Advanced Business Mathematics and Statistics

NCBA

Ram Krishna Ghosh
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ADVANCED
BUSINESS MATHEMATICS
AND STATISTICS

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ADVANCED BUSINESS MATHEMATICS AND STATISTICS

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Group C: Statistics

Preface

The contents of this book consist of three Groups A, B, and C. Group A [Algebra (15 marks)] contains Matrix Algebra including Determinants and Quadratic Forms. Group B [Calculus (35 marks)] contains Functions, Limit and Continuity, Differentiation, Applications of Derivative, Rolle's Theorem including Taylor's and Maclaurin's Theorems, and Integration including Definite Integrals. Group C [Statistics (50 marks)] contains Introduction (Definition, Importance and Scope, Types and Sources of Data, Limitations), Analysis of Univariate Data, Analysis of Bivariate Data, Set Theory, Probability, Index Numbers, and Time Series Analysis.

We have tried our best to make the book more attractive to our beloved students and their esteemed teachers by giving various illustrations, diagrams, and a large number of typical solved problems.

The book is written in such a manner as Commerce students may not find any difficulty in understanding the subject-matter, and also in solving problems given in the exercises. We express our gratitude to Shri Dipan Roy for pointing out various misprints in the previous edition of the book. We also convey our sincere thanks and gratitude to the respected teachers of Vidyasagar University for their cordial response to the previous editions of the book.

We shall consider our labour amply rewarded if the book is of considerable help to those for whom it is intended.

Finally, we earnestly thank Shri Amitabha Sen, Director, New Central Book Agency (P) Ltd, for the patient cooperation he has extended to us in bringing out this treatise within a short period.

We look forward to receiving the cordial and earnest cooperation of the respected teachers, educationists, and our beloved students for improvement of the book.

Authors

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Important Formulae and Results

ALGEBRA

1. Determinants

(i) The value of a second-order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

(ii) The value of a third-order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{or,} \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\text{or,} \quad -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{or,} \quad -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad \text{etc.}$$

(iii) The cofactors A_1, B_1, C_1, A_2, B_2, etc. of a_1, b_1, c_1, a_2, b_2, etc. in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ are respectively,

$$+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, +\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, +\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \text{etc.}$$

(iv) **Properties**

(a) The value of a determinant remains unaltered by changing *its rows into columns and columns into rows.*

(b) If two adjacent rows (or columns) of a determinant are interchanged, the numerical value of the determinant remains the same, but its sign is changed.

(c) If two rows (or columns) of a determinant are *identical*, the value of the determinant is zero.

(d) If every element in a row (or a column) of a determinant is multiplied by the same constant k, then the original determinant is multiplied by that constant k.

(e) If A_1, B_1, C_1, etc. are the respective cofactors of the elements a_1, b_1, c_1, etc. in

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } \begin{aligned} a_iA_j + b_iB_j + c_iC_j &= D, \text{ when } i = j(=1,2,3) \\ &= 0, \text{ when } i \neq j \end{aligned}$$

Also $a_1A_1 + a_2A_2 + a_3A_3 = D, b_1B_1 + b_2B_2 + b_3B_3 = D, a_1B_1 + a_2B_2 + a_3B_3 = 0$, etc.
The adjoint or adjugate of a given determinant D is the determinant whose elements are cofactors of the corresponding elements of D and is denoted by D'. Here $D' = D^2$.

(f) $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(g) The value of a determinant remains unaltered if each element of a row (or column) is increased or decreased by the same multiple of the corresponding element of another row (or column), i.e.,

$$\begin{vmatrix} a_1+ma_2+na_3 & b_1+mb_2+nb_3 & c_1+mc_2+nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(h) If two rows (or columns) of a determinant become identical when $x = a$, then $(x-a)$ *is a factor* of the determinant.
If r rows (or columns) of a determinant become identical when $x = a$, then $(x-a)^{r-1}$ is a factor of the determinant.

