotic activity and the mathematical structures they attend to. I consider two related questions that this case study motivates:

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 69–87.

1. How does the nature of one's visualisation change for these three purposes?
2. What kinds of mathematics does one attend to during these different types of visualisation?

I use a new analysis tool called SPOT (Structures Perceived Over Time) diagrams (Yoon, submitted) to portray the different kinds of mathematical structures the participants attended to as their visualisations changed. The use of this tool constitutes a kind of meta-visualisation – the SPOT diagrams enable researchers to visualise changes in the participants' visualisations.

VISUALISATION IN CALCULUS

Visualisation has been promoted as a powerful tool for enhancing students' conceptual understanding of calculus. Researchers caution that overemphasising algebraic procedures in calculus can lead to students applying rules without understanding their meaning (Aspinwall, Shaw, & Presmeg, 1997; Thompson, 1994). Eisenberg and Dreyfus (1991) argue that the overemphasis on algebraic techniques in teaching has led to students being reluctant to use graphical or visual techniques to solve calculus problems, even when the problems are presented visually. Instead, students typically resort to more familiar algebraic methods, despite the problem being much simpler to solve through visualisation. Recent studies (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010) advocate teaching calculus using graphical and visual representations in addition to the algebraic and numerical representations that are often heavily favoured in practice.

Some studies have specifically shown that visualisation can help students understand relationships between a function and its antiderivative in the graphical domain, which is the focus of the calculus task in this chapter's case study. Berry and Nyman (2003) found that students who exhibited an algebraic symbolic view of calculus had trouble constructing antiderivative graphs within the graphical domain. However, when they experienced the "physical feel" of "walking" a displacement time graph using graphic calculators, they developed a deeper understanding of the relationships between a graph of a function and a graph of its antiderivative. Another study (Haciomeroglu, Aspinwall, & Presmeg, 2010) presented students with derivative graphs, and observed the visual and analytical techniques that the students used to sketch corresponding antiderivative graphs. They found that students who synthesised both visual and analytical approaches were more adept at the task than those who favoured one approach exclusively. Yoon, Thomas and Dreyfus (2011) showed that when participants were presented with a graphical representation of a function without recourse to algebra or numerical methods, they used gestures to build a virtual visualisation of an antiderivative in their mathematical gesture space. This chapter expands the results from Yoon et al. (2011) to show how the participants used other visualisation tools to generalise and communicate specific aspects of their method for building an antiderivative.

SEMIOTIC BUNDLES

The first question in this chapter, *how* one's visualisation changes for different purposes, is supported by the theoretical framework of semiotic bundles for analysing semiotic activity (Arzarello, Paola, Robutti, & Sabena, 2009). Semiotic bundles are built on the notion of a semiotic system, which Ernest defines as consisting of three components:

> First, there is a set of signs, each of which might possibly be uttered, spoken, written, drawn, or encoded electronically. Second, there is a set of rules of sign production, for producing or uttering both atomic (single) and molecular (compound) signs… Third, there is a set of relationships between the signs and their meanings embodied in an underlying meaning structure. (Ernest, 2006, p. 69)

The above excerpt emphasises that a semiotic system is more than just an individual sign taken in isolation—it is a system of related signs that are linked in their production and in the way they give rise to meaning. Arzarello et al. (2009) extend the notion of a semiotic system to include gestures and glances, pointing out that semiotic activity may draw on many different modes, including:

> …words (orally or in written form); extra-linguistic modes of expression (gestures, glances,…); different types of inscriptions (drawings, sketches, graphs,…); various instruments (from the pencil to the most sophisticated information and communication technology devices); and so on. (p. 97)

Arzarello et al. (2009) introduce the term *semiotic bundle* to refer to the multiple semiotic systems that one may employ for a particular task, together with the multiple relationships that occur between this wider set of systems and mathematical representations, such as tables, graphs, formal symbols, diagrams and so forth. The framework of semiotic bundles requires one to examine not only distinct instances of semiotic systems that students may produce, but also the relationships between different semiotic systems and mathematical representations produced in different modes. These relationships are both synchronic, where multiple signs are connected and coordinated while being used simultaneously, and diachronic, where a related collection of signs is transformed over time. When applied to analyses of students' visualisation, the framework encourages researchers to focus not only on semiotic activity produced through modes and representations commonly associated with visualisation, such as gestures, diagrams and graphs, but also on the written and spoken words and symbols with which they are linked. Such a holistic approach aligns well with current cognitive research that shows thinking is distributed across multiple modalities. For example, research has shown that gestures co-occur with speech in their production and signification, and should be interpreted in conjunction with the speech that accompanies them (Goldin-Meadow, 2003; McNeill, 2005).

MATHEMATICS AS THE STUDY OF STRUCTURE: SPOT DIAGRAMS

The second question of this chapter considers the mathematical content of students' visualisations. My approach is influenced by the view that mathematics is the study of structures (Shapiro, 1997; Mason, 2004), which are made up of mathematical objects (such as counts, measures, sets), attributes (such as few, large, open), operations (such as combine, enlarge, invert), and relationships (such as greater than, equivalent to, isomorphic). This view is expressed in characterisations of pure mathematics:

> Group theory studies not a single structure but a type of structure, the pattern common to collections of objects with a binary operation, an identity element theoreon, and inverses of each element. Euclidean geometry studies Euclidean-space structure; topology studies topological structures, and so forth. (Shapiro, 1997, p. 73)

And of applied mathematics:

> Mathematical models are distinct from other categories of models mainly because they focus on structural characteristics (rather than, e.g., physical, biological, or artistic characteristics) of systems they describe. (Lesh & Harel, 2003, p. 159)

It is important to clarify that I am using the term "structure" to describe the *mathematics* students perceive or attend to, rather than students' *mental representations* of mathematical knowledge, as it has sometimes been used in the learning theory literature. For example, Skemp (1987) describes schemas or relational understanding as interconnected networks or structures, and Piaget (1970) describes conceptual development as the growth of children's cognitive structures. The terms "perceive" and "attend to" are not meant to suggest that these mathematical structures are embedded "in" the problem, hidden for students to find. Instead, I use the terms in the sense of Mason (2004), to reflect an emphasis on the personal, idiosyncratic mathematical structures that students themselves construct, manipulate and bring to bear on the problem, rather than any formal mathematical structure that students should find.

My analytical approach uses SPOT (Structures Perceived Over Time) diagrams to portray the mathematical structures that students construct and perceive as relevant to the task at hand (see Yoon (submitted) for more detail on this methodological approach). SPOT diagrams are a methodological tool for visualising the objects, attributes, operations and relationships that students attend to, and how these structural components change over time. Although I will describe the process for creating SPOT diagrams in more detail in the methodology section, it is relevant to mention now that these diagrams employ a graph theory paradigm to represent the structural components. Attributes of objects are represented using dots or nodes, while relationships between these objects are represented as connecting lines or edges. Distinct objects are indicated by the colour and location of the dots, and operations are described in words next to the operand.

THE ANTIDERIVATIVE TASK

The antiderivative problem used in the study was set in the context of tramping (a term used in New Zealand to describe hiking), and was created using Model Eliciting Activity design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000). Accordingly, it begins with a newspaper article that describes how a person died while tramping a dangerous track because he was misled by the track's vague difficulty rating, and calls for tracks to be described more carefully in terms of how steep they become. Afterwards, participants are given warm-up activities in which they calculate gradients of a given distance-height graph of a tramping track, and then sketch the gradient graph (derivative) of the track. This sets up the problem statement, which asks participants to design a method that can be used to find the distance-height graph of any tramping track from its gradient graph. Students are asked to explain their method in the form of a written letter to hypothetical clients, the O'Neills, and to apply their method to find features of a specific track whose gradient graph is given in Figure 1.

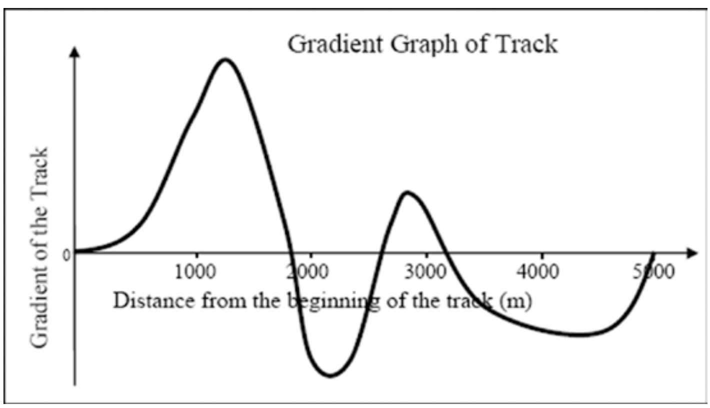

Figure 1. The graph of the tramping track's gradient

> Design a method that the O'Neills can use to sketch a distance height graph of the original track (like the one given in the warm up question). You can assume that the track begins at sea level. Write a letter to the O'Neills explaining your method, and use your method to describe what the tramping trail will be like on the day. In particular, you must clearly show any summits and valleys in the track, uphill and downhill portions of the track, and the parts of the track where the slopes are steepest and easiest. Most importantly, your method needs to work not only for this tramping track, but also for any other tramping track the O'Neills might consider. (Yoon, Dreyfus, Miskell, & Thomas, in press)

This task is mathematically equivalent to creating instructions for finding the graphical antiderivative of a function presented graphically. However, the task doesn't mention the term "antiderivative", nor does it require formal prior knowledge of antiderivatives. In order to encourage students to reason visually, the gradient graph is intentionally given without a scale on the vertical axis, so that students are not inclined to compute the values of the distance-height graph numerically. Additionally, the function of the gradient of the track is presented graphically without an algebraic formula, and is not an immediately recognisable function (such as a quadratic, cubic, or sine or cosine function). These measures were taken to discourage students from solving the question algebraically.

The tramping task engages students with a number of mathematical concepts such as maxima and minima, points of inflection and the area under the curve and its relationship to vertical displacement. In order to compare visualisation in three stages of problem solving, I restrict my focus to the first of these concepts, which was the development of rules for finding maxima and minima in antiderivative graphs. The design of the tramping task encouraged visualisation not just during the discovery or idea generating stage, but also in the generalisation and communication stages. Specifically, the students were first required to describe the tramping track (the antiderivative)—consequently, they had to discover the maxima and minima for the given track. They were also required to generalise their method for finding graphs of any tramping tracks and identify the maxima and minima from any gradient graph. Finally, they were asked to communicate their generalised method in the form of a nontechnical written letter to layperson clients.

METHOD

The participants in this study were two female New Zealand secondary school mathematics teachers: Ava and Noa. Both teachers had studied calculus, but neither had taught it at the Year 13 level (the last year of secondary school in New Zealand). In this study, Ava and Noa functioned as students working on calculus tasks for their own professional development, rather than as teachers teaching in a classroom. The data were collected as part of a larger study that looked at students' construction of calculus concepts (see Acknowledgements).

Ava and Noa worked together on the tramping problem for an hour in the presence of a researcher who clarified the task instructions but refrained from directing them mathematically. Their work on the problem was videotaped and audiotaped to produce verbal transcripts, which were then annotated to include the gestures and nonverbal cues that Ava and Noa performed during the problem. The annotated transcripts were then coded to identify three aspects of their mathematical activity: (1) the semiotic bundles Ava and Noa created using drawings, symbols, graphs, speech, written language, and gestures; (2) the mathematical relationships, objects, attributes and operations that Ava and Noa described through these semiotic bundles; and (3) the goals Ava and Noa set for themselves during the course of the problem.

The author performed the initial coding. Two research assistants then used the same three coding schemes to code parts of the annotated transcript independently (see Acknowledgements). These were compared with the author's codes, and consensus was reached on how to revise the coding schemes.

SPOT diagrams were created to portray the mathematical content of three of the participants' visualisations: when they were discovering maxima and minima, when they were generalising a rule for identifying maxima and minima, and when they were communicating their rule. These diagrams were developed by the author, using the coding of the mathematical relationships, objects, attributes and operations, and they were then checked by an independent research assistant who had not been involved in the coding, but had access to the raw data (transcript, images and written work). The independent research assistant corroborated that the SPOT diagrams accurately portrayed the mathematical content of the participants' visualisations.

VISUALISATION IN THREE STAGES OF MATHEMATICAL PROBLEM SOLVING

Ava and Noa engaged in visualisation during three stages of mathematical problem solving: while discovering, generalising, and communicating maxima and minima properties of antiderivatives. In each case, Noa initiated the signs that were associated with visualisation (e.g., gestures, graphs and inscriptions), and Ava copied and appropriated these signs, integrating them with spoken language and mathematical notations. Ava often did so consciously and deliberately as a way of understanding and contributing to the visualisations (see Yoon, Thomas, & Dreyfus, in press for more detail on Ava's conscious gesture mimicry).

1. Visualising While Discovering Maxima and Minima

Noa (and later Ava too) used gestures to visualise the slope of the tramping track by tracing with her right hand finger along the curve of the gradient graph (segments indicated in Figure 2) and performing gestures with her left hand to describe the corresponding gradient of the tramping track (see Yoon, Thomas, & Dreyfus, 2011 for more detail on the gestures).

Figure 3 shows the path traced out by Noa's gestures, which has been superimposed onto the snapshot image.

Noa describes the journey along the tramping track from the point of view of walking the track herself. She takes into consideration the continuously changing y-values of the gradient graph, and varies the tilt of her hand smoothly to reflect the corresponding changes in the track's gradient, thereby discovering the first maximum on the track.

Noa 154: … and then starting to get not so steep (traces right index finger along section (1) in Figure 2 and gestures corresponding uphill slope that is flattening with her left hand)[1] up to the summit (points with right index finger to (2) in Figure 2 and gestures a flat maximum with her left hand).

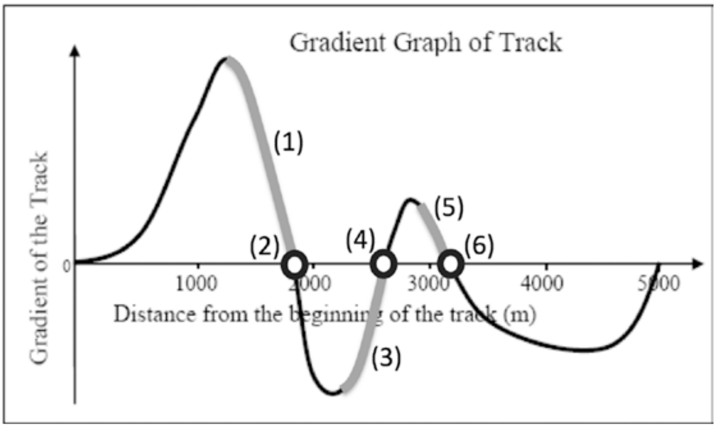

Figure 2. Noa traces her right index finger along sections of the gradient graph

The SPOT diagram in Figure 4 shows the structure Noa attends to through this visualisation. She begins by noticing that the positive and decreasing *y*-value in the gradient graph corresponds to the track slope going uphill but flattening. This is followed by a zero *y*-value, which corresponds to a flat point on the track. The

Figure 3. Noa gestures with left hand the slope of the tramping track (curve indicates gesture path)

combination of the track going uphill, flattening, and then becoming flat implies to her that it reaches a summit. Noa encounters this maximum on the tramping track not by deducing its presence from basic principles, but by noticing it as she experiences it, much like trampers would notice reaching a summit while walking the terrain of the track. She does not discover the maximum on the track directly from the positive, decreasing, then zero *y*-values on the gradient graph, but relies on the intermediary interpretations that the positive, decreasing then zero *y*-values on the gradient graph imply the track slope is uphill, flattening, then flat to identify the maximum on the track. Thus, she identifies the maximum by experiencing the changing slope of the track through the embodied gestures, as if she were walking it herself.

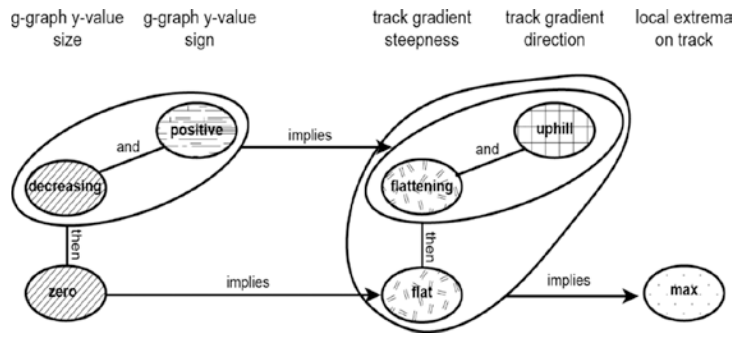

Figure 4. SPOT diagram showing Noa's identification of the first maximum

The subsequent minimum and maximum are identified in much the same way, using an isomorphic combination of semiotic systems. The corresponding SPOT diagrams (Figures 5 and 6) show that Noa also attends to an isomorphic set of objects, attributes, relationships and operations.

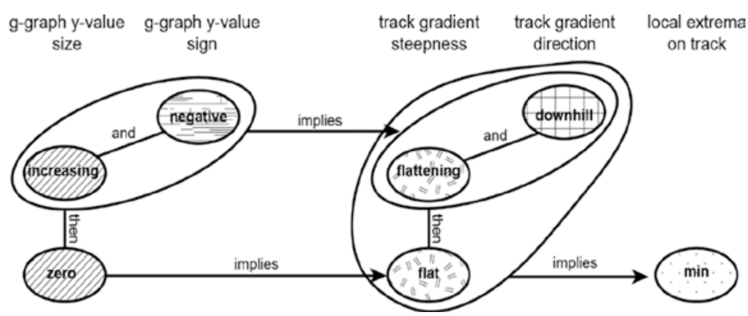

Figure 5. SPOT diagram showing Noa's identification of the first minimum

Noa 158: … it starts levelling off and getting easier again (traces right index finger along section (3) in the gradient graph shown in Figure 2 and gestures corresponding downhill slope that is flattening on an imagined tramping track with left hand) until you get to like a bottom (points with right index finger to (4) on Figure 2 and gestures with left hand a flat minimum).

Noa 160: We're still going up but we're flattening off (traces right index finger along section (5) in the gradient graph shown in Figure 2 and gestures corresponding uphill slope that is flattening on an imagined tramping track with left hand) and then we reach another point where we've got to the top (points with right index finger to (6) on Figure 2 and gestures with left hand a flat maximum).

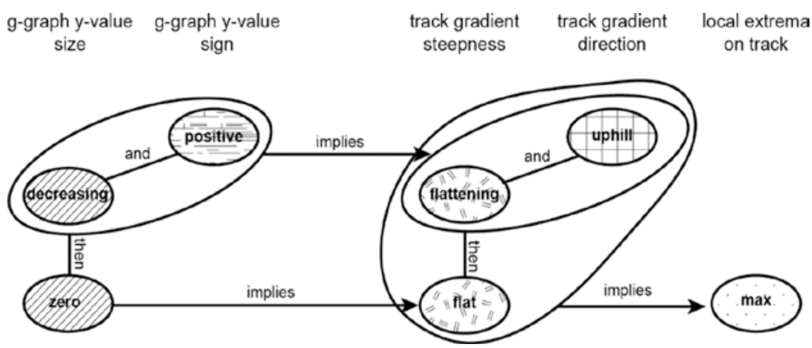

Figure 6. SPOT diagram showing Noa's identification of the second maximum

After visualising the track thus, Ava draws these maxima and minima in a graph of the track shown in Figure 7.

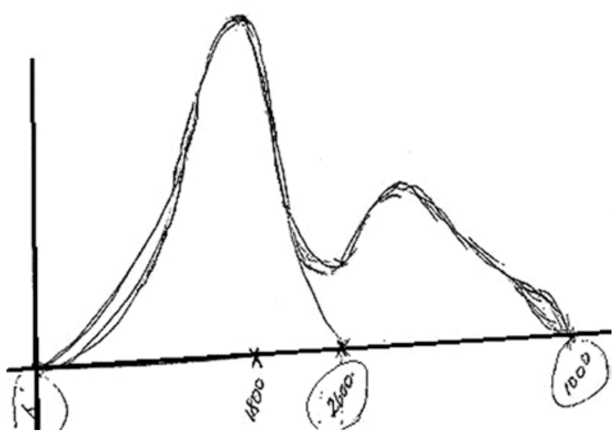

Figure 7. Ava's graph of the tramping track

2. Visualising While Generalising a Rule for Finding Maxima and Minima

After drawing the graph, Ava asks Noa "how are we going to generalise that?" At this question, Noa shifts her attention from discovering maxima and minima by encountering them while visualising the track as a whole, to examining more closely the mathematical properties that give rise to maxima and minima. She compares the incidence of maxima and minima in the graph of the tramping track they have drawn, with the behaviour of the gradient graph immediately before and after the corresponding *x*-axis intercepts.

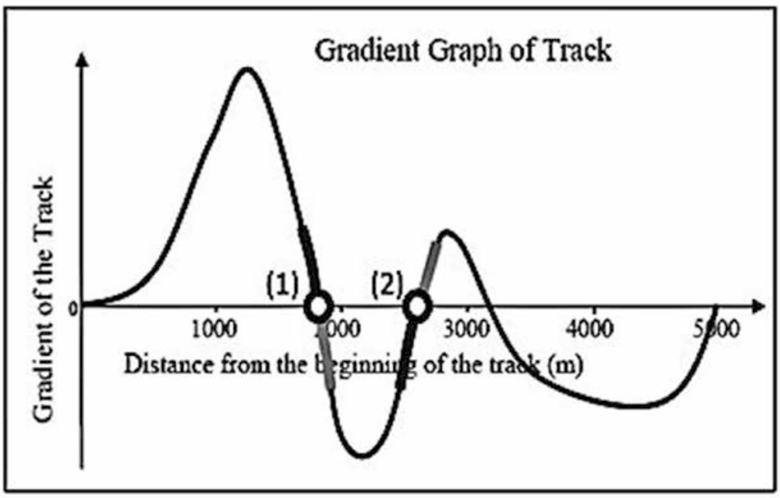

Figure 8. Noa points to the positive and negative segments of the graph adjacent to the two x-axis intercepts

Noa 236: I was just wondering if we were going to be able to say, the first one, you know, when it goes from a positive (*points to positive segment above (1) in Figure 8*) to a negative (*points to negative segment below (1) in Figure 8*), that's going to be a maximum (*points to (1) in Figure 8*), when it goes from negative (*points to negative segment below (2) in Figure 8*) to positive (*points to positive segment above (2) in Figure 8*), that's going to be a minimum (*points to (2) in Figure 8*).

Noa notices that in the given gradient graph, a change from positive to negative *y*-value corresponds to a maximum in the track, whereas a change from negative to positive *y*-value corresponds to a minimum in the track, and she wonders whether this pattern applies to any gradient graph. The SPOT diagram in Figure 9 shows that here, Noa considers a more streamlined structure than previously, skipping over interpreting the track slope and conjecturing directly about the local extrema on the track from the order of positive and negative values on the continuous gradient graph.

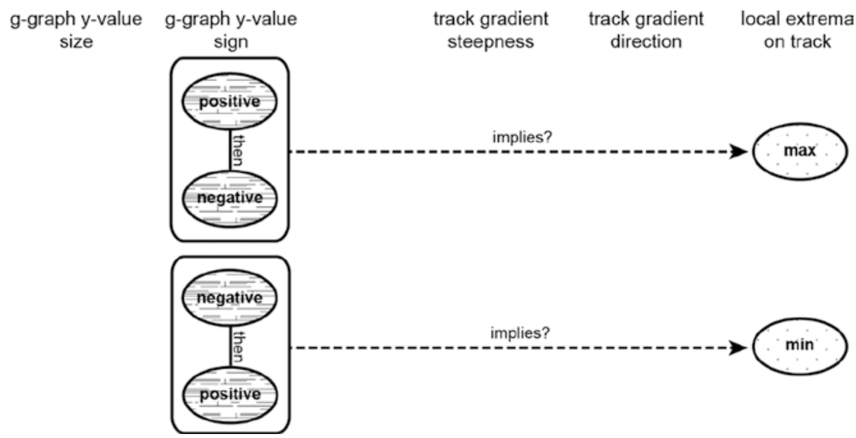

Figure 9. SPOT diagram of Noa's conjectures about positives and negatives and extrema

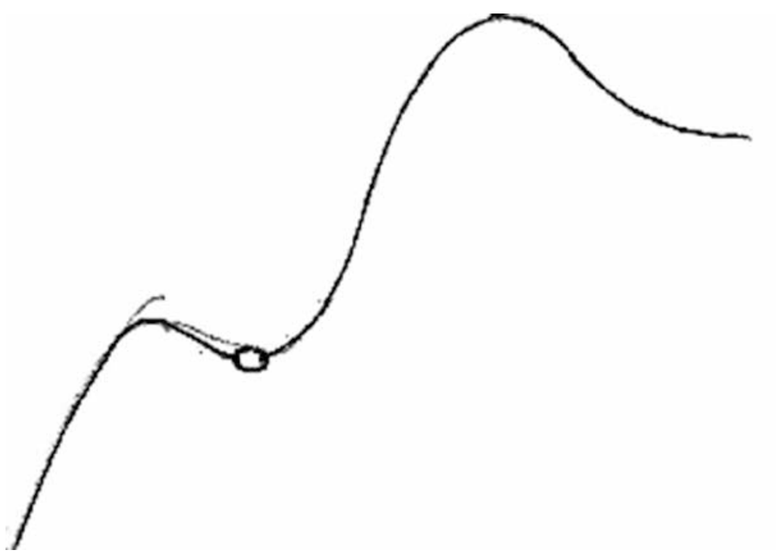

Figure 10. Noa's drawing of a new curve with two local maxima and one local minimum

Noa then draws a new curve with two local maxima and one local minimum (Figure 10) to explore the behaviour of peaks more generally. She reasons:

Noa 239: For that to be a peak (*points to first maximum in the curve in Figure 4b*) and not just a variation in the positive gradient – an actual peak, it has to have a downhill (*points to the downhill section to the right of the first maximum in Figure 10*). So you will have to cross over (*traces with her finger, a curve crossing over an x-axis on the gradient graph in Figure 8*). Do you agree?

As shown in the SPOT diagram in Figure 11, this reasoning differs from Noa's previous thinking in that it starts from the opposite side and works backwards. Noa considers the maximum first, then the duo of implications in the backwards direction: A downhill then uphill sequence in the track is necessary for a maximum, and in turn, a positive then negative sequence in the gradient graph is necessary for the downhill then uphill sequence in the track. Ava and Noa agree with this interpretation and convert it into the forward implications:

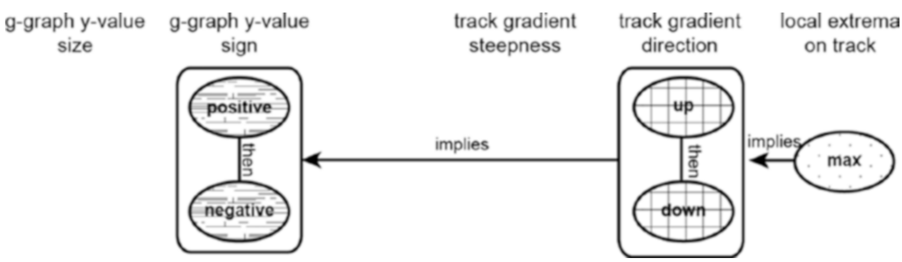

Figure 11. Noa's drawing of a new curve with two local maxima and one local minimum

Ava 240: Yeah, yeah. So you mean going from above (points to positive section before (1) in Figure 8) to below (points to negative section after (1) in Figure 8).

Noa 241: Yes. So it goes from above to below (traces the curve from the positive section before (1) to the negative section after (1) in Figure 8).

Ava 242: All right.

Noa 243: Well that should be a positive to negative (points with right index finger to (1) in Figure 8 and gestures in left hand a curve going from positive to negative in the air, as in Figure 3) a minima (shakes her head) – a maxima.

Here, Noa generalises a rule: A peak in a graph will occur at the corresponding x-value where its gradient graph changes from positive to negative y-values. Interestingly, she describes the change from positive to negative y-value in the gradient graph as an embodied experience of *crossing over* the x-axis, similar to the embodied experience of walking the track. Ava agrees, and applies the generalisation back to the gradient graph they were given, saying, “if it goes from positive to negative then it’s a maxima”, while analysing the behaviour around (1) on Figure 8. The SPOT diagram in Figure 12 shows that Ava and Noa’s rule for identifying a maximum shifts away from considering the continuous variation in the *y*-value of the gradient graph as in Figure 10 to a much simpler structure that considers the discrete changes in the *y*-value (positive to negative) only.

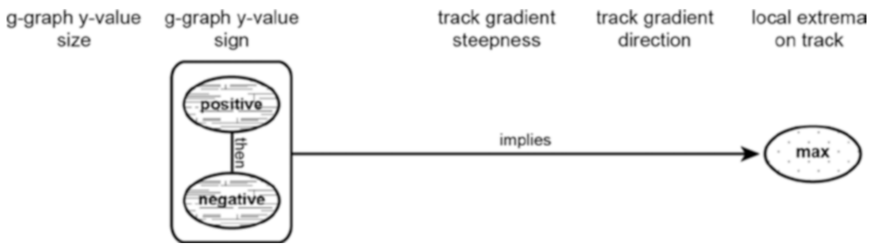

Figure 12. Noa’s drawing of a new curve with two local maxima and one local minimum

3. Visualising While Communicating a General Method for Finding Maxima and Minima

Towards the end of the activity, Ava and Noa switch back and forth between generalising the method and communicating their method to the O’Neills—the layperson clients described in the problem statement. Ava begins by writing the following instruction to the O’Neills: “Find where the graph cuts the horizontal axis. This is either a valley or a peak!” She reflects on the vagueness of this instruction saying, “That’s very helpful [*laughter*], you’re either up high or you’re down low. You’ll know when you’re there.” Her sarcasm suggests that she is aware of the inadequacy of these instructions, and that she and Noa need to provide clearer directions on how to identify peaks and valleys from the gradient graph, rather than relying on the O’Neills identifying them experientially.

Noa constructs a visual tool for communicating to the O’Neills the maxima / minima rule that they had previously generalised. She suggests they tell the O’Neills to draw a rough sketch of the track first, showing uphill, flat and downhill portions of the track. She demonstrates this by drawing straight lines on the graph of the tramping track they had drawn (see Figure 13), which correspond to positive, zero, and negative gradients indicated in the gradient graph in Figure 1. Noa explains that this rough sketch can be a useful step for the O’Neills to visualise the maxima and minima in the track easily, without having to construct the whole graph first.

Noa 250: Yeah, so this is a postive gradient (points to first entire positive portion on gradient graph in Figure 1), so it’s basically up (draws straight line with positive gradient on the tramping track in Figure 13), this is zero (points to (2) on Figure 2), it’s across (draws

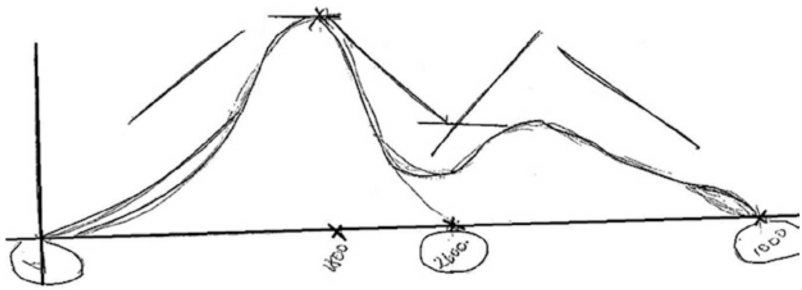

Figure 13. Noa draws straight lines on the graph of the tramping track to indicate positive, negative and zero gradients

> straight line with zero gradient on the tramping track in Figure 13), this one is negative (points to first entire negative portion on gradient graph) it must be down in that whole portion (draws straight line with negative gradient on the tramping track in Figure 13). This one is zero (points to (3) on Figure 2) it has to be flat (draws straight line with zero gradient on the tramping track in Figure 13). This one is positive (points to second entire positive portion on gradient graph) it has to be up (draws straight line with positive gradient on the tramping track in Figure 13) and this bit from here in this section is negative (points to second entire negative portion on gradient graph), it has to be down (draws straight line with negative gradient on the tramping track in Figure 13). So then you've got a bit of a sketch already of where the hills and valleys are.

This rough sketch only describes the direction of the track's slope, and ignores many of the other features they had considered previously, including the variation in the track's steepness and the height of valleys and summits. As the SPOT diagram in Figure 14 shows, Noa attends to the very simple implications between the discrete segments of the *y*-value on the gradient graph (positive, negative, and zero) and the corresponding uphill, downhill or flat portions of the track. Next, she considers trios of up—flat—down in the track to identify maxima and minima appropriately. Yet, this very simplicity makes the sketch a compelling, visual tool that the O'Neills can use to identify the location of summits and valleys easily.

DISCUSSION

Although Ava and Noa drew heavily on visualisation throughout the activity, the nature of the visualisation they used and the kinds of mathematical structure they attended to varied depending on the purpose for which it was employed. While they were engaged in discovering maxima and minima on the antiderivative graph, their

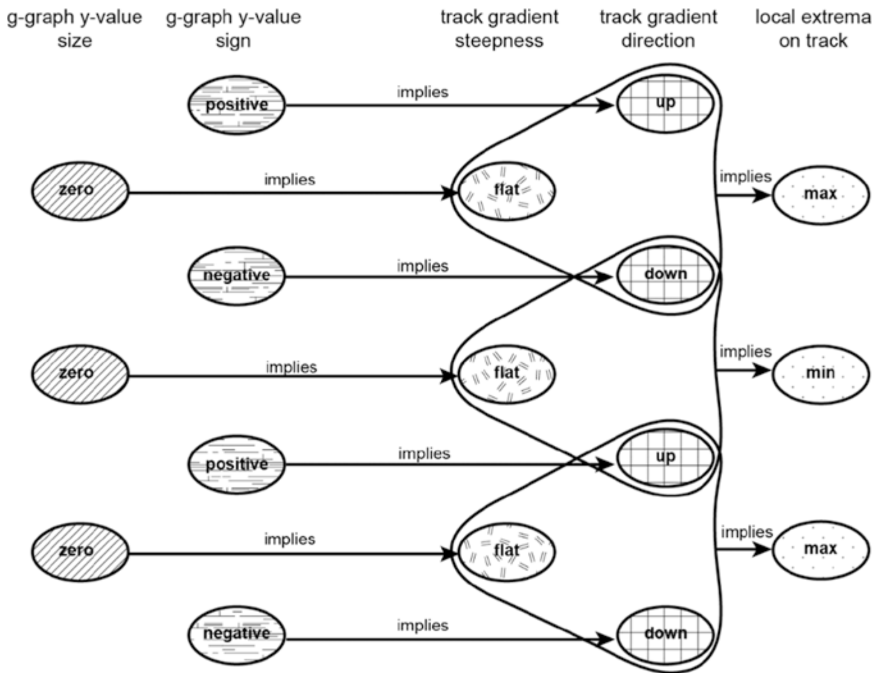

Figure 14. Noa draws straight lines on the graph of the tramping track, which correspond to uphill, downhill and flat portions of the track

visualisation took the embodied, contextual form of gestures, which simulated the experience of actually walking along the track. These holistic, enactive gestures enabled them to imagine the physical exertion involved in walking up or down a hill, to notice the continuous variation in the incline, and to encounter summits and valleys along the way. This "physical feel" is similar to what Berry and Nyman (2003) described when observing students use graphic calculators to make sense of gradient graphs in the context of speed. In both instances, the students' visualisations enabled them to imagine and experience the physical feelings associated with the contexts in which the gradient graphs were set.

The nature of Ava and Noa's visualisation changed when they began to generalise a rule for determining whether an x-axis intercept in a gradient graph corresponded to a maximum or a minimum in the original graph. Previously they had visualised the entire tramping track, but their attention turned toward the behaviour of the gradient graph (and the tramping track) immediately before and after the x-axis intercepts. In order to focus their attention on this narrower set of mathematical objects and relationships, they drew a portion of a tramping track (Figure 10) that emphasised maxima and minima, and began pointing at small regions of the gradient graph surrounding x-axis intercepts. This semiotic activity enabled them to shift their structure of attention (Mason, 2004) from the global graph to the local behaviour of the track and its gradient graph in these smaller regions. In doing so, they focused less on the continuous variation in the gradient, and more on the discrete change in the sign of the gradient, which formed the foundation of their rule for determining maxima and minima.

Ava and Noa continued to visualise discrete components of the tramping track when their goals changed from generalising a rule for determining maxima and minima, to communicating that rule to layperson clients. The sketch of uphill, downhill and flat sections constituted a simple tool for visualising the peaks and valleys in the track. Again, the change in semiotic activity is indicative of their shifts in attention: whereas the gestures used during the discovery phase led to a graph of the complete track (Figure 7), the rough sketch in the communication phase (Figure 13) distilled only those components of the track that would help the clients identify maxima and minima. In both cases, Ava and Noa used graphs of the track, but the graphs focus on different mathematical features of the track, due to the different purposes for which they are built. In the first instance, the detailed accuracy of the drawing (and gestures) was important for discovering maxima and minima in the track, as it was the faithful simulation of the experience of walking the track that counted as evidence of maxima and minima in the track. Once the general rule had been developed, however, Ava and Noa abandoned the detailed graph as it did not fit their new goal of communicating to clients how to find maxima and minima. It became more important to develop a simple way of visualising the maxima and minima, and the rough sketch was more appropriate for this goal.

One could wonder whether changes in semiotic activity lead to changes in the structure one attends to, or whether changes in the structure one attends to necessitate a change in semiotic activity. This case study suggests that the source of changes in semiotic activity and perceived structure is in fact due to changes in the participants' goals. When they moved from wishing to discover, to generalising, and then to communicating, this led to a change in both semiotic activity and the kinds of structure perceived. Thus, educators may wish to design and implement tasks that encourage a range of mathematical goals in order to encourage students to engage flexibly in mathematical visualisation.

CONCLUSION

This case study supports calls for visualisation to be used in calculus teaching and learning to provide an alternative way of making sense of calculus concepts. It demonstrates that visualisation can be used for reasons other than simply helping students generate or discover mathematical ideas—visualisation can also help students generalise and communicate those ideas. However, this doesn't mean any kind of visualisation will be successful automatically in each case. Rather, different semiotic activity enables different types of visualisation, depending on the desired goal.

This chapter showed that initially, Ava and Noa used complicated gestures to engage in a kind of embodied visualisation, which helped them make mathematical discoveries. When it came to generalising those discoveries, however, Ava and Noa turned to local pointing gestures and simple diagrams that facilitated a visual analysis of key mathematical objects and relationships. In order to communicate their generalisation, Ava and Noa created a simplified graph that could be used as a visualisation tool by layperson clients. Ava and Noa's case study should not be interpreted as suggesting a trajectory of visualisation approaches, or a guide for matching specific semiotic systems or modes to visualisation needs. Instead, it highlights the importance of being aware of the purposes for which students may engage in visualisation, and challenges educators to create and implement tasks that encourage a variety of these purposes.

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NOTE

[1] The information provided in parentheses are inferences made by the researcher about the participants' non-verbal actions and gestures.