(v) Product of Two Determinants

(a) If $D_1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$, then $D_1D_2 = \begin{vmatrix} ap+bq & ar+bs \\ cp+dq & cr+ds \end{vmatrix}$.

[Row-by-row multiplication is performed]

(b) If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$, then

$$D_1D_2 = \begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_1x_2+b_1y_2+c_1z_2 & a_1x_3+b_1y_3+c_1z_3 \\ a_2x_1+b_2y_1+c_2z_1 & a_2x_2+b_2y_2+c_2z_2 & a_2x_3+b_2y_3+c_2z_3 \\ a_3x_1+b_3y_1+c_3z_1 & a_3x_2+b_3y_2+c_3z_2 & a_3x_3+b_3y_3+c_3z_3 \end{vmatrix}.$$

(vi) Cramer's Rule

(a) The solutions of $\left.\begin{aligned} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{aligned}\right\}$ are given by

$$\frac{x}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ where } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

(b) The solutions of $\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3\end{aligned}\right\}$ are given by

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}=\frac{y}{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}$$

or, $\dfrac{x}{D_1}=\dfrac{y}{D_2}=\dfrac{z}{D_3}=\dfrac{1}{D}$ (say),

where $D\neq 0$, which give

$$x=\frac{D_1}{D}, y=\frac{D_2}{D}, z=\frac{D_3}{D}.$$

2. Matrices

(i) A *matrix* is a rectangular array of numbers enclosed by a pair of brackets [ ] or ( ) subject to certain rules of operation.

(ii) Two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are said to be *equal* if, and only if, they are of the *same order* and $a_{ij}=b_{ij}$ for all i and j.

(iii) *Addition* and *subtraction of two matrices* $A=[a_{ij}]$ and $B=[b_{ij}]$ are defined if, and only if, they are of the *same order.* If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then $A+B=[a_{ij}+b_{ij}]_{m\times n}$ and $A-B=[a_{ij}-b_{ij}]_{m\times n}$.

(iv) A matrix A is said to be *singular* if $|A|$ or $\det A=0$, and *non-singular* if $|A|$ or $\det A\neq 0$.

(v) The transpose A^T or A' of a matrix $A=[a_{ij}]_{m\times n}$ is obtained from A by interchanging its rows and columns. The order of A^T is $n\times m$.

(vi) A square matrix $A=[a_{ij}]$ is said to be *symmetric* if $a_{ij}=a_{ji}$, i.e., if $A^T=A$, and *skew-symmetric* if $a_{ij}=-a_{ji}$, i.e., if $A^T=-A$.

(vii) If k be a scalar, then $k[a_{ij}]=[ka_{ij}]$.

(viii) The product AB of two matrices A and B is defined, if the number of columns in A is equal to the number of rows in B. If $A=[a_{ij}]_{m\times p}$ and $B=[b_{ij}]_{p\times n}$ be two matrices, then the product matrix AB is of order $m\times n$, whose element in the ith row and jth column is the sum of the products obtained by multiplying the elements of ith row of A by the corresponding elements of the jth column of B.

If

$$A=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix},$$

then

$$AB=\overrightarrow{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}\times\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\Bigg\downarrow=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}.$$

In general, $AB\neq BA$, and $CA=CB$ does not imply $A=B$; $AB=O$ does not imply that $A=O$ or $B=O$.

(ix) For a square matrix A, $AI = IA = A$, where I is the unit matrix of the same order.

(x) The *adjoint* (or *adjugate*) of a given square matrix is the transpose of the matrix whose elements are the cofactors of the elements in $|A|$ and is denoted by adj. A.

(xi) $A^{-1} = \frac{\text{adj.} A}{|A|}$ and A^{-1} exists when A is non-singular. Also, $A^{-1}A = AA^{-1} = I$.