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5

CAROLINE YOON AND TESSA MISKELL

5. VISUALISING CUBIC REASONING WITH SEMIOTIC RESOURCES AND MODELLING CYCLES

ABSTRACT

Diagrams and physical manipulatives are often recommended as useful semiotic resources for visualising area and volume problems in which nonlinear reasoning is appropriate. However, the mere presence of diagrams and physical manipulatives does not guarantee students will recognise the appropriateness of nonlinear reasoning. Three case studies illustrate that the effectiveness of such semiotic resources can depend on whether they enable students to visualise, test and examine their existing incorrect mathematical approaches as they progress around the modelling cycle. Some students used diagrams and multilink blocks to test and reject incorrect linear and quadratic reasoning, whereas others who created diagrams did not use them to test their ideas, and persisted with incorrect linear or quadratic reasoning.

INTRODUCTION

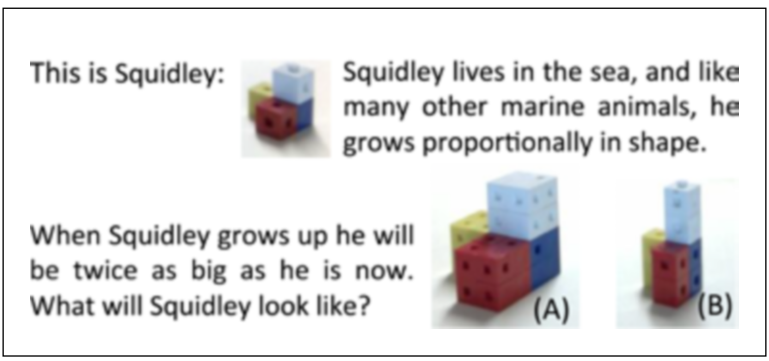


Figure 1. The Squidley warmup activity

Consider the following problem: *Baby humans' bodies are not proportional to adult humans' bodies: a baby's head is large in relation to its squat limbs, whereas an adult's head is smaller in relation to its long limbs (Thompson, 1992). In contrast, baby seahorses resemble miniature replicas of their parents, with their scaled down body part shapes in similar proportions to each other as those of adult seahorses. The following "marine animal" named Squidley (see Figure 1), was constructed*

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 89–109.

from multifix cubes. Like seahorses, Squidley grows proportionally in shape. When he grows up, he will be twice as big as he is now. What will Squidley look like?

The answer of course, depends on what we mean by "twice as big". If we mean that Squidley's *dimensions* of length, width and height are doubled, then option A could be the answer, although option A's volume is not twice as big, but eight times bigger than young Squidley, containing 32 multifix cubes to young Squidley's four. If we mean that Squidley's *volume* is doubled, then option B could be the answer as it contains twice as many multifix cubes as young Squidley, although option B is not proportional in shape to young Squidley, having grown only in one dimension (height). In fact, neither of these two options are correct if we intend to double Squidley's volume while retaining the proportions in his shape: this would require multifix cubes with side lengths that are times bigger than those of young Squidley.

The above question highlights an important mathematical concept: that increasing the dimensions of a three-dimensional figure by a given scale factor does not yield a linearly proportional increase in volume. Instead, the change in volume obeys a cubic pattern as shown in option A above, which is $2^3 = 8$ times bigger in volume when its dimensions are multiplied by a scale factor of 2. Many students and adults find this concept counterintuitive and misapply linear reasoning, saying for example, that option A must be twice as big in volume as its dimensions have been doubled. The misapplication of linear proportional reasoning to situations where non-linear reasoning is required has been described as "the illusion of linearity" (De Bock, Verschaffel, & Janssens, 2002). It occurs in many areas of mathematics (see for example, Shaughnessey, 1992), but is particularly prevalent in problems about scaling up or down the volume (and area) of geometrical figures, where students often assume that multiplying the dimensions of a three (two) dimensional figure by a factor of x will result in a new volume (area) that is also scaled up by a factor of x, rather than x^3 (x^2) (De Bock et al., 2002; Modestou, Elia, & Gagatsis, 2008; van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005).

A common recommendation for overcoming misconceptions like the illusion of linearity is to visualise the scenario using diagrams (Polya, 1957; Schoenfeld, 1994). In this chapter, we consider how students used diagrams and other semiotic resources (van Leeuwen, 2005) within a modelling activity about the scaling up of volume of fish. The mere presence of diagrams alone did not guarantee success in identifying the correct cubic relationship. Instead, our case studies suggest that the effectiveness of diagrams and other semiotic resources depends on whether they enable students to visualise, test and examine their existing incorrect mathematical approaches as they progress around the modelling cycle.

FACTORS IN OVERCOMING THE ILLUSION OF LINEARITY IN AREA AND VOLUME

Students' difficulties with reasoning about scale and proportion in linear, area and volume problems are well documented (Lamon, 2007) and resistant to change, continuing even into adulthood (De Bock et al., 2002). Researchers have investigated the effect of three factors in overcoming the illusion of linearity, with mixed results: the use of diagrams, problem contexts and metacognitive prompts (De Bock et al., 2002; De Bock, Verschaffel, Janssens, van Dooren, & Claes, 2003; Modestou et al., 2008).

Diagrams are often credited with helping students succeed in mathematical problem solving by enabling students to discover and examine underlying relationships (Pantziara, Gagatsis, & Elia, 2009) and generate new ideas (Diezmann, 2005; Nunokawa, 2006), while reducing students' cognitive load (Gibson, 1998; Koedinger, 1994). De Bock, van Dooren, Janssens, and Verschaffel (2002) report a study in which they found only slight advantages in presenting students with diagrams in problems dealing with scaling up and down length, area and volume. Secondary school students received questions like the following:

> Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m? (p. 69)

Half of the students were allocated to a diagram treatment group, in which they also received scale drawings on grid paper for each question; the diagram accompanying the above question showed two squares representing the two pieces of land. Students in the diagram treatment performed better on questions that required non-linear reasoning (area and volume questions), but this difference was very small: success in 17% of the non-linear scaling up problems by the drawing group, compared to 13% correctness in the non-drawing group. In addition, this slightly higher performance on non-linear questions was mitigated by a similarly small but significantly lower performance on questions that required linear proportional reasoning by students who had been given diagrams, suggesting that the diagrams they presented to the students did not help students determine when non-linear reasoning was inappropriate.

In a follow up study, De Bock et al. (2003) switched from giving students ready-made diagrams to encouraging students to construct their own for a similar set of problems. Students in the diagram treatment group were given partial diagrams to complete for each problem. For example, students were given a diagram of square *Q*, and were asked to draw a scale diagram of square *R* for the following question:

> The side of square *Q* is twelve times as large as the side of square *R*. If the area of square *Q* is 1440cm^2, what's the area of square *R*? (p. 448)

Surprisingly, the students in the drawing condition performed significantly worse on the non-linear problems than those without the drawings, which De Bock et al. (2003) attributed to the very process of creating a scale drawing. They reasoned that when students drew a reduced copy of a geometrical figure, they would have measured a linear element such as its height or length, and divided that element by a linear scale factor, effectively activating a linear thought process. This could have enhanced the students' inclination toward a linear model, rather than the quadratic or cubic model that was required. De Bock et al.'s two studies highlight that diagrams are not in themselves effective or ineffective in helping students overcome the illusion of linearity as their success depends on the ability of the person viewing the diagrams to recognise the relevant structure portrayed. The diagrams in De Bock et al.'s two studies (2002, 2003) may have been ineffective in helping students overcome the illusion of linearity because the students did not know what mathematical structure to look for in the diagrams they were given or constructed.

Modelling tasks which use real world contexts have been promoted as potentially useful means for developing linear proportional reasoning (e.g., Lamon, 2007) and non-linear reasoning (Treffers, 1987). However, De Bock et al. (2003) caution that merely setting a routine problem in a real world context doesn't necessarily constitute a modelling task. De Bock et al. (2003) gave one group of students scaling up/down problems set in the context of Gulliver's Travels to the Isle of Lilliputians, a world where all lengths are 12 times as small as those in Gulliver's world, and another group solved mathematically equivalent problems presented as standard textbook formulations with no real world context. Students in the standard textbook group received questions like the one described above about the area of squares *Q* and *R*, whereas students in the Gulliver's Travels group received questions like the following:

> Gulliver's handkerchief has an area of 1296 cm^2. What's the area of a similar Lilliputian handkerchief? (p. 448)

On finding that students in the Gulliver's Travels group performed worse on the test than those in the standard textbook group, DeBock et al. (2003) reason that the Gulliver's Travels problems were simply standard textbook questions that had been "dressed up" in a real world context (Blum & Niss, 1991). They suggest that greater success may be possible with modelling tasks that require more authentic performance-based assessment, such as filling a Lilliputian's wine glass or making a Lilliputian handkerchief.

Metacognitive prompts and scaffolds are a third means for helping students overcome the illusion of linearity as they can encourage students to become more conscious of their misapplication of linear reasoning. Students are often unconscious that they are misapplying linear models whereas others knowingly apply them without realising they are not appropriate (Esteley, Villarreal, & Alagia, 2004). De Bock et al. (2002) gave one group of students a metacognitive prompt intended to provoke cognitive conflict to problems such as, "A wooden cube with an edge of 2 cm weighs 6 grams. How heavy is a wooden cube with an edge of 4 cm?" (De Bock et al., 2002, p. 71). The prompt offered two possible solutions for the students to choose between, one of which misapplied a linear model, and the other used appropriate nonlinear reasoning. For example, the two solution options accompanying the above question were (a) since the edge doubled, the weight also doubled, and (b) a cube with an edge of 4 cm will contain eight cubes with edges of 2 cm so the weight needs to be multiplied by eight. The study yielded significant, positive results but did not enable students to overcome their misconceptions completely as some students in the metacognitive treatment group continued to misapply the linear model afterwards. Moreover, the study found that students who originally applied the linear model "everywhere" started to do the same with the non-linear model, generalising it to inappropriate situations and effectively replacing one model with another.

Modestou et al. (2008) used a different metacognitive prompt to encourage students to question their spontaneous application of the linear model. Students were given sets of three questions comprising one that required nonlinear reasoning, one that required linear reasoning, and an unusual question that could have more than one correct answer. After solving all three questions, the students were asked to identify which (one) of the three questions yielded a given numerical answer. In each case, the question requiring linear reasoning was the correct match, but if students had misapplied linear reasoning to the nonlinear question, they would have obtained the same numerical answer (though incorrect). Almost half of the students who had initially misapplied a linear model to the nonlinear question ended up selecting the correct problem for the given answer, which suggests that the metacognitive prompt forced them to reconsider and correct their initial misapplication. However, a quarter of the students who misapplied a linear model selected the nonlinear question (which is incorrect), which suggests that the metacognitive prompt also led to mistakenly rejecting a correct application of the linear model.

This section has reviewed three factors (diagrams, metacognitive prompts and problem contexts) that may help students overcome the illusion of linearity. In the next section, we consider how all three factors can be incorporated into a theoretical framework based on modelling cycles.

THEORETICAL FRAMEWORK: MODELLING CYCLES AND SEMIOTIC BUNDLES

The theoretical framework used in this chapter draws on two constructs: modelling cycles (e.g., Lesh & Doerr, 2003; Niss, Blum, & Galbraith, 2007; Stillman, Galbraith, Brown, & Edwards, 2007) and semiotic bundles (Arzarello, Paola, Robutti, & Sabena, 2009). Mathematical modelling involves the complex coordination of processes that can be depicted around the modelling cycle as shown in Figure 2. The

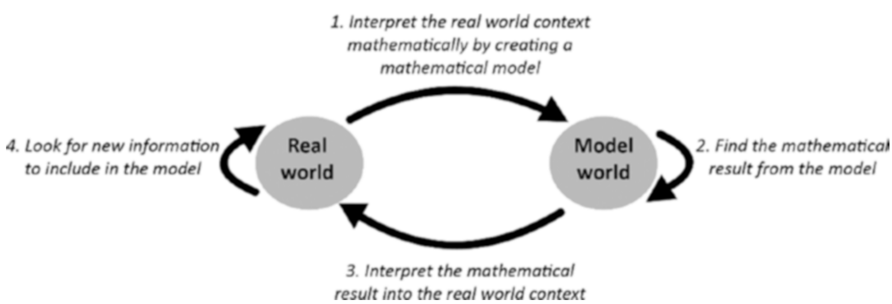


Figure 2. The modelling cycle

modelling cycle begins in the real world, where one determines which features of the real context are mathematically relevant to the problem, and incorporates these relevant features from the real world into a mathematical model. This model is then used to find a mathematical result, which is in turn interpreted back into the real world context. The fitness of the model is then assessed, and if necessary, the cycle is entered into again in pursuit of a model that incorporates more relevant information from the real world. Such cycling continues until the modeller is satisfied with the mathematical model that has been created.

The creation of the mathematical model can be regarded as the development of a semiotic bundle (Arzarello et al., 2009), which consists of signs that mathematically express relevant real world information from the problem situation. The notion of a semiotic bundle is predicated on Peirce's notion of a sign, which is something that "stands to somebody for something in some respect or capacity" (Peirce, 1931/1958, vol. 2, paragraph 228), and is defined as follows:

> A semiotic bundle is a system of signs—with Peirce's comprehensive notion of sign—that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. (Arzarello et al., 2009, p. 100)

Mathematical semiotic activity involving such signs are not necessarily confined within strict boundaries of separate modalities, but are spread across speech, inscriptions, gestures, glances and so forth (Radford, 2009; Arzarello et al., 2009). Consequently, a semiotic bundle includes not only instances of signs and sign systems, but also the coordination of and interrelationships between sign systems across multiple modalities.

Arzarello et al. (2009) often uses the term "semiotic resource" has in place of the terms "sign", "sign system", or "representation". van Leeuwen (2005) clarifies the idea of a semiotic resource as emphasising the semiotic potential of a sign or sign system:

> In social semiotics resources are signifiers, observable actions and objects that have been drawn into the domain of social communication and that have a *theoretical* semiotic potential constituted by all their past uses and all their potential uses and an *actual* semiotic potential constituted by those past uses that are known to and considered relevant by the users of the resource, and by such potential uses as might be uncovered by the users on the basis of their specific needs and interests. (p. 4)

We also adopt the term "semiotic resource" in this chapter to emphasise the semiotic potential of the diagrams and physical manipulatives students employed during problem solving to visualise mathematical structures.

The semiotic bundle approach enables us to consider the diagrams and physical manipulatives students use not as independent semiotic resources, but in relation to other inscriptions and semiotic resources that they also develop. We use both a synchronic analysis (which considers the relationships between semiotic resources activated at the same time) and a diachronic analysis (which considers the evolution of semiotic resources activated over time) (Arzarello et al., 2009) to study how students' diagrams and physical manipulatives evolve in conjunction with other semiotic resources. The evolution of a semiotic bundle over time is similar to Duval's (2006) notion of conversion, where a representational transformation involves a change in register (e.g., from graphical to algebraic), but not in the mathematical object. A number of researchers (e.g., Kaput, 1989; Thomas, 2008) have highlighted the ability to translate fluently between and sometimes within different semiotic resources as an important component of mathematical meaning making.

This theoretical framework of modelling cycles and semiotic bundles encompasses all three factors (diagrams, problem contexts and metacognitive prompts) that were previously identified in the literature as potentially productive ways of overcoming the illusion of linearity. In the first step in the modelling cycle (Figure 2), the real world context encourages students to create a mathematical model (via some semiotic bundle) that has a meaningful real world purpose. The diagrams (and other semiotic resources) that may be created to describe the mathematical model in step 2 may encourage students to test their model in step 3. And this testing and subsequent comparison of the output from the model in light of the real world context may lead students to re-examine their mathematical reasoning in steps 3 and 4. Thus, modelling activities give students the opportunity to experience the potential benefits of all three factors by going through the modelling cycle in a more holistic way than in the "dressed up" textbook problems (Blum & Niss, 1991) used in the studies by De Bock et al. (2002, 2003) and Modestou et al. (2010).

DESCRIPTION OF THE MODELLING ACTIVITY

The Snapper problem (Yoon, Radonich, & Sullivan, in press) is a modelling activity concerned with the fair division of snapper fish of different sizes. It begins with a warmup involving the Squidley question (see Figure 1) to engage students in using physical images related to scale, proportions and volume. After the warmup, students read the Snapper problem statement (see Figure 3) and work on the problem in groups of three for about 45 minutes.

The Snapper problem was designed to encourage students to overcome the illusion of linearity and to develop non-linear (in this case, cubic) models of reasoning about the fair distribution of snapper fish. Its design was influenced by six principles for designing Model-Eliciting Activities (Lesh, Hoover, Hole, Kelly, & Post, 2000), which are a type of problem noted for encouraging students to go beyond their initial, primitive ways of thinking, to develop more sophisticated mathematical interpretations of real world situations (Lesh & Doerr, 2003). The Snapper problem satisfies the *reality* principle as it is set within the realistic context of dividing up a catch of fish. It satisfies the *model construction* and *model generalisation* principles by requiring students to create a generalisable mathematical model (in this case, an argument) that can be used to solve the problem, rather than merely a single numeric solution, such as "8 small fish = 1 large fish". By giving students the physical manipulatives of multilink cubes to test out their ideas, the problem also satisfies the *self-assessment* principle. It also satisfies the *model documentation* principle as students are required to document their mathematical model in the form of a letter to Joe. Finally, the problem elegantly maps the context of a fishing trip to the need for a mathematical model about cubic reasoning with volume, so that it satisfies the *simple prototype* principle.

CLASSROOM IMPLEMENTATION AND DATA COLLECTION

We implemented the Snapper problem in four classes at a large New Zealand tertiary institution. The students in all four classes were taking an elementary mathematics course for foundation studies—that is, they were studying towards tertiary degree entrance qualification as they had not achieved the tertiary level entry requirements

The Loverich family and the Borich family went fishing together. They caught nine Snapper. Zoe Borich caught the biggest one, which was 54 cm long.

Joe Borich took the job of dividing the fish up fairly between the two families so that they had the same amount of fish each. He gave himself the big snapper as his daughter Zoe caught it, and said that it was worth two of the smaller fish (27 cm each). Peter Loverich thought that the flesh from the big fish was probably more than four times that of the smaller ones but decided not to say anything to avoid a scene.

Peter needs your help!

Your job is to work out a mathematical argument for deciding how many little fish the big fish is worth. **Write a letter** to Peter, describing your mathematical argument clearly, using diagrams if you wish. Peter wants to be able to use your argument for future fishing trips, so explain in your letter how he can make your argument work for fish of any size.

Figure 3. The problem statement for the fishing trip MEA

from secondary school. The students worked in groups of three on the Snapper problem during a 90-minute class: about 60 minutes of the time involved the students working on the problem in their groups, and the remaining 30 minutes involved students presenting their group solutions to the class. A researcher and the class lecturer were present in the classroom for each implementation. They interacted with the students to facilitate group discussion and encourage them to use physical blocks and drawings to test out their ideas but they refrained from telling that or explaining why one large fish was worth 8 small fish.

During the in-class presentations, at least one group of students in each of the four classes gave correct reasoning that showed that one fish of length 54cm was worth eight smaller fish of length 27cm with proportional dimensions. Students and the lecturer were given the opportunity to ask presenting groups questions about their solutions: some question and answer interchanges occurred in each class, but there was no in-depth whole class discussion. After the session, the students then completed written individual solutions to the Snapper problem in their own time. Forty-six students handed in their individual solutions one week after the 90-minute class. We collected the written work from the 19 groups in the four classes, the 46 individual written solutions, and researcher field notes of the student presentations. In this chapter, we present three case studies of three groups (comprising three students each) from one of the classroom implementations: Case study 1 involves Liv, Liz and Pania's group; Case study 2 involves Dee, Jan and Lea's group; Case study 3 involves Del, Lyn and Mac's group. We use both the data from their in-class group work and their individual letters to analyse their modelling cycles.

DATA ANALYSIS

Field notes and data from the students' written group work were used to construct a description of the groups' progress during the classroom implementation. We analysed the group and individual solutions to assess the effectiveness of the mathematical argument, taking into consideration the written language, diagrams, tables, numerical examples, and algebraic expressions used by students. We also

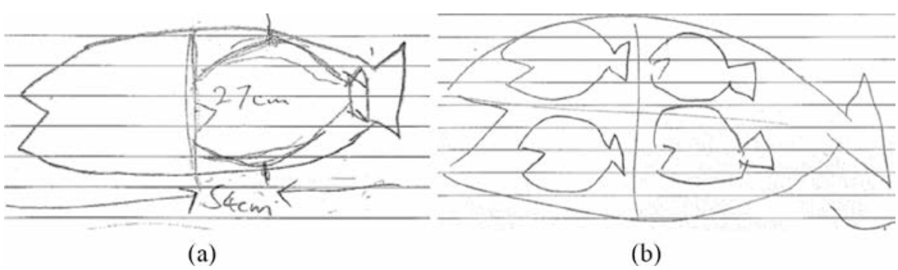


Figure 4a & 4b. Two drawings showing a linear and quadratic model drawn by Liv, Liz and Pania during in-class group work sorted the individual and group letters into three categories. The first category included those that demonstrate *no understanding* of the cubic relationship between snapper volume and linear scale factor. The solutions in this category do not use a cubic model to describe the relationship between the volume of the large fish and the small fish – instead they use either linear reasoning, saying the large fish is worth 2 small fish, or they consider some aspect of area instead of volume. The second category of letters demonstrate a *partial understanding* of the cubic relationship by correctly reporting that one large fish is worth eight small fish, but have weak or incorrect arguments to support or explain why this was so. The third category of letters demonstrate *conceptual understanding* of the cubic relationship between snapper volume and linear scale factor by articulating a convincing argument based on correct cubic reasoning for why one large fish is worth eight small fish.

For each diagram, we analysed how the students had incorporated the diagram into their written argument, the accuracy of the dimension proportions represented in the diagrams, and the mathematical understandings expressed.

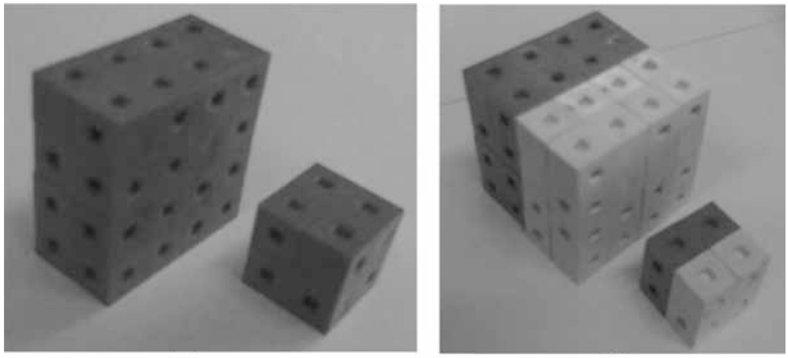
RESULTS

Three case studies show how students used diagrams and physical manipulatives to visualise fish while working on the Snapper modelling activity. For each case study, we describe how the semiotic resources they created facilitated (or not) their progression around the modelling cycle during classroom groupwork and subsequent individual written work.

Case Study 1: Diagrams and Multilink Cubes Generate Multiple Modelling Cycles

During the classroom implementation, Liv, Liz and Pania initially argued that one large fish of length 54cm was indeed worth two small fish of length 27cm, and drew one long fish that was 54cm in length, with another shorter fish only 27cm in length inside it (see Figure 4a). They then realised that their diagram showed two fish that were not proportional in shape: the larger looked liked a stretched out version of the smaller, whose width was the same. This led them to revise their argument to saying that the larger fish was worth four small fish, and they drew a diagram of four small fish fitting into the area of the large fish (Figure 4b). They began writing up their solution, thinking they had found the correct solution.

They informed the researcher that they were finished, at which the researcher asked them to articulate their argument to each other using the multilink blocks. Liv Liz and Pania initially used the configuration of blocks shown in Figure 5a to show, that one large fish was worth four small fish, in accordance with the diagram they had drawn in Figure 4a. However, Pania soon noticed that the two "fish" they had constructed were not proportional in shape in the 3-dimensional representation. Pania realised in an Aha! moment (Liljedahl, 2005) that there are "two sides to each fish", which she explained as flesh on both sides of the bones, and that they had



(a) (b)

Figure 5. Two configurations of multilink cubes for representing the volume of fish used by Liv, Liz and Pania during the in-class group work

neglected to consider the thickness of the fish. This led to a new configuration of blocks shown in Figure 5b, which demonstrated that when one considers the third dimension of thickness, the volume of one large fish is worth eight small fish. In Liv, Liz and Pania's final letter, they justified their final recommendation of eight fish with the statement, "We used the cubes to help us work this out".

The diagrams and physical multilink cubes helped Liv, Liz and Pania to go through the modelling cycle more than three times, as each successive visualisation led to testing then revising their mathematical model or argument (see Figure 6). We have constructed diagrams showing the extent to which students in each case study progressed around the modelling cycle in Figures 6, 12 and 14. In each case, the students begin in the real world context of the fishing trip, and mathematise the problem by creating a model: the arrow from "real world" to "model world" indicates this process. As shown in Figure 2, progression around the modelling cycle ideally involves all four processes between and within the real and model worlds, often with multiple iterations. However, the students in our case studies did not always complete full cycles, and often only carried out a subset of these four processes. The labels on the arrows in each of the modelling cycle diagrams are numbered to indicate the chronological order of the processes.

The individual written solutions that were handed in one week later revealed that the individual students had different levels of understanding of their group's final argument. Pania's individual solution incorporated diagrams of the multilink cubes to show the three dimensions of the fish, which she then used to devise a numerical example to illustrate how to apply her mathematical argument (see Figure 7).

Liz also articulated in her individual solution that they had to take into consideration the thickness of the fish by doubling the amount of flesh on each side. She drew the following diagrams to illustrate this point (see Figure 8).

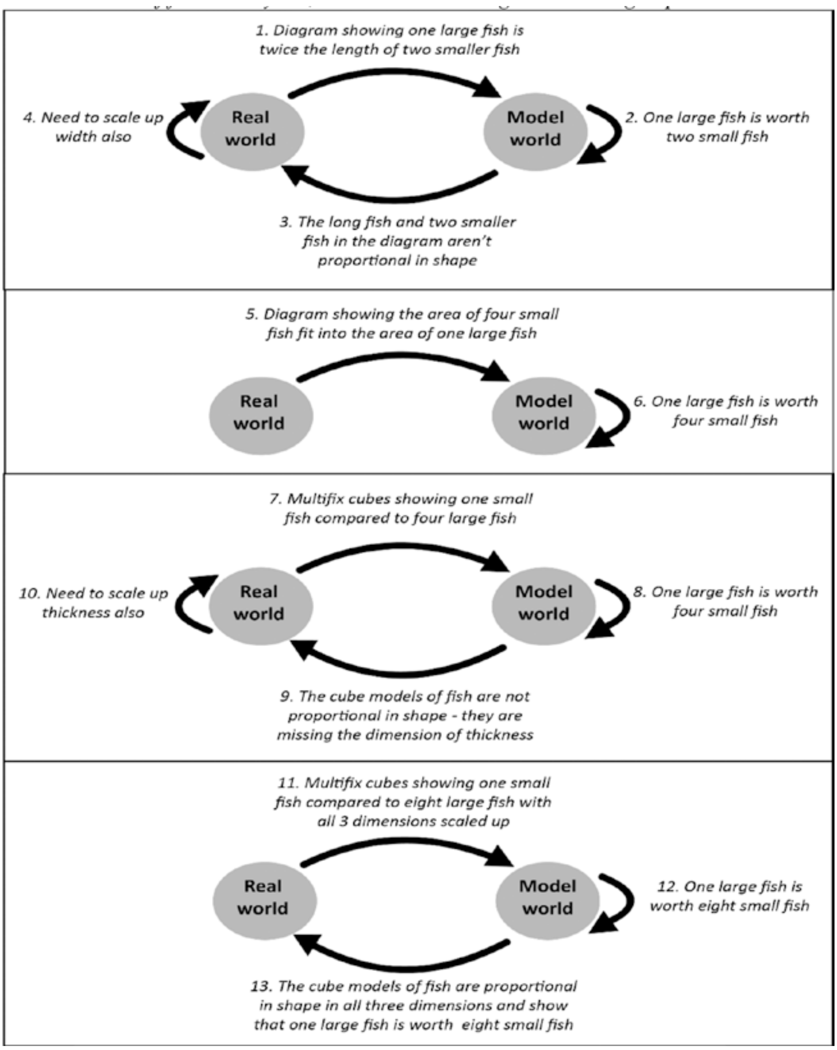


Figure 6. The modelling cycles entered into by Pania, Liv and Liz during group work

In contrast to Pania and Liz, Liv's individual solution reveals a limited understanding of the reasoning for why one large fish was worth 8 smaller fish. She drew scale diagrams of two cube configurations that were meant to represent the two fish (see Figure 9a).

Liv's diagram of the "cubes" in fact only shows a 2-dimensional representation of squares, rather than cubes. These diagrams have the correct number of squares appropriate for the argument, but do not portray the correct proportions in terms

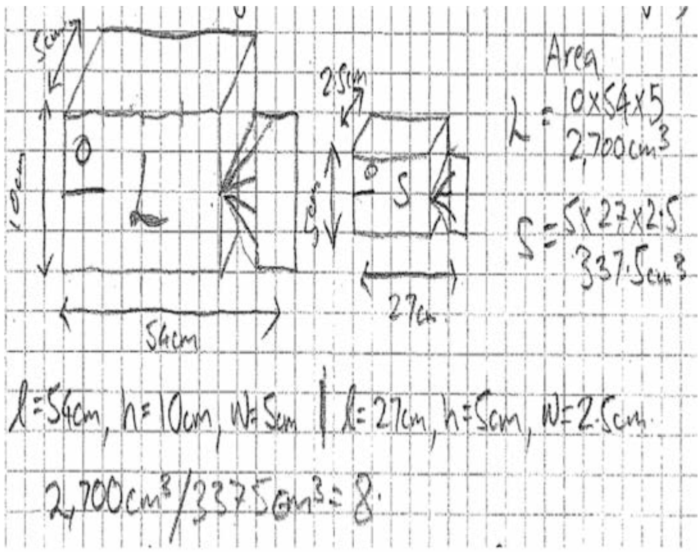


Figure 7. Pania's diagrams in her individual final letter

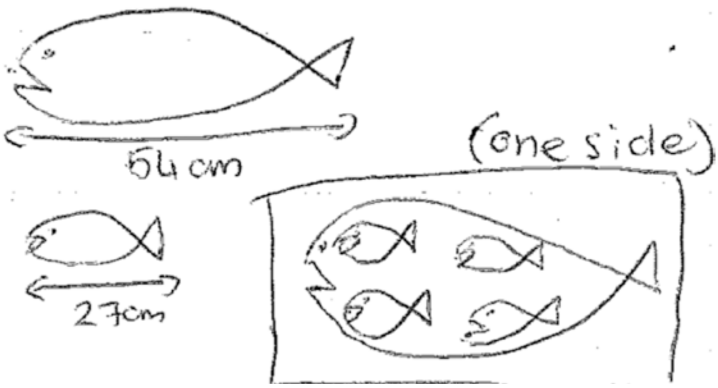


Figure 8. Liz' diagrams in her final individual letter

of shape of the collections of squares, as the diagram of the larger "fish" has doubled in width, but quadrupled in length, while the dimension of depth (which isn't shown) presumably stays the same. Figure 9b shows a redrawn version of her diagram to identify the four parts she drew using different colours more clearly. Liv's diagram suggests that she remembered the group's agreement that 8 was the mathematical result, but did not understand or remember the group's argument as to why it was so.

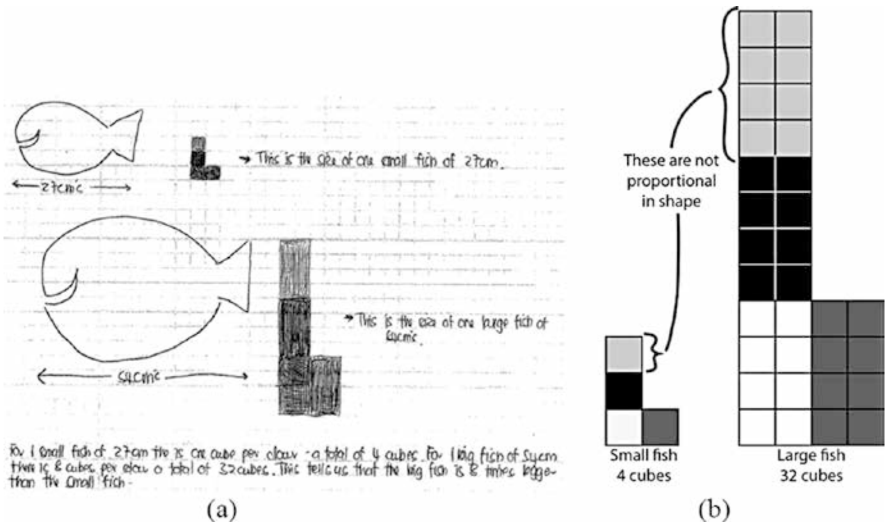


Figure 9. Diagrams (and redrawn version) in Liv's individual final letter

Case Study 2: Algebraic Equation and Diagrams Lead to Different Modelling Cycles

During the classroom session, Dee, Jan and Lea created a mathematical model (see Figure 10) that used a score combining the fish's length, width and height using additive relationships: *Score* = *Length* – (*Height* + *Width*). When they applied this score to two hypothetical fish, one whose three dimensions are double that of the other, they found that their model gave scores of 39 and 19.5 respectively, indicating that the large fish is worth twice that of the small fish.

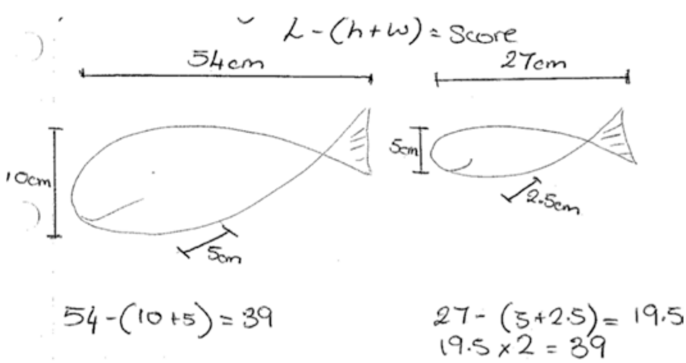


Figure 10. Excerpt from Dee, Jan and Lea's group letter written during the classroom implementation

During the group presentations at the end of the classroom presentation, Dee, Jan and Lea were exposed to three other groups' solutions that argued that one large fish is worth eight small fish in terms of volume—a result that was at odds with their solution. Dee's individual solution that was handed in one week later presented the same mathematical score as the group's, although this time, she did not demonstrate the score on hypothetical fish dimensions, nor did she communicate how many small fish the large fish was worth under this scoring system.

In contrast, Jan's individual solution was markedly different to the group's solution. She wrote that reviewing other solutions led her to believe that one large fish is worth eight small fish, and drew the diagram in Figure 11 to explain why.

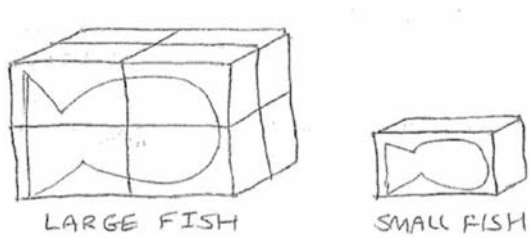


Figure 11. Diagram from Jan's individual written solution

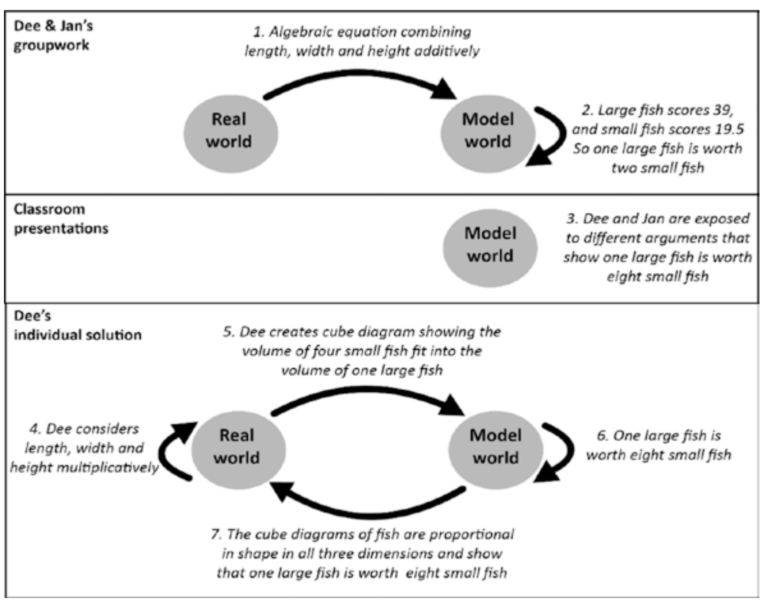


Figure 12. Dee and Jan's modelling cycles during group work and in Dee's individual solution

Note that this type of diagram, which combined the shape of the fish with the multifix cubes was not drawn by any of the other students in Jan's class, either during the in-class groupwork nor in individual written letters that were handed in one week later. Lea didn't hand in an individual solution.

Dee and Jan's individual solutions indicate different experiences in the extent to which they engage in the modelling cycle (see Figure 12).

During the classroom implementation, Dee and Jan only engaged in half of a modelling cycle, as they developed the linear score, then ran it to find a result. They were exposed to different arguments that yielded different results to their model, but only Dee used this information to revise her model and test and interpret it again.

Case study 3: Algebraic Equation and Diagram with a Quarter Modelling Cycle

During the classroom implementation, Del, Lyn and Mac's group created a mathematical argument that relied on the product of the fish girth and length: *Score* = *Girth* × *Length* (see Figure 13). They argue that multiplying the two measures (girth and length) of each fish, then dividing the product of the larger by that of the smaller will reveal how many small fish the big fish is worth.

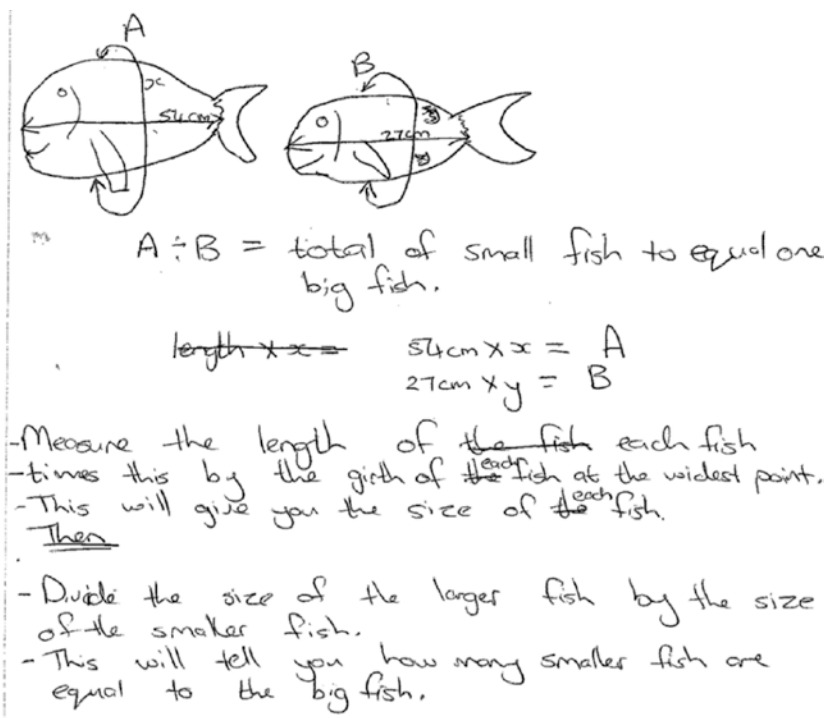


Figure 13. An excerpt from Del, Lyn and Mac's group letter

This model is not useful as it compares the *surface areas* of the two fish rather than the *volumes* of the two fish, and thereby yields the result that a fish whose dimensions are double that of a smaller fish is worth only four smaller fish. Although Del, Lyn and Mac acknowledged that this method compares surface areas, they did not test their method on any fish, and thus, did not realise that their method claims that one large fish is worth four small fish, rather than the eight small fish argued by most of their classmates.

Thus, even when they were exposed to the correct answer of eight, they did not have a point of reference to compare this amount with their own. Del, Lyn and Mac all handed in individual written letters with the same argument about *Girth* × *Height*, and none of the letters ran the model to find out how many small fish the large fish was worth. The group's modelling process can be described as undergoing only one-quarter of a modelling cycle (see Figure 14), in that they developed a mathematical model, which they visualised through diagrams and an equation, but they never went beyond this initial step.

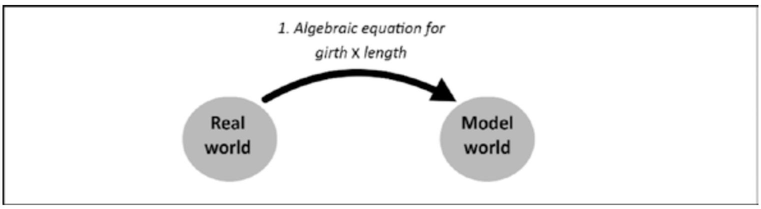


Figure 14. The quarter of a modelling cycle entered into by Del, Lyn and Mac during group work and subsequent individual written work

DISCUSSION

Liv, Liz and Pania's case study supports findings in the literature that students tend to misapply a linear model in scale problems when nonlinear reasoning is more appropriate (e.g., De Bock et al., 2002; De Bock et al., 2003; Modestou et al., 2008). Indeed, they began by assuming that one large fish was worth only two smaller fish as its length was doubled. However, it also offers a different slant into De Bock et al.'s (2002, 2003) finding that students' use of diagrams has little effect in overcoming the illusion of linearity: Liv, Liz and Pania's use of diagrams helped them test and reject their linear model, by enabling them to visualise the second dimension of width, and thereby adopt a quadratic model. Their subsequent use of the multilink cubes enabled them to test and reject a quadratic model and develop an argument for adopting cubic reasoning.

However, Del, Lyn and Mac's groupwork supports De Bock et al.'s (2002, 2003) finding: Del, Lyn and Mac used diagrams to construct their inappropriate model of *Girth* × *Length*, but they never applied this to a specific instance, so never experienced the cognitive perturbation from seeing a different result to those presented by their peers (unlike Jan in the second case study) that could have led to them revising their mathematical model. This case study emphasises that the use of diagrams to construct a model doesn't guarantee the testing of that model. In fact, an appealing diagram (especially when accompanied with an algebraic equation) may lull one into a false sense of security that one has something that "looks right", even if it is not.

The second case study of Jan and Dee adds a further insight—that experiencing cognitive perturbation from comparing one's results to different results from other models doesn't guarantee one will revise one's model. Both Jan and Dee experienced cognitive perturbation at the end of the classroom presentations, as the result from their model stated that one large fish was worth two small fish, which was at odds with most of the other group presentations that stated it was worth eight small fish. Only Jan responded to this dissonance by creating a new model; Dee's individual approach was to ignore the dissonance by removing the result (1 big fish = 2 small fish) from her letter, and simply handing in the inappropriate, untested model.

The 3-dimensional nature of the multifix cubes seemed to be most effective in helping students articulate an argument based on cubic reasoning. The multifix cubes helped Liv, Liz and Pania's group appreciate the third dimension of depth (or thickness) of the fish, which they had previously ignored in their drawings. In contrast, the 2-dimensional diagrams were often limiting in this regard, as they lend themselves to portraying two dimensions of length and width, thereby activating an area, rather than volume thought process. Jan's revised model described in her individual letter used a diagram that superimposed the 2-dimensional shape of the area of a fish onto a 3-dimensional representation of multilink cubes to reason about why eight small fish was worth one big fish. This diagram suggests that the multilink cubes used by other groups in their class presentations were particularly effective semiotic resources for helping Jan see that one large fish was not worth two small fish, but eight.

However, just like diagrams, the physical manipulatives of multifix cubes do not automatically guarantee that students will be able to perceive the 3-dimensional structure portrayed. Indeed, Liv's individual letter suggests that she remembered the presence of multifix cubes in her group's argument as to why one large fish is worth eight small fish, but she couldn't reconstruct the mathematical argument on her own. In distorting the diagram of the "blocks" to fit her assertion that there are 32 blocks in the large fish, compared to 4 blocks in the small fish, Liv reveals that she doesn't truly appreciate the impact of the the fish's third dimension (depth or thickness) on its volume. Thus, the effectiveness of physical manipulatives, like diagrams, partly lies in whether students can use them to attend to the mathematical structure that is appropriate for the problem.

Together, this trio of case studies suggests the theoretical semiotic potential of diagrams and physical manipulatives in overcoming the illusion of linearity lies in whether or not they enable students to visualise, test and examine the mathematical structures they describe in their mathematical models. In modelling terms, these semiotic resources are potentially useful if they enable one to progress through of the steps in a modelling cycle, beginning with formulating a mathematical model using the semiotic resources, running the model and examining the results, then comparing the results to information in the real world and if necessary, developing another model. Lesh et al. (2000) advocate designing activities that have some form of "self assessment", whereby students can determine for themselves whether their solution is on the right track, without having to appeal to the teacher or textbook for confirmation. Our case studies suggest that one way of fulfilling the self assessment principle may be to encourage students to construct and manipulate semiotic resources that have the semiotic potential for enabling students to visualise, test and examine their mathematical approaches.

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6

GERT KADUNZ

6. DIAGRAMS AS MEANS FOR LEARNING

ABSTRACT

A challenging task when doing research in mathematics education is the comprehensible description of activity shown by students and their construction of new knowledge when doing mathematics. The semiotics of Charles S. Peirce seems to be a promising tool for fulfilling this task. For several years, Peirce's semiotics has been well known and extensively discussed in the scientific community of mathematics education. Among the numerous research reports grounded on semiotics, several of them deal with Peirce's semiotics on the meaning of diagrams as a tool for gaining new knowledge. The aim of this chapter, where a case study will be presented, is to offer the usefulness of such a view on diagrams. In this study the diagrams made by two students who solve a problem from elementary geometry are analysed. The question presented to them asked for a mathematical description of the movement of a rigid body. To answer this question they started experimenting with this rigid body and afterwards invented and used diagrams in manifold ways. Video-based data show these diagrams to be the source of new mathematical knowledge for these students. Therefore, this chapter offers Ch. S. Peirce's semiotics as a successful theoretic frame for describing and interpreting the learning activity of the students and their use of diagrams to solve the given mathematical task.

INTRODUCTION

Mathematics is a science, which is always interested in inventing and using signs (e.g., symbols, diagrams etc.). Since the end of the 1990s, numerous scientists in mathematics education have been investigating theoretical approaches to such signs, with the aim of using these theories as a tool for answering research questions in the field of learning and teaching mathematics. In order to become a "tool science" in mathematics education, semiotics has to prove its usability for establishing new and fruitful views on questions beside those sciences like psychology, pedagogy or sociology already used and accepted in mathematics education.

One of these questions to which the semiotics of Ch. S. Peirce offers an answer (Hoffmann, 2005) will be focused of this chapter. How does new knowledge come into being? Using Peirce's answer, I will analyse the activity and the results of two young students when solving a geometrical task. To do so, I will accompany them through their solution of the task, focusing mainly on everything they draw and

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 111–126.

write. Following the sociologist Bruno Latour, I call "inscriptions" their drawings and their written signs. On the evidence provided by these inscriptions, Peirce's semiotics will be introduced as a successful instrument for describing aspects of learning mathematics. It will be shown that certain kind of inscriptions, which are diagrams, in Peirce's sense, are valuable means for constructing new knowledge (see also Dörfler, 2005; Hoffmann, 2005a; Stjernfelt, 2000).

My considerations are divided into five sections. The first gives a short review of the literature on semiotics and mathematics education. The second introduces the students and the geometrical problem given to them. The third concentrates on some concepts of Peirce's semiotics in order to analyse the students' activity. The fourth focuses on the video-based data, which shows a surprising solution. Finally, the fifth applies the introduced semiotic concepts as a tool to analyse the students' activity.

SEMIOTICS AND MATHEMATICS EDUCATION

In his "Handbook of Semiotics" (Nöth, 1995), Wilfried Nöth presents a thorough and comprehensive review of the mainstreams of modern semiotics. Nöth's handbook demonstrates that there is no universal semiotics, but a number of quite different ones. In addition to his presentation of well-known semioticians—from Peirce to Eco—Nöth also shows their semiotics to be valuable tools in different research areas. We can find semiotics, for example, in the fields of linguistics, aesthetics, or media theory, just to name only a few. Semiotics seems to be a very "broad" concept. The use(s) of semiotics in mathematics education seem similarly "broad" as we will now see.

If we look at papers in mathematics education we can find numerous articles in journals and edited books treating questions from a semiotical point (Cobb, 2000; Anderson, 2003; Hoffmann, 2003, 2005; Educational Studies in Mathematics Education (ESM), special issue 2006). Because of its topicality, I have chosen the special issue of ESM on semiotics from this list. Adalira Sáenz-Ludlow and Norma Presmeg (Sáenz-Ludlow & Presmeg, 2006a), the editors of this special issue, note the founding of a PME "discussion group on semiotics" at the 25th PME conference at Utrecht, and being continued on the following PME conferences (Norwich, 2002; Honolulu, 2003; Bergen, 2004). The outcome of this discussion group was the basis for this special issue. Among other topics presented in this issue the reader finds semiotics as a means of studying epistemological questions, or of planning mathematics lessons (in a very wide sense), or of interpreting classroom communication. A range of research questions are approached from a range of semiotics perspectives, Ch. S. Peirce, Ferdinand de Saussure, and Michael Halliday, to mention only a few.

In the reviewing paper of this ESM issue, Michael Hoffmann (Hoffmann, 2006) presents the highlights of all articles. He closes his text with an answer to the following question: "… is there a shared conception of "semiotics" behind all the "semiotic perspectives" delivered here, and should there be one?" (Hoffmann, 2006, p. 290). Hoffmann denies that one universal semiotics can be established and warns against blending different semiotics.

This variety, however, is not necessarily a problem. As long as the terminology is consistently defined and used so that communication and understanding are possible, several semiotic approaches can be used side by side. … If we are interested in epistemological problems of learning and communicating mathematics, and if we need a highly differentiated semiotic terminology that allows very precise discussions of problems such as meaning, cognition, interaction, and interpretation in mathematics, Peirce's semiotics is by far the best tool (Hoffmann, 2006, p. 290).

In my use of semiotics, I will follow Hoffmann's suggestion. In order to achieve a thorough and precise deliberation of a single problem in mathematics education, I will focus on one semiotic approach. In the third section of this chapter, I will concentrate on Peirce's semiotics as he not only developed a differentiated semiotic terminology but also used his semiotics to answer epistemological questions about mathematics. I will give a view on Peirce's famous classification of sign vehicles (i.e., representaments) into icons, indices and symbols. In particular, I will concentrate on the role of icons and diagrams in constructing new knowledge. Before illustrating my view on icons and diagrams, I will now move to the first presentation of video-based data.

CASE STUDY PART I

In spring 2005 Manfred Katzenberger—a mathematics teacher at the Gymnasium of St.Paul/Carinthia—started a series of learning experiments with some of his students (7th, 8th, and 9th graders). He investigated the impact of free hand drawings and sketches when solving a mathematical problem. Along the way, he produced numerous videos. I am grateful to him for making one video available to me. This video shows two 8th graders solving a geometrical task. In the following I will call them A and B. The students had been asked to describe the movement of given objects on the table. They were to use their mathematical/geometrical knowledge. Among the objects the teacher presented to them we can find cylinders, cones, spheres, and wooden objects similar to drumsticks. Figure 1 shows both students with the object they had to investigate. The question to describe the movement of the object was formulated in a very open way. This openness was intended. The researcher's aim was to establish a context where both students found themselves acting like researchers. The study was designed in such a way that they could write down all their attempts without looking for an algorithmic solution. A more narrowly formulated task would have produced such a straight forward tactic rather than a thoughtful strategy. The number of tools that both participants were allowed to use mirrored the openness in which the task was formulated. In addition to the tools for doing geometry (ruler and compass), they could also use different measuring tools such as a measuring tape, a Vernier callipers (a tool for measuring a circle's diameter), and computer software (spreadsheet software and software for dynamic geometry).

Figure 1. The object

The joint activity of the students was captured with two video cameras. To support the evaluation of their strategies, the two video pictures were incorporated into a single picture (Figure 1). One camera was fixed in one position, while the observer focused the other to capture interesting details. The students were given 90 minutes time to answer the questions presented to them. The video was taken in the afternoon when school classes had finished. Following my own research interest, I will focus on the students' inscriptions, i.e., icons and diagrams they invented and used. Therefore, I will only reproduce their spoken comments whenever they seem necessary for explaining their activity.

The video data shows that both students investigated the objects presented to them in various ways during the first 40 minutes (starting phase). Among others things, we can observe: rolling the different objects on the table, sketching the rolling paths, estimating the radii of rolling paths, and making other measurements.

In the documentation that follows, I will present the students' activity after this starting phase. In what follows, I will start with the presentation of a new task when a body to be rotated (Figure 1) was introduced to the students. Again they had to investigate the movement of this object to find out the rolling paths. The first successful attempt took about 17 minutes. It started from the observation of the movement of this object on the table. Several times the students in our case study, like young children playing with a toy, pushed the object to roll on the table and observed this rolling with great attention. Thereby, they focused their interest on the points of contact where the rolling object touched the table. As the movement of the object obviously describes circles—the contact points between object and table defined two circles with one common centre. Student B suggested marking some contact points with his pencil to get some details about these circles. However, the students failed because they could not agree on contact points that had to be marked.

The video data also revealed that neither of the students ever thought about investigating any other curves except the contact curves, though this would have been easy to do. If we were to fix a single point at the surface of the rolling object, then this point will describe a curve similar to a cycloid, which is an important curve in connection with technical problems. During our experiment the object was always seen as one whole body. It never entered the students' minds to define the object as a set of different points. As both students failed to locate points to draw the contacting circles, they estimated the position of the common centre. Again this attempt was not successful. Therefore, the question of how to record the history of this rolling object remained. A clever strategy, invented by student B, brought them a step further. This strategy determined their remaining activity.

B: (He holds the object with two fingers and looks at that paper the object had rolled on.) Hey, we could just try to press it down firmly, couldn't we?

A: Where?

B: There (He points at the paper.), now you can see it!

What had happened? When trying to find the circles the students had placed several sheets from a stock of paper on their table. While rolling their object on these sheets traces of the rolling object had come into existence. Student B recognized this fact and it became the starting point of their new strategy. They took some more sheets from the stock of paper to let their object roll on a "soft plane". While rolling the object on the sheets, one student pressed it with great force into the paper (Figure 2). The result was a partially visible but completely "sensible" engraved on the sheets. To strengthen this tactile impression and to make it more utilizable for their visual senses, student B coloured the engraving of the smaller circle with his pencil (Figure 3).

Figure 2. Rolling

Figure 3. The shadow

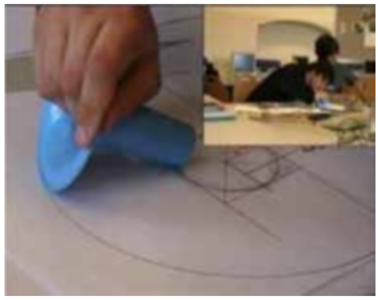

Figure 4. A proof

In the minutes that followed, the students started processing the curves they had found. As there was no question that the discovered curves were circles, the students began to look for the circle's common centre. They opened their geometrical toolbox and a circumscribed square to the engraved circle was drawn. The square's diagonals led to the circle's centre immediately (Figure 4). It is worth noting that our students had never used the theorem of the circumscribed circle to a given triangle, although they had learned about it in their geometry lessons.

What I want to show is not only our students' success but also the way they found their solution. When watching the video one recognizes that the inscriptions which the students had already produced or which they had invented and drew on the fly heavily influenced many steps on the way to a solution. Starting from their practically unsystematic playing with the given object, they followed a strategy which enabled them to literally feel the curves they were looking for. To strengthen this first tactile impression and to make it more functional to their visual senses one student coloured the engraving with his pencil. Both from the "seen" and the "felt", students conjectured that the curves they were looking for had to be the circles. This conjecture paved the way to their first solution to the given problem.

Thus our students used their inscriptions to achieve the target. In addition to this clever use of inscriptions, it is worth mentioning that the students had a thorough working knowledge of how to do a geometrical construction. They acquired this working knowledge during their last two school years. We can call it *contextual knowledg*e or, following Ch. S. Peirce, we call this knowledge *collateral knowledge*. This knowledge has its origin in the particular form of geometrical socialization of the students. Although this collateral knowledge was relatively insignificant for finding the first solution, it will be at the heart of the students' second solution.

Finally, looking at the students' first attempt at solving the given problem, it is easy to recognize a connection between hand and eye or between inventing and using inscriptions. The use of such inscriptions, their meaning for constructing new knowledge, and the importance of the already mentioned collateral knowledge, all viewed from the Peircean semiotic perspective, are at the centre of the next part of my chapter.

DIAGRAMS AS MEANS FOR THINKING

Like other sciences, mathematics education also deals with the concept of "representations". For illustrative examples I refer to papers and research reports presented by researchers like Gerald G. Goldin and James J. Kaput (Goldin & Kaput, 1996; Goldin, 1998, 1998a). Generally speaking, they investigate internal or mental representation and external or physical representation. This kind of separation between the mental and the physical brings up some epistemological and psychological difficulties that I will not discuss here. For a detailed explanation, I refer the reader to Falk Seeger (Seeger, 2000) or Michael Hoffmann (Hoffmann, 2005a, 4th chapter). A remarkable development during the last 20 years is the notion of *turn,* where those "representations" which are perceptible to our senses step into the centre of interest.

> Before attributing any special quality to the mind or to the method of people, let us examine first the many ways through which *inscriptions* are gathered, combined, tied together and sent back. Only if there is something unexplained once the networks have been studied shall we start to speak of cognitive factors. (Latour, 1987, p. 258, italics added)

From the perspective of art theory rather than from sociology, Thomas Mitchell (1994) diagnosed a *pictorial turn* and Gottfried Boehm introduced in 1994 his *iconic turn* (Boehm, 1994, p. 13). With these *turns* Mitchell, Boehm and other researchers express their interest in the epistemological importance of "representations" available to our senses. Similarly, Frederik Stjernfelt formulated the importance of icons in semiotics: "…this return of the iconic in semiotics is probably the main event in semiotic scholarship during the recent decades…" (Stjernfelt, 2000, p. 357). In my deliberations I will concentrate on such perceptible signs such as icons and diagrams. These diagrams will be introduced later in detail.

With his semiotics, Ch. S. Peirce introduced a far-reaching project to demonstrate the importance of signs. I will point at a "trademark feature" of his semiotics. I only mention Peirce's view of signs as a triadic relation. This relation consists of an *object*, a *representamen*, and an *interpretant*. They are the corners of Peirce's semiotic triangle: " … a 'sign' is *integrated* in a triadic relation whose most important feature is what he called the sign's 'interpretant'" (Bakker, 2006, p. 336). As I will concentrate on the second "trademark feature", I refer to papers which elaborate this triadic concept of sign (e.g., Hoffmann, 2003; Bakker, 2006; Presmeg, 2006; Sáenz-Ludlow, 2006). Peirce's "trademark feature" which I will examine, is his famous classification of the sign's representamen, sometimes call by Peirce sign vehicle, into icon, index, and symbol.

Icon. An icon is a representamen that stands for a relation of similarity. By definition it is a sign vehicle which has some similarities with the object of the sign. This similarity can lead to some misunderstanding (Stjernfelt, 2000, p. 358).

Critical remarks dealing with the concept of similarity can also be found in Nelson Goodman's *Language of Art* (Goodman, 1976). As Stjernfelt indicates, it seems that Peirce himself recognized some of the difficulties connected with similarity. The impression of similarity comes into existence from possible activity we can do with the icon: "The icon is not only the only kind of sign involving a direct representation of qualities pertaining to its object; it is also – and this amounts to the same – the only sign by the contemplation of which more can be learnt than lies in the directions for its construction." (Stjernfelt, 2000, p. 358)

These constructions and the activity with them may be the source of new knowledge. As I will show in the following section when students use diagrams. Diagrams, as we will see, are intimately related to icons.

Index. Following Peirce, a representament is an index which focuses the attention of a person using this sign. We can find indexes in our everyday language when we use words indicating something. If we think of geometrical drawings, then the labels on these drawings are indexes as they point to certain parts of the construction.

Symbol. A symbol is a sign, the use of which is given by definition. We can find symbols in words of a language as the meaning of a word, which has to be learned by definition. In mathematics, symbols are widely used. We can think of *e* or *pi* to name the most famous ones. But also letters used as variables in an equation are symbols in this sense.

Diagram. Icons can be further classified, following Ch. S. Peirce, into images, diagrams, and metaphors. Among these three, diagrams will have the greatest importance for the rest of this chapter. Diagrams are icons, which are *constructed* following certain rules and may thereby show relations. When we look for diagrams, we can find them in geometry. Every drawing obeying the rules of geometry is a diagram. In the same sense, a written sentence is a diagram if it follows the grammar. On the other hand, the reader reading this sentence has to know the grammar to decide whether it is a diagram. Therefore, a diagram is not a diagram by itself!

However, diagrams are in most cases very complex signs. If we again take a diagram from geometry, we see in it symbols, indexes and even other diagrams. As an example, we can imagine the drawing of a triangle and its circumscribed circle. The labels of its corners are indexes and symbols too. If we label the circle with "solution" then we have another symbol. The triangle itself is a diagram, as it is constructed using segments connecting three points in a special way.

Alongside this use of rules in constructing diagrams, the operational view on diagrams I mentioned previously for icons (Stjernfelt) will now be discussed. This operational view will be used in the interpretation of students' activity to be presented in the fifth section. With diagrams as a special kind of icons, we can perform experiments when learning mathematics. Doing experiments and constructing new knowledge is called diagrammatic reasoning (Hoffmann, 2003; Bakker, 2006). How can we imagine such reasoning when learning mathematics?

In the first step, a diagram has to be constructed. To give some examples, this may be an equation from algebra, a geometrical drawing using software, or pencil and paper, or designing a graph to solve a problem from graph theory. In the second step, once the construction has been finished, we can start experimenting. The algebraic equation may be transformed following the rules from algebra. If we have used software (DGS) for constructing a geometrical drawing, we can use the drag mode (Arzarello, 2002) to change the construction without destroying the geometrical relations of the drawing. However, we could also implement a new line or segment or even a new label into the drawing to gain a new view. This also means that when performing experiments, we have to obey the rules governing the system. "What makes experimenting with diagrams important is the rationality that is immanent to them… The rules [of the system] define the possible transformations and actions, but also constraints of operations on diagrams" (Bakker, 2006, p. 340).

In the final third step, the results of the experiment are explored. In the observers' eyes new relations can become visible. A new configuration may show "itself". A new pattern (Oliveri, 1997) may be visible within the algebraic equation. Making use of DGS drag mode the continuous movement of parts of the drawing may raise the idea of the equality of areas. As Peirce wrote, the diagram constructed by a mathematician "puts before him an icon by the observation of which he detects relations between the parts of a diagram other than those which were used in the construction" (NEM III, 749, cited by Bakker, 2006, p. 341).

With the following citation, I close my remarks on diagrams and diagrammatic reasoning. I will finish this part with some hints on two further concepts Peirce presented. I will use them as a tool to "measure" the creativity of our students. In his semiotics, Peirce introduced two interesting concepts to describe logical deduction.

> There are two kinds of Deduction; and it is truly significant that it should have been left for me to discover this. I first found, and subsequently proved, that every Deduction involves the observation of a Diagram (whether Optical, Tactical, or Acoustic) and having drawn the diagram (for I myself always work with Optical Diagrams) one finds the conclusion to be represented by it. Of course, a diagram is required to comprehend any assertion. My two genera of Deductions are first those in which any Diagram of a state of things in which the premises are true represents the conclusion to be true and such reasoning I call Corollarial because all the corollaries that different editors have added to Euclid's Elements are of this nature. To the Diagram of the truth of the Premises something else has to be added, which is usually a mere May-be, and then the conclusion appears. I call this Theorematic reasoning because all the most important theorems are of this nature. (Peirce, A Letter to William James, EP 2: 502, 1909)

As we see, the corollarial deduction is the simplest form of deduction. It describes the logical activity we have to do when we draw a conclusion from observing a diagram without changing this diagram. Take, for instance, an isosceles triangle with its axis of symmetry drawn in. Then we can deduce corollarially that the base angles of this isosceles triangle are equal. If we draw a new or change a given diagram and we deduce a conclusion, then we have done a theorematic deduction. Mathematical argumentations or the proving of theorems are, in most cases, examples of theorematic deduction. The following data will present an instance for this kind of deduction. I return now to our two students and their interesting second solution to their geometrical problem.

CASE STUDY PART II

The video data I will present now offers a new solution of a very different kind. The way to this solution can be seen from three positions. From the first, we see free hand drawing where collateral knowledge, which I have mentioned above, plays a crucial role. From the second, I mention the collaboration between the students where they use one diagram together and from this diagram develop the main solving strategy. From the third, we will find in the students' activities different kinds of inventing and using diagrams, in particular, the rule governed transformations of an algebraic equation. We can now examine all three positions in detail.

After they had finished their first solution, the observer Manfred Katzenberger asked the students to search for a second way to answer the given problem. Thereby, they were requested to apply the given measuring tools. After about two minutes of observing and measuring the object, student B starts with a sketch of this object, which differs greatly from the already marked activity that led them to their first "engraved" solution. After B had finished his sketch, he started to label it with measured values obeying labelling rules he had learned in school. Among the measured values we find: the diameter of the base and the top circle of the given object, the outer distance of these two circles, and the approximated height of the whole object. During sketching and labelling, our three-dimensional object becomes an object on the drawing plane. A problem from geometry in the three dimensional space is transformed into a problem of plane geometry (Figure 5). Video data show that during the next few minutes both students observe the object and the visible sketch very carefully. We should keep in mind that the students were investigating circles looking at diameters and centre points. Suddenly student A presents a far-reaching suggestion.

A: Ah, mmh, I know how to reach the centre.
B: Reach?
A: The way we can calculate it.
B: How? (*He looks at the paper with the sketch*)
A: (*Turns the paper*) Look! If we have there 10,5 (*radius of the base circle*), if we have this (*A starts a new sketch*), 10,5 there.

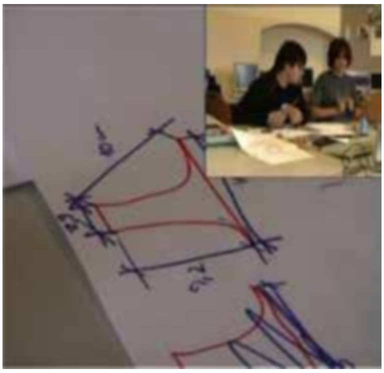

Figure 5. 3D figure

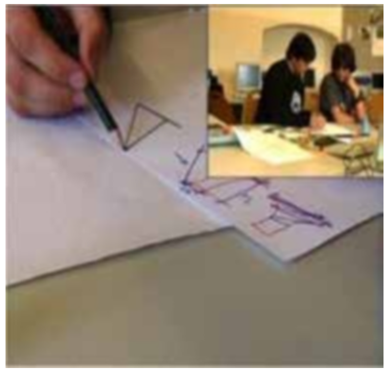

Figure 6. Hypotenuse

B: Mhm (*affirmative*).
A: (*A draws a base line*) This is a kind of a triangle with a right angle.
B: Yeah. (*affirmative*)
A: We can make this longer (A *draws the hypotenuse*), OK? If we take the axis from this one (A *points to the sketch* B *had drawn and describes this axis with a movement of his pencil*) then (A *draws the second short side of the right-angle triangle*) they have to meet. That is the distance…
B: Yes?
A: …from the centre point.

The visible result of this activity can be seen in Figure 6. During the next few minutes, both students compare the given object and their sketches. This common activity ends with a comment by student A which will be remembered later in another context by student B. Student A want's to draw a vertical projection of the object to see it with "true measures" on the paper. He formulates:

A: If we draw it obliquely then we need a 1:1 angle.

If we remember descriptive geometry, we know what he has in his mind. Student A wants to draw the vertical projection of the object lying on the table (the object's axis is oblique to the table plane). After this comment, the students' common activity, which was dominated by A, ends. It takes some minutes before student B begins a further attempt.

B: Just wait a moment. *(In the meantime, student* A *had begun to draw his vertical projection)*
B: Now let me draw. Do you know what I have thought? It is the intercept theorem that means the relation!
A: (*looks doubtful*)
B: Now, look!

B starts his explanation with the aid of his labelled sketch. Then he begins to draw a new inscription. He labels it with all the measurements (Figures 7 and 8) and uses this new inscription as a means to establish an algebraic equation (Figure 8). Without any delay student B transforms this equation, even though he did not recognize the error he had made. In this way, he gets a result, which he compares with the already existing "engraving" solution. As the result of his calculation differs from the measurements, he makes another attempt using the intercept theorem. To do so, he concentrates now on the inscription, which A had drawn. As the drawing seems to be too "pale" to B, he draws it again and labels the construction with measurements (Figure 9). He obtains a second equation from this drawing, which leads after some transformations to another numerical solution (Figure 10) which fits with the "engraved" solution.

STUDENT ACTIVITY FROM A SEMIOTIC PERSPECTIVE

I now return to the goal of my chapter, to discover how new knowledge come into existence. Peirce's semiotics, as introduced in this chapter, will now provide a successful theoretical frame for describing and interpreting the learning activity of the students and their use of diagrams in order to solve a given mathematical task. If we remember the first data, I presented in section 2 the successful idea for finding a solution started from rolling the object on the table. The students had already used such a kind of movement when they investigated other bodies of revolution. Finding their interesting strategy of pressing the object into the sheets of paper emerged from a rather random observation. The students' achievement was their connecting of the engraving and the given task. This engraving was just a necessary requirement for finding a step to the solution. Memorizing the colouring of the engraved curve, we can say that the first solution was determined by their senses. Hand and eye, the sense of touch and the visual sense organize the students' activity. Beginning from the first inscriptions (Figures 3 and 4), which are diagrams from geometry, the solution developed step by step. The students' collateral knowledge, in this case knowledge from geometry, was a handicraft-like prerequisite for their first solution. In some sense, the activity after colouring the engraved curve seems mechanical because, by the students' geometrical knowledge, their activity seemed like an algorithm. One might say that the geometrical construction was not drawn but was written Comparing the data given in section 2 with the data from part 4 seems to be more profitable for my enterprise. After having done a series of measurements, B started to draw a sketch from the axial section of the object, which he labelled carefully. The labelling with all its details was an easy job for student B. This ability has its root in his geometrical socialization. On the other hand, this construction of the sketch and all related activity were in some sense, as the already mentioned, "mechanical" activities—they are done collaterally. These activities offer, on the one hand, no direct support for gaining a new idea to the students but, on the other, the constructed diagram provided a fertile ground for the new ideas that followed.

How did student B invent the idea of using the intercept theorem? To begin with, we could suggest that B could read this theorem from his diagram. If B had offered an argument for his suggestion then, in Peirce's words, he would have made a corollarial deduction. However, he could not even formulate a suggestion from his diagram. Using Peirce's words, again, student B could not even create an *abduction*. On the contrary, B needed support from his colleague, student A. Similarly to the "engraved" solution, where the starting point to the solution arose from marks on the paper, which emerged unintentionally, something unintentional was again the source of a successful idea. We can find this source in students B's activity, when he labelled his sketch with measurements (Figure 5). Labelling a sketch or any other geometric drawing was a well-known practice for both students. This is another example of the use of collateral knowledge leading to unexpected results. Student A did not see just a section of the given object when he looked at the labelled sketch. His engagement with the given object and the observation of the measurements labels—which does not belong to Euclidian Geometry—caused A's intention to draw a right-angled triangle (Figure 6). We can say that A *abused* these measurement labels. When *(ab)*

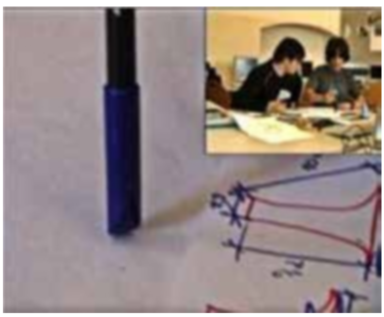

Figure 7. Showing the idea

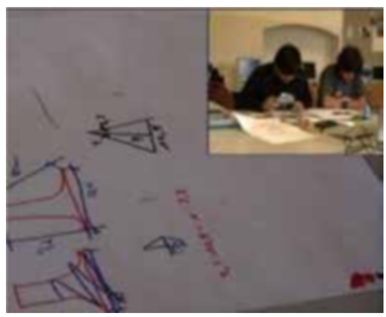

Figure 8. Next step

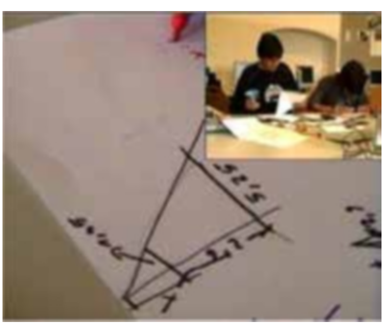

Figure 9. A new diagram

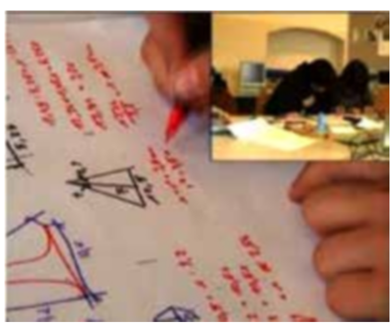

Figure 10. Calculation

using these labels, A always had the context in mind as he referred to all sides of his right-angled triangle to the given object. However, A lost interest in this diagram. The video data show that in the meantime student B followed A's activity very carefully. Now two diagrams are drawn on the paper. There is the right-angled triangle as the result of A's "abuse" and B's own sketch. If one lays the first diagram over the other and additionally knows the intercept theorem then it is imaginable that such a person would have the idea of using this theorem. That is exactly what B did. One remark may be necessary here. In the interaction between A and B, A might have influenced B. A uttered the intention to draw vertical projection and he uses the wording "1:1". This wording meant "something to something", which is part of the formulation of the intercept theorem.

However, that was not all! Not only did B suggest the intercept theorem by abduction but he was also able to give some arguments for why it was correct to use it. In Figure 7, we see the tip of his pencil. With this tip, B marks the imagined corner of a right-angled triangle above his sketch. At this moment, he explains to A why the intercept theorem was to be used. This means that by combining two diagrams a deduction was done. This was a theorematic deduction in Peirce's sense.

The remaining activity can also be seen in the light of diagrammatic reasoning. In a first attempt, student B formulated an algebraic equation, which is a diagram. He used it to explore his solving strategy and to prove it empirically. As B had deduced this theorem with the aid of two geometrical diagrams, the geometrical intercept theorem had to pass the test. And this did not happen! In his first try, student B made a mistake when establishing his equation. We can see it in Figure 8. However, as the calculation of one variable was the only task B had to fulfil, he could easily test his calculated result against the already existing "engraved" solution. We can say that a rule-governed transformation of a diagram supported the exploration. When he recognized his error, B constructed a new diagram and with a correct equation, he succeeded (Figures 9 and 10).

CONCLUSION

The case study presented here indicates the importance of inscriptions and, in particular, diagrams when solving a mathematical problem. Constructing and using diagrams can be seen as a possible source of new knowledge. At several points of the students' attempts to find a strategy and to answer the task, the students invented and transformed diagrams. These diagrams heavily influenced their learning activity. Thus we can say that the concepts of icon, diagram, theorematic and corollarial deductions found in Peirce's semiotics are valuable tools for describing the learning of mathematics.

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7

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7. INSTEAD OF THE CIRCLE… WHAT?

ABSTRACT

Using a semiotic perspective based on Peirce's triadic sign theory, we try to capture part of the complexity that teacher and students encounter during the transition from an empiric procedure used to solve a geometric problem to a mathematical procedure needed to validate the construction, within a theoretic system for Euclidean geometry. Such a step implies substituting the use of a circle as a tool to transfer length measurements for the postulate that permits establishing a bijective correspondence between the points of a line and the real numbers, and through it, determine a point with a certain distance condition. We analyze a class episode that took place in a geometry course of a pre-service mathematics teacher program.

INTRODUCTION

The university level geometry course that is the setting for the study documented in this chapter bases its curricular design on the assumption that meaning-making and autonomous and significant student participation is favored with empirical experimentation articulated with work within a theoretical system that is gradually conformed. Two related curricular decisions can be detached from the assumption. The first is the early introduction into the theoretical system of a postulate that permits the establishment of a bijective correspondence between the points of a line and the real numbers, here designated as Line-Real Numbers Postulate. This is in agreement with George Birkhoff's (1932) proposal, which establishes, for plane geometry, axioms based on the use of a ruler. The second is to promote the use of dynamic geometry software with which the students can empirically explore geometrical situations in a useful way. This is possible because the programs embody sufficiently well the Euclidean geometry postulates to present a model that offers information that is very close to the theoretic reality.

The Postulate is introduced to justify, initially, that the line has at least two points, and later, that it has an infinite number of points. Further on, having also established the existence of a correspondence between pairs of points and positive real numbers, it is used to answer the question: *how can the distance between two points be found?* With the Postulate, other facts are justified: (i) given three points of a line, one of the points must be between the other two; (ii) there exists, on the line, points between

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 127–153.

and on a side of two given points; (iii) the midpoint of a segment exists (details of the didactic proposal can be found in Samper & Molina, 2013).

After this trajectory with respect to the Postulate, a process to abandon its *direct use* as theoretical warrant that permits placing points with special conditions on the line is begun. It is replaced by a theorem that makes locating points on a ray more expeditious, here designated as Point Localization Theorem. The process begins by solving the Four Points Problem, in which a construction is made and validated, in order to determine the existence of a point with special properties. The didactical value of solving the problem is that each step in the construction procedure corresponds to a step of the proof of the point's existence, which is based on the Postulate, so it prefigures the proof of the theorem. In other words, the problem's solution should allow students to evidence once more the Postulate's utility but also, glimpse at the usefulness of a new theoretical resource that condenses the Postulate's power.