(xii) A square matrix A of order n is said to be *orthogonal* if $A \cdot A^T = A^T \cdot A = I$, where I is the unit matrix of order n. In this case, $A^T = A^{-1}$.

(xiii) A matrix A is said to be of *rank* r if (a) there is at least one minor of A of order r, which is non-zero, and (b) all the minors of A of order $(r+1)$ or more are zero.

3. Quadratic Forms

(i) If each term of a polynomial in two or more variables has the *same degree* 2, then the polynomial is called a *quadratic form*.

(ii) A quadratic form $q \equiv ax^2 + 2hxy + by^2$ is (a) *positive definite* if, and only if, $a > 0$ and $ab - h^2 > 0$, (b) *negative definite* if, and only if, $a < 0$ and $ab - h^2 > 0$.

(iii) A quadratic form $q \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is (a) *positive definite* if, and only if, $a > 0, ab - h^2 > 0$ and $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} > 0$, and (b) *negative definite* if, and only if, $a < 0, ab - h^2 > 0$, and $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} < 0$.

CALCULUS

4. Function

If x and y be two real variables, i.e., $x, y \in R$, where R is the set of all real numbers, such that there exists a rule f defined by $f : X \to Y, x \in X$ and $y \in Y$, i.e., for every value of $x \in X$, we get a unique value of $y \in Y, X, Y$ being subset of R, then the rule f is called a *function* or f maps X into Y. In this case, we write $y = f(x)$. The set X is called the *domain of the function* and Y, the *codomain*. The set Y_1 of all values of $y = f(x)$ in Y is called the *range of the function*. Clearly, the range Y_1 is a subset of Y, i.e., $Y_1 \subseteq Y$.

5. Limit and Continuity

(i) R.H. limit = $\lim_{x \to a+} f(x)$ and L.H. limit = $\lim_{x \to a-} f(x)$. $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ both exist and are equal.

(ii) $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, if $a > 0$ and n is rational.

(iii) $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$.

(iv) $\lim_{x \to 0} \dfrac{1}{x} \log(1 + x) = 1$.

(v) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a \, (a > 0)$.

(vi) $\lim_{x\to\infty} x^n = 0$, if $-1 < x < 1$ and $x^n \to \infty$ as $n \to \infty$, if $|x| > 1$.

(vii) (a) $f(x)$ is *continuous* at $x = a$, if $f(a)$ exists and $\lim_{x\to a} f(x) = f(a)$;

(b) $f(x)$ is *continuous* at $x = a$, if $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$, where $f(a)$ exists;

(c) $f(x)$ is *continuous* at $x = a$, if $\lim_{h\to 0} f(a+h) = f(a)$.

6. Derivatives or Differential Coefficients

(i) *Derivative from first principle (or definition)*

(a) $f'(x)$ or $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

(b) $Rf'(x) = \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}$ and $Lf'(x) = \lim_{h\to 0-} \frac{f(x+h) - f(x)}{h}$. $f'(x)$ exists if both $Rf'(x)$ and $Lf'(x)$ exist and are equal.

(c) $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ and $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$, provided each limit exists.

(ii) *Standard Derivatives*

(a) $\frac{d}{dx}(x^n) = nx^{n-1}$ (b) $\frac{d}{dx}(x) = 1$

(c) $\frac{d}{dx}(e^x) = e^x$ (d) $\frac{d}{dx}(e^{mx}) = me^{mx}$

(e) $\frac{d}{dx}(a^x) = a^x \log_e a \, (a > 0)$ (f) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(g) $\frac{d}{dx}(a^{mx}) = ma^{mx} \log_e a \, (a > 0)$ (h) $\frac{d}{dx}(c) = 0$, where c is constant.

(iii) *Rules of Differentiation*

(a) $\frac{d}{dx}(Cu) = C \cdot \frac{du}{dx}$, where C is a constant and u is a function of x.

(b) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$, where u, v are functions of x.