In this document,[1] we analyze a class episode in which the Postulate is used to validate the construction needed to solve the Four Points Problem. We try to capture the complexity, which teacher and students confront when they must pass from an empirical procedure, to the corresponding theoretical account. Naturally, the empirical procedure and its theoretic account have a common mathematical background.

In what follows, we present the theoretical framework, which we use to analyze the episode, some methodological aspects of the research study, the analysis itself, and some concluding remarks.

THEORETICAL FRAMEWORK

Semiotic Perspective of Teaching and Learning Mathematics

We adhere to the idea that human cognition is inevitably mediated by different systems of sociocultural signs. Therefore, the social interaction that takes place in the classroom, between teacher and students, to construct mathematical meaning is semiotic activity. To describe and interpret such activity, in the research here reported, we use the semiotic perspective for teaching and learning, based on Charles S. Peirce's theory of triadic sign that Sáenz-Ludlow and Zellweger (2012) develop. Peirce considers semiosis as a communication or thought activity in which "signs" are created or used.

Peirce's "sign", denoted as SIGN (all in capital letters) by Sáenz-Ludlow and Zellweger, refers to a triadic relation that is a result of the inseparable integration and unification of three dyadic relations in which an object, a representation of the object (representamen) and an interpretation of the object through its representation (interpretant) are articulated. The diagram in Figure 1 is an iconic representation of the general structure of the SIGN as a whole. The inverted "Y"[2] permits capturing the three components of the SIGN and its three dyadic relations (i.e., object-representamen, representamen-interpretant and object-interpretant), which are represented in different colours.

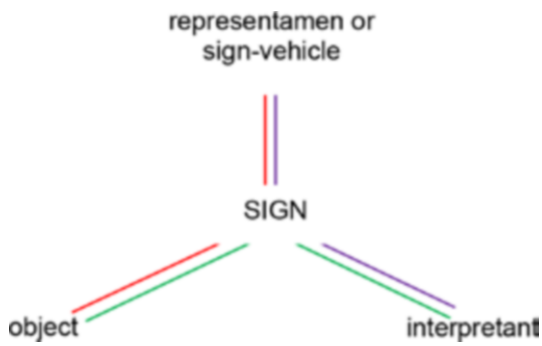


Figure 1. Diagram of the general structure of SIGN

In a more precise way: communication with others or oneself is a semiotic act that focuses on an *object* represented in a *sign-vehicle* (e.g., gesture, word, graph, mental image) with which that which wants to be communicated is made explicit; what the sign-vehicle produces in the mind of whoever perceives and interprets it is an *interpretant*. Note that the terms "interpretant" and "interpreter" do not refer to the same thing: the first term alludes to one of the components of the SIGN while the second one to the person that interprets a sign-vehicle.

The interpretation model proposed by Saénz-Ludlow and Zellweger is essentially a communication model based not only on the idea of SIGN but also on the differentiation that Peirce makes of the object of the SIGN. This distinction focuses on the aspects of the object that are indicated and transported in the sign-vehicle, and on the characteristics of the object constructed by the interpreter once he has received and interpreted the sign-vehicle. Peirce refers to three objects: the Real Object, the dynamic object and the immediate object. The Real Object (RO) is the object that is accepted by the discourse community[3] in which the semiotic act occurs. In the present situation, we refer to the Mathematical Real Object (MRO); object that is of a social, cultural and historic nature. The dynamic object (do) is a representation of the Real Object, an idiosyncratic interpretation, generated in the mind of the interpreter when he receives a sign-vehicle and interprets it. The immediate object (io) is a representation of the Real Object that refers to one or more specific aspects of it that are encoded and expressed in a sign-vehicle.

Peirce's distinctive contribution to the traditional notion of "sign" is the fundamental inclusion of the mind that interprets. This inclusion highlights that communication is not an immediate process that permits passing a specific message in a direct manner from one person to another with supposedly "objective" meanings and associated with those objects on which the sign-vehicles that mediate communication focus. Instead it is a mediated indirect process in which the construction of interpretants of those who are involved is essential and plays a preponderant role. Conscious of the role that interpretation plays in communication, and understanding that the teaching-learning of mathematics is, in essence, communication, Saénz-Ludlow and Zellweger (2012) model this process considering the double character of interpretation, that is taking into account the interpretation in two realms, the personal and interpersonal. The first one, the *intra-interpretation*, is activated when the person, in a semiotic act, interacts with himself and assumes consecutively the roles of utterer and receptor. The second one, the *inter-interpretation*, is carried out when the persons assume the roles of utterer and receptor, alternatively.

What follows is a description, according to the model, of how semiosis occurs with respect to a determined Real Object, in a verbal interchange constituted by two turns. Figure 2 is our diagram of the model. In an intra-interpretation act, person A generates an immediate object, selecting from his interpretant related to the Real Object, some specific aspect on which he wants to focus his communication, encodes it and expresses it in a sign-vehicle directed to person B. In an inter-interpretation act that takes place in the context of his knowledge and experience, B decodes the sign-vehicle emitted by A and generates an interpretant from which emerges a dynamic object that can be in greater or lesser accordance with A's immediate object. Then, in an intra-interpretation act, B generates his immediate object and encodes it in a sign-vehicle directed to A, who likewise decodes and interprets it to generate another dynamic object. Please observe that the relation through which we compare the dynamic object of the interpreter with the immediate object of the utterer of the sign-vehicle is not equality but consonance. It is practically impossible that in some moment they will be equal, given the provisional and therefore changing nature of dynamic objects, and, above all, due to the fact that they are influenced by the previous experience and knowledge of the interpreter. It is not possible to think of equality of previous experiences of two persons.

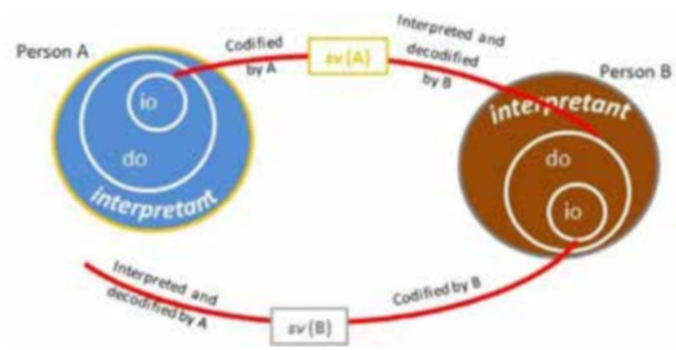


Figure 2. Model of verbal interchange

In a dialogical interaction of various turns in a mathematics classroom, whose purpose is student learning with the support of the teacher, who represents the mathematical discourse community, a collective semiosis occurs. The intention of such semiosis is meaning-making of a *situated* Mathematical Real Object, that is, it is immersed in a specific didactic situation; we denote this as the *Teacher's Real Mathematical Object*. For example, in a specific semiosis, the Real Mathematical Object could be the geometric object ray while the Teacher's Real Mathematical Object could be the geometric object ray as a tool to locate, in a plane, a point at a certain distance from another.

Dynamic Objects, Meaning-Making and Teacher Semiotic Mediation

With respect to mathematics education, the notion of meaning-making has been understood by various authors (Godino & Llinares, 2000; Radford, 2000; Robles, Del Castillo, & Font, 2010) as the search for compatibility between the ideas an individual has about a mathematical object, i.e., personal meaning, and those of the cultural community of reference, i.e., objective or institutional meaning. From the perspective set in this text, we shall try to precise such notion. We understand as *meaning-making* the intra-interpersonal process of interpretation through which the convergence of the students' dynamic objects towards the teacher's intended immediate objects is looked for. This convergence is informed during the communication act by the immediate objects that the students carry in their sign-vehicles. As Saénz-Ludlow and Zellweger (2012) point out, the meaning of each SIGN is found in two worlds: that of the *intended meanings* and that of the *interpreted meanings*.

It is expected that the teacher's intended meanings have as reference the objective or institutional meanings of some Mathematical Real Object. An *objective meaning* is the integration of consensus of meanings that have been historically constructed in the professional community of mathematical discourse. A *subjective or personal meaning* that a student gives to a Mathematical Real Object is the integration of partial and provisional personal meanings that have been primarily constituted in a collective manner in the classroom with the teacher's semiotic mediation. When a person interprets a sign-vehicle that carries an aspect of a Real Object, that is, when he reads a sign-vehicle to try to understand what the utterer expressed, an interpretant and a dynamic object are generated. Once the interpretant of such a sign-vehicle has emerged, the interpreter's subjectivity is activated as he makes an effort to construct his own meaning of the Real Object, probably influenced by the interpretations mentioned before. That is, the personal meaning is related to the dynamic objects that the interpreter has been constructing throughout his interaction with others. This *provisional meaning* that a person confers to a specific Real Object, in a semiotic process that could probably not finish if the person keeps working on it, is constituted by all the ideas related to it that he keeps on forming, reforming, specifying, modifying, and the uses he can give not only to the definition but also to the object itself.

Summarizing: with this perspective of meaning-making, in the treatment of a specific mathematical idea in the classroom, it is possible to distinguish *interpretation cycles* (separated or not in time, and in the same or different contexts) through which the student dynamic objects are produced and refined. With this, interpreted meanings are constructed, reconstructed, specified and widened in such a way that in a medium or long term they will be consonant with the teacher's expected meaning, and thus, consonant, in some extent, with the respective Mathematical Real Object's meaning. The diagram in Figure 3, models a cycle of interpretation.

In general, in the midst of a dialogic interaction in the classroom, the dynamic objects of the teacher and the students are constituted in a different manner. This is something we want to emphasize here. In both cases, the formation of the dynamic objects is determined by the interpretation that the interpreter has of the sign-vehicle, interpretation that is perceived in the light of the mathematical knowledge and experience of each one. But, we wonder what type of effects, related with his role as guide in meaning-making in the classroom, the student sign-vehicles could have in the teacher's mind. Some of these are: (a) evoke his meanings of certain aspects of the MRO that is the focus of the conversation and use them as a reference for specific actions that can aid him or his students to reach greater compatibility with respect to the mathematical discourse community; (b) identify aspects of the MRO on which he must focus to increase or clarify corresponding student meaning-making; (c) recognize that his comprehension of a specific aspect of the MRO can/should improve; (d) produce hypotheses with respect to the students' meaning-making; (e) recognize whether or not the student meaning-making is developing in an acceptable manner; (f) decide how to continue guiding the conversation with a specific didactical purpose; etc.

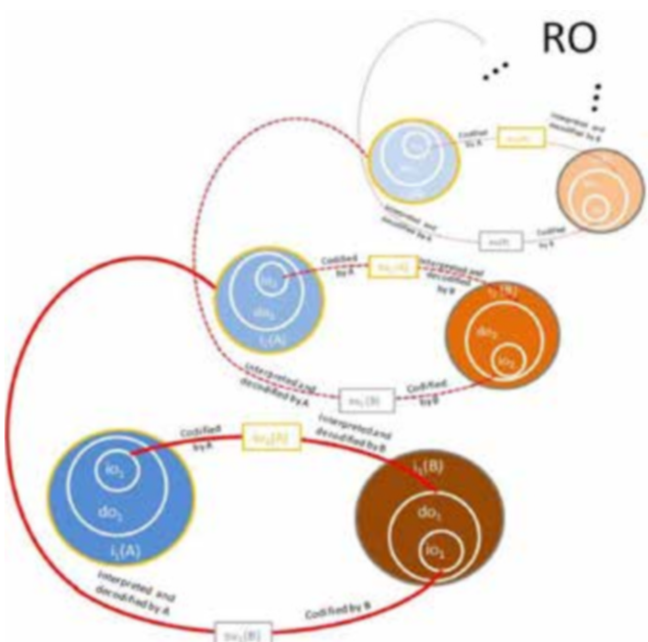


Figure 3. Model of an interpretation cycle

From the above considerations we can infer that, in an interpretation act, the teacher must contemplate the RO from two perspectives: one, mathematics itself (accepting that the teacher's Mathematical Real Object is close to the object that the community of mathematical discourse accepts and therefore represents it) and the other, that of the object in construction, in which the principal focus is teaching and learning (didactic perspective of the mathematical object in construction). So it is clear that the teacher's dynamic objects are also based on his inferences about the students' dynamic objects. Likewise, they are greatly influenced by his interpretation of the development stage of the meaning-making process he wants to support, on the mathematical aspects that could be fundamental to impulse the process, and the utility of student contributions to impulse the individual and collective process.

That is, in the classroom, the majority of the dynamic objects that the teacher constructs are not his "genuine" mathematical dynamic objects because the interaction objective is not centered on the teacher's advancement in meaning-making of the OR (even though occasionally he may do so), but on contributing to his students' respective advancement. These dynamic objects, fundamental in the semiotic activity in the classroom, have a didactic nature, reason why we distinguish them and call them *didactical dynamic object* (*odd*) (Perry, Camargo, Samper, Sáenz-Ludlow, & Molina, 2014). The qualifier "didactical" alludes to the fact that these are results of didactical decisions made to facilitate the evolution of student dynamic objects towards immediate objects that approximate the intended immediate object.

As has been suggested various times in this text, the teacher, as representative of the mathematics discourse community, plays a special role in meaning-making. We denote, as *teacher semiotic mediation,* the interpretative and deliberate actions that he realizes with the purpose of attaining the convergence of student dynamic objects to his intended immediate object. The teacher's actions reveal the effects on his interpretants as he infers the students' interpretants in a specific moment. In this communicative interchange, the teacher adjusts his *dynamic objects* to those aspects he has interpreted, when he acts as a receiver of the students' sign-vehicles. His constructed dynamic objects are intended to guide the evolution of the students' dynamic objects so that they approximate his initial immediate object, which is the goal of the communicative interchange with the students. That is, the teacher constructs integrated didactical dynamic objects with an intended didactical goal.

RESEARCH METHODOLOGICAL ASPECTS

The episode that is analyzed in this chapter is part of a set of episodes that gather the communicative interaction concerning the Point Localization Theorem.[4] It took place in an implementation of the plane geometry course in a secondary-school pre-service teacher program, at the Universidad Pedagógica Nacional (Colombia). The course is located in the second semester of the program. Its purpose is to offer the students opportunities to learn how to prove in geometry, via their participation in solving open-ended geometrical problems from which conjectures must be formulated and then validated in the theoretic geometric system that is gradually conformed in the course. The students use the dynamic geometry program Cabri to solve the problems. Fourteen students, with ages between 18 and 24 years, made up the group. The teacher, co-author of this article, has ample experience with the respective curricular development.

The information about the semiotic activity that is analyzed comes from five sources. (i) Videos recorded by two cameras that either focused on the teacher, the students, the blackboard, or the computers the students were using, to capture communicative interaction and reproduce it as faithfully as possible. (ii) Audio recordings from two recorders—one placed close to the teacher and the other set before whoever was speaking. (iii) Class notes prepared by a chosen group of students with the intention of reconstructing the main aspects treated in class (e.g., emergent issues, theoretic elements introduced, proofs, etc.) and revised by the teacher before making them public for use by all the students in the group. (iv) Notes taken, during the class session, by a member of the research-team, with observations *in situ* of the classroom interaction; these notes were discussed in the weekly meetings of the research-team. (v) The teacher's narrative of class events during the research-team weekly meetings to evaluate the teaching episodes.

The video recording of the teacher complemented with the video recording of the students and the complete transcription of the class that corresponds to the episode constitutes the data here analyzed. In the first part of the analysis, three members of the research-team identified the teacher and student sign-vehicles considered relevant for the reconstruction of the semiosis focused on meaning-making of the Line-Real Numbers Postulate through its use in the validation of an empirical construction procedure. They also identified the immediate objects in the sign-vehicles, and proposed inferences with respect to the interpretants, the student dynamic objects, and the teacher didactic dynamic objects. In a second part of the analysis, the interpretation cycles that allow making sense of the complete semiotic activity were identified. In the third part of the analysis, the interpreting cycles of teacher and students and their meaning-making were refined to identify the components of the teaching-learning semiosis.

EPISODE CONTEXTUALIZATION

The class session begins by recalling the problem solved in the previous class:

> *Four Points Problem:* Given three non-collinear points *A*, *B* and *C*, does a point *D* exist such that segments *AB* and *CD* bisect each other? Describe the construction made and formulate a conjecture of the geometric fact associated to the construction.

With respect to a conjecture examined the previous session, the teacher questions the fact that it presupposes the existence of point *D*. Based on this concern, he proposes the task of validating the construction done in Cabri to solve the problem, that is, justify the construction steps that led to the determination of the point *D* requested in the problem. Juan recounts to the class the construction steps his group carried out: (1) construct non-collinear points *A*, *B* and *C*, (2) construct segment *AB*, (3) construct the midpoint, *M*, of segment *AB*, (4) construct ray *CM*, (5) construct circle with center *M* and radius *CM*, (6) determine the intersection point of the circle and ray *CM* that is not *C*, (7) identify such point as the point *D* looked for. And he proceeds to give the respective theoretical warrants: (1) the points are given, (2) line *AB* exists by the Two-Points Line Postulate,[5] and segment *AB* exists by the Line-Ray-Segment Theorem;[6] (3) point *M* exists by the Midpoint Existence Theorem;[7] (4) ray *CM* exists by the Two Points-Line Postulate and the Line-Ray-Segment Theorem. The justification process is interrupted at this point because no element of the theoretical system the students can use at that moment refers to circles. This circumstance prompts the teacher to ask how to substitute the action performed with the circle in the construction, that is, how else, justifiable in the available theoretical system, could the point be obtained. This is the moment when the episode, object of the analysis in this chapter, begins.

EPISODE ANALYSIS

We identified five interpretation cycles: 1) outline of proposals to determine point *D* without using circles, 2) use of a particular case to highlight the difference in the application of the two items of LRNP, 3) beginning of the formalization of the second step of the procedure to determine point *D*, 4) presenting the strategy that simplifies the procedure to determine point *D*, 5) use of the simplified procedure in the validation with which the episode starts.

In the episode that is analysed here, the Real Mathematical Object is the Line Real-Numbers Postulate (LRNP). This postulate establishes two relations that are converses of each other; relations that students find difficult to differentiate.

> Given a line, it is possible to establish a correspondence between the points of the line and the real numbers such that: (i) to each point on the line there corresponds exactly one real number; (ii) to each real number, there corresponds exactly one point on the line.

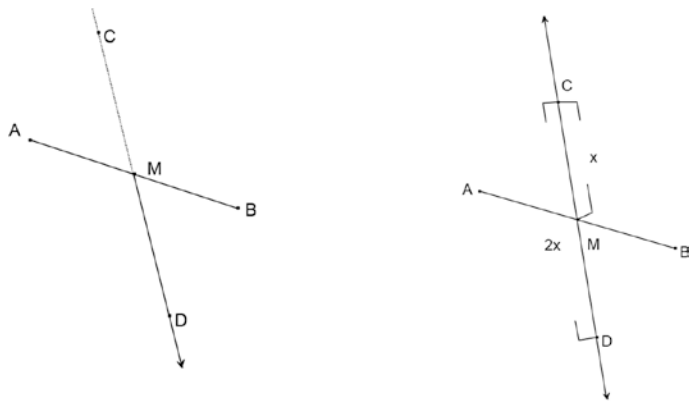
The Teacher's Mathematical Real Object is LRNP as a theoretical tool to validate the mathematic procedure that will replace the use of the circle in the setting of the Four Points Problem, that is, in determining point *D* that satisfies the conditions *C-M-D*[8] and *CM* = *MD*. We affirm this because the procedure to determine point *D*, in a valid way within the available theoretic system, consists in:

1. assigning coordinates to points *C* and *M* to calculate *CM*, distance with which point *D*'s distance to *M* is described (*CM* = *MD*);
2. calculating the number *z*, in terms of the coordinate of *M* and of *CM*, with which it is possible to determine the searched point, *D*;
3. assigning to the number *z* a point on the line that is precisely the wanted point, *D*.

LRNP is the warrant for steps (1) and (3). A variant of this procedure is to consider the relation *CD* = 2*CM* and calculate the number *z* in terms of the coordinate of *C* and of 2*CM*.

As an answer to the teacher's question about how to attain the appearance of point *D* without using the circle, two procedures are outlined: (i) transporting the distance from *C* to *M* on a ray starting at point *M*; (ii) using coordinates on line *CD* and the relation that twice the distance from *C* to *M* should be equal to the distance from point *C* to point *D*.

Cycle 1: Outline of proposals to determine point D without using circles



Procedure (i) Procedure (ii)

Let's take a look at the transcription:[9]

Teacher: Remember that *the circle is not yet an object of our theoretical system* […] *Give me ideas to replace it. When I say 'replace the circle' I am looking for something that produces, the same as the circle, the existence of a point* D *with special conditions. How can we make point* D *appear without using the circle?*

María: There *we already have a ray* [*CM*]… *We could take measurements* (with the index and middle fingers of her left hand extended she simulates an open compass and she slightly moves the index finger as if marking a small arc) right?

Teacher: Ah, okay. How?

María: First take *a measurement from…* C *to* M (with the index finger of her right hand extended she marks off what could be the endpoints of a segment) and…

Teacher: Yes.

María: *then we take the measurement from* M *to…* (while she talks, her hands are curved and one in front of the other, keeping a certain distance), *take another measurement and up to where that measurement takes us, (with the fingers of the extended hands, and these in perpendicular planes, she lets the right hand fall over the left one twice, gesturing cutting), there put point* D.

Teacher: Ah, that's a very good idea, right? And *if we take measurements, what do we automatically need?*

Ángela: *Coordinates*

Dina: We can say […] that *twice the distance from* A *to* M *is equal to the distance from* A *to* B, right? Because it is a midpoint. So, using Ángela's idea, *with coordinates, we can say that twice* CM *is equal to the distance from* C *to a point* D *that I am going to place somewhere*. That way we are placing…

[a] *The proper names that appear in the transcription are pseudonyms.*

In the teacher's sign-vehicle, "Remember that the circle is not yet an object of our theoretical system […] Give me ideas to replace it. When I say 'replace the circle' I am looking for something that produces the same as the circle, the existence of a point *D* with special conditions. How can we make point *D* appear without using the circle?," we can identify as the immediate object of the teacher (io-T) a procedure to determine a point *D* that satisfies two conditions (*C-M-D* and *CM* = *MD*). This procedure is not formulated, just alluded to through the expression "how can we".

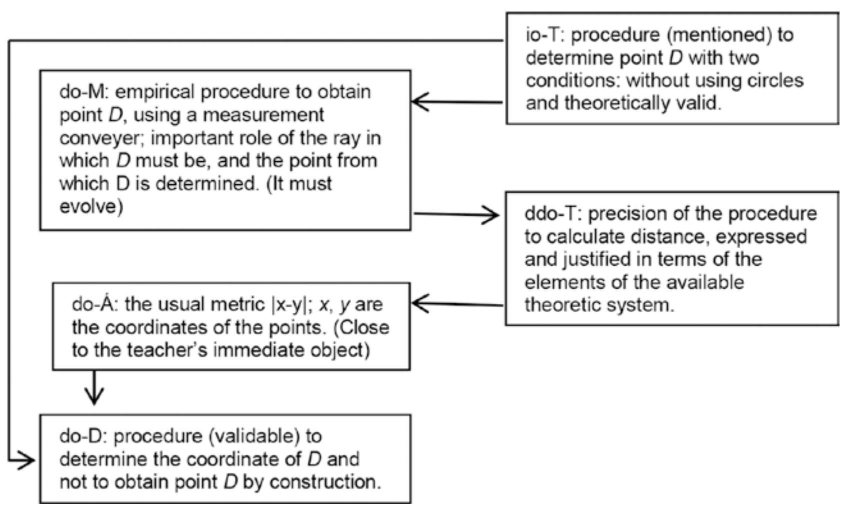
The implicit insinuation of the teacher is that it must be possible to guarantee the validity of any other procedure they can think of instead of using the circle.

María's sign-vehicle, "we already have a ray [*CM*]… We could take measurements […] a measurement from… *C* to *M* […] then we take the measurement from *M* to… take another measurement and up to where that measurement takes us, there put point *D*", together with the hand gestures used to reinforce her verbalization indicates that her immediate object is an empirical procedure to obtain point *D*, stated with few details and even imprecisely. It consists in taking the measure *CM*, transport it from point *M* onward and, whichever point is reached becomes *D*. We say it is imprecise because two conditions are not explicitly set and which are probably taken for granted in her interpretant: the distance is transported on the ray *CM* (or on the ray opposite to ray *MC*) and *C-M-D* must be satisfied. Her interpretant was possibly influenced by a previous discussion to this fragment and by an iconic image of the situation. We infer that she could have the following idea in mind: The use of a circle to obtain the required equidistance ($CM = DM$) could be replaced by taking the measurement *CM* and transporting it with a compass; the use of a circle to find point *D* as the intersection of the circle and the ray *CM* could be replaced with the measurement *CM* transported on ray *CM*, starting at *M*, so that *M* becomes the mid-point of the segment with endpoints *C* and *D*. This is to say that María's actual dynamic object (do-M) probably is an empirical procedure for obtaining point *D* without using circles; this procedure uses a transporter of length measurement that could be a ruler or compass. It seems that in such a procedure the ray on which *D* is to be found and the point *M* from which point *D* is to be determined play an important role. Although Maria's dynamic object is in consonance with the teacher's immediate object (i.e., it is a procedure to obtain point *D*, without using circles) it has to evolve to become a procedure that can be validated within the available theoretical system. This issue requires changing "transporting a length" for the assignment of a point to a specific real number.

The teacher's answer to María, "And if we take measurements, what do we automatically need?" points out his immediate object, in a non-explicit manner: geometrical and theoretical objects involved in determining distances (i.e., line, coordinates, item (i) of LRNP, and definition of a metric). We infer that the teacher sees in María's proposal an appropriate ground to start determining and justifying the first step of the procedure, where the distance is calculated. The teacher's dynamic didactical object (ddo-T) seems to be the procedure to calculate distances, expressed and justified in terms of the elements of the available theoretical system.

It is Ángela who answers the teacher's question. Her sign-vehicle, "Coordinates", leads us to infer that maybe she is thinking about the procedure used to calculate distance between two points, that is, assigning coordinates to the points and using the established metric. Probably, her dynamic object (do-Á) is the usual metric $|x - y|$, where x and y are the coordinates of the points, in which case it would be relatively close to the teacher's immediate object.

Immediately, using what Ángela has said, Dina intervenes, "twice the distance from *A* to *M* is equal to the distance from *A* to *B*, right? Because it is a midpoint. So, with coordinates we can say that twice *CM* is equal to the distance from *C* to a point *D* that I am going to place somewhere". We can see that her sign-vehicle carries two immediate objects: on the one hand, a distance relation that can be inferred from the midpoint definition ($2CM = CD$, *M* midpoint of segment *CD*), and on the other hand, a statement in natural language about the distance relations with which it is possible, using coordinates, to determine point *D*. In her interpretant she could be considering the idea of posing and solving an equation (be it of a particular case or for the general case: $2|y - x| = |x - z|$, with *x*, *y*, *z* the respective coordinates of *C*, *M* and *D*, and *z* the unknown). We also infer that in her interpretant the following idea could be present: The condition $MD = MC$, due to the fact that *M* must be the midpoint of segment *CD*, can be replaced by the condition $2CM = CD$, and this relationship does not correspond to a circle with centre *M* and radius *MC*; in this way, point *D* can be determined without that circle. Her dynamic object (do-D) seems to be a procedure to determine the coordinate of point *D* and not for locating *D* by construction. Under this perspective, her dynamic object would be very close to the teacher's intended immediate object since the procedure can be validated within the available theoretical system.



Summary diagram that includes only some of the semiotic objects

In this Cycle, two proposals of procedures to determine point *D* are outlined, neither one alluding to a circle. This suggests that María and Dina, the proponents, are trying to meet the teacher's petition to substitute the use of a circle. However, María does not seem to capture the reason why the teacher requests such substitution because her proposal of transporting a measurement has the same problem: it cannot be validated in the theoretical system for the same reason the use of a circle cannot.

Let's look, in greater detail, at the problem that is being attended to when soliciting the substitution of the use of a circle in the Four Points Problem context. In the initial construction, the use of the circle has two empirical actions that are implicit: (i) *the distance* from C to M is *captured*, and (ii) *that distance* is also *transported* an infinite number of times, in the plane, from point M. This leads us to accept that the intended meaning of the expression "substitute the circle" must go beyond the avoidance of mentioning it in the procedure (which was what María and Dina did). Also, it must refer to something else than the result obtained with the use of the circle (teacher's clarification when he formulates the task). Specifically, it must refer to the actions that are being done with the circle in the initial construction. The "objective" meaning of the expression "substitute the use of the circle" could refer to, on the one hand, changing the empirical action of capturing a distance for the mathematical action of calculating a distance, and, on the other hand, changing the empirical actions of transporting a distance and localizing a point for the mathematical actions of choosing a convenient number and assigning to it a point. The three mathematical actions mentioned can be validated within the theoretical system available; this makes the mathematical procedure that replaces the use of the circle can be validated in the theoretical system on hand. In addition to the actions mentioned before, the circle – drawn after having ray CM – has another function: it determines the searched point. However, the students may not be necessarily aware of all that is implicitly involved. When eliminating the circle in the present context, students must understand two things of fundamental importance in the meaning-making of the expression "substitute the circle": (a) the point D must be *on a specific ray*, and (b) it also must be at a particular distance *from a specific point*.

Dina's proposal could be heeding the teachers' petition in so far that she centres her procedure in finding a number with which point D could be determined. Although vaguely outlined, this proposal does not seem to refer to an empirical procedure but to an algebraic one due her allusion to the use of coordinates to establish the relation between CM and CD. If such a proposal were to be developed, that is, if the corresponding equation for z, $2|y - x| = |x - z|$, where $c(C) = x$, $c(M) = y$, is set and solved for z, then the number z would permit determining the point D. How? By assigning the point D to the number z. In this case, the two actions implied are: first, *calculating distance*, action guaranteed by LRNP and the metric; and second, *assigning point* D *to a certain number*—the number that becomes its coordinate and which is obtained under the tacit assumption of the existence of the point –action guaranteed by the second item of LRNP.

Cycle 2: Use of a Particular Case to Highlight the Difference in the Application of the Two Items of LRNP

We are not going to present an analysis of the verbal exchange that took place in this Cycle, because we consider it does not contribute information of interest for this article. We give a summary that will allow the reader to form a panoramic idea of what happened. Also we present and analyse a teacher intervention in which, with a particular case, he treats a highly important issue for meaning-making of LRNP.

The teacher supports Dina's proposal in which she alludes not only to Ángela's idea of using coordinates, but also to the use of a distance relation between points *C*, *M* and another point, *D*, whose location is not given, to determine its coordinate. He expresses his support, providing a general reference to the purposeful use of coordinates in the context of the problem. Although Dina's proposal does not mention the use of numerical coordinates and, on the contrary, could be suggesting a general treatment, the teacher attributes to her the intention of using numerical coordinates. Thus, in an instructional conversation, the teacher and some students produce a particular case of the determination of point *D*. First they assign numerical coordinates (2 and 4) to points *C* and *M*, and then they obtain the coordinate (6) that point *D* should have if *M* is to be the midpoint of segment *CD*. That is, they carry out a procedure to determine the point *D* using the number that should be its coordinate. However, they do not say how to calculate such a value.

At the beginning of the particular case's construction, the teacher writes on the board the expressions c(*C*) = ____, c(*M*) = ____, c(*D*) = ____ and he completes them with the particular values of the case: c(*C*) = 2, c(*M*) = 4, c(*D*) = 6. Based on this deliberate action that could have been unnoticed by the students, he comments:

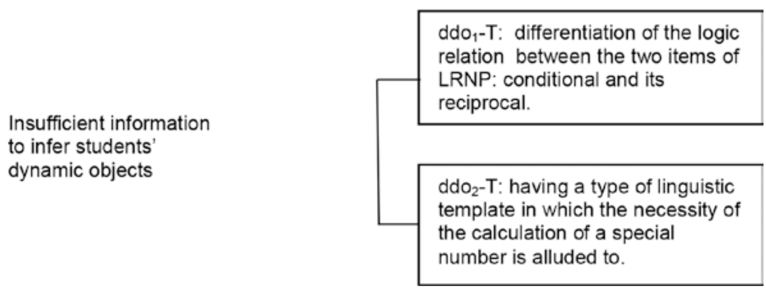
> Here we are doing something a little wrong. These points (in c(*C*) = 2, c(*M*) = 4 signals to *C* and *M*) *already exist, we can give them those coordinates*, right? *After that we would have to say: this number exists*, *six*, (pointing to the numeral in c(*D*) = 6) *and what do we do to this number six*?

In the above sign-vehicle, we can identify as the immediate object of the teacher the difference in the direction in which the correspondence between points (*C*, *M* and *D*) and their respective coordinates is established in the procedure to determine point *D*: *C* and *M* are given points and to them coordinates can be assigned; instead, *D* is the point assigned to a chosen number. It seems that the teacher's interpretant includes a reflection about the difficulty students usually have to differentiate under what conditions it is possible to apply each of the items of LRPN, and the respective effects. We infer two didactic dynamic objects. The first one (ddo_1-T) is the differentiation of the logical relation between the two items of the Postulate: conditional and its reciprocal. The second (ddo_2-T) is to have on hand a linguistic template, kind of an already made phrase ("this number ___ exists and to it we assign point *D*") to allude to the number with which point *D* is determined.

To the teacher's question, "what do we do to this number six?", Juan and another student coincide in their response, "assign to it point *D*". It is clear that the immediate object of their sign-vehicle satisfies what the teacher expected. But what can we say about the dynamic object of those that answered correctly? In this verbal exchange, the question and the answer are very precise and are completely contextualized in the discourse that the teacher is conducting, so in the students' correct answer we recognize that they are paying close attention to what is being said but we do not have sufficient information to infer some dynamic object.

Once the two steps of the procedure to determine *D* were exposed, the teacher points out that, in effect, the point *D*, which corresponds to the conveniently chosen coordinate, satisfies the wanted distance condition.

It could seem that the teacher's intention to trigger the production of a particular case was to illustrate a procedure to obtain a number with which to determine point *D*, vaguely suggested in Dina's proposal. However, the intention was another one: to have numerical data to illustrate the erroneous form of proceeding and later suggest one free of error. To warn the students about a typical erroneous practice, the teacher deliberately incurs in it. He decides to make the mistake so he can later call students' attention about it and explain what it consists of.



Summary diagram

He uses the same format to symbolize the *point-number* assignation (c(*C*) = 2), and the *number-point* assignation (c(*D*) = 6) where point *C* is given while point *D* is not. To show how he would not have made that mistake, he starts using a linguistic template in which he wants to highlight that what is being looked for is not the coordinate of *D* but a number with which it is possible to determine *D*. This way of proceeding indicates his interest in mediating the meaning-making of the students respect the application of the two items of the LRNP to determine point *D*. However, what can the teacher say without referring to the procedure that permits the determination of point *D*? The answer has to do with the considerations made in Cycle 1, specifically in relation to the action of "transporting a measurement" or "solving an equation." With this we want to call your attention about the meanings that are in play in the following three expressions as representatives of actions: $z = 4 + 2$; $z = 2 + 2(2)$; $2(4 - 2) = z - 2 \rightarrow z = 6$.

Cycle 3: Beginning of the Formalization of the Second Step of the Procedure to Determine Point D

Once the correct way of acting in the given particular case concluded, the teacher proposes the task of "generalizing" what they have done. They assign the coordinates x and y, respectively, to points C and M ($c(C) = x$, $c(M) = y$). Then, they initiate the formalization of the second step of the procedure so that the point D can be determined. That is, the formulation and symbolization of a general statement that refers to the required real number.

Teacher:	And *now what do we do?*
Antonio and others:	*The coordinate of D would be…*
Teacher:	Then, *first the number… what would it be?*
Antonio:	(Almost inaudible) *x minus y*
Teacher:	A number would appear … then, *there exists a number z such that* (he writes on the board: ∃! *z such that z* = ...)

The teacher's sign-vehicle, "And now what do we do?", has as immediate object the action that should follow after assigning coordinates in the procedure to determine, validly, point D (io-T). With this question, the teacher could be testing the hypothesis that the students will incur in the error he forewarned about in the previous cycle. Antonio's answer, "The coordinate of D would be…", although unfinished, seems to give the teacher information that supports his hypothesis, probably based on a didactic dynamic object (ddo_1-T) that restricts the use of the expression "coordinate of D" to the case in which D already exists. The teacher decides not to refer to any part of Antonio's answers and, instead, he describes, verbally and written, what would be the desirable answer to his initial question. He concentrates first on "a number" and, he continues, "[…], there exists a number z such that". His verbal expression permits us to infer his interpretant, i.e., the necessity of assuming the "existence" of a number to formalize the second step; in that sense, we go back to the already mentioned teacher's didactic dynamic object. This is, in fact, (ddo_2-T) what we consider to be a type of linguistic template in which the existence and oneness of a special number, as a number and not as a coordinate, is alluded to (i.e., "there exists one and only one number z such that $z = \ldots$).

Since Antonio's answer, "The coordinate of D would be…", is not complete, we do not have information to identify the immediate object, much less to try to infer his interpretant or dynamic object. Yet, what seems to be Antonio's interpretation of the teacher's reaction to his mentioning point D's coordinate? That is, what interpretation could he be giving to the sign-vehicle uttered by the teacher, "first, the number”, without any commentary allusive to the student’s answer? Antonio’s sign-vehicle, almost inaudible, that he later repeats more decisively, “x minus y”, gives us some insight. Antonio’s interpretant could include an idea as the following one: the teacher wants to reconstruct the procedure carried out in the particular case; therefore after assigning coordinates, the determination of the distance between points C and M follows, which contributes to determine point D. Associated to such an interpretant, we infer Antonio’s dynamic object to be the usual metric without the need to consider the absolute value. The clear cognitive distance that separates Antonio’s dynamic object from the teacher’s intended immediate object, as formalized by the teacher in his last intervention, permits us to see an apparent disagreement in the communication. We consider that the *quid* of this disagreement lies on the student’s lack of awareness about the problematic behind the use of the expression “the coordinate of a point” when one doesn’t have the point.

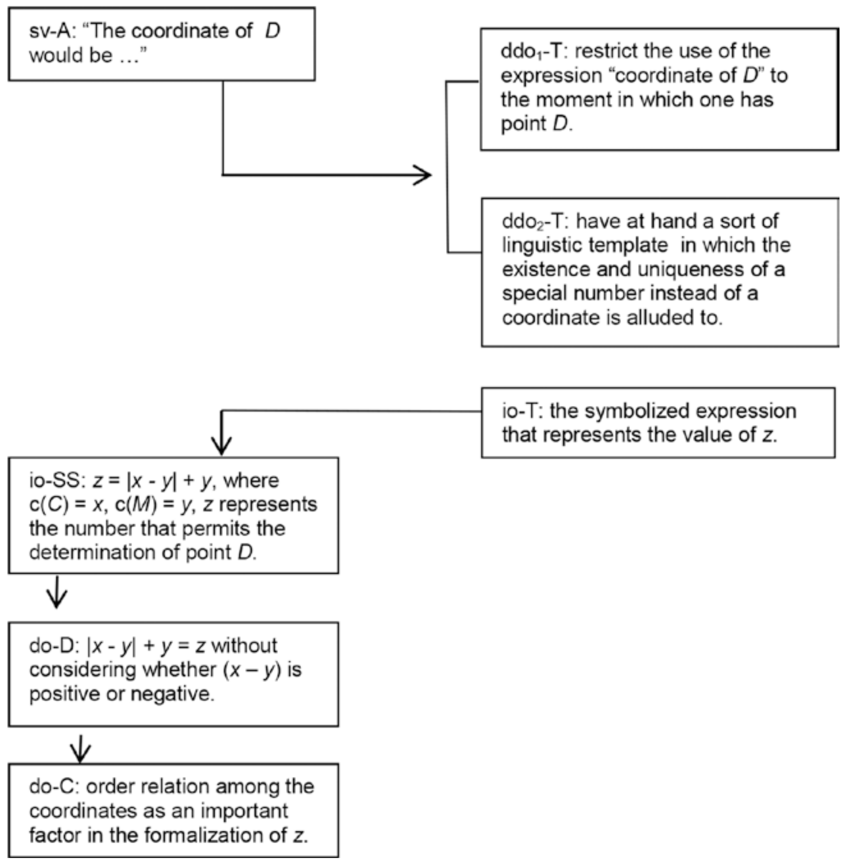
With the intention of establishing the algebraic expression that defines the number z to fill out the template for the case of the determination of point D, a verbal exchange takes place between the teacher and some students—exchange that we do not analyse here because we consider that it does not contribute significantly to the cognitive issue of interest in this chapter. It suffices to know that they reached the following proposal: $z = |x - y| + y$, where $c(C) = x$, $c(M) = y$ and z represents the number that permits determining point D (io-SS); it was later rejected because it does not take into account that the order relation between x and y affects the value of z. Let’s analyse the following verbal interchange:

Dina:	[Referring to the proposal $z = \|x - y\| + y$] Yes, because *the absolute value will give me a positive number; if I add it to the coordinate y, I will obtain the coordinate z.*
Teacher:	Okay. *Do it with some particular cases to see if it works.*
Dina:	(Talking to Antonio.) *Why not?*
Teacher:	To see if it works.
Camilo:	No.
Teacher:	Why do you say no, Camilo?
Camilo:	*The problem is that you can obtain point* C *itself.*
Teacher:	*Why can it be point* C *itself?*
Various students:	Wait… It doesn’t work.
Camilo:	*If x equals two, and y equals zero, the distance is two. Adding* y, *it would be two.*
Teacher:	And, you add y… so z would be two, and two was the coordinate we assigned to point C, therefore it does not work. Notice, whenever one works with coordinates and absolute values, one must be careful with the things. *So it does not work.*

Dina's sign-vehicle to support her agreement with the stated proposal, "the absolute value will give me a positive number; if I add it to the coordinate y, I will obtain the coordinate z", has as immediate object the correspondence between the algebraic expression $|x - y| + y = z$ and the second step of the empirical procedure proposed by María at the beginning of the episode. This is, in fact, the translation of Maria's step from mathematical language to natural language. The following idea could be present in Dina's interpretant: the number with which D will be determined is the mentioned sum which in turn will make M the midpoint of segment CD. We infer that her dynamic object (do-D) is $|x - y| + y = z$ for which she does not consider the possible result of the expression inside the absolute value. The teacher invites her to consider particular cases to see whether the proposal works, which surprises her. Camilo shows an example for which the proposal does not work. In his sign-vehicle, "The problem is that you can obtain point C itself. If x equals two, and y equals zero, the distance is (positive) two. Adding y, it would be two", the immediate object is the lack of correspondence between the obtained formalization and the wanted value that is represented by it. The following idea could be in Camilo's interpretant: Given that the expression for the value of z depends on an absolute value, there are two possibilities to consider. We infer that his dynamic object (do-C) could include the order relation between the coordinates as a factor that must be taken into account for the formalization at hand. The teacher closes this interpretation Cycle with the following sign-vehicle, "It does not work." However, after describing a modified proposal, which we will see in the next section, the teacher invites the students to formulate the general case using the coordinates x and y and the number $|x - y|$, and yet suggesting to them to think about the conditions for x and y.

In Cycle 3, the formalization of the second step of the procedure to determine point D has begun. The teacher again uses the template, this time in general terms (i.e., "there exists a unique z such that $z = \ldots$"). The exercise of expressing, in a general form, the number that will determine D is done, but the proposal is discarded maybe because it does not consider whether $x - y$ is positive or negative, or maybe because it takes for granted a certain order relation between the coordinates of C and M.

An incident that we want to point out has to do with Antonio's answer when the teacher asks "And now, what do we do?". Carefully considering Antonio's sign-vehicle, "The coordinate of D would be…", we see that it does not necessarily indicate that the student has the intention of assigning a coordinate to D, although it is the expression's most natural meaning, because to talk about "the coordinate of D" insinuates that the operator "coordinate" is applied to the element "point", and therefore the point must have been given. To show that there is another situation that is not included in the given interpretation, we speculate about what Antonio's sign-vehicle could have been if the teacher would have let him continue expressing his idea. Probably he would have generated the following sign-vehicle:



Summary diagram

> *The coordinate of* D would be… has to be the difference between x and y if what we want is for it (the number) *to determine the point D* that satisfies the condition of having M to be the midpoint of segment CD.

Even though "the coordinate of D" has been mentioned and this was done at the beginning of the sentence, the phrase "for it (the number) to determine the point D" sets a relation in which point D depends on the number, reason why we consider that the immediate object of this imagined sign-vehicle refers to the assignment real number-point; this could come from a meaning that is consistent with the teacher's prior sign-vehicle, "(with respect to point D) first the number". This brief analysis lets us uphold that, at this point in the classroom semiotic activity, the teacher opts for a mediation that recurs to the use of expressions made by him rather than to the idiosyncratic expressions used by the students.

Cycle 4: Presenting a Strategy that Simplifies the Procedure to Determine Point D

After given students the opportunity to tackle the formalization of the condition for the number that permits the determination of point D, and the opportunity to note the subtle details that must be taken into account when working with coordinates and distances expressed in a general form, the teacher presents and substantiates a strategy that considerably simplifies the procedure to obtain the number needed. Since the involvement of the students was limited, we only present a summary of the content of the teacher's exposition.

The essence of the strategy lies upon assigning, respectively, zero and y as coordinates to the points C and M, with the condition $y > 0$. It is possible to make such correspondence without assuming that one is working with a particular case; in addition, it is convenient because it simplifies the search for the number with which point D can be determined.

> Assuming the conditions $y > 0$ and $c(C) = 0$ implies the equality between the distance from C to M and the coordinate of M: $CM = |y - 0| = y = c(M)$. Therefore, since the distance from C to the point D that is being determined must be $2CM$, that is, $2y$, and since $2y$ is a positive number, then the number, z, which permits the determination of point D should be $2y$.
>
> From the above, the first two steps of the procedure to determine D, are:

1. $c(C) = 0$, $c(M) = y > 0$
2. $\exists!\, z$ such that $z = 2y$.

Cycle 5: Use of the above Procedure in the Validation of the Initial Proposal

At this point of the teaching-learning episode, after formalizing the first steps of the procedure to determine point D, they go back to the pending validation, that of the existence of point D, without using circles. In the two columns format in which they are writing the proof, in the column that corresponds to assertions, the teacher writes the seventh step: $c(C) = 0$, $c(M) = y$, $y > 0$. Let's take a look at the transcription:

Teacher:	*What are we using to be able to assign zero as coordinate of C and y as coordinate of M?*
Various students:	*The Line Real-Numbers Postulate, both items.*
Teacher:	Okay. Coordinates for points C and M appeared. *What data are we using?*
Various students:	*The line… That line (CM) already contains the points… Data 5.*
Teacher:	*What do we have to do afterwards?*
Joaquín:	*Create the number z.*

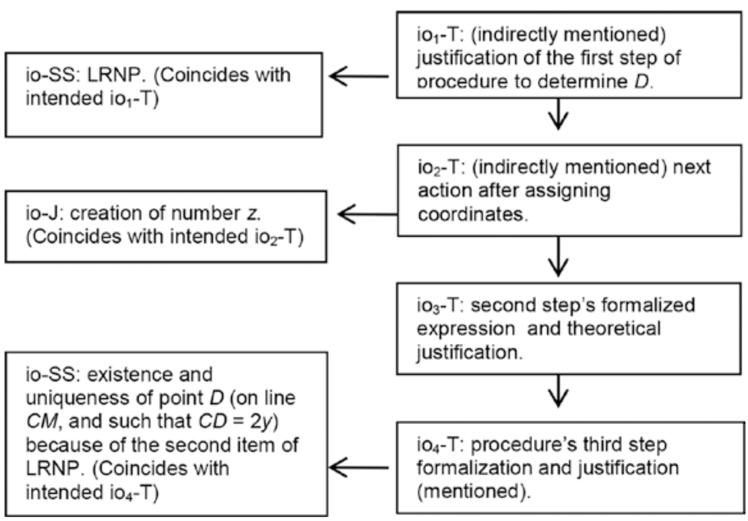
Teacher:	*Create the number z. Then, there exists… besides, this number [z] is unique. There exists z such that z is going to be very special*. This *z*, we said, was twice *y* (*in the proof template he writes: 8. There exists a unique z such that z = 2y*) Okay. The warrant, set of real numbers property, applied to step 7, that is where we have the number *y*. *What would we do with that number z afterwards?*
Various students:	*Assign a point to it.*
Teacher:	We assign a point to it. *How do we write this part?*
Ángela, Juan and others:	*There exists… a unique point, D, such that D belongs to line CM and the coordinate of D is 2y … z.*
Teacher:	The coordinate of *D* is equal to *z*. That is the correct way to write it. (In the proof template, he writes: 9. There exists a unique *D* such that *D* ∈ line *CM* ^ c(*D*) = *z*.) *And what are we using here?*
Antonio:	*The Line Real-Numbers Postulate.*
Various students:	*Item two.*
Teacher:	Item two, only.

The verbal exchange starts by focusing on the theoretical justification of the first step of the procedure. In the teacher's sign-vehicle, "What are we using to be able to assign zero as coordinate of *C* and *y* as coordinate of *M*?", we recognize as immediate object (io_1-T), expressed in an indirect manner, not only the theoretical warrant that allows assigning coordinates to the points *C* and *M*, but also the data that makes its use viable in the particular situation. Therefore, the teacher's intended immediate object is the first item of LRNP that can be used because line *CM* is given. The students' answer largely satisfies the teacher's expectations, since they mention the adequate warrant. However, they do not allude to the data. In his next question, "What data are we using?", the teacher pinpoints his immediate object with which he attains from the students an answer consistent with the expected one: "The line… That line (*CM*) already contains the points… Data 5." The teacher's next question, "What do we have to do afterwards?", has as immediate object, again expressed in an indirect form, (io_2-T) the second step of the procedure; that is, the search of the number with which point *D* can be determined. The idea of verifying whether the students have followed the previous development and whether they bear in mind the difference that the teacher pointed out with respect to the first two steps of the procedure could be in his interpretant. It seems like Joaquin's correct answer, "Create the number *z*", although not complete because he does not mention how to obtain the number *z*, is sufficient for the teacher as indication for the verification he wants; so, instead of asking for the precision of the condition that the number z must satisfy, he is the one that says and writes the formalized expression that represents the second step of the procedure together with the corresponding theoretical justification (io_3-T). The teacher's following questions, "What would we do with that number z afterwards?", "How do we write this part?", and "What are we using here?", added to the opportunity that he gives the students to respond one by one, permits us to infer his intention of having the students participate in the formalization and justification of the third step of the procedure to determine D (io_4-T). In the sign-vehicle integrated by the interventions of various students, "Assign a point to it. […] (the statement that corresponds to such an action is) There exists… a unique point, D, such that D belongs to line CM and the coordinate of D is $2y$ … z […] (The warrant is the) LineReal-Numbers Postulate, item two", we recognize that the immediate object is in agreement with the teacher's intended immediate object.

With respect to the previous analysis it is necessary to clarify something. The reason why we do not allude to the student's interpretants nor to their dynamic objects is due to the meager information we have to make the respective inferences

In Cycle 5, the conversation is dedicated to finish formalizing the procedure to determine D and theoretically justify each step.

After the conversation, an event takes place that we present and comment here due to its relation to considerations about the meaning of the expression "substitute the circle" that we exposed in the discussion of Cycle 1. Let's take a look at the transcription of the verbal interchange between the teacher and Elisa, which was promoted by her after the validation was finished.



Summary diagram

Elisa: Teacher, a question: *When we create point z don't we have to say that it is collinear with C and M?*

Teacher: The point… Say that again, Elisa.

Elisa: *When we create the coordinate z (she laughs)*

Teacher: The coordinate z

Elisa: *don't we have to indicate that the… (she smiles), how do I say it?, that the coordinate replaces the… ? (she laughs)*

Teacher: Let's see… Create the number is create the number; there you are in the world of the real numbers. Okay? This number (indicates what is written in step 8 on the board: there exists a unique z such that $z = 2y$) appears so that afterwards we can assign a point to this number (indicates $2y$). But where are we going to assign that point? *We are not going to assign it on any line that passes through C. No, we assign it on the line that makes sense to us. Which is the line that makes sense to us? Line CM*. Okay? Then, what do we do? We say that a point appears on this line such that we assign the number to this point. Do you see what I am talking about? Here you are already in the world of lines. Okay?

Elisa: (Nods her head.)

We integrate Elisa's three interventions to analyze only one sign-vehicle: "When we create point z, don't we have to say that it is collinear with C and M? When we create the coordinate z (she laughs) don't we have to indicate that the… (she smiles), how do I say it?, that the coordinate replaces the…? (she laughs)". Trying to paraphrase this sign-vehicle that is a bit confusing, we propose the following:

> When we create the number z that lets us determine D, when we create the number that will be the coordinate of D, is it necessary to say that point D is collinear with C and M?

If our proposal is a reasonable interpretation of Elisa's question, we recognize that she is taking into account a very important issue for the solution of the initial task formulated by the teacher "How can we substitute the circle?" In her interpretant, Elisa can be imagining that the number $z = 2y$ could be determining an infinite number of points and therefore it is necessary to restrict the situation setting as condition that the point D be collinear with points C and M. There remain two doubts with respect to her legitimate and pertinent preoccupation. The first is whether or not she realizes that the condition she is referring to is expressed in the statement of the third step. The second is whether or not she realizes that having points C, M and D collinear is equivalent to saying that D belongs to line CM.

From the teacher's sign-vehicle, "We are not going to assign it on any line that passes through C. No, in the line that makes sense to us. Which is the line that makes sense to us? Line CM", we infer that in his interpretant it was of no importance for the teacher to explicitly answer the question as to why and how the LRNP substitutes the circle in the validation of the construction of point D. Elisa's intervention indicates that it was necessary.

CONCLUDING REMARKS

Due to the requirement, imposed by the semiotic perspective for teaching and learning, to pay close attention to the communicational objects that are present in a communicative exchange, we were able to carry out the analytic exercise presented in this article. It helped us recognize, in greater depth and in a more conscious way, the complexity involved in the passage from an empirical procedure to its theoretical counterpart. This is to say, the passage *from* the use of an empirical procedure to determine a special point *to* the mathematical procedure conformed by actions that can be validated within the available theoretical system.

Briefly, in the case here analyzed, we see that two aspects must be heeded in the elaboration of the mathematical procedure. On the one hand, the differentiation in the use of the two items of the LRNP, and on the other, the search for the number with which to determine point D. But also, there are two tasks: one is "converting" the empirical procedure into a mathematical one, and the other is the election of the optimal conditions for the search of the number that permits the determination of the point. All this together is equivalent to the introduction of a "ruler," with an implicit unit, for which the distance from the origin to a point on a ray coincides with the point's coordinate.

The subtle details that must be taken into account when working with coordinates and distances lead us to think carefully, from the curricular design point of view, about the way the number that will determine point D is looked for. Accepting that the calculation of that special number must be in terms of adding (or subtracting) a coordinate and a number that represents a distance is conciliating a mathematical action that corresponds to the empirical action of "transporting a distance," action which cannot be validated within the theoretical system. What could then be the repercussions in the meaning-making of LRNP if the number with which D is determined has to be found in the *world of coordinates* and not in *the world of distances*? That is, what would be the effect when students think something like $c(M) = [c(C) + c(D)]/2 \rightarrow c(D) = 2c(M) - c(C)$ instead of something like $z = x + 2|x - y|$?

The above considerations have made us realize that we cannot take for granted the meaning of expressions like "substitute the circle" with which the teacher promotes participation, even though it seems that the students move comfortably between the empirical and theoretical worlds. Consequently, mediating semiotic actions of the teacher are required so that students can really take advantage of the articulation of empirical experimentation with work within a theoretical system, to favor meaning-making. For example, one could ask the students to explain what they understand by that phrase, facilitate consensus over the meaning and, also, evaluate in the light of such consensus which of the proposals given satisfies the imposed condition of substituting the use of the circle.

The analysis carried out brings out the importance of promoting collective semiotic processes around the ideas that are developed in class, via communication. Be it that the students respond to the teacher's expectations or not, semiotic mediation must go further guiding the development of ideas, based on the belief that the students are following the line of conversation. It is necessary to promote the explicit development of their ideas, advance in the interpretation of them to infer the students' dynamic objects, and thereof favour communication so that they will evolve towards the intended mathematical objects.

NOTES

1 The translation to English of the document in Spanish was done by Carmen Samper.

2 This iconic representation, as Perry (2009) tells us, refers to the only figure apt to represent the general structure of Peircean semiosis, and he mentions that it was rediscovered by the mathematician Robert Marty.

3 According to Sfard (2008, p. 91), the different types of communication that group together some individuals while leaving others outside is called discourses. With this definition, any human society can be divided into discourse communities that partially overlap. To be members of the same discourse community does not require relating to each other face to face. Membership to an ample discourse community is attained by participating in communication activities of any collective that practices that discourse, regardless of how small the group is.

4 Point Localization Theorem: Given ray CT and a real number z, $z > 0$, then there exists only one point X that belongs to ray CT such that the distance from C to X (CX) is equal to z. (Note that this is the theorem that will substitute the Line-Real Numbers Postulate).

5 Two-Points Line Postulate: Given two points, there exists one and only one line that contains them.

6 Line-Ray-Segment Theorem: Given line AB, then ray AB and segment AB exist. Likewise if either segment AB or ray AB are given then line AB exists.

7 Midpoint Existence Theorem: Given segment AB, its midpoint exists.

8 Betweeness definition: Point M is between points C and D if: (i) M, C and D are collinear and (ii) $CM + MD = CD$. C-M-D is the notation with which we indicate the betweeness relation among points C, M and D. It is read as: the point M is between points C and D.

9 To be able to read the transcription comprehensively, it is necessary to clarify the following. We underline the teacher's and student's oral or written expressions and their gestures where their immediate objects are evidenced. Parenthesis, ( ), are used to include our descriptions of the teacher's or student's actions that are carried out when they are talking; inside brackets [ ] we include an expression not verbalized by whom is talking, that refers to the object which is being talked about; […] indicate that we have deleted part of the intervention of whoever is speaking at that moment; (…), (… …), (… … …) indicate moments of silence of lesser to greater duration.

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8

ADALIRA SÁENZ-LUDLOW

8. ABDUCTION IN PROVING

A Deconstruction of the Three Classical Proofs of "The Angles in Any Triangle Add 180°"

> What the rigorous proof of a theorem, say the proposition about the sum of the angles in a triangle, establishes is not the truth of the proposition in question but rather a conditional insight to the effect that the proposition is certainly true provided that the postulates are true.
>
> Carl G. Hempel, Geometry an Empirical Science, 1956, p. 1637

ABSTRACT

This chapter is framed both within the Kantean notions of sensible and intellectual intuitions and within the Peircean notion of collateral knowledge and classification of inferential reasoning into abductive, inductive, and deductive. An overview of the Peircean notion of abduction is followed by a sub-classification of abductions according to Thagard and Eco. The constructive nature of the process of proving seems to involve not only deductive reasoning but also abductive reasoning. The later plays an essential role both in the anticipation of auxiliary constructions and in the construction of geometric arguments. The chapter presents a summary of Kant's classification of the proposition "the angles in any triangle add 180°" as a synthetic proposition. It also presents a deconstruction of the three classical proofs of this proposition—the Pythagorean proof, Euclid's proof, and Proclus' proof. This deconstruction discloses both the Greek analysis-synthesis method of proving and the role of abduction in the analysis phase. It also argues that the deconstruction of classical proofs has pedagogical and epistemological value in the teaching-learning of geometry.

INTRODUCTION

The proposition "the angles in any triangle add 180°" plays a fundamental role in Euclidean geometry. It appears to state a simple fact, but this simplicity hides an intrinsic complexity. Kant classifies this proposition as a synthetic proposition rather than an analytic one. There are only three classical proofs of this proposition—the Pythagorean proof, Euclid's proof, and Proclus' proof. Reading these proofs entails no difficulty. Constructing them is another matter. Reading and constructing proofs are two different processes. The main goal of this chapter is to deconstruct these three classical proofs to have an insight into the role of abduction in the creation of auxiliary lines to construct valid geometric arguments. This deconstruction illustrates the analysis-synthesis method of proving employed by the Greek mathematicians. This method is in essence a working-backwards strategy effective not only for proving geometric propositions but also for problem-solving (Proclus, 1970). It illustrates that proving is a constructive process and, at the same time, it helps to answers questions often asked by students: Where do proofs come from? How do I start a proof? These questions indicate the cognitive need to have a general strategy, a heuristics to guide the proving process.

The chapter is divided into seven sections. The first presents a theoretical rationale to support the assertion that proving is a constructive process. The second argues that the production of proofs is the mind's activity rooted in observation, sensible intuition, intellectual intuition, collateral knowledge, and inferential reasoning. The third presents, in a simplified form, Peirce's classification of inferential reasoning into inductive, abductive, and deductive and focuses on the process of abduction. The fourth presents Kant's examination of the proposition "the angles in any triangle add 180°" as a synthetic proposition. The fifth presents the Greek analysis-synthesis method of proving. The sixth presents a deconstruction of the three classical proofs and an examination of the role of abduction in the creation of auxiliary constructions and geometric arguments. The last puts into perspective the essential role abductive reasoning has in the analysis phase of the analysis-synthesis method of proving. This section also brings to the fore the pedagogical value of deconstructing classical proofs to learn about the analysis-synthesis method of proving and to have them as paradigmatic illustrations of proving as a constructive process.

PROVING AS A CONSTRUCTIVE PROCESS

Hersh (1997) synthesizes new and old philosophical perspectives of mathematics into two essential ones—the absolutist and humanist perspectives. Under the first, mathematics is seen as a system of absolute truths independent of human involvement, and mathematical proofs are seen as external and eternal only to be admired and accepted. Consequently, the purpose of proofs is to certify the admission ticket for theorems and propositions into the catalogue of absolute truths. Under the second, mathematics is seen as a system of truths that are the product of playful, consensual, social, cultural, and historical human activity.

What is the relation between these two philosophical perspectives about the nature of mathematics and the actual teaching-learning of proof and proving? The belief on either perspective is, consciously or unconsciously, transmitted from teachers to students. On the one hand, a teacher with an absolutist perspective will present students with the shortest and/or the more general proofs. These proofs are aesthetically pleasing and obvious only to those who have a holistic knowledge of the subject matter and who can appreciate their aesthetic value and conceptual significance. The role of these proofs is mathematical persuasion and the acceptance of mathematical rituals (Hersh, 1993).

On the other hand, a teacher with a humanist perspective will analyze given proofs and construct new ones with the purpose of understanding mathematical propositions and their interrelations. The humanist teacher will choose and accept more enlightening proofs and not necessarily the more general and sophisticated. For this teacher, proving is a thought experiment, an inquiry process by which and through which valid logical arguments are constructed. The role of proofs is to develop reasoning and mathematical conviction (Hersh, 1993). Research studies on proof and proving in geometry, implicitly or explicitly, support and promote the humanist perspective of mathematics (e.g., Hanna, 1989, 1995; Mariotti, Bartolini, Boero, Ferri, & Garuti, 1997; Garuti, Boero, & Lemut, 1998; Douek, 1999, 2007; Duval, 2007; Mariotti, 2007).

The humanist perspective is extended when it is acknowledged that students often experience abductive reasoning. This reasoning is often reported as the students' "Aha! moments." Abductive reasoning is at the root of the construction of conjectures and the construction of mathematical arguments. It seems that it appears at young ages in arithmetical thinking (e.g., Sáenz-Ludlow, 1997; Reid, 2002; Norton, 2009), in proving processes (e.g., Arzarello, Andriano, Olivero, & Robutti, 1998; Ferrando, 2000; Reid, 2003; Rivera, 2008), and in problem solving (e.g., Cifarelli & Sáenz-Ludlow, 1996; Cifarelli, 1999; Rivera & Becker, 2007). This type of reasoning sheds light not only on the process of proving and problem solving but also on the process of teaching and learning.

Problem solving and proving rooted in the construction of logical arguments with the purpose of understanding and convincing oneself and others was an idea advanced by the ancient Greeks (cf. Kadunz chapter on argumentation, this volume). For example, Proclus asserts that every problem and every geometric theorem contains in itself five elements: (1) the enunciation states which premises are given and the conclusion sought; (2) the specification states axioms, known theorems, and definitions; (3) the construction and machinery adds what is needed in order to draw the conclusion sought; (4) the proof deduces the truth of the conclusion from the premises; and (5) the closing returns to the enunciation, confirming what has been demonstrated (Heath, 1956, vol. I).

Polya's heuristics (1945/1973) for solving problems is in tune with Proclus' insights about the process of proving mathematical propositions: (1) understand the proposition or problem, what is given and what is asked; (2) devise a plan, construct a diagram, make an orderly list, eliminate possibilities, use direct reasoning, work backwards; (3) carry out the plan, work carefully, discard it if it did not work and choose another; and (4) look back and reflect on what worked and what did not, and on the significance of the problem in the context of other problems. When we consider proving as a particular case of problem-solving, Polya's heuristics can also be useful in the deconstruction proofs as well as in the production and reproduction of proofs.

Polya (1945/1973,1962/1985), Freudenthal (1973), and Hempel (1956) argue that *proving* is, at the same time, a process and a product. This view permeates their mathematical and pedagogical works when they motivate and guide the reader to construct logical arguments and to validate mathematical propositions. Hempel argues that proving, as a process, is essentially a conceptual analysis that discloses the assertions *concealed* in a given set of premises and the commitment one makes when they are accepted. Freudenthal argues that geometry, more than any other mathematical subject, disciplines the mind because of its closest relation to logic, and that it can only be meaningful when its relations are explored in the experiential space. For him, geometry offers opportunities to mathematize reality and to make discoveries.

In general, when problem-solving or proving, it is useful to have a heuristics, a method, a general procedure. Proclus' and Polya's heuristics are like road maps. They help to anticipate the territory and allow for the preparation of a plan to explore it. Road maps do not induce anyone to follow any major highway or any secondary road. They only insinuate different possibilities to get to the final destination. Heuristics, like road maps, only insinuate a plan of action to construct one or more arguments from which the conclusion of a proposition follows from the premises in a logical and valid manner. A heuristics may also facilitate the emergence of abductive reasoning.

Both Proclus and Polya consider the construction of geometric diagrams an essential step in the understanding geometric propositions because they unveil what is explicit or implicit in the premises. Another important step that naturally follows is the observation of geometric diagrams in order to coordinate and integrate geometric relations. Similar ideas about the observation of geometric diagrams are also expressed by Peirce and Mander. Peirce argues that "the geometer draws a diagram…and by means of observation of that diagram…he is able to synthesize and show relations between the elements which before seemed to have no necessary connection" (CP 1.383, emphasis added). Mander (1947), in his book "Logic for the Million", argues that the observation of geometric diagrams complements perception and inference to give rise to recognition and differentiation.

Both heuristics and geometric diagrams co-exist with abductive reasoning. This kind of reasoning aids the creation of conjectures, the conceptualization of auxiliary constructions, and the creation of novel ways of combining premises and collateral geometric knowledge. This is to say that heuristics, geometric diagrams, and abductive reasoning have a great epistemological value which is often not emphasized.

Actively producing a proof in contrast to passively reproducing a proof requires an insightful playing of the mind to conceptualize and re-conceptualize geometric diagrams in order to "see" geometric relations that facilitate a logical passage from the given premises to the conclusion. In the following section, we make an effort to comprehend the mind's activity in the process of proving. To do this, we borrow from the epistemological perspectives of Aquinas, Kant, and Peirce.

MIND'S ACTIVITY IN THE PROCESS OF PROVING

Centuries ago, Thomas Aquinas (1266/2003, Summa Theologica, q. 85, a. 2) recognized that the mind performs two kinds of activity—internal and external. The internal activity is that activity that remains within a Person such as seeing with the mind's eye. In this activity, the mind formulates to itself a model of something seen or never seen before. In contrast, the external activity is that activity that passes over to a "thing" outside the mind. For example, pointing, moving, manipulating, and encoding thoughts into external representations. The internal and external activities of a Person are interrelated and the latter somewhat manifests the former. Moreover, in the interaction with others, a Person constructs and co-constructs cycles of internal-external activity in a synergistic manner.

This internal-external activity of the mind is not independent of the relation between the mind and the object of thought. According to Kant, the mind could create an object or be influenced by an object.

> Theoretically, there are two ways in which a mind, or mode of knowledge, can be directly related to an object. If the object depends upon the mind, then the mind is active with respect to [the object],…such a relation is given the title of 'intellectual intuition'. Alternatively, the mind may wait passively upon the object, and establish a relation to [the object] only in so far as [the object] affects the mind. This capacity of the mind to be affected by objects is entitled "sensibility," and the product of such affection is "sensible intuition." (Wolff, 1973, p. 73, emphasis added)

According to Kant, when the mind creates an object, this object depends on the activity of the mind (the mind is in a creative mode) and he calls this relation *intellectual intuition*. When the mind is influenced by an object, this object is received by the mind (the mind is in a receptive mode) and he calls this relation *sensible intuition*. That is, when an *object* affects the senses directly, it produces a variety of sensible intuitions—a manifold of sensations and perceptions. This manifold carries with it two kinds of elements: (i) a subjective or material element (colours, taste, hardness, etc.), which has no cognitive value; and (ii) a formal or knowledge-giving element, which is the spatiotemporal organization and ordering of sensations that facilitates the formation of perceptual judgments (Wolff, 1973). Then the internal-external activity of a Person, mathematical or not, is intimately connected with intellectual intuitions, sensible intuitions, and perceptual judgments.

For Kant, a *judgment*, in general, is an act of the intellect in which two ideas, comprehended as different, are compared for the purpose of ascertaining their agreement or disagreement (Wolff, 1973). Judgments are usually expressed in propositions composed by subject, predicate, and copula (i.e., a word or set of words that act as a connector between the subject and the predicate of the proposition).

Borrowing from Kant, Peirce argues that *perceptual judgments* on the particular and concrete contain general elements from which one can intuit general patterns, universal propositions, and principles (CP 5.180–212). Perceptual judgments, he says, are also related to the more deliberate and conscious processes of inferential reasoning, and this reasoning is continuous and carries with it the vital power of self-correction and refinement (Peirce, 1992, Vol. 1).

For Peirce, *all* knowledge is a self-corrective process of continuous refinement. He contends, following Kant, that there is nothing in the intellect that has not been first in the senses (CP 8.738). He argues that realities compel us to put some things into very close relation and others less so. But in the end, it is only the genius of the mind that takes up all those *hints of sense*, adds immensity to them, makes them *precise*, and shows them in intelligible forms of intuition of space and time (CP 1.383).

Both Kant and Peirce deeply value the epistemological power of *observation.* They consider that observation is tied to judgment, and that judgment is tied to intentionally planned reasoning. Peirce contends that any inquiry activity fully carried out by a Person is rooted in observation and perceptual judgment. For example, he argues, that when different people observe a geometric diagram, they are able to "see" different relations, some *perceived* by the senses and some *inferred* with the aid of collateral knowledge. He also considers that this collateral knowledge is a prerequisite in the apprehension and the construction of new meanings (Peirce, 1992, Vol. 2). Consequently, it can be said that geometric diagrams, observation, sensible intuitions, intellectual intuitions, collateral knowledge, and inferential reasoning (induction, deduction and abduction) are essential components in the process of proving.

There is no doubt that visual imagination, visual observation, and visual thinking play an epistemic role in the observation of geometric diagrams (Arnheim, 1969; Giaquinto, 2007). These diagrams are in essence icons of *possible* relations. They have the potential to bring to the fore logical connections between the explicitly or implicitly given in the premises of a geometric proposition and the Person's collateral geometric knowledge. These connections are essential in the conceptualization and re-conceptualization of geometric arguments to reach, in a convincing and valid manner, the conclusion of the proposition. Thus any given proof of a geometric proposition is the product of the internal-external constructive thinking process of the mind.

PEIRCE'S CLASSIFICATION OF INFERENTIAL REASONING

Peirce, logician and mathematician himself, argues that one of the tasks of logic is the classification of inferences. He also argues that inferences and logical arguments are at the very heart of mathematical inquiry and that inferences are also at the very heart of the proving process. By *inference* he means any cognitive activity that could be internal or external, not merely conscious abstract thought (Davis, 1972).

Peirce retraces inferential reasoning from the simplest forms of sensation and perception to the most elaborated forms of semiotic activity. He considers that *each* inference draws upon former ones making logical inferences a historical process that requires *continuity* and *time* (Davis, 1972; Sheriff, 1994; Colapietro, 1989).

In general, Peirce's classification of inferential reasoning borrows from Kant's notion of perceptual and intellectual judgments. Figure 1 shows his classification of inferences into *ampliative* (*synthetic*) and into *explicative* (*analytical*). He subdivides *ampliative* reasoning into *inductive* and *abductive*, while *explicative* reasoning is classified only as *deductive.*

For centuries, inductive and deductive reasoning were known as the only forms of inferential reasoning. Less than two centuries ago, Peirce recognized a new form of reasoning that was neither inductive nor deductive. He called it *abductive reasoning*. He describes it as an inference through which and by which the mind, indirectly, comes to know the existence of an object by means of the active relation of the mind with the object (material or conceptual), relation that is based on intellectual intuition. This intellectual intuition regards "the abstract in concrete forms by the realistic *hypostatization* of relations" (CP 1.383, emphasis added).

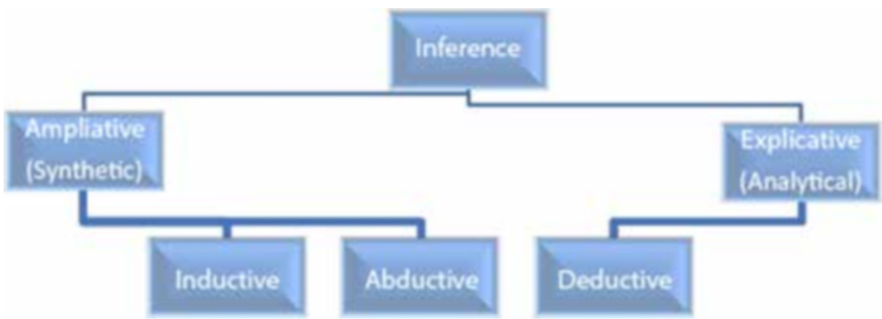


Figure 1. Peirce's classification of inferential reasoning
(diagram adapted from Peirce 1878)

For Peirce, to interact with the world, in any way, is to make judgments of inductive, abductive, and deductive nature. Induction *evaluates* and shows that something *is* actually operative; abduction merely suggests that something *may be* (may-be or may-not-be); and deduction *explicates* and shows that something *must be* (Fann, 1970, p. 51).

Prior to Peirce's recognition of abduction as a form of inference, the meaning of 'abduction' was encoded in syllogisms in which the minor premise was *only probably true* (better known as *apagoge*). In his early work, Peirce also focused on syllogisms and on the role of the character of specific cases and classes (CP 2.508, 511). A case *S* might be a member of a class *P* and have a number of characters *M*. See Chart 1.

Induction	*Abduction*	*Deduction*
S' S'' S''', etc. taken at random as M's	Any M *is*, for instance, P'P''P''', etc.	Any M *is* P
S' S'' S''', etc. *are* P	S *is* P'P''P''	S *is* M .
:. Any M *is probably* P	:. M *is probably* P	:. S *is* P

Chart 1. Peirce's induction, abduction and deduction in terms of syllogisms

Later, he describes his classification of inferential reasoning in terms of *rule*, *case*, and *result* (CP 2.623–625). Chart 2 pulls together the three types of inferences with the illustrative example given by Peirce. While in induction the *general rule* is *deduced* and in deduction the *general rule* is *given*, in abduction the *general rule* is *temporarily chosen*. In other words, abduction is the step between a fact (*case*) and its cause or origin (*general rule*). Therefore, abduction, for Peirce, is the *provisional entertainment of a rule* or *hypothesis* (that must undergo further testing) to explain that the particular *case* will follow by *deductive inference*.

Induction	*Abduction*	*Deduction*
Case These beans *are* from this bag.	*Rule* *All* the beans from this bag *are* white.	*Rule* *All* the beans from this bag *are* white.
Result These beans *are* white.	*Result* These beans *are* white.	*Case* These beans *are* from this bag.
:. *Rule* *All* the beans from this bag *are* white.	:. *Case* These beans *are* from this bag.	:. *Result* These beans *are* white.

Chart 2. Peirce's inferential classification in terms of rule, case and result

In his early work, Peirce emphasizes the differences, in logical form, between induction, abduction, and deduction. In his later work (e.g., "Lectures on Pragmatism", CP 5.14–212), he shifts the emphasis to the function satisfied for each kind of reasoning. The logical form of abduction is then reduced to

C	*The surprising fact, C, is observed*
A implies C	*But if A were true, C would be a matter of course*
A	*Hence, there is a reason to suspect that A is true*

At this point in time, abduction becomes, for Peirce, and "explanatory hypothesis". Then his criteria for a good abduction comes to include, at least, that which "must explain the facts" (CP 5.197). He then argues the difference between his three forms of inference as follows: *abduction* explains the case by introducing a new rule; *induction* evaluates the consequent by comparing the conclusion drawn from it to experience; and *deduction* draws necessary conclusions from the consequent of the abduction

In recent years, philosophers and semioticians had come to the realization that not all abductions are of the same nature. Some would require a higher level of creativity and intellectual sophistication while others require a higher level of thinking to see intellectual connections. Chart 3 presents recent sub-classifications of abductive reasoning.

Thagard (1978) classifies abduction into *overcoded abduction/hypothesis* and *abduction* proper. By *overcoded abduction* he means an abduction for which *the hypothesized rule* is not a genuine creation of the mind, but rather it is automatically or semi-automatically encoded in the *case*. That is, when a Person proposes an *overcoded abduction* his effort is *in the isolation of an already encoded rule* to which the *case* is correlated. In contrast, by *abduction* (proper) he means that the Person's effort is in the novel creation of a *rule*.