(c) $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$. [Product Rule]

(d) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. [Quotient Rule]

(e) If $y = f(z)$ and $z = f(x)$, then $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$. [Chain Rule]

(f) $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ and $\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}$, where $\frac{dy}{dx} \neq 0$.

(g) If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, where $\frac{dx}{dt} \neq 0$.

(iv) *Second and Higher Order Derivatives*

The derivative of

$$f'(x) \text{ or } \frac{dy}{dx}$$

w.r.t. 'x' (if it exists) is the *second derivative* or *second-order derivative* of $y = f(x)$, and is denoted by

$$f''(x) \text{ or } \frac{d^2y}{dx^2} \text{ or } y_2.$$

The derivative of $f''(x)$ or $\frac{d^2y}{dx^2}$ w.r.t. 'x' is the *third derivative* or *third order derivative* of $f(x)$ and is denoted by

$$f'''(x) \text{ or } \frac{d^3y}{dx^3} \text{ or } y_3$$

and so on.

7. Partial Differentiation

(i) *The partial derivative* of $u = f(x,y)$ w.r.t. x is the derivative of u w.r.t. x, treating y as a constant, and is denoted by

$$\frac{\partial u}{\partial x} \text{ or } \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}.$$

(ii) The *partial derivative* of $u = f(x,y)$ w.r.t. y is the derivative of u w.r.t. y, treating x as a constant, and is denoted by

$$\frac{\partial u}{\partial y} \text{ or } \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}.$$

(iii) *Euler's Theorem:* If u is a homogeneous function of degree n in two variables x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u.$$

(iv) **Envelope:** The equation of the envelope of the family of straight lines $f(x,y,m) = 0$, where m is a parameter, is obtained by eliminating m between $f(x,y,m) = 0$ and $\frac{\partial f}{\partial m} = 0$.

APPLICATIONS OF DERIVATIVES

8. Geometrical Interpretation of Derivative

(i) The derivative $\frac{dy}{dx}$ or $f'(x)$ at $x = x_1$ represents the rate of change of $y = f(x)$ w.r.t. x at $x = x_1$.

(ii) Geometrically, $\frac{dy}{dx}$ represents the slope (or gradient) of the tangent to the curve $y = f(x)$ at the point (x,y), i.e., if ψ be the inclination of the tangent with the positive direction of the X-axis, then $\frac{dy}{dx} = \tan\psi$.

(iii) The differential of a function $y = f(x)$ is given by $dy = f'(x)dx$.

(iv) The equation of the tangent to the curve $y = f(x)$ at (x_1, y_1) is

$$y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1).$$

(v) The equation of the normal to the curve $y = f(x)$ at (x_1, y_1) is

$$(x - x_1) + (y - y_1)\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 0.$$

9. Maxima and Minima: Second Derivative Test

(i) If $f'(c) > 0$, then $f(x)$ is increasing at $x = c$, and if $f'(c) < 0$, then $f(x)$ is decreasing at $x = c$.

(ii) A function $f(x)$ has a maximum value at $x = c$, if $f'(c) = 0$ and $f''(c) < 0$.

(iii) A function $f(x)$ has a minimum value at $x = c$, if $f'(c) = 0$ and $f''(c) > 0$.

(iv) If $f'(c) = 0, f''(c) = 0$ and $f'''(c) \neq 0$, then $f(x)$ has neither a maximum nor a minimum at $x = c$. In this case, $x = c$ is a *point of inflexion.*

Cost Function. The cost function, denoted by $C(x)$, is the total cost of producing x units of an item by a firm (or a company). It consists of Fixed Cost and Variable Cost.

If F be the fixed cost and $V(x)$ be the variable cost of x units, then

$$C(x) = F + V(x).$$

Average Cost (AC) $= \dfrac{C(x)}{x} = \dfrac{\text{Total Cost}}{\text{Output}}$.