<table>
<tr><td>Peirce
1878</td><td colspan="3">ABDUCTION
Provisional hypothesis suggesting that something may-be or may-be-not</td></tr>
<tr><td>Thagard
1978</td><td>Overcoded Abduction
Hypothesis implicitly encoded</td><td colspan="2">Abduction (proper)</td></tr>
<tr><td>Eco
1983</td><td>Overcoded Abduction
Hypothesis implicitly encoded</td><td>Undercoded Abduction
Hypothesis selected from a set of equiprobable possibilities</td><td>Creative Abduction
Hypothesis invented ex novo</td></tr>
</table>

Chart 3. Thagard's and Eco's sub-classification of Peirce's abduction

Eco (1983/1988) continues Thagard's sub-classification and further subdivides *abduction* (proper) into *undercoded abduction* and *creative abduction*. By *undercoded abduction* he means an abduction in which the Person's effort is *in the selection of a rule* from a series of *equiprobable rules* put at his disposal by his current knowledge about the world. By *creative abduction* he means those abductions in which the Person's effort is *in the ex-novo creation of a rule*; for example, Copernicus' new conceptualization of the relation between the motions of the sun and the earth. These abductions are revolutionary discoveries that change established scientific paradigms (Kuhn, 1962).

In geometry, abduction, in any of its forms, plays a role in the conceptualization of auxiliary constructions, in the observation and visualisation of relations implicit in geometric diagrams, and in the conceptualization of geometric conjectures. It also plays a role in the selection, coordination, and organization of collateral knowledge to generate geometric arguments to prove geometric propositions in a logical, valid, and convincing manner.

KANT'S ANALYSIS OF THE PROPOSITION "THE ANGLES IN ANY TRIANGLE ADD 180°"

In this section we present Kant's analysis of two geometric propositions: (1) a triangle has three sides, and (2) the sum of the angles in any triangle is 180°. His examination of these two propositions illustrates the distinction between analytic and synthetic propositions and between a priori and a posteriori propositions. We acknowledge the philosophical debate about the usefulness of this distinction in different fields of knowledge. Nonetheless, in this chapter, this differentiation brings to the fore insights into the nature of the proposition about the sum of the interior angles of any triangle. Kant's philosophical analysis provides us with a mathematical insight into the complexity imbedded in the mathematical simplicity of this fundamental proposition of plane geometry. It also sheds light onto the question often asked by students, "Where do definitions and theorems come from?" According to Kant, they come from intellectual intuitions.

Kant contends that geometry, being a branch of mathematics, contains *a priori* analytic and *a priori* synthetic truths about space and things in space (Wolff, 1973). For him, mathematical propositions are the result of judgments and intellectual intuitions *a priori* to experience. The analytic-synthetic and the *a priori-a posteriori* distinctions, combined, yield four types of propositions: analytic *a priori* and analytic *a posteriori*; synthetic *a priori* and synthetic *a posteriori*.

Kant argues that analytic propositions depend on the actual meaning of the words that constitute them. Therefore, these propositions cannot be considered *a posteriori* to experience. The predicate of an analytic proposition is inherent to the subject of that proposition. Thus, *all* analytic propositions are *a priori* since we only need to consult the meanings of the words used.

He also argues that new knowledge is possible only through synthetic *a priori* propositions. The predicate of a synthetic *a priori* proposition is *not* inherent to the subject of that proposition. Nonetheless, some knowledge can also be achieved through synthetic *a posteriori* propositions because concrete cases allow us to gain insight into the general pattern.

Kant analyzes the proposition "*a triangle has three sides*" (1) as an *a priori* analytic proposition, and the proposition "*the angles in any triangle add 180°*" (2) as an *a priori* synthetic proposition. These two propositions seem simple and straightforward to students but not so to philosophers and mathematicians. Chart 4 summarizes Kant's analysis.

Kant considers that the *truth* of propositions (1) and (2) is known prior to any physical experience. Proposition (1), he says, is by necessity analytic because it merely reveals logical relations between the meaning of the words, and its denial involves a contradiction. Proposition (2) is a synthetic universal proposition because it reveals something substantive about the character of space (Wolff, 1973). This means that the predicate of the proposition (i.e., 180°) is not inherent to the subject of the proposition (i.e., the angles in a triangle), and its denial does not result in a contradiction.

Both propositions are independent of experience in the minds of mathematicians who construct, or some would say, discover them. However, for school students of different ages, the first proposition is simply a definition to be accepted. Some students believe the truth of the second proposition only after measuring angles of triangles in the real world or after proving the proposition. Consequently, it can be said that for students, who encounter geometry for the first time, the truth of the second proposition is *a posteriori* to experience. Nonetheless, one thing is certain. Students *inherit* this *a priori* synthetic proposition from the mathematicians.

Proposition (2), whether it is synthetic *a priori* or *a posteriori*, is fundamental to Euclidean geometry. This proposition and its proof were first credited to the Pythagoreans. Later, Euclid presented a different proof—Proposition 32, Book 1 of The Elements. Even later, Proclus presented another proof in his Commentaries to the Book 1 of The Elements. In Section 6 we present a deconstruction of these classical proofs. This deconstruction not only brings forward the Greek analysis-synthesis method of proving but it also sheds light into the role of abduction in the process of proving.

Reading and understanding a given written proof of a mathematical proposition is a linear deductive process. However, one thing is to *read* and *understand* a written proof and quite another is to *produce* it. To produce a proof is to engage in a nonlinear process of thinking which interconnects abductive, inductive, and deductive inferences. This process seeks to generate, at least, one geometric argument to logically justify the conclusion. Thus a written proof is only the product of a thinking process—the process of proving.

ANALYSIS-SYNTHESIS METHOD OF PROVING

The justification of a mathematical proposition can be done directly or indirectly. The direct method starts with the given premise P and then arrives at the conclusion Q using inferential reasoning and appropriate collateral knowledge. This method is symbolically expressed as $(P \rightarrow Q)$. When done indirectly one could either use the contrapositive method or the contradiction method. The contrapositive method negates the conclusion and then arrives at the negation of the premise. This method is symbolically expressed as $(-Q \rightarrow -P)$. The contradiction method starts with the acceptance of the premise P and the negation of the conclusion $(-Q)$ and, from this conjunction a contradiction of a statement or principle within a mathematical

NATURE OF PROPOSITIONS	*ANALYTIC PROPOSITION* A proposition known to be true by knowing *only* the meanings of the words, i.e., justified by virtue of meaning.	*SYNTHETIC PROPOSITION* A proposition known to be true by knowing *not only* the meanings of the words *but also* something about the world.
	Proposition 1 A triangle has three sides (a necessary proposition)	*Proposition 2* The angles in any triangle add 180º (a universal proposition)
A PRIORI PROPOSITION A proposition whose justification *is not* grounded in experience, but it can be validated through experience.	• A proposition whose truth value is independent of experience. • A proposition necessarily true because the predicate (*three sides*) is inherent to its subject (*tri-angle or better tri-lateral*). • If negated, it does result in a contradictory proposition.	• A proposition whose truth value is independent of experience. • A proposition which predicate (*180º*) is not inherent to its premise (*the angles in any triangle*). • If negated, it does not result in a contradictory proposition.
A POSTERIORI PROPOSITION A proposition whose justification *is* grounded in experience and can be validated through experience.		• A proposition whose truth value can be justified and validated by observation of concrete triangles.

Chart 4. Analytic-synthetic and a priori-a posteriori nature of propositions

system is pursued ($C \wedge -C$). This method is symbolically expressed as $[(P \wedge -Q) \rightarrow (C \wedge -C)] \leftrightarrow [P \rightarrow Q]$.

Then it is not by chance that Mariotti, Bartolini, Boero, Ferri, and Garuti (1997) propose a system-definition of *mathematical theorems* as the triad (statement, proof, theory within which the statement makes sense). This is to say that a proof of a mathematical proposition does not happen in isolation but in the context of a system of mathematical concepts. This system contains, among other things, principles, axioms, definitions, and theorems that, in one way or another, are associated with one another (Hempel, 1956).

Producing a proof of a mathematical proposition is to produce a mathematical argument to prove that once the premise is accepted as true in a mathematical system, then the conclusion that follows need to be true in that system. There is no doubt that some mathematical arguments are more difficult to produce than others. One of the reasons is that some mathematical propositions, for example universal propositions, are stated in a single sentence (subject-verb-predicate), and the predicate is not inherent to the subject of the proposition.

They are *a priori* synthetic propositions in Kant's sense. For example, "*Prime numbers are infinite*", "$\sqrt{2}$ *is an irrational number*", or "*The angles in any triangle add 180°*."

In order to produce a mathematical argument to prove any of these propositions, the mind is forced either: (i) to generate *ex novo* a mathematical contradictory argument, or (ii) to start "backwards" from the predicate and construct reversible *inferences* in order to arrive at some general mathematical principle and, then, reverse the inferences. This backwards method of proving was conceptualized by the Greeks and was called the *analysis-synthesis method.* Proclus (1970) contends that even the more obscure problems in mathematics can be pursued through this method. He also contends that Plato taught this method in his Academy even though it does not mean that he discovered it.

Heath (1921/1981, vol. 2) explains that the analysis-synthesis method has two well differentiated phases: (a) the backwards phase or *analysis* and (b) the forward phase or *synthesis*. The *analysis* phase traces back an acknowledged fact or principle starting from the desired conclusion. The *synthesis* phase reverses the steps of the analysis. In order to do this, each step of the chain of inferences in the analysis phase *has to be unconditionally reversible*.

> In *analysis* we assume that which is sought as if it were already *admitted*, and we inquire what it is from which this results, and again what is the antecedent cause of the later, and so on, until by so *retracing our steps* we come upon something *already known or belonging to the class of the first principles*, and such a method we call *analysis* as being solution backwards. However, in the process of *synthesis* we *reverse* the process. That is, we take as already done that which was last arrived at in the analysis and, by *arranging* in their natural order as *consequences* what *before* were *antecedents*, and *successively*

> *connecting them one with the other*, we arrive finally at the *construction of what was sought.* (Heath, 1921/1981, vol. 2, p. 400, emphasis added)

The analysis-synthesis method is different from the indirect methods of proving either by contrapositive or by contradiction (Heath, 1921/1981, vol. 2). This method of proving seems to have been used in the classical proofs of the proposition "the angles in any triangle add 180°."

The Pythagorean proof of this proposition is often presented in geometry textbooks. It starts by giving the auxiliary construction and then the deductive argument follows. Reading and understanding this proof entails almost no effort because of its aesthetic simplicity. After all, the written proof (the final *product* of the proving process) is only a linear deductive organization of abductive and deductive inferences that were previously generated to construct a viable and logical geometric argument. However, the abductive nature of the auxiliary construction is anything but linear, and it is left unexplained. Thus the creative nature of the proving process is left untouched and implicit.

Constructing or producing a geometric proof, in contrast to *reading* or *re-producing* a proof, requires an active playing of the mind (internal and external) to bring into play auxiliary geometric constructions, to make geometric diagrams, and to observe and interpret them. In this process, it is also essential to bring into play appropriate *collateral geometric knowledge* to provide for the emergence of necessary logical relations from which the validity of the geometric argument follows.

The next section presents a *deconstruction* of each of the three classical written proofs of the proposition "The angles in any triangle add 180°." It is argued here that to prove this proposition the analysis-synthesis method was used by the Pythagoreans, Euclid, and Proclus. This deconstruction highlights the important role played by both abduction and collateral geometric knowledge.

A DECONSTRUCTION OF THE THREE CLASSICAL PROOFS

It could come as a surprise that there are only three wellknown proofs of this fundamental proposition: the Pythagorean' proof, Euclid's proof, and Proclus' proof. This small number of proofs is in sharp contrast to the large number of proofs constructed for another fundamental geometric proposition "*The Pythagorean Proposition*" (Loomis, 1940/1972). This contrast between the number of proofs for each proposition points to their different nature. In the first, the predicate (180°) is not intrinsic to the subject of the proposition (the angles in any triangle). In the second, the predicate (the sum of the square of the sides of a right triangle equals the square of its hypotenuse) is intrinsic to the subject of the proposition (a right triangle).

Pythagorean Proof and Euclid's Proof

There are differences and similarities between these two proofs. Figure 2 presents them side by side. The Pythagorean proof (Figure 2a) uses the parallel postulate to construct one-and-only-one auxiliary parallel line to one of the sides of the triangle which passes through the opposite vertex. Let's notice that this parallel line is external to the triangle (Proclus constructs parallel lines interior to the triangle). To construct their logical argument, the Pythagoreans choose a side (BC) and its opposite vertex (A). Then, the other two sides of the triangle (BA and CA) are re-conceptualized as transversals to the parallel lines (BC and xy). Afterwards, the congruence of the alternate interior angles formed between parallel lines and their transversals is used to show the straight angle xAy is, in fact, congruent to the angles of the triangle. Therefore, the logical conclusion is that the sum of the angles inside the triangle is the same as the measure of the straight angle—180°.

The *analysis phase* of this proof is rooted in three *overcoded abductions*: (i) the selection of a side of the given triangle and the *appropriate point* through which the parallel line should pass (the vertex opposite to the side into focus); (ii) the construction of a parallel line to form a straight angle which is known to measure 180°; and (iii) the congruence of the straight angle with the three interior angles of the triangle.

Once the side of the triangle and the vertex was chosen, the properties of the angles formed between parallel lines and transversals accounted for the relation between the measure of the sum of the interior angles of the triangle and that of the straight angle.

The *synthesis phase* is the linear and deductive organization that captures the reverse order of the argument produced in the analysis phase. The synthesis phase starts with the hypothesized construction in the analysis phase and ends up justifying the hypothesized congruence of the straight angle and the sum of the interior angles of the triangle.

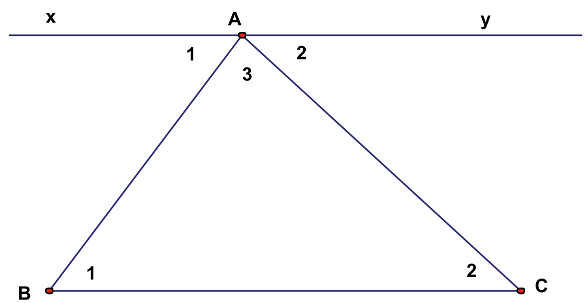
It is obvious that the written proof in Figure 2a leaves silent the creative part of the proving process—the abductive reasoning that accounted for the auxiliary construction of the parallel line and the relation between the 180° measure of the straight angle and the sum of the interior angles of the triangle.

Figure 2b presents the Euclidean proof (Proposition 32, Book 1 of The Elements). In this proposition Euclid presents not one but two propositions. The first introduces the notion of *external angles* of triangles in contrast to the notion of *interior angles*. He states that an external angle is equal to the sum of the two opposite (remote) interior angles. The second states that the sum of the interior angles of a triangle is equal to two right angles.

In the proof of this proposition, Euclid introduces two auxiliary constructions. First, he extends one of the sides of the triangle (side BC) to construct the straight angle BCD. Second, he uses the parallel postulate to construct one-and-only-one line CE parallel to side BA and passing through vertex C. He, then, re-conceptualizes sides AC and BC as transversals to the parallel lines BA and CE. Subsequently, he uses the congruence of the angles formed between parallel lines and transversals to justify the relation between the external angle (<ACD) at vertex C and the sum of the two remote interior angles (<A1 and <B1). In addition, he justifies the straight

THE PYTHAGOREAN PROOF

Prove that the sum of all three angles of any triangle is 180°.



We construct a triangle ABC and from the vertex A we construct a parallel line xAy to the side BC.
Then, $\angle A_1$ is congruent to $\angle B_1$ as alternate interior angles. Similarly, the $\angle A_2$ is congruent to $\angle C_2$ as alternate interior angles.

But $\angle A_1 + \angle A_2 + \angle A_3 = 180°$ (1) because they form a straight angle.
We proved that $\angle A_1 = \angle B_1$ and $\angle A_2 = \angle C_2$

Then we replace in the relationship (1) the congruent angles and we have $\angle B_1 + \angle C_2 + \angle A_3 = 180°$ (2)

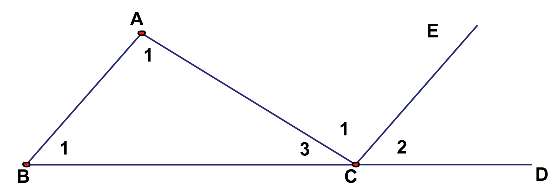
From the figure we have $\angle B_1 = \angle B$, $\angle A_3 = \angle A$, and $\angle C_2 = \angle C$

If we replace in the relationship (2) the congruent angles then we have
$\angle A + \angle B + \angle C = 180°$

Figure 2a. Pythagorean proof

EUCLID'S PROFF - Proposition 32 BOOK 1 in the "ELEMENTS"

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.



Let be a triangle ABC and let one side of it BC be produced to D
Let CE be drawn through the point C parallel to the straight line AB.

Then, since AB is parallel to CE, and AC has fallen upon them, the alternate interior angles A_1 and C_1 are congruent to one another.
Thus $\angle A_1 = \angle C_1$.

Again, since AB is parallel to CE, and the straight line BD has fallen upon them, the corresponding angles B_1 and C_2 are equal to one another. Thus $\angle B_1 = \angle C_2$
So, $\angle A_1 + \angle B_1 = \angle C_1 + \angle C_2 = \angle ACD$

Therefore, the exterior angle ACD is equal to the two interior and opposite angles A_1 and B_1.

But $\angle C_1 + \angle C_2 + \angle C_3 = 180°$ (1) because they form a straight angle.
We proved that $\angle A_1 = \angle C_1$ and $\angle B_1 = \angle C_2$

Then we replace in the relationship (1) the congruent angles and we have $\angle A_1 + \angle B_1 + \angle C_3 = 180°$ (2)

From the figure we have $\angle A_1 = \angle A$, $\angle B_1 = \angle B$, and $\angle C_3 = \angle C$ Thus,
$\angle A + \angle B + \angle C = 180°$.

Figure 2b. Euclid's proof

angle (<BCD) as the sum of the external angle ACD and the vertex angle C3. With this justification, he proves that the 180° measurement of the straight angle is also the measurement of the three interior angles of the triangle.

The *analysis phase* contemplated the auxiliary constructions that aided the formation of the geometric argument. These constructions were anticipated and hypothesized by means of abductive reasoning. The first overcoded abduction was the relation between the 180° measure of the straight angle and the sum of the measures of the angles of the triangle. A subsequent overcoded abduction was the extension CD of the side BC to construct a straight angle with vertex at C. Still another overcoded abduction was the construction of line CE parallel to the side AB and justified by the parallel postulate. These auxiliary constructions and appropriate collateral knowledge (the parallel postulate and the properties of the angles formed between parallel lines and their transversals) aided the formation of the geometric argument to justify both the measure of the exterior angle and the measure of the three interior angles of the triangle.

The *synthesis phase* was the linear and deductive organization of the reverse argument nonlinearly created in the analysis phase by means of abductive reasoning. This phase starts with the construction of an exterior angle and also with the construction of a parallel line to the side AB passing through its vertex C. Then the argument follows in a deductive manner.

Again, it also goes without saying that the written proof (Figure 2b) leaves silent the creative part of the process of proving—the abductive reasoning that led the construction of a parallel line and of a straight angle congruent to the sum of the three interior angles of the triangle.

It is worthwhile to observe two details about the Euclidean proof. First, the exterior angle and its measurement as the sum of the two remote interior angles were not absolutely necessary for the argument of the proof. Observing Figure 2b, the argument could have been made as follows:

- <BCD = 180° (measure of the straight angle)
  $<BCD = <C_3 + <C_1 + <C_2$.

  Then $<C_3 + <C_1 + <C_2 = 180°$ (transitivity property of equality)

- Since $<C_1 = <A_1$ (alternate interior angles between parallel lines AB and CE and the transversal AC)
  $<C_2 = <B_1$ (corresponding angles between parallel lines AB and CE and the transversal BD)

  Then, $<C_3 + <A_1 + <B_1 = 180°$

- Since $<C_3 = <C$, $<A_1 = <A_,$ $<B_1 = <B$, then $<C + <A + <B = 180°$.
  Then, the sum of the measures of the interior angles of any triangle is 180°

Second, instead of using the exterior angle to justify the sum of the interior angles, the argument could have been made in the reverse order. This is to say that the 180°

measure of the interior angles could have been proved first and then the measure of the exterior angle as the sum of the two remote interior angles could have ensued. Observing Figure 2b, the argument could have been made as follows:

- <ACD + $<C_3$ = 180° (measure of the constructed straight angle)
- $<A_1 + <B_1 + <C_3$ = 180° (the sum of the interior angles of the triangle is 180°)
- Then <ACD + $<C_3 = <A_1 + <B_1 + <C_3$ (transitivity of equality)
- <ACD = $<A_1 + <B_1$ (when equals are subtracted from equals, the remainders are equal)
- Then, the measure of an exterior angle of the triangle is the same as the sum of the measures of the two remote interior angles

The Euclidean notion of exterior angle, although not indispensable for the proof of the 180° measure of the angles inside the triangle, is an important notion that can be extended to any polygon. He not only stated the property of the sum of the *interior angles* of triangles but also the property of the *exterior angles* of triangles. In other words, he not only classified the angles of triangles into *interior* and *exterior* but also established a relation between these two kinds of angles. This creative abduction seems to have existed in the minds of the Pythagoreans. Heath (1921/1981, vol. 1) argues that we should not infer that the notion of external angle was not known to the Pythagoreans. He also asserts that more general propositions are also credited to them: (i) if n is the number of sides of a polygon, then the sum of the interior angles of a polygon is equal to $(2n–4)$ right angles, and (ii) the sum of the exterior angles of any polygon is equal to 4 right angles.

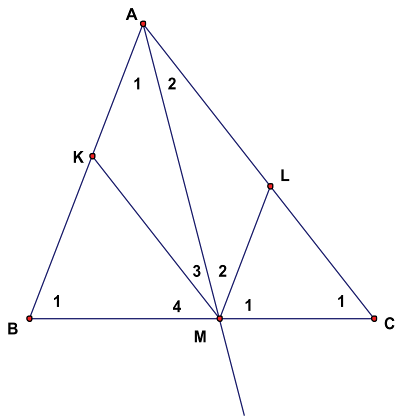
Proclus' Proof

Proclus' geometric argument has some similarities and differences with the Pythagorean and the Euclidean arguments. Figure 3 presents Proclus' proof. All three arguments are similar because they are based on the measurement of the straight angle and on the construction of parallel lines. Proclus' argument is different from the other two because he makes the construction of parallel lines to two different sides of the triangle, rather than to only one side. Moreover, these lines fall inside the triangle rather than outside and they pass through an interior point on the remaining side rather than through a particular vertex.

Proclus first chooses a point M *interior* to a side of the triangle (BC) and then constructs parallel lines to the remaining sides (AB and AC). He joins M with A (opposite vertex to the side BC) and constructs line AM. He also uses point M to construct lines MK parallel to side AC and ML parallel to side AB. Then he re-conceptualizes lines AM and BC as transversals to two pairs of parallel lines (MK//AC and ML//AB). Finally, he uses the congruence of the angles formed between parallel lines and transversals to prove that the straight angle BMC (of

PPOCLUS' PROOF

Prove that the sum of all three angles of any triangle is 180°.



We construct any ray AM (M lies on the side BC). Then we construct from the point M a parallel line to the side AB that intersects the side AC at the point L.
We also construct from the point M a parallel line to the side AC that intersects the side AB at the point K.

Since ML//AB and they are intersected by the transversal BC, then $\angle B_1$ is congruent to $\angle M_1$ as corresponding angles.

Since ML//AB and they are intersected by the transversal AM, then $\angle A_1$ is congruent to $\angle M_2$as alternate interior angles.

Since MK//AC and they are intersected by the transversal BC, thent $\angle C_1$ is congruent to $\angle M_4$ as corresponding angles.

Similarly, MK//AC and they areintersected by the transversal AM, then $\angle A_2$ is congruent to $\angle M_3$ as alternate interior angles.

Therefore,
$\angle A_1 + \angle A_2 + \angle B_1 + \angle C_1 = \angle M_2 + \angle M_3 + \angle M_1 + \angle M_4$
but $\angle M_2 + \angle M_3 + \angle M1 + \angle M_4 = 180°$
Thus $\angle A_1 + \angle A_2 + \angle B_1 + \angle C_1 = 180°$
But from the figure $\angle A_1 + \angle A_2 = \angle A$, $\angle B_1 = \angle B$, and $\angle C_1 = \angle C$
Finally, $\angle A + \angle B + \angle C = 180°$

Figure 3. Proclus' proof

measurement 180°) is also congruent to the sum of the angles in the triangle. Fundamental to Proclus' geometric argument were the auxiliary constructions of the lines AM, ML (ML//AB), and MK (MK//AC). These auxiliary constructions were *overcoded abductions* to construct a straight angle congruent to the angles in the triangle. Appropriate collateral knowledge (the parallel postulate and the congruence of the angles formed between parallel lines and their transversals) aided in the justification of the congruence between the straight angle BMC and the angles in the triangle.

In the analysis phase of this proof, abductions, auxiliary constructions, and collateral knowledge were essential. The first overcoded abduction anticipated the possible relation between the 180° measure of the straight angle and the sum of the angles in a triangle. The second overcoded abduction was the construction of two parallel lines, through an interior point of one side, and parallel to the other two sides. The third overcoded abduction anticipated a straight angle, with vertex at the above mentioned interior point, and congruent to the angles of the triangle. The properties of the angles between parallel lines and transversals accounted for this congruence.

In the synthesis phase, the nonlinear argument produced in the analysis phase was reversed to capture the argument in a deductive manner. Thus the written proof starts with the auxiliary constructions to arrive at the conclusion sought. Without the analysis phase it would have been impossible to imagine how to start the proof and how to incorporate viable auxiliary constructions. Therefore, it goes without saying that the written proof in Figure 3 also leaves silent the creative aspect of the proving process—the abductive reasoning that allowed the emergence of auxiliary constructions.

SUMMARY AND CONCLUSIONS

The deconstruction of the three classical proofs indicates that, given the nature of the proposition, these geometers ingeniously called upon a working backwards strategy or what they called the analysis-synthesis method. Any other method of proving would have been impossible due to the universal nature of this proposition. We argued that abductive reasoning played a fundamental role in the construction of a straight angle and the relationship between its measurement and that of the sum of the three interior angles of the triangle. This is to say that abductive reasoning played a key role in the analysis and then the synthesis phases of each proof. Chart 5 summarizes the analysis and synthesis phases that we argued were essential in the proving process of this proposition.

The *analysis phase* of each proof was based on overcoded abductions grounded in collateral knowledge (the 180° measure of straight angles and the congruence of angles between parallel lines and transversals). The first overcoded abduction was the connection between the measure of the straight angle and the sum of the interior angles of any triangle. The second overcoded abduction was the possibility of constructing a straight angle with angles that were congruent to the angles of the triangle. This construction was abductively implied from the parallel postulate. The third overcoded abduction was the actual construction of a straight angle using parallel lines and the congruence of angles formed between parallel lines and transversals.

It is important to note that an infinite number of parallel lines to one side of a triangle can be constructed due to the fact that there are an infinite number of points outside the line containing any side. Which point should be chosen? The Pythagoreans and Euclid anticipated the strategic point to be the opposite vertex to the side chosen first. Through that point they constructed a parallel line to the side into focus. Proclus anticipated the strategic point to be any point *between* the two vertices of the side first chosen (thus he excludes vertices). From that point, he constructed parallel lines to the other two sides of the triangle (forming a parallelogram); he also constituted this point into the vertex of the straight angle.

The *synthesis phase* was pursued after the *analysis phase* has produced a viable and logical geometric argument. This phase starts with the auxiliary constructions and pursues a chain of deductions by reversing their abductive reasoning in the analysis phase.

This is to say, they started with the auxiliary construction—a parallel line to an arbitrary side(s) of the triangle and passing through a particular point. What was the end goal of the construction? To form a straight angle with angles congruent to the (interior) angles of the triangle. Finally, they use the fact that the measure of the

Analysis phase of the proofs (overcoded abductions)	*Synthesis phase of the proofs (deductive reasoning)*
First overcoded abduction: Association between a geometric fact and the conclusion sought out • Could there be a connection between the 180º measure of the straight angle and the sum of the angles in a triangle?	*Auxiliary Construction (second and third abductions)* • Construct a parallel line to one side of a triangle and passing through the opposite vertex to that side. • Determine the straight angle that can be formed with angles congruent to the angles in the triangle. • Construct two parallel lines to two sides of a triangle and passing through a point between the two vertices of the third side. • Determine the angles between the parallel lines that are congruent to the angles in the triangle.
Second and third overcoded abductions: Possible Auxiliary Constructions • Could angles congruent to the three angles in a triangle form a straight angle? • Could parallel lines to one side of a triangle and through its opposite vertex guide the construction of the desired straight angle? • Could parallel lines to two sides of a triangle and passing through a point interior to the third side guide the construction of the desire straight angle? • Is this constructed straight angle congruent to the angles in the triangle?	*Geometric facts* • Every straight angle measures 180º (two right angles). • A straight angle, congruent to the three angles of a triangle, can be constructed.
Plausible Conclusion • The sum of the three angles in a triangle *should be* 180º because a straight angle can be constructed with angles that are congruent to the three angles of any triangle.	*Conclusion* • The addition of the measures of the three interior angles of any triangle *is* 180º.

Chart 5. Outline of the analysis-synthesis geometric argument of the three proofs

straight angle is 180° to conclude that the sum of the interior angles of any triangle should also be 180°.

Why to deconstruct the three classical proofs of one of the most fundamental propositions of plane Euclidean geometry? Certainly it could appear to be a useless exercise. After all, the proofs are there; they are not too long; and they can be easily followed once you are given the auxiliary constructions. However, one could ask questions like "Why are these auxiliary constructions appropriate?" "Where do these auxiliary constructions come from?" or "Is there any other way to prove this proposition?"

Being aware of the very essence of the proving process also entails being aware of *how* a proof is constructed. This awareness may encourage one's mind to imitate similar thinking strategies or to generate new ones in other geometric situations. There is no wonder why some students ask, "Where do proofs come from?" How do I start a proof?" Intentionally or unintentionally, these students are asking for a method to direct their own thinking during the process of proving.

The deconstruction here presented brings to the fore a cognitive issue that seems to be also a pedagogical issue—the key role of proving in the development of mathematical thinking and mathematical understanding. A research question that could be investigated is whether or not the deconstruction of classical proofs could help students to become aware of the role of abductive reasoning in the process of proving and in mathematical thinking.

Mathematicians like Polya argue that students should not only be given proofs to be memorized but also the knowledge of "how" to go about proving. This, he says, will encourage the formation of habits of thinking and methodical work. He also encourages teachers and students to *learn by guessing* (i.e., abductive inference or hypothesis) and to learn by proving (1962/1981, vol. 2).

Hanna (1989) also contends that learners should become aware of the need to reason carefully when building, scrutinizing, and revising mathematical arguments. She asserts that proving deserves a prominent place in the curriculum "because it continues to be the central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding" (1995, pp. 21–22).

The document *Principles and Standards for School Mathematics* (NCTM, 2000) advocates the teaching and the learning of geometry and, in particular, the teaching and the learning of proving in order to improve the development of students' systematic reasoning. It also advocates the teaching of geometry in such a way that allows students to explore geometric figures, to generate geometric conjectures, and to construct logical arguments and counterarguments.

More recently, the document *Common Core State Standards for Mathematics* (2012) also advocates the development of critical thinking, systematic reasoning, and habits of thinking. It argues that these are the most important competences to be developed in *all* students K-12 (Hirsch, Lappan, & Reys, 2012).

Proving, as a special type of problem-solving, is among the most powerful means to develop habits of thinking in students' minds. Memorization of proofs, alone, has less chance of developing these habits. Some proofs are great constructions (some say discoveries) done by mathematicians. These proofs should be analyzed before being memorized to serve as paradigmatic examples. However, less sophisticated proofs, like those of simpler propositions which are constructed by students themselves, will have the greatest impact on their mathematical thinking. Polya emphasizes this point in a clear and simple manner.

> Your problem may be modest, but if it challenges your curiosity and brings into play your *inventive faculties*, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. (Polya, 1945/1973, preface of the first edition, p. v)

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PROBLEM SOLVING

9

CHRISTOF SCHREIBER

9. SEMIOTIC ANALYSIS OF COLLECTIVE PROBLEM-SOLVING PROCESSES USING DIGITAL MEDIA

ABSTRACT

This article illustrates the analysis of chat sessions using Charles Sanders Peirce's triadic sign relation. The episodes presented here are from a project called 'Math-Chat', which is based on the use of mathematical inscriptions in an experimental setting. One characteristic of this chat setting is that it requires the pupils to document all their attempts of solving mathematical problems as mutual inscriptions in written and graphical form. In order to analyse the outline, as well as the use and development of the jointly-devised inscriptions, a suitable instrument for analysis has been developed by combining an interactionist approach with a semiotic perspective. Through incorporating this semiotic perspective into an empirical study on mathematic learning, the development and use of such an instrument can be demonstrated. Furthermore, the development and the structure of 'Semiotic Process Cards' will be explained. In conclusion, the findings related to the use of inscriptions in general as well as in primary-classroom problem-solving processes will be presented.

INTRODUCTION

Particularly in mathematics, the learning process depends highly on written-graphical communication. Morgan (1998) describes the widespread significance of writing in mathematical learning processes. Pimm (1987) mentions that mathematics depends on written forms of communication. This is due to the fact that the depiction and description of many mathematical operations are seen as the mathematical idea or procedure itself, which does not necessarily have to be its sole representation in the form of a symbol or sign. Writing and written presentations are integral elements of mathematical communication. Krummheuer (2000b) refers to the fleetingness of spoken utterances in learning situations in mathematical education and suggests the following:

> […] the quick evaporation and the situational uniqueness of verbal accomplishments impedes reflection on such interactive procedures […]. Complementing such reflections with a written presentation of the result (especially of the work process) seems helpful. (p. 31)

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 183–208.

Comparing spoken and written language Donaldson (1978) notes:

> The spoken word […] exists for a brief moment as one element in a tangle of shifting events, […] and then it fades. The written word endures. It is there on the page, distinct, lasting. We may return to it tomorrow. (p. 90)

When ideas and solving methods pertaining to mathematical problems are recorded in written form, their status becomes more explicit and negotiable (see also Bruner's "externalization tenet," 1996, pp. 22–25). There is very little empirical research that examines the importance of a written record of the resolution of mathematical problems in relation to this process.[1]

In order to access the written products of a problem-solving process, pupils were asked to solve mathematical problems in a specific setting, namely using two tablet PCs that were connected to an online chat. As any form of oral communication between the participants was not possible, it was necessary to write down questions, tips, suggestions and different approaches in order to communicate with the other participant. By analysing the written elements theoretical and methodological questions on solving mathematical problems could be studied. The aim was to encourage pupils to engage in written communication, in order to analyse the meaning and importance of their communication while inventing collective problem solving strategies. As there was a lack of instruments to analyse the pupils' written products, in the 'Math Chat'[2] project, a semiotic instrument of analysis was developed which enabled an accurate examination of the jointly produced written problem-solving processes and communication. This instrument also can be used to analyse any occurring oral utterances of the participants during this process. Thus, this instrument was even more versatile than initially intended (Schreiber, 2013/ 2010/ 2006).

In this article, the technical and organizational requirements as well as the specifications of the Math-Chat project will be described first. Thereafter, an explanation will follow that explains what is seen as mathematical inscriptions by Latour and Woolgar (1986). Moreover, the useful aspects of Peirce's semiotics for the development of the instrument of analysis will be presented. Hereafter, the 'Semiotic Process Cards' that were developed based on these theoretical approaches will be described. Then, an example from the chat project will be analysed through the mentioned methods and depicted by a Semiotic Process Card will be described. In the final section some results and findings will be presented (see also Schreiber, 2013/ 2010).

THE 'MATH CHAT' PROJECT

The main focus of the 'Math-Chat' project lay on the examination of the fundamental problem of the written depiction of collective strategies for the solving of mathematical problems in an experimental setting. The research concentrated on the type of inscription pupils compiled during the joint problem-solving process, the way these inscriptions were used and developed as well as the role they played in structuring the problem-solving process.

The fourth-grade pupils, who took part in the project, used two tablets in two different rooms to communicate during the chat sessions. There were one or two pupils each to one tablet. Special pens or markers were used to enter information on the screens. The two tablets were situated in different locations, connected by a wireless internet connection. No oral communication between the chat participants was possible as they were situated in two different rooms. Hence, in order to communicate with the other participants, they had to enter questions, tips, methods and proposals in written form during the joint problem-solving process. The program NetMeeting was used to facilitate the chat settings.[3] This program enables participants to enter data in two different forms: While alphanumeric data, which appears in the 'chatbox', is entered by using the keyboard, the marker is used to enter data on the 'whiteboard' (see Figure 1 for use of terms). The chatbox and the whiteboard appear on the screen, side by side, as two separate windows.

The aim was to induce pupils to communicate the problem-solving process non-orally, so as to be able to investigate the importance of the written product of the collective problem-solving process, as described above. In this way, the chat setting offers a new perspective on a number of fundamental questions concerning the teaching and learning of mathematics. Chatting is a form of interaction that is based on the written word and graphics. However, it is similar to oral interaction due to its interactive nature. Thus, through the medial written form of communication it

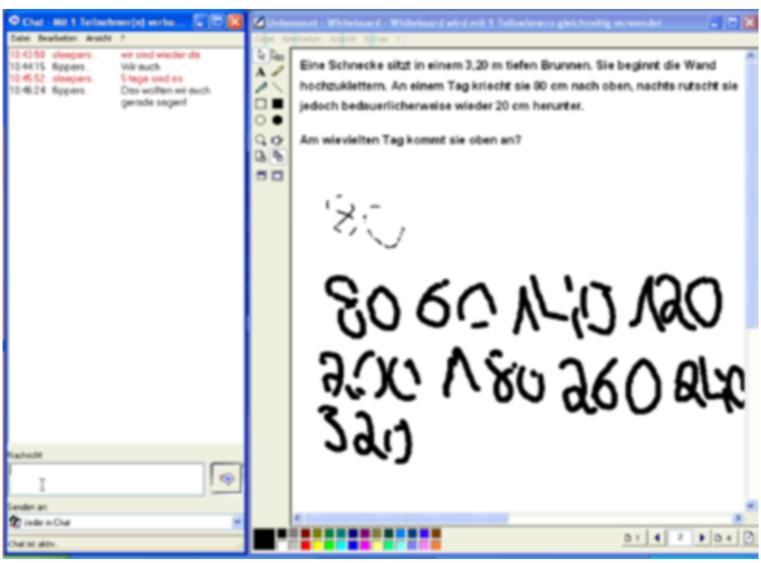


Figure 1. Screenshot of a 'net meeting'

is possible to gain insights into a conceptually oral situation. Both, theoretical and methodological questions on mutually created, written aspects of a mathematical problem-solving process can be examined, since no oral communication between the participants was possible.

The communication in the chatbox is, according to Dürscheid's (2003) terminology, "quasi-synchronous" (p. 44): Messages had to be typed and sent in order to reach the other participant. Up to this point, they have not been visible to the counterpart. Messages are typed into a window and can be edited or deleted by the sender before being sent. Once a message has been sent it appears in all the chatboxes of the participating computers (Figure 1, left). Both, the author's name and the time of sending are displayed. Also, any further changes to the message are not possible.

However, all communication on the whiteboard area of the screen takes place simultaneously (Figure 1, right). This means that every operation or action carried out on the whiteboard area of a computer appears simultaneously on the other participants' whiteboard. Any changes to the message or graphics by the author would be visible to all participants, who have permission to edit them.

During the research project 'Math-Chat' all operations that were carried out on the computer screens and all utterances spoken by the participants were recorded. By doing so, scenes, which seemed relevant to the research question, were transcribed in order to enable a detailed analysis.

MATHEMATICAL INSCRIPTIONS

In terms of solving problems based on the research of mathematical information and correlations, the participating pupils mainly worked on the written products in the setting described above. From here onwards, the written-graphical products generated by the pupils in the chat setting will be referred to as "inscriptions" (Latour & Woolgar, 1986). Latour and Woolgar studied the development and evolution of knowledge in laboratories. They classified the different kinds of models, pictures, icons and notations used in the laboratories as 'inscriptions'. They described several characteristics of inscriptions (Latour & Woolgar, 1986; Latour, 1987):

- Inscriptions are mobile because they are recorded in materials and can be sent by mail, courier, facsimile or computer networks.
- They are immutable during the process of moving to different places. Inscriptions remain intact and do not change their properties.
- The fact that they can be integrated in publications just after a little cleaning up is described as one of the most important advantages of inscriptions.
- The scale of inscriptions can be modified without changing internal relations.
- It is possible to superimpose several inscriptions of different origins.
- They can be reproduced and spread at low cost in an economical, cognitive and temporal sense.

Latour and Woolgar see inscriptions as a very ductile means of representation that is continuously changing and improving. As such, they represent aspects of the conceptual development during the research process (see also Schreiber, 2004). Latour (1990) talks about "cascades of ever more simplified" (p. 20) inscriptions. Latour and Woolgar's definition of the term 'inscription' applies exactly to the subject matter of the 'Math-Chat' project. The interest is centred on a detailed analysis of the inscriptional aspect during mathematical interactions, both on the interactive origin of the inscriptions as well as the meaning and importance of the developing inscriptions for the interaction process.

Roth and McGinn (1998) pointed out that the use of inscriptions is closely connected to the social practice they originated from:

> Inscriptions are pieces of craftwork, constructed in the interest of making things visible for material, rhetorical, institutional, and political purpose. The things made visible in this manner can be registered, talked about and manipulated. Because the relationship between inscriptions and their referents is the matter of social practice … students need to appropriate the use of inscriptions by participating in related social practices. (p. 54)

Herein lays the basis for the here used interactionistic approach in relation to the learning of mathematics, which forms the foundation of this project. What is unique about this approach, is that the focus lies on the genesis of individual inscriptions: pupils externalize their ideas in a chat-based dialogue using alphanumeric and/or graphic notations. Their chat partners' reactions enable the gradual development of a single inscription into a joint or mutual inscription. The internet chat method is conducive to this process of text compilation, as it (the process of compilation) becomes both collective and interactive. This process can be viewed as an important component of the (chat-based) interaction as it generates the "taken-as-shared-meaning" of the chat partners (Cobb & Bauersfeld, 1995). There are many other publications that deal with interactively created inscriptions (see Roth & McGinn, 1998; Lehrer, Schauble, & Carpenter, 2000; Sherin, 2000; Meira, 1995, 2002; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Gravemeijer, 2002; Fetzer, 2007), although all of these focus on face-to-face situations. The focus however, was solely placed on the inscription-based communication between the two sides of the chat setting, which was facilitated by the experimental design.

ANALYSIS OF INTERACTION

Interaction analysis is a method that was developed on the basis of the ethnomethodological conversation analysis of Bauersfeld, Krummheuer and Voigt at the IDM Bielefeld. It deals with processes of interaction that take place in a school setting. This form of analysis is based on symbolic interactionism:

> The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate

> to define the thing for the person. Thus, symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact. (Blumer, 1969, pp. 4–5)

The meaning of a thing is thereby negotiated through interaction. This negotiation occurs during processes of social interaction from which understanding and cooperation emerge on a semantic level.

In order for this negotiation of meaning to happen, the participants' interpretations of a situation must accommodate one another. Their definitions of the situation do not necessarily have to be identical, but must sufficiently harmonize to continue and further the development of the interaction. In this context, therefore, the participants' products are not intended to have a *shared meaning* but rather a "taken-as-shared-meaning" (Krummheuer & Fetzer, 2005, p. 25). Such an "interim product" of the interaction is generated by the continuous process of negotiation of meaning and signals a thematic openness toward the continuing progress of the interaction (see Naujok, Brandt, & Krummheuer, 2004). Through the reciprocal interpretation attempts, a process of 'clarification' takes place during the attribution of meaning by the participants.

By using the interaction analysis the way in which individuals create and negotiate taken-as-shared-meaning is reconstructed (Krummheuer & Naujok, 1999; Krummheuer, 2000a). The aim is to reconstruct any operations in the situation that are meaningful for the participants and to construct as many interpretations of these actions as possible. These initial interpretations are then reinforced or rejected in order to ensure the most convincing interpretation of the episode.

A number of scenes were selected from excerpts of lessons, which were recorded as screen videos and transcribed. These scenes were then interpreted in detail using the interaction analysis. Such an analysis is illustrated below as a summarized interpretation and further analysed using the semiotic aspects described in the following section.

ASPECTS OF CHARLES SANDERS PEIRCE'S SEMIOTICS

Peirce's sign model was used to analyse the jointly produced inscriptions in the chat-based problem-solving processes. The Peircean sign model is a very differentiated classification and applied by some researchers of mathematical didactics (e.g., Volkert, 1990; Hoffmann, 1996, 2003; Dörfler, 2004, 2006; Seeger, 2011; Otte, 2011) as well as by pedagogical researchers (i.e., Zellmer, 1979). In comparison to other semiotic approaches (e.g., Saussure & Lacan: see Gravemeijer, 2002), this model seems to be a more suitable instrument for the analysis of inscriptions. Among other arguments, the Peircean approach is oriented less towards language than the ones of Saussure or Lacan (see a detailed discussion in Hoffmann, 2003, pp. 7–12), but it integrates the individual as an interpreting instance. Therefore this approach can be combined with the interactional theory of mathematical learning and teaching. Also Eco (1976) points out the advantages of Peircean semiotics in "A Theory of Semiotics" (see the discussion by Eco, 1976, pp. 14–16, where he describes the Peircean approach as being more complete and semiotically useful; see also Schreiber, 2004).

Peirce's Triadic Sign Relation

Peirce's (1931–1935) triadic sign relation consists of a "triple connection of sign, thing signified and cognition produced in the mind" (CP 1.372). The three correlates in this triadic relation can be described as seen in Figure 2:

> A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of the representamen. (CP 2.228)

When Peirce (1931–1935) refers to 'sign' or 'representamen' it can be understood as the external, visually, aurally or otherwise perceptible depiction of a sign, while the 'interpretant' is a sort of inner sign, which the observer associates with an external perceptional sign. The 'object' is understood as what the observer of an external sign believes to be the creator's intention. For Peirce, these three correlates are integral parts of a sign and none of the three are superfluous. The sign itself only becomes a sign when it is perceived by an observer to be such. According to Peirce, that which is not interpreted as a sign is not a sign (CP 2.228). The terms 'representamen' and 'interpretant' for the correlates and the term 'sign-triad' for the complete 'triple connection' described above will be used in order to avoid potential confusion when using the term 'sign'. Peirce's definition of the term 'object' will be retained but it should be noted that the term does not necessarily refer to an object in the material sense.

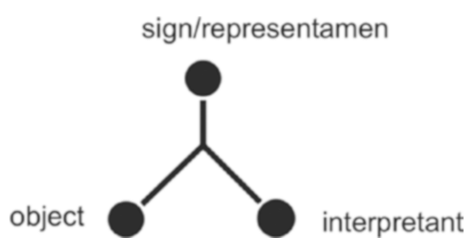


Figure 2. Peirce's triadic sign relation

The Foundation of Peirce's Triadic Sign Relation

Applying and developing the Peircean approach, Hoffmann (e.g., Hoffmann, 1996) focuses on the 'idea' or 'ground' of the Peircean sign model, which he refers to as 'das Allgemeine', meaning 'generality' (translation by Schreiber). Hoffmann mentions the following examples for 'generality': Concepts, theories, habits and competences etc., which are given mentally or physically. The concept of 'generality' was initially fundamental for the analysis of the excerpts of the 'Math-Chat' project. As the interpretant is determined by the concepts, theories, habits and skills of the observer, the element of generality is the foundation of the triadic sign relation (see also Schreiber, 2005a; 2006). An appropriate term has to be found, as the terminology is quite misleading.

Looking at the way the concept of 'generality' was used in the first semiotic analyses (Schreiber, 2006), it would be useful to adopt Goffman's term "frame" (1974, p. 7) with regard to Bateson (1955). By doing so, the "ground of the representamen" is integrated into an interactionistic perspective: Each individual creates interpretants against the background of his or her own subjective interpretation experiences and under a specific perspective. Goffman (1974) points out the importance of standardization and the formation of a routine during the "definition of the situation" (p. 1 f.) and introduces the term "frame" to describe interpretation processes (Goffman, 1974, p. 7). Krummheuer uses the term coined by Goffman in a content-based setting, and relates it to curriculum-based educational theory.

The terms 'generality' and 'frame' largely correspond with one another, each being integrated in their respective theoretical fields. The preferred term for this particular research is 'frame', which is well established in interpretative educational research, as it has proven to be relevant to many areas of reconstructive social research. The term 'generality' has otherwise not been mentioned further in literature on semiotics in the didactics of mathematics.

All parts of verbal statements or operations of a given individual during an interaction can be identified as a representamen. Frames are 'activated' by a known representamen. These framing procedures can be taken as the 'ground of representamen', as defined by Peirce (see Figure 3).

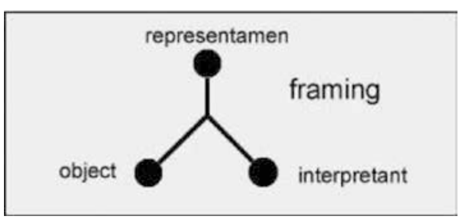


Figure 3. The frame as the basis of Peirce's triadic sign relation

The variance of the interpretant is limited to the frame, which is triggered within each individual by the representamen. The representamen does not stand for this object in all regards, but only in regard to an activated frame. The connection of Peirce's semiotics with Goffman's frame analysis makes the empirical analysis of frames at the detailed level of Peircean triads possible.

The Chaining Process

Peirce describes meaning as a continuous developing process, in which the interpretant of a given triadic sign relation becomes the representamen of another triad: "Anything which determines something else (it's *interpretant*) to refer to an object to which itself refers (its *object*) in the same way, the interpretant becoming in turn a sign, and so on *ad infinitum*" (CP 2.303; italics Peirce's own). Peirce believed that every interpretant within a triad could also be interpreted within another (Figure 4). This continuous process of semiosis is infinite. It cannot be brought to an end but can be interrupted (CP 5.284).

Peirce noted that it is not possible to identify a first or final sign in this context.

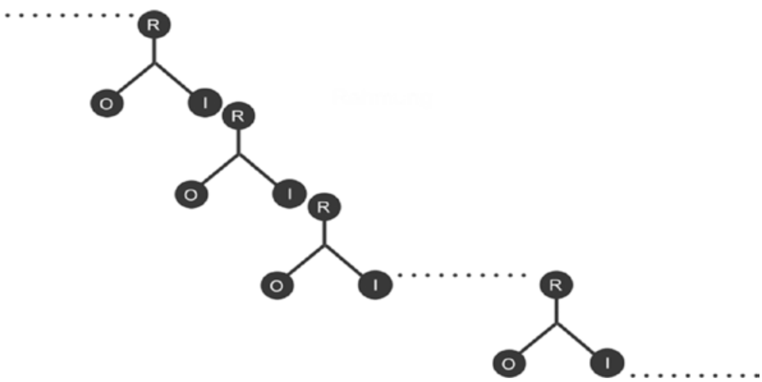


Figure 4. The infinite process of semiosis (see also Schreiber, 2006, p. 248)

Saenz-Ludlow (2006) also alludes to "unlimited semiosis" (p. 188; see also Eco, 1979, p. 198) and describes an illustration similar to Figure 4 as "meaning emerging in the translation of signs into new signs" (ibid.). Presmeg (2006) describes and compares this chaining process first, based on Saussure's dyadic sign model and later, based on Peirce's triadic sign model (p. 165 ff.). She points out, that the "dyadic chaining conceptual model was not completely adequate as an explanatory lens" (ibid., p. 168). Furthermore, she suggests that a "chain is not the best metaphor" (ibid., p. 170) because of its "nested quality" (ibid.). She illustrates this continuous process by using the image of the Russian nested dolls (ibid., p. 171; also 2001, p. 7). She describes this continuous process of semiosis as

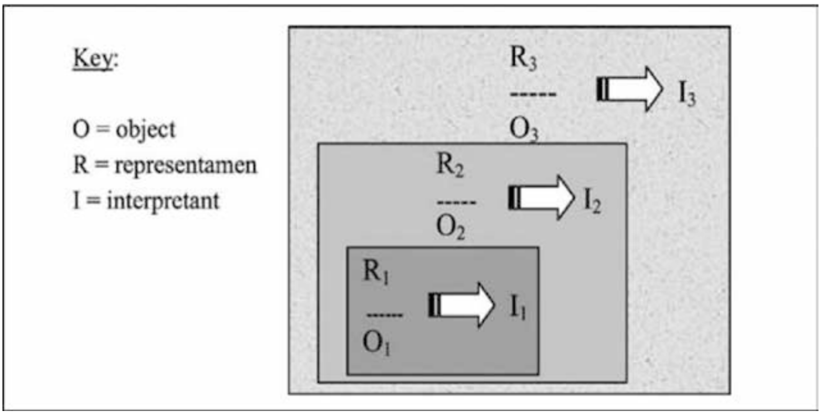


Figure 5. A triadic representation of a nested chaining of three signs (Presmeg, 2006, p. 170)

a three-step linear process in which a triad is simultaneously the object of a further triad. The "nested model of semiotic chaining" is described below (2006, p. 169 ff.).

However, the processes that were reconstructed are not in all cases linear. According to the analyses, there are interpretants that serve as representamen in the following triad, and groups of sign triads that serve as representamen in a new sign triad. Furthermore, there are sign triads that are connected with one another because they correspond with the same representamen. This is abstractly depicted in Figure 6 and corresponds with the example in Figure 7, which is a detail of the semiotic process (see Figure 8).

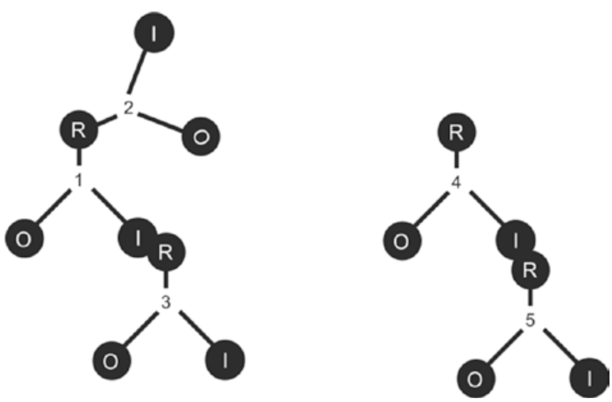


Figure 6. The complex semiotic process

Due to the non-linear alignment of the process in the example, the term 'chaining' is rejected and the term 'complex semiotic process' has been chosen instead. This

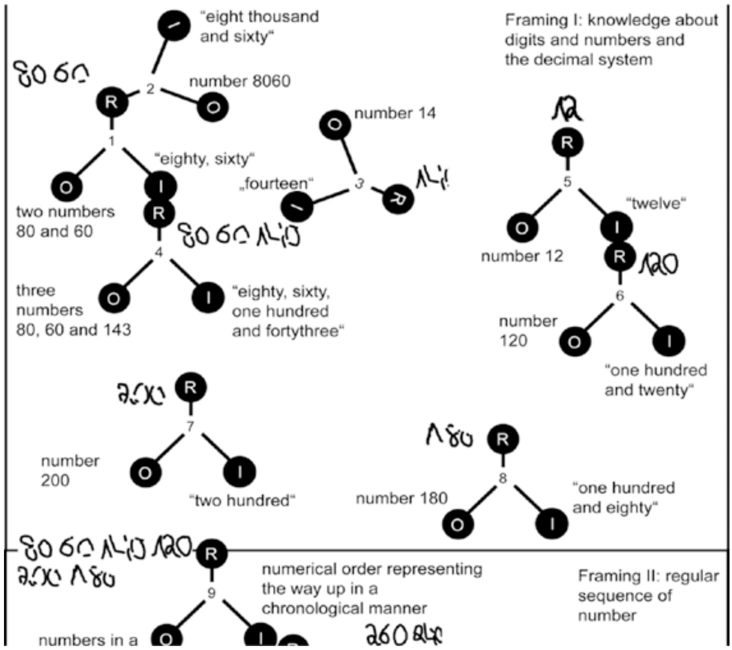


Figure 7. Detail from a semiotic process card (Figure 8)

term reflects the development of the interpretation process accurately, although these processes are partially linear. Hoffmann (2006) describes this use of the phrase 'complex semiotic process' as an additional development to the 'nested chaining model', devised by Norma Presmeg, with reference to the construction of meaning in educational settings (p. 175).

The complex semiotic process comprises of the following: The semiotic equivalent of the negotiation of meaning, which takes place during interaction, provides a first impression of the communication that takes place and leads to the creation of 'taken as shared meaning'. It characterizes a process leading to the construction of meaning, which can be roughly described as going from the "direct" via the "dynamic" and ultimately to the "final interpretant" (see Nöth, 2000, p. 64). Hoffmann (2002) notes that after the conduction of a complex semiotic process, the interpretant can become "the general meaning of a sign" or a "change of habit" (p. 62),[4] just as at the end of a process of constructing meaning, where the result can lead to the creation of a new frame.

Diagrams as Particular Signs

Many publications point to the special role that Peirce's definition of diagrams has on educational settings in general and particularly in mathematics education

(Dörfler, 2006; Hoffmann, 2003; Kadunz, 2006 & Krummheuer, 2008). Dörfler (2006, p. 200 ff.) elaborates especially on viewing the learning of mathematics as participation in diagrammatic practices (see also Dörfler, 2004). His position is characterized by the idea that 'diagrams' are objects of mathematical activities, which are devised, created and described, and thus lead to the formation of mathematical concepts.

Using numerous examples, Dörfler (2006, 2004) criticizes that abstract objects consistently form the central point of interest in mathematics education and that activities using illustrations and other representations are only viewed as supporting resources. This view of mathematics, based on the "immaterial nature of mathematical objects" (Saenz-Ludlow, 2006, p. 183), is also dominant in semiotic articles on mathematics didactics (see Steinbring, 2006; Hoffmann, 2006). Based on viewing mathematics as "the science of abstract objects", Dörfler advocates a shift of focus towards mathematical activities as an activity with material, perceptive and manipulative inscriptions (2006, p. 203; see also 2004). In summary, here are some important aspects of the diagrammatic thinking as referred to by Dörfler (2006, p. 210 ff.; 2004, p. 5ff.):

- Diagrams are a kind of inscription, not isolated, single inscriptions, but rather part of a system of very structured depictions that provide the means by which inscriptions can be constructed and read.
- A diagram is determined by conventions and requires a legend for comprehension. The legend need not be explicit, but can be learned through exposure to diagrams (2006). So diagrams are "imbedded in a complex context and discourse which better is viewed as social practice" (2004, p. 8).
- "Diagrams are extra-linguistic signs. One cannot speak the diagram, but one can speak about the diagrams" (2004, p. 8; see also 2006, p. 210).
- "Intensive and extensive experience with manipulating diagrams (…) supports and occasions the creative and inventive usage of diagrams" (2004, p. 7).
- Diagrams are the underlying objects of research of diagrammatical thinking, making mathematics a perceptive empirical and not only mental practice (2006, p. 211).
- Diagrams can be analysed and manipulated "irrespective of what their referential meaning may be. The objects of diagrammatical reasoning are the inscriptions themselves" (2004, p. 7; see also 2006, p. 211).
- The possibility of observing, describing and communicating diagrammatic thinking in the form of operations with inscriptions makes the materiality of mathematical activities explicit (2006).

The manipulation of formulae, numerals and figures as well as the "intimate experience with several diagrammatical inscriptions, their structure and operations" (2006, p. 213) are mentioned as examples of the basic form of diagram utilization.

Dörfler (2006) views the next stage as being mainly concerned with experimentation with diagrams and the exploration of their specific characteristics (p. 213). The theory then proceeds with the relationship between different diagrams, which pupils at primary level encounter as different forms of representation. At this point, Dörfler (2006) once again criticizes the fact that these variations of representation forms are only used for learning “abstract objects” (p. 214). He points out the particular role of pupils’ own creations and designs of diagrams. All of these diagrammatic activities are embedded in the social practices of a group. Such social practices are constitutive of the meaning and significance of diagrams. Dörfler (2004) emphasizes that the meaning is not the result of a pre-existing “referential meaning” (p. 7). Thus, through operating with the diagram within social processes, its meaning can emerge and also change.

SEMIOTIC PROCESS CARDS

Interaction analyses were carried out based on written transcriptions of the ‘Math-Chat’ sessions (see Tables 1 and 2). These analyses allow a detailed description of the sessions, which in turn facilitate a combined or summarized interpretation (see Krummheuer & Naujok, 1999). These interpretations were then used to describe the complex semiotic process. The results of these descriptions are presented below as ‘Semiotic Process Cards’ (henceforth SPC). In the SPC the elements described previously are all accounted for: Peirce’s triadic sign relation, embedded in an underlying ‘frame’, and its development as part of a complex semiotic process. The format of the SPC will be illustrated first; the specific SPC used here for demonstration purposes will be analysed later in greater detail.

The SPC should be read from top to bottom and generally from left to right. In order to facilitate the orientation, the triads are numbered chronologically. The letters “R” for representamen, “I” for interpretant and “O” for object are used to indicate the part that is referred to on the triad. Labels and images will be used to display and demonstrate the three correlates. In some cases the interpretant of a triad will be supplemented by further aspects and thus become a new representamen (e.g., Figure 7, triad 3 & 5).

The frame, which is recreated through the interaction analysis, is referred to in the semiotic analysis and indicated in the SPC. After the compilation of several SPCs, the various reconstructed frames have been able to be classified as ”mathematical”, “argumentative”, “formal” and “social” frames (see Schreiber, 2010).

The complex semiotic process is represented by the configuration of the triads. As Figure 7 shows, the process can progress very differently. Where the progress is linear, the subsequent triad with the correlate ‘representamen’ is positioned at the correlate ‘interpretant’ of the preceding triad (see Figure 7, triads 5 and 6). Where two parts of the process relate to the same representamen, the representamen is assigned to two triads (see Figure 7, triads 1 and 2). If the representamen of a triad corresponds with the entire previous process, from an accumulation of sub-processes, then the correlate representamen is placed on a line of the box, which underlies the hitherto existing process (e.g., Figure 7, triad 9).

Table 1. Transcription: Part A

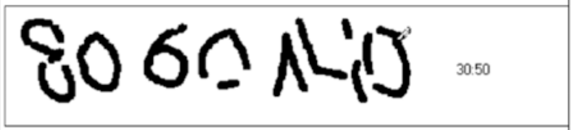
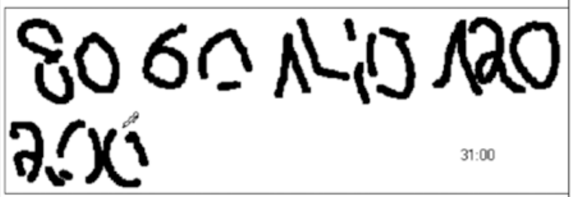
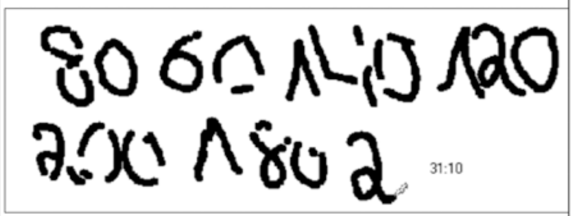
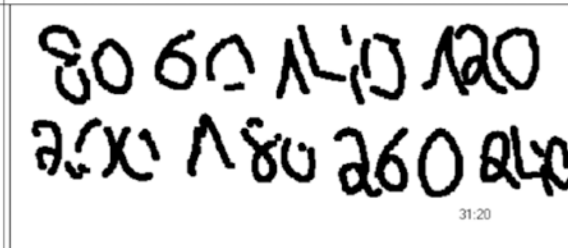
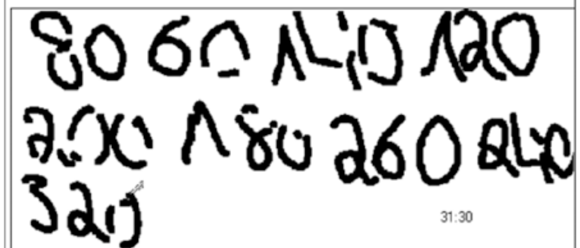
line/ time	oral utterances Sleepers	activities Sleepers	whiteboard	activities Flippers	oral utterances Flippers	time/ line
30:40						**30:40**
14	**S2**: this is [#5 s i x t e e n\ #5] 60	[#5 **30:41** writes 60 onto the white-board]			**F1**: 80 60\ 8060 yes\ (..) 14 80 60 143	6
15	**S2**: <plus 80 is					
16	**S1**: < 140\					
17	**S2**: [#6 140 #6]	[#6 **30:46** writes 140]			(..) yes\ 12 120\ (...)	**30:50**
30:50						
18	**S1**: (..) em\ (..)					
19	**S1**: <[#7 120\ (.)	[#7 **30:52** writes 120]				
20	**S2**: < 120\					
21	**S1**: #7] Plus 80\ is [#8 200\ #8] minus 20\	[#8 **30:58** writes 200]			200\ (5sec.)	**31:00**
31:00 22	**S2**: is [#9 180\ #9] plus 80\ is	[#9 **31:04** writes 180]			**F1**: 180\ (...) I see\ 80 is minus, so it is 60\ 60 plus 80\	**31:10** 7
31:10	[#10	[#10 **31:10** writes 260]				
23	**S2**: < 260\ #10]					
24	**S1**: < 60\					
25	**S2**: >[#11 240\	[#11 **31:15** writes 240]				
26	**S1**: > 240\					
27	**S2**: <#11] Plus 80					
	S1: < 80					

Table 2. Transcription: Part B

line/ time	oral utterances Sleepers	activities Sleepers	whiteboard	activities Flippers	oral utterances Flippers	time/ line
31:20 29 30 31 32 33 34	**S1**: > is two- **S2**: > is\\ **S1**: <hundred- **S2**: <hundred and **S2**: twenty **noo**\\ **threehundred-** twenty\\ 320\\ **S1**: *whispering* [#12 320\\ #12]	[#12 **31:26** writes 320]			is 140\\ (.) minus 20 120\\ **F1**: <120 plus 80 **F2**: < mmh\\ **F1**: is 200\\	**31:20** 8 9 10
31:30 35 36 37 38 39 **31:40**	**S1**: **yes**\\ to say **S1**: < 1 2 3 4 **S2**: < 1 2 3 4 **S1**: **5** days\\ **S2**: **Noo**\\ 4, 1 2 3 4 noo let us calcu-late \\ **noo**\\ wait				**F1**: plus (.) minus 20 is 80 260 240 320\\ how many/ how many days does it take to crawl up / 1 2 3 4 fifth\\ (7 sec.)	**31:30** 11 **31:40**
40 41 42 43 **31:50** 44	**S2**: < 1 2 3 4 5 **S1**: < 1 2 3 4 5 **S1**: yes\\ it is 5 days\\ **S2**: 5 days or/ 5, 5 **yes 5**\\ **S1**: 5 d a y s \\		Whiteboard does not change		**F2**: should we write right/	**31:50** 12

AN EMPIRICAL EXAMPLE

In the chat episode presented here, two pupils, one on either side of the chat connection, solve mathematical problems. By using a transcription and a summarized version of its interpretation, the following episode could be described: Two pupils on either side of the chat connection attempted to solve mathematical problems. The semiotic analysis will be presented through the SPC.[5] During the episode two groups, nicknamed *Sleepers* and *Flippers* respectively, worked on the following mathematical problem:

"A snail sits in a well which is 3.2 m deep. It begins to climb up the wall. During the day it climbs up 80 cm. During the night it slides down 20 cm. How many days does it take for the snail to climb to the top of the well?" This type of mathematical problem is well-known and describes a classical problem. It can not only be found during Adam Ries'[6] times (Krauthausen & Scherer, 2001, p. 107; Ries, 1574, p. 73), but also appear in modern mathematics textbooks. These types of exercises are also used in teacher training courses for the topic on "improving problem solving though drawings" (Kelly, 1999). This mathematical problem was chosen, as it could be solved using pictures, sketches or tables.

Interpretation of the Episode

In the interpretation presented here, all the information in brackets refers to parts A and B of the transcription. The utterances have been abbreviated as follows: For example, (S1, 3) stands for member 1 of the Sleepers and utterance 3.

It is possible to divide the communication that took place in the chat into three interconnected processes; namely, the creation process of the inscriptions on the Sleepers' side, the process of chat communication on the basis of these inscriptions and the interpretation process of the inscriptions on the Flippers' side. Solely the Sleepers constructed the inscriptions. The construction of the Sleepers' individual inscription parts took place, hand in hand, without further consultation (S1 and S2, 14–34): Both pupils understood the meaning and composition of their jointly created inscription. Although the meaning of the numbers' sequence was not easily accessible to an observer outside, it was possible for both pupils to continuously predict, pre-calculate, suggest and adjust the next part of the inscription found on this page. The result (in days), which cannot be directly read, can be gathered from both pupils (S1 and S2, 35–44).

The Flippers' interpretation is characterized by the reception of the Sleepers' created and 'published' inscription. The Flippers understood and reconstructed the writing process. However, they could not accurately identify the individual numbers from the arrangement of numerals. From their point of view, the numbers stood adjacent to one another without being connected and thus, their relation to the problem was not clear. This changed after the Sleepers wrote the number 180. From this point on, Member 1 of the Flippers presumed that a legitimate sequence of numbers was being presented and commented on the inscription and their meaning (F1, 7; F1, 8). He could reassess this assumption through the algorithmic continuation of the numeric sequence, and appeared to be able to validate his assumption (F1, 8; F1, 10; F1, 11). The result of the exercise was thereby readable, at least for the Flippers. Member 2 of the Flippers approved this outcome and suggested giving the Sleepers a positive response (F2, 12).

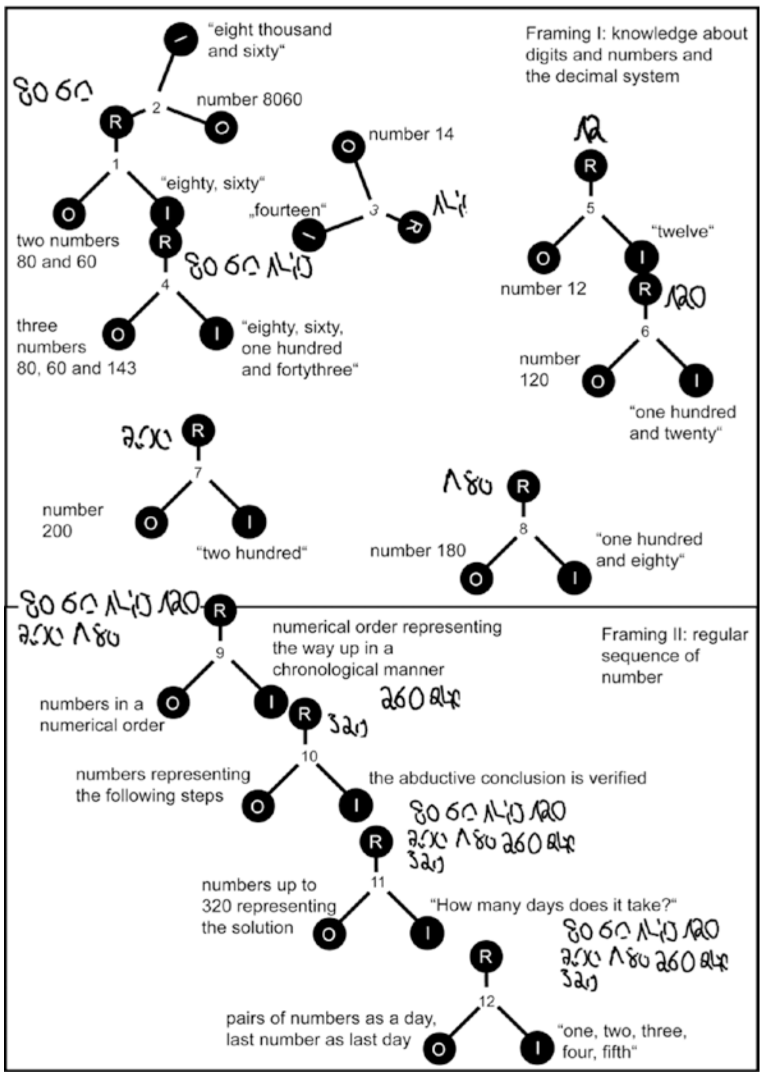


Figure 8. Semiotic process card from the perspective of the flippers

Semiotic Analysis from the Flippers' Perspective

This type of summarized analysis serves as a basis for the reconstruction of the complex semiotic process. For this reason, the so-called "Semiotic Learning Cards" (Semiotische Lernkarten; Schreiber, 2005b, see also 2005a) were first developed. During the later stages of the analysis process, these were renamed to "Semiotic Process Cards" (Schreiber, 2010), as the processes described were predominantly semiotic and not necessarily learning processes. The information in brackets alludes to parts A and B of the transcription (see Tables 1 and 2). The triads correlate with the semiotic process cards (see Figure 8), which graphically reflect the analysis. The presented semiotic analysis of this scene is from the perspective of the Flippers.

The representamen in triad 1 is the beginning of the inscription and generates two different interpretants, namely "eighty, sixty" and "eight thousand and sixty" (in F1, 6). The first interpretant shows that the inscription is being read as a representamen for the object "two two-digit numbers", namely 80 and 60. The same inscription is also read as the object "one four-digit number", namely 8060. Both understandings imply a similar frame: the "frame as a decimal depiction of numbers" (Framing I). This frame is based on socially shared knowledge of the decimal system and the correlation between digits and numbers (Framing I). The digits "1" and "4", which are read as "fourteen", follow (in S2, #7) a two-digit number (F1, 6). When the Sleepers extend the inscription by "0" (S2, #7), the Flippers refer to the interpretant of the first triad "eighty, sixty" (F1, 6) again. They construe the entire inscription up to this point "80 60 140" (whiteboard excerpt at 30: 50) as the representamen of the object "80, 60 and 143", which is therewith made up of three separate numbers (F1, 6). The last digit is read as "3" (F1, 6). This, however, is not discussed further. Framing I still forms the basis of the interpretation. Subsequently, the Flippers only refer to that particular part of the inscription as seen in triad 5. On the basis of Framing I, this representamen elicits the interpretant "twelve" (F1, 6). When the digit "0" is added to the inscription, the representamen in triad 6 produces the interpretant "one hundred and twenty" (F1, 6), which represents the three-digit number 120. In the same way, the representamen in triads 7 and 8 produce interpretants based on Framing I.

In the following triad, a change of focus, crucial to the problem-solving process, occurred: The Flippers no longer related solely to the individual parts of the inscription, but rather to the inscription as a whole (see Figure of triad 9). In other words, they related to the interpretant that had been generated by the inscription so far. One of the Flippers came to the following abductive conclusion: He recognized the representamen from triad 9 to be a representation of regularity in the development of a sequence of numbers. The basis of this comprehension is assumed to be the "frame as a regular sequence of numbers", Framing II. The pupil has grasped the concept of consecutive numbers as a sequence defined by its regularity, and the idea that this sequence of numbers reflects the stages of the mathematical problem.

Further examples were chosen specifically due to their partially analogue development or, in other cases, precisely due to their oppositional development. After they were analysed, they were presented as SPCs and the complex semiotic processes were compared. The abductive conclusions that the pupils reached as well as the associated adjustment of the framing caught my interest in particular. These conclusions and the adjustments in the framing proved to be central to the utilization of inscriptions. In some episodes, these could then be used diagrammatically after the adjustment. The findings, which are presented briefly in the following section, were the result of the analysis and the comparisons of numerous examples.

FINDINGS

The findings covered three areas: The use and development of inscriptions, the use of diagrams as a particular sign and the complex semiotic processes within the developing frames. It should be noted that the research described here included inscriptions, which were designed and used as mutual inscriptions and used by the pupils. Whenever the inscriptions were used collectively, they formed mutual inscriptions, even if they were generated from only one side of the chat setting, as seen in the given example. By continuing with this example, the way in which this inscription can be used mutually, be reconstructed as a mutual inscription and applied productively to an analogue situation will be demonstrated.[7]

In this example, an abductive conclusion had been reached in order to interpret the inscription. This conclusion is key to understanding the utilization of the inscription. The abductive conclusion is the prerequisite for the conversion of one group's inscription into the mutual inscription of both participating groups. As Meira (2002, p. 95) described, the inscription is applied in a minimalist fashion; only what is absolutely necessary is contained therein.

Diagrams were developed at this point, facilitating a productive problem-solving process. In reference to Dörfler (2006, p. 210 ff.), some characteristics of diagrams, which can be found in the examples, are listed below:

- The inscriptions described here are not single and isolated, but form part of a system of diagrams.
- There is a type of "legend" (ibid., p. 210) which is not present, but which was developed out of experience through the exposure to diagrams and their use.
- The diagrams are discussed orally, but they appear in written form, as inscriptions on a computer screen.
- As the diagrams in this form are usually not familiar in educational settings (example 1), it is assumed that new diagrams are constructed, which are then formalized.
- The inscription of the participants on one side of the chat setting becomes an "object of research" for the participants on the other side (ibid., p. 211).

The various semiotic processes are clarified in the Semiotic Process Cards. At the same time, the individual triads represent the analysis on the smallest possible micro-level. During the process, a sign, a representamen is analysed in detail in order to ascertain which interpretants it generates. The interpretant, which Peirce referred to as an inner sign, can only be determined by the utterances that follow the representamen. The object that stands for the relation between representamen and interpretant can be defined during the analysis. What the observer believed that the creator of the representamen intended to demonstrate can be identified. In this context, all three parts of Peirce's sign relation in their entirety are considered to represent the sign. The perspective of the person who perceives the representamen as a sign is crucial, thus setting the process of interpretation in motion.

Illustrating the progression of the complex semiotic process becomes possible through the configuration of the triads. These are arranged singly or in strands and are examples of either linear processes, due to the 'chaining' process described above, or non-linear processes, in terms of a complex semiotic process. The triads correspond partly with the same representamen, yet they create different interpretants, either despite or because of the fact that they have the same frame. In this way, the interpretant of a triad can become the representamen of a further triad. Only the interpretant generated and the oral or written statement resulting from it can serve as representamen of further triads. In some cases, it is not a single interpretant but a summation of tested interpretants that make up the representamen of the following triad. Therefore, it is the catalyst for the continuation of the process.

The respective framing that has been activated, which was also graphically based in the SPC process, determines these processes considerably. The reconstruction of the frame is the result of detailed interaction analyses, and is referred to in the semiotic analysis that follows each interaction analysis. These frames sometimes reappear at a later point in time during the processes, while other frames develop further. This development takes place partly as a result of new representamen, which are perceived as such by the interpreter. However, in some cases the changes in the frame are due to a new interpretation of prior knowledge.

What adds to the uniqueness of this chat is the much higher time and effort needed for the coordination of the process as compared to the face-to-face situations in which a group clarifies social processes orally and through gestures. The time invested in the *argumentative* situations is also notably higher in the chat settings. What remains remarkable in those situations, where diagrams were created and used, is that the *arguments* are already inherent within these diagrams. These arguments are evident to the user of the diagram.

An abductive conclusion leads to the change in the frame, forming the basis of the process in SPC 1. These are precisely such changes in the frame, which generate the special attention. They do not occur very often, yet they constitute very clear steps in the development of the problem-solving process. It is possible to recreate one of the abductive conclusions in the example above. As a result of this reconstruction, it became clear how the conclusion supports the interpretation of a diagram, ultimately leading to the mathematical problem's solution. The diagram was used for the solution of another example by becoming a mutually used diagram. This appeared to be a step towards a "final interpretant" (CP 4.536), which could then serve as a frame in subsequent examples.

The above-mentioned example showed how the successful development of the pupils' own inscriptions in a written or graphical based problem-solving process can progress. What takes place in this example is what Hoffmann (2002) refers to as the "change of habit" ("Veränderung einer Gewohnheit") (p. 62; CP 5.476) with reference to the development of an interpretant. From an interactions-theoretical point of view, this corresponds with the (preliminary) end of a 'negotiation of meaning', leading to a new frame.

As shown above, using the inscriptions created and developed by the pupils has proven to be very productive for problem-solving processes. The inscriptions in question were exclusively effective when they did not constitute solutions or parts of solutions, but rather significant elements, ultimately leading to the solution. The use of the inscriptions showed to be especially useful when mutually created by the participants or a joint production was at least possible. The observation of the starting point of the process was particularly helpful facilitating its further formation and utilization.

In this way, the use of inscriptions in collective problem-solving processes should also be enabled through working together on these inscriptions in a way that all participants can observe and contribute to the construction of these inscriptions. The requirements for the creation of mutual inscriptions are highly advantageous under these conditions. In this way, the use of inscriptions in collective problem-solving processes becomes feasible.

Once inscriptions have been constructed and used productively, they can be retrieved for subsequent problem-solving processes. At the same time, a gradual development, in terms of an enhancement or formalization of the inscription, took place. The mutually created and successfully implemented inscriptions can be activated as (part of) a new frame.

In this way, the inscriptions can be self-created or in a team, whereby the value for the problem-solving process and mathematical significance was extremely high. In this way inscriptions are very productive for mathematical solving processes. Working with diagrams as a central mathematical activity can lead to the creation of what Dörfler (2006) calls "successful diagrammatical thinking" (p. 216). The necessary "intimate experience" (Dörfler, 2004, p. 8) is given through the involvement in its generation, the observation of its generation, its amendment etc. It is precisely this type of diagram, embedded in social practice that makes the problem-solving process fruitful for learning. The empirical evidence presented in this chapter greatly corresponds with Dörfler's theses on the importance of diagrams for academic mathematics and mathematical learning processes.

The following conclusions, pertaining to the inscriptions generated by the pupils and the diagrams that were subsequently generated, should be taken into consideration when having to deal with problem-solving situations initiated by teachers:

Pupils

- work in a manner that allows all of them to participate in the compilation of a graphical or written representation;
- put the problem-solving process into writing without being bound by formality;
- make the written form a central topic of the process and discussion;
- have a purpose and are motivated to try out their own customary methods anew;
- are given the opportunity to optimize the inscriptions, thus developing diagrams of higher quality;
- may realize the importance of self-generated diagrams.

The above presented Semiotic Process Cards were developed as a tool to describe and present the analysis of collective problem-solving processes in their individual elements on a micro level and also to display their progress graphically. An analysis based on Peirce's semiotics is relatively effective for comparing processes, in which representamen are available in written from, and thus, proofing to be an appropriate form of presentation.

The changes in relation to the frames, which are triggered by new representamen, other focusing or abductive conclusions, provide very interesting insights into the progress of the process. The pupils' use of their self-created diagrams as well as the activation of the same frame, as one for analogue problems, demonstrate the learning curve of the participants.

NOTES

1 For example Meira (1995) "discusses the production and use of mathematical notations by elementary school students" (p. 87).

2 This study was supported by Müller-Reitz-Stiftung (T009 12245/02) entitled Pilotstudie zur Chat-unterstützten Erstellung mathematischer Inskriptionen unter Grundschülern (Math-Chat: pilot study of chat-based creation of mathematical inscriptions among primary pupils").

3 NetMeeting is Freeware from Microsoft.

4 Hoffmann refers also to Peirce's term "habit change" in CP 5.476.

5 This and other contrasting examples are described in detail in Schreiber 2010.

6 Adam Ries (*1492, +1559) was a German mathematician.

7 See Schreiber, 2006 and Schreiber, 2010.

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APPENDIX: TRANSCRIPTION RULES

1. column and 7. column
   line numbers and time

2. column and 6. column
   shortnames of interacting persons on the left hand side (Times New Roman 12 pt bold).
   oral utterances on the right hand side (Times New Roman 12 pt); incomprehensible utterances are marked as (incomprehensible).
   paraverbal information, (special characters see below), for example *emphasizing, whispering* etc. (Times New Roman 12 pt italics)
   # refers to actions on the computer

3. column and 5. column
   actions marked with # are actions on the computer of each chat-participant.

4. column
   part of the screenshot with time information (here every 10 seconds)

Special characters:

,	short break in an utterance
(.)	break (1 sec.)
(..)	break (2 sec.)
(…)	break (3 sec.)
(4 sec.)	duration of a break longer than 3 sec.
/ - \	rising, even, falling pitch
yes	*bold*: accentuated word
s i x t e e n	s p a c e d: spoken slowly

(“<”) and (“ > ”) two participants are talking both at the same time, for example:

8 S2: < plus 80 is 140/
9 S1: < 140\ ok

10

10. THE IMPORTANCE OF ABDUCTIVE REASONING IN MATHEMATICAL PROBLEM SOLVING

ABSTRACT

Charles Sanders Peirce (1839–1914) made a distinction between formal and informal reasoning, and argued that the formal reasoning processes of induction and deduction were not sufficient to explain those instances when individuals entertain new ideas to explain surprising facts. Peirce asserted the existence of another kind of reasoning, abduction, through which the individual generates a novel hypothesis to account for or explain surprising facts under consideration. The hypothesis represents an initial explanation that is both plausible, in the sense that it is the best explanation under the circumstances, and also provisional in the sense that it is open to further exploration. While research in mathematics learning has acknowledged the importance of hypothetical reasoning, few studies have identified the prominent role that Peirces's theory of abductive reasoning may play in problem solving, and fewer still have acknowledged how we as educators might help nurture and support abductions that our students make. This chapter addresses two key questions. (1) Why is it important that our students be able to make abductions when they solve mathematics problems? (2) How should educators help students develop reasoning habits that include abductive reasoning?

INTRODUCTION

Accounts of mathematics learning have long acknowledged the need for learners to develop autonomous cognitive activity, with particular emphasis on the learner's ability to initiate and sustain productive patterns of reasoning in mathematical problem solving situations (Burton, 1984; Cobb, 1988; NCTM, 2000; Schoenfeld, 1985). Nevertheless, explanations of problem solving have often focused on the application of objective strategies and processes, providing little explanation of the subjective actions solvers often generate prior to introducing formal algorithmic procedures into their actions. For example, cognitive models of problem solving (Reed, 1999), while useful in providing microscopic analyses of cognitive processes, have been challenged because "they fail to recognize the need to place cognitive functioning in a broader perspective that takes into account aspects such as affect, motivation, attitudes, beliefs and intuitions, as well as social and cultural factors" (Verschaffel & Greer, 2003, p. 62). In particular, they seldom address aspects of the

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 209–225.

solver's idiosyncratic reasoning activity such as the solver's selfgeneration of novel hypotheses, intuitions, and conjectures, even though these processes have been documented as crucial tools through which mathematicians ply their craft and thus are goals in the teaching of mathematics (Anderson, 1995; Burton, 1984; Carlson & Bloom, 2005; Mason, 1995; National Council of Teachers of Mathematics, 2000; Schoenfeld, 1985). Moreover, several researchers have documented that subjective actions play an important role in mathematics learning and have called for additional studies to examine the novel actions of learners (Cai, Moyer, & Laughlin, 1998; Cifarelli, 1998; Mason, 1995; Reid, 2003; Rivera, 2008; Sáenz-Ludlow & Walgamuth, 1998).

The chapter begins by developing a rationale for how Peirce's theory can be considered to examine problem solving processes. The second part of the chapter summarizes the mathematics education research that has been conducted on abduction. The third part examines the episodes of a college student solving a mathematics problem that involved a visual array, documenting and explaining the important role that abduction played in her solution. The fourth part discusses instructional implications for mathematics education.

ABDUCTIVE REASONING IN MATHEMATICS EDUCATION

Charles Sanders Peirce (1839–1914) made a distinction between formal and informal reasoning, and argued that the formal reasoning processes of induction and deduction were not sufficient to explain those instances when individuals entertain new ideas to explain surprising facts. Peirce asserted the existence of another kind of reasoning, *abduction*, through which the individual generates a novel hypothesis to account for surprising facts under consideration (Fann, 1970). The hypothesis represents an initial explanation that is both plausible, in the sense that it is the best explanation under the circumstances, and also provisional in the sense that it is open to further exploration. In contrast, Peirce viewed *deduction* as a process that explicates and clarifies hypotheses, deducing from them the necessary consequences; and *induction* as a process through which hypotheses are explored and tested for their explanatory merit and usefulness (CP 7.202–207; CP 8.209).[1] According to Peirce, abduction is the only logical operation which introduces new ideas (CP 5.171).

Peirce's theory of hypothesisbased reasoning is helpful to explain how learners develop plausible explanations to address 'surprising situations' they find themselves faced with. This chapter thus takes to heart Cobb's (1988) assertion that solvers actively construct new knowledge in problem solving situations when "their current knowledge results in obstacles, contradictions, or surprises" (Cobb, 1988, p. 92). Hence, genuine problem solving situations can be viewed as opportunities for problem solvers to reason abductively as they generate problem solutions.

The view that abduction may play an important role in mathematics learning and problem solving is not new. Abduction has been mentioned within various theoretical perspectives. For example, von Glasersfeld (1998) described abductions as accommodations that help stimulate and structure the learner's novel actions. According to von Glasersfeld, "abduction appears in accommodations of action schemes on the sensorimotor level as well as in subsequent levels of concrete and formal mental operations", calling them "the mainspring of creativity" (p. 9). Hence, a focus on the learner's abductions in problem solving situations may help provide an explanation for the formation and modification of the learner's schemes.

The idea that the solution of a problem may involve hypothesisbased reasoning of the type theorized by Peirce is useful if one adopts a constructivist broadbased view of problem solving in which solvers continually buildup their mathematical knowledge. For example, while solving a problem, the solver might experience an unanticipated difficulty that requires further reformulation of the original problem. Silver referred to this process as *withinsolution* problem posing (Silver, 1994) where the essence of the problem, as viewed through the eyes of the solver, has undergone a change and must be reformulated in order for the solver to proceed. The solver may reformulate the original problem as a collection of several 'smaller problems' that can be addressed and solved individually, and then organize his/her actions accordingly 'to break the problem up'. In this example, the solver's reformulation derives from their changing perceptions of what is problematic and awareness of the need to reorganize their goals and purposes for action. Hence, the reformulation indicates a plausible yet provisional action on the part of the solver to solve the original problem. If the solver's reformulation is hypothesized-based and has as its goal the explanation of some aspect of the problematic that requires further investigation, then the reformulation may involve abductive reasoning.

The work of Polya (1945) is consistent with the view that problem solvers may engage in abductive or hypothesis-based reasoning while in the course of solving a problem. Specifically, Polya identified heuristic reasoning as "reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" (Polya, 1945, p. 113). Further, Polya cited the usefulness of varying the problem when solvers fail to achieve progress towards their goals because the solvers' consideration of new questions serves to "unfold untried possibilities of contact with our previous knowledge" (Polya, 1945, p. 210). Hence, solvers who engage in hypothesis-based reasoning are: (1) cautious in their reflections about appropriate courses of action to carry out; (2) always looking to monitor the usefulness of the activity they plan to carry out; and, (3) willing to adopt a new perspective of the problem situation when their progress is impeded.

A good example of students demonstrating abductive reasoning in solving a problem is found in Reid (2003). Reid illustrated how two students, Jason and Sofia, solved the Handshake Problem (determine the number of handshakes exchanged from among n individuals) by reformulating the problem to examine a particular case. From this particular case, Jason then hypothesized a rule to solve the general case. Specifically, the students solved the problem for the case of six people, first using a diagram to count the number of handshakes (15) and then finding that they could get the solution by summing the numbers from 1 through 5 (Figure 1).

$1 + 2 + 3 + 4 + 5 = 15$ handshakes

Figure 1. The students' diagrams (adapted from Reid, 2003)

Then Jason and Sofia tried to solve the problem for N=26 people. Jason got the correct answer of 325 using his calculator to compute the sum 1 + 2 + …+ 24 + 25 = 325. Sofia claimed to "know an easier way", noticing that the sum could be computed more efficiently by first grouping numbers that sum to 26, and made a diagram to sum the numbers (Figure 2).

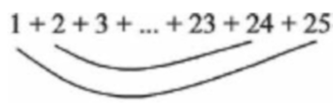


Figure 2: Sofia's grouping strategy

By grouping the numbers in this way, Sofia is demonstrating the Gauss method for summing n consecutive integers. Sofia reasoned that there were 13 such sums, an assertion that is incorrect. There are actually only 12 such sums, totalling 312, and a middle number of 13, so that the total sum is 325.

Sofia: So it is. $26 * \frac{26}{2} = 338$. (Reid, 2003, p. 6)

Sofia's calculation, $26 * \frac{26}{2} = 338$ was incorrect. However, Jason focused on her result of 338, comparing it to what he knew to be the correct answer, 325.

Jason: That can't be right. But you were close.

Jason: Maybe it's the number times half the number, hmm subtract half the number?

Sofia: You lost me.

Jason: Because that would work, 338 subtract 13, which is half of 26, is right.

Jason's use of the word "maybe" indicates the beginning of a hypothesis about a more general rule, ("Maybe it's the number times half the number, hmm subtract half the number?"). His use of the word "because" suggest the beginning of an explanation of why Sofia's calculation was close ("Because that would work, 338 subtract 13, which is half of 26, is right.")

According to Reid, Jason used abductive reasoning to arrive at the general rule

$$h(n) = n \times \left(\frac{n}{2}\right) - \frac{n}{2} \tag{1}$$

> [The number of handshakes is] the number [of people] times half the number, subtract half the number. (Reid, 2003, p. 6)

From the specific case:

> Because that would work, [the number of handshakes for 26 people is] 338 subtract 13, which is half of 26, is right. (Reid, 2003, p. 6)

In other words, Jason hypothesized a general rule that helped explain how Sofia's result was "close". He verified the rule in the specific case (by revising Sofia's calculation accordingly) and then tested the rule for other cases.

The preceding example, while showing that abduction may play an important role in problem solving situations, also indicates the intricacies of assessing abductions as examples that fit with Peirce's definition, a point that has been echoed by Mason (1995). According to Reid, the difficulty lies in the fact that Peirce focused on different aspects of abduction at different times in his writings. Hence, trying to identify precisely the particular components to Peirce's theory can be challenging. For the example provided, "The abduction is used (as the later Peirce would suggest) to explore (in finding a formula) and to explain (why Sofia's method gave an answer that was close)" (Reid, 2003, p. 6).

The following section will elaborate on these challenges and also summarizes the various ways that abduction has been interpreted by researchers in mathematics education. This discussion will help provide further context and rationale for studying the role of abduction in mathematical problem solving.

STUDIES OF ABDUCTION IN MATHEMATICS EDUCATION RESEARCH

Reid (2003) examined the writings of Peirce and noted how he emphasized different aspects of abduction at different times. Reid found that Peirce focused on the logical form of abduction in his earlier writings (CP 2.508; 2.623), emphasizing syllogisms and the role of characters of specific cases and classes to summarize the process. Reid then documented how in his later writings (CP 5.197), Peirce emphasized abductive reasoning in terms of the purposes and needs satisfied by the reasoning, thereby providing a more elaborate description of how abductions, though provisional, explain the surprising facts under consideration. The two characterizations of abduction proposed by Reid (2003) provide a useful lens through which to view the research that has been conducted.

Some of the studies of abduction within in geometry microworlds such as Cabri and Geometer Sketchpad (Arzarello, Olivero, Paola, & Robutti, 2002; Baccaglini-Frank, 2009) exemplify the first category of abduction as described by Reid. These studies focus on the logical form of abduction, considering abduction as a logical modality that supports the development of conjectures (Hoffman, 1999).[2] For example, Arzarello et al. (2002) examined dragging practices in the Cabri geometry environment and how, through continued feedback, they support the solver's emerging conjectures about the problem being solved. In this context, abduction is viewed as a logical operation that mediates a hierarchy of various dragging routines and thus "rules the transition" in cognitive focus that occurs when the solver moves between actual experiences (exploringconjecturing) and emerging theoretical ideas (proving results) (p. 67). While Arzarello et al. focused their attention on subjects' use of dragging schemes during the development of conjectures, Baccaglini-Frank (2009) documented how the subjects' use of particular dragging schemes induced patterns of abductive reasoning, thus suggesting a source of how abductions originate in the Cabri environment.

Studies that fit Reid's second category include those that focus on the structure of abductions and its role in inquirybased activity (Rivera, 2008; Ferrando, 2006). For example, Rivera (2008) characterized *complete* abductions as those hypotheses that undergo a series of developmental transformations that eventually result in generalized rules. Similarly, Ferrando (2006) characterized students' learning of calculus concepts in terms of abductive cycles of reasoning.

This second set of studies appear more useful to interpreting Peirce's theory to examine problem solving since they focus on how individuals form and transform their actions as needed while solving a problem. In particular, the abduction is viewed as a source for generating and organizing the exploration that follows. In this way the individual modifies his or her solution activity so that subsequent explorations become opportunities to develop new goals that reformulate the original problem. The individual can then express (or carry out) this reformulation to examine particular cases.

The comments above suggest that being aware of abduction in the context of problem solving enables a focus on the evolving structure of one's activity as he or she elaborates and extrapolates his or her ideas. Designing studies that focus the individual on these structuring processes would seem to provide a means to examine not only the interconnections among the individual's abductions but also among his or her inductions and deductions. According to Peirce, abductions interconnect with deductions and inductions. Once the explanatory hypothesis has been generated the individual must develop and formulate the hypothesis so that it can be tested (CP 7.202–207; CP 8.209). This is the deductive phase, which may involve slight modification of the original hypothesis through clarification and refinement, to render it testable (CP 7.202–207; CP 8.209). Once the hypothesis has been conformed, the hypothesis can then be tested through further action to determine its usefulness (CP 7.202–207; CP 8.209). This is the induction phase, the result of which places a degree of acceptance on the hypothesis.

Viewing a problem solver's actions under the lens of Peirce's theory of abduction may provide a useful framework with which we might be able to clarify and make better sense of the seemingly meandering actions that solvers sometimes demonstrate. However, we need to be careful in adopting only one point of view. There are many views of hypothesisbased reasoning, not all of which are compatible with Peirce's definition of abduction. For example, Magnani (2009) argues for the inclusion of nonexplanatory hypotheses in his definition of abduction. Hypotheses and conjectures made by individuals have always been acknowledged as important processes in problem solving. It is thus important to keep in mind Peirce's notion of abduction and its interconnections with inductive and deductive reasoning as a powerful theoretical lens through which we can view the problem solving activity of individuals in a coherent manner.

The following section examines the episodes of a college student solving a mathematics problem that involved a visual array of numbers. The analysis focused on the student's solution activity from initial problem formulation through eventual solution, highlighting episodes where she appeared to demonstrate abductive reasoning.

PROBLEM SOLVING INTERVIEWS

Sarah came from a graduate class in Mathematics Education taught by the researcher, at a southern university in the United States. Observing college students solving mathematics problems has proven to be an effective way of modelling the processes of problem solving (Carlson & Bloom, 2005; Cifarelli, 1998; Schoenfeld, 1985). In addition, studying the problem solving of graduate students can be useful in explaining a developmental range of actions (Carlson & Bloom, 2005; Cifarelli & Cai, 2005). Observing such solution activity is important to capture in view of the broad range of processes that appear to encompass abductive reasoning.

Sarah was interviewed by the researcher on 3 occasions during the semester. During the interviews, she solved a variety of word problems while 'thinking aloud'. Sarah worked individually in solving the problems and was given as much time as she wished to complete the tasks. Interviews were videotaped for subsequent analysis.

Sarah's Solution of the Number Array Task

During the second interview, Sarah solved the Number Array task (Figure 3). The Number Array task is discussed extensively in Becker and Shimada (1997), including a detailed description of typical patterns students will see in the array. Samples of the various mathematical relationships students typically construct are provided in Table 1.

Find as many relationships as possible among the numbers

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 3. Number array task

Table 1. Samples of relationships constructed by students solving the number array task

1. All numbers on the left-to-right diagonal are squares (1, 4, 9., …, 100)
Relationships about the spatial arrangement of numbers.
2. The numbers are symmetrically arranged about the left-to-right diagonal numbers
Relationships about the sums of numbers.
3. Sum of numbers in any row is a multiple of 55
4. Sum of two numbers in a row or column that located symmetrically about a pivot number is two times the pivot number.
Relationships about the products of numbers.
5. The number in the m^{th} row and n^{th} column is $m \times n$
6. For any rectangle or square array, the products of the end numbers are equal.
7. For any square array, the products of the numbers on the two diagonals are equal

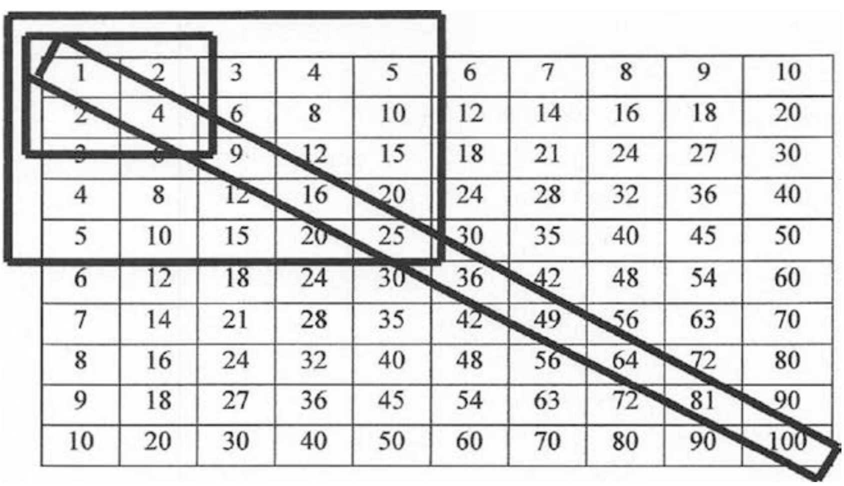
Sarah began by focusing on simple relationships that had to do with the symmetry of the numbers. Sarah explored several of the fairly simple patterns such as those drawing from the symmetry of the arrangement of numbers, and simple arithmetic relationships. For example, she noticed that any entry in the table can be found by multiplying the row number by the column number, relationship #5 in Table 1 (e.g., 12=3×4). In addition, in any 2×2 block, the product of the diagonal entries are equal, and that the result holds true for any square block, N×N, N>2.

After identifying several additional simple patterns, she focused on finding more mathematically sophisticated relationships. Episodes of her verbal statements are presented to refer to and support the assertions made by the researcher. (Italicized comments within the episodes indicate inferences of the researcher regarding the nonverbal gestures made by the student.)

> *Sarah:* Let's see … (long reflection) … I was wondering about those square numbers on the diagonal going from left to right (points to the sequence 1, 4, 9, 16, …, 81, 100). They seem to relate to the dimension of the square blocks, … I don't know, … Maybe they relate to the sums of these blocks I had earlier (points to the 2×2, 3×3, 4×4 blocks). So, let's check it.

Sarah proceeded to examine the sum of the entries of each NxN block that contained the square numbers on the diagonal (Figure 4). From her analysis she developed an informal method to find the sums of the entries of the NxN blocks going down the main diagonal (Figure 5).

> *Sarah:* So, for a 1×1, I get a sum of 1 *(points to the sequence of square numbers on the diagonal)*. For a 2×2 *(points to block [1, 2 : 2, 4])*,[3] I get a sum of 9 … but what happened to 4? It appears to have been *skipped*! *(several seconds of reflection)*. Okay, let me try this, I will write down the sequence of squares of all numbers, all in a row *(writes the following sequence of square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225)*. So, the first number, 1, tells the sum of the very first matrix, a 1×1. And the first 2×2 has a sum of 9. …. So, I *skipped* over 4 to get the next sum *(crosses out the 4 in the sequence)*, going from 1×1 to a 2×2, a sum of 9. The 4 gets *skipped*? Interesting!



1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 4. Examples of 2×2 and 5×5 blocks on the diagonal

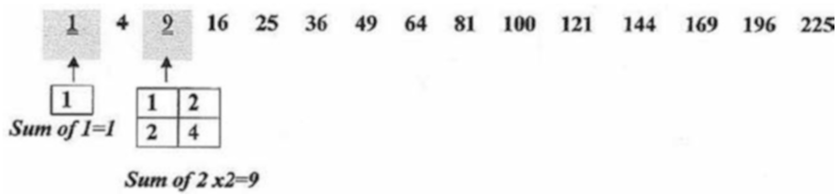


Figure 5. Sarah's skipping to find sums of block entries

With her actions, Sarah sensed a new problem to solve – she thinks that there could be a relationship between the sequence of square numbers on the diagonal of the array and the successive sums of the entries of N×N blocks. Sarah was able to continue her 'skip' method to generate the sequence of sums of the entries of all NxN blocks.

Sarah: So, for the first 3×3 *(points to [1, 2, 3 : 2, 4, 6 : 3, 6, 9])*, I already did this over here, so it is 36. So, in going from the 1×1 to the 2×2 to the 3×3, we go from 1, to 9, to 36 – so we skipped over the 16 and the 25 *(she crosses out the 16 and 25 in the square number sequence)*, a skip of 2 in this sequence!! So, okay, if this is true, then it looks like we will skip over the next 3 square numbers, and that should tell us the sum for a 4×4 should be equal to 100 *(crosses out the 49, 64, 81 in the square number sequence)* – that is what I have over here!! Cool! So, for a 5×5, we skip over the next 4 numbers in the sequence, *(points to the sequence 121, 144, 169, 196)* and get 225 – yes, I got that one earlier for the 5×5. (Figure 6)

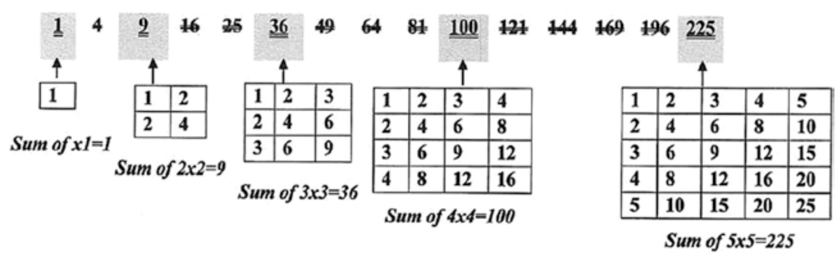


Figure 6. Sarah's skipping to find sums of block entries

Sarah then looked to make sense of her method with some further exploration.

Sarah: I wonder why this skipping works? Let's see it another way, for the 6×6, we add the entries in the rows to get 21+42+…+126 = 21(1+2+3+4+5+6) = 21×21 = 441. Do we get 441 by skipping the next 5 in the square sequence? *(Sarah extended her original sequence beyond 225, crossed out* *the corresponding 'skips,' and got a result of 441 as the next number in the sequence)* (Figure 7). But also, I notice that 21 over here *(points to the factored form 21• (1+2+3+4+5+6))* is the sum of the first 6 numbers in that first row. Yes!

In the last statement she makes, Sarah noticed that the sum of the row entries is the sum of the numbers from 1 through 6. She then makes a projection in her thinking to a general case:

Sarah: So to find the sum of these N×N blocks, I bet you just need to look at the sum of 1 to N and then square that total to get the sum.

This is the first evidence that Sarah had made an abduction, that she had hypothesized the calculation she had carried out for the 6×6 block could be generalized to N×N blocks. However, the abduction appeared to have its source in her earlier comments:

Sarah: Let's see it another way, for the 6×6, we add the entries in the rows to get 21+42+…+126 = 21(1+2+3+4+5+6) = 21×21 = 441. Do we get 441 by skipping the next 5 numbers in the square sequence?

So Sarah had a sense of the general in the particular and her hypothesis about summing the numbers from 1 to N resulted from her deductions made by reflecting on the results of her factoring of the sums:

Sarah: I notice that 21 *(points to product 21• (1+2+3+4+5+6))* is the sum of the first 6 numbers in that first row. Yes!

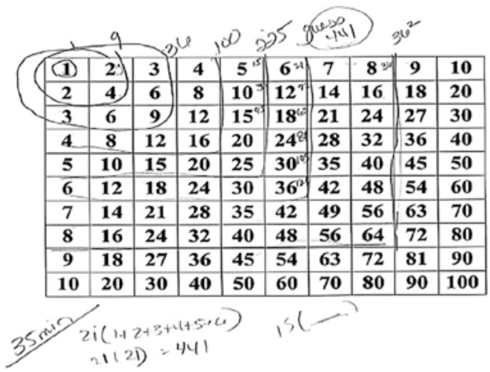


Figure 7. Sarah's diagram of her computation of sums in a 6×6 block

This enabled Sarah to re-state her hypothesis: In order to find the sum of entries in an N×N block, she needed to sum the numbers from 1 through N, and square the result. Sarah then looked to test her hypothesis on an 8×8 block (Figure 8).

1	2	3	4	5	6	7	8	36
2	4	6	8	10	12	14	16	
3	6	9	12	15	18	21	24	
4	8	12	16	20	24	28	32	
5	10	15	20	25	30	35	40	
6	12	18	24	30	36	42	48	
7	14	21	28	35	42	49	56	
8	16	24	32	40	48	56	64	

Skipping in the sequence: Finding the sum for the 8x8 block

225	~~256~~	~~289~~	~~324~~	~~361~~	~~400~~	441	**Skip next 6 for 7x7 case**	784	**Skip next 7 for 8x8 case**	1296
15^2	16^2	17^2	18^2	19^2	20^2	21^2		28^2		36^2

Figure 8. Sarah's computation of the sum for the 8×8 block

Sarah: Let's try a big one, say 8×8. So, I guess that it would be …. 1+2+…+8 = 36, I don't know why I am adding these individual numbers since I know that the sum is (8×9)/2, and then I take 36^2? So that comes out to be … 1296. And does it check with my skipping over here? Let's see, so for 8×8, I first skip 6 over 21 to get 28^2 for 7×7, and then skip 7 more to get the one for 8×8, … so 7 more is 35, and the next one is 36! So my algorithm seems to work! The algorithm is pretty efficient for larger numbers, beyond all of these *(pointing to the array)* – how about a 100×100 grid! – But I thought that the skipping relationship was pretty cool!

DISCUSSION

This chapter addressed two questions. (1) Why is it important that our students be able to make abductions when they solve problems? (2) How should educators help students develop reasoning habits that include abductive reasoning?

The results help provide an answer to the first question. Sarah developed an informal method to find the sum of entries in NxN blocks and then transformed her method into a more general method that both explained the results for the particular cases she had solved and also could be used to solve the problem for larger values of N extending beyond the array. Sarah's solution activity is important for the following reasons. First, Sarah's development of her informal method to compute the sums made use of a metaphor (Saenz-Ludlow, 2004), 'skipping', that named and explained her method for finding sums of entries in the various blocks, by 'skipping' through a sequence of square numbers. With these idiosyncratic actions, she had constructed an informal method. She verified that the method appeared to work for other cases that could be generated from the array. This finding is consistent with research that identifies informal methods as playing a prominent role in the development of formal algorithms (Cai, Moyer, & McLaughlin, 1998; Sáenz-Ludlow, 1995).

Second, abduction played a prominent role in her actions, and came about from her goal to explain why the 'skipping' method worked for computing the particular sums. With her abduction Sarah hypothesized a general method (rule), that then explained not only the particular cases within the array that she had already verified with 'skipping', but that could be used to compute cases that went beyond the actual array (e.g., "how about a 100×100?"). Specifically, her subsequent development of the general method involved her first making a subtle shift in her attention from validation and verification of the 'skipping' method for blocks of dimension 2×2, 3×3 and 4×4, to efficacy considerations (why it appeared to work for the cases she generated). Exploring issues of efficacy for one's problem solving actions is an important though under-utilized activity in most instructional settings. In Sarah's case, this exploration with a view to explain the usefulness of her actions made possible her abduction. Her goal to examine her action in a new light provided for her an opportunity to unfold the process, and relate her informal 'skipping' method to operations on the row and column numbers. Her re-writing of the sum of row entries into factored form $21+42+\ldots+126 = 21(1+2+3+4+5+6) = 21\times21 = 441$ appeared to be the first indication of her abduction, hypothesizing that the results of applying her 'skipping' could alternatively be found by operating on the row and column numbers. Her reflection on the factored form to conclude that the sum of the numbers in parentheses represented the sum of the column numbers in the particular row indicated that she had made a deduction because these actions led her to state the hypothesis in a form that made possible further testing ("I bet you just need to look at the sum of 1 to N and then square that total to get the sum."). In this way, she was able to generalize her method from skipping within a simple sequence to a formal algorithm that was more efficient for finding the sums of entries in N×N blocks beyond the 10×10 array. She proceeded to test her hypothesis (the rule) for cases she could verify (with 'skipping') within the 10 × 10 array.

Sarah's abduction appeared to be an example of a creative abduction (cf. Sáenz-Ludlow chapter on abduction in proving, this volume; Eco, 1983) for the following reasons. First, her abduction of the general rule did not draw from consideration from among several equally probable hypotheses; rather, her hypothesis drew from her creative actions performed by reformulating her original problem of finding sums of entries blocks, to determining why the particular 'skipping' method worked. Second, while Sarah's abduction was based on her stated goal to determine why the "skipping' worked, she was quick to value the efficiency of the general rule over the 'skipping' method. Sarah's consideration of efficiency in making her hypothesis would appear to be an example of a type of 'aesthetic value' that is a basis on which creative abductions are formulated (Eco, 1983).

TEACHING AND LEARNING IMPLICATIONS

The results do not suggest an easy answer to the second question and must be treated with care in making particular recommendations for the teaching and learning of mathematics in K-12. It may be useful to reformulate the question as two separate related questions: 1. Can abductive reasoning be taught explicitly? and 2. Do certain kinds of tasks induce the solver's use of abductive reasoning?

Can Abductive Reasoning be Taught Explicitly?

The question of whether abductive reasoning can be taught explicitly is not easy to answer. Sarah demonstrated conceptual growth in her problem solving because she was able to selfgenerate and selfregulate most all of her solution activity with little prompting, skills that many students in K-12 find difficult to develop. Moreover, as Sinclair (2006) has remarked, abductions, with their air of uncertainty, can be risky for students to make in K-12 mathematics classrooms because it leaves them vulnerable to ridicule by peers (N. Sinclair, personal communication, 2006). However, there are some recommendations that might be useful.

Promote reflection and discussion in classroom discourse. Abductions can occur only if the student has a secure sense of his or her role as a problem solver and is not afraid to express their ideas even if they may be incorrect. In order for students to become secure in their role as a mathematical problem solver, they must be provided with ample problem-solving opportunities that enable them to explore their understandings. Ferrando (2006) voiced the concern that students are often unwilling to explore the mathematics problems they are faced with and more often than not, give up working on a problem if they do not see an immediate strategy to pursue. Hence, we must carefully listen to students and observe what they do rather than conduct classroom activities based on our expectations of what we think they will say and do. Due to large class sizes, it is difficult for teachers to engage in the lengthy discussions represented in the interviews conducted in the study. However, more one-on-one communication can be facilitated using small group problem solving that invites students to share their thoughts about both the decisions they make and difficulties they face while solving problems. This in turn provides teachers with opportunities to respond to the problems and questions that students formulate.

Encourage proactive agency in problem solving. Students must not only be able to develop ideas about the problems they face, they must be willing to present and defend them in classroom discussions. Sarah viewed herself as in control and aggressively switched course whenever unexpected problems arose. Instructional activities that allow students opportunities to share and defend their ideas for solving particular problems prior to actual solving help develop self-advocacy in students and contribute to a proactive sense of agency.

Do certain kinds of tasks induce abductive reasoning? While Sarah performed well with the Number Array task and all of the other non-traditional task that she solved in other interviews, we must be careful in concluding that abductions can be stimulated through the use of particular tasks and problems. For example, one approach that has gained prominence in recent years involves the use of 'open ended' tasks to stimulate problem posing and solving (Becker & Shimada, 1997). The results suggest that our focus should be on the students' mathematical thinking and learning, and helping them to open up and explore their own interpretations of mathematical situations. The ideas generated by Sarah were their own, self-generated to help her 'make sense' of the situations she faced, and seen by them as plausible explanations of the problems. While it is true that Sarah's solution of the Number Array task involved problem posing and solving in an unfamiliar context, her initial ideas evolved into conceptually rich ideas that included new problem formulations and re-formulations, and conjectures about how potential solution activity would work out. In this way, she developed novel structures for her solution actions as she saw fit. In other words, the external structure of the task was less important for Sarah than her evolving of structure within her actions. These results suggest that while there can be a degree of novelty designed into the tasks we give students, the greater need is for mathematics educators to broaden their view of problem solving as learning opportunities and incorporate problem solving tasks that provide abundant posing and solving opportunities to our students so that they stretch and broaden their understandings as they solve problems.

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This chapter is dedicated to the loving memory of Margaret A. Cifarelli who was a constant source of inspiration to me during our life together and who, most every day, put the smiles in my heart…

NOTES

1. Some of the Citations of Peirce that appear in this chapter are taken from *The Collected Paper*, Volumes 1–6, edited by Charles Hartshorne and Paul Weiss, Cambridge, Massachusetts, 1931–1935; and volumes 7–8 edited by Arthur Burks, Cambridge, Massachusetts, 1958. The standard format for citing Peirce has been used. For example, CP 5.172 refers to Volume 5 of *The Collected Papers*, paragraph 172.
2. Since Hoffman (1999) argued that there is no logic of abduction in the sense of syllogistic logic when it comes to the generation of hypotheses, and that "logic" should be understood in the broader sense of "methodology", these studies might be better described as studies that involve the methodological understanding of abduction.

[3] A bracket notation is used to list the top to bottom rows of the block being considered. For example, the 2×2 is indicated by the sequence [1, 2 : 2, 4] and a 3×3 block is indicated by the sequence [1, 2, 3 : 2, 4, 6 : 3, 6, 9].

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