Average Variable Cost (AVC) $= \dfrac{V(x)}{x} = \dfrac{\text{Variable Cost}}{\text{Output}}$.

Revenue Function. The revenue function, denoted by $R(x)$, is the total amount of money generated from the sales of x units of an item by a firm (or a company).

If x units be sold at ₹ p per unit, then

$$R(x) = px, \text{ where } p > 0 \text{ and } x > 0.$$

Average Revenue (AR) $= \dfrac{R(x)}{x}$ = price per unit.

Profit Function. Profit function, denoted by $P(x)$, is the amount of money available to a firm (or a company) from the sale of its product of x units (say) after all costs have been deducted, and is defined by

$$P(x) = R(x) - C(x).$$

For *break-even point,* $P(x) = 0$ and $R(x) = C(x)$.

Marginal Cost. The marginal cost (MC) $= \dfrac{dC}{dx}$, where $C = C(x)$.

Marginal Revenue. The marginal revenue (MR) $= \dfrac{dR}{dx}$.

Curvature. The curvature (κ) of the curve $y = f(x)$ at any point $P(x, y)$ is given by

$$\kappa = \frac{y_2}{\left(1+y_1^2\right)^{3/2}}, \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

$$\text{or, } \kappa = \frac{x_2}{\left(1+x_1^2\right)^{3/2}}, \text{ where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}.$$

We shall consider numerical value of curvature. Radius of curvature $= \rho = \frac{1}{k}$.

10. Indefinite Integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c\,(n \neq -1)$ and $\int dx = x + c$.

(ii) $\int \frac{1}{x} dx = \log|x| + c$.

(iii) $\int e^{mx} dx = \frac{e^{mx}}{m} + c$ and $\int e^x dx = e^x + c$.

(iv) $\int a^{mx} dx = \frac{a^{mx}}{m \log_e a} + c$ and $\int a^x dx = \frac{a^x}{\log_e a} + c\ (a > 0)$.

(v) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$.

(vi) (a) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c;$

(b) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c;$

(c) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|x + \sqrt{x^2 \pm a^2}\right| + c.$

(vii) $\int u \cdot v dx = u \cdot \int v dx - \int \left\{\frac{du}{dx} \cdot \int v dx\right\} dx.$ [Integration by parts]

(viii) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c.$

(ix) $\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2+a^2}\right| + c.$

(x) $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2-a^2}\right| + c.$

11. Definite Integrals

(i) $\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$, where $nh = b-a$,

or, $\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$, where $nh = b-a$.

(ii) $\int_0^1 f(x)dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(rh)$, where $nh = 1$; here $a=0, b=1$.

(iii) If $f(x) = F'(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

12. Definite Integral as an Area

(i) Area bounded by the curve $y = f(x)$, the X-axis, and two ordinates $x = a$ and $x = b$ is

$$\int_a^b y\,dx = \int_a^b f(x)dx.$$

(ii) Area bounded by the curve $x = f(y)$, the Y-axis, and $y = c$ and $y = d$ is

$$\int_c^d x\,dy = \int_c^d f(y)dy.$$

(iii) Area between two curves $y = f(x)$ and $y = \phi(x)$ bounded by $x = a$ and $x = b$ is

$$\int_a^b (y_1 - y_2)dx, \text{ where } y_1 = f(x) \text{ and } y_2 = \phi(x).$$

Here $x = a$ and $x = b$ are the x-coordinates of the points of intersection of $y = f(x)$ and $y = \phi(x)$.

STATISTICS

13. Measures of Central Tendency

(i) Simple AM $(\bar{x}) = \dfrac{\Sigma x}{n}$.

(ii) Weighted AM $(\bar{x}) = \dfrac{\Sigma wx}{\Sigma w}$ or, $\dfrac{\Sigma fx}{\Sigma f} = \dfrac{\Sigma fx}{N}$, where $N = \Sigma f$.

(iii) In case of a frequency distribution, AM $(\bar{x}) = \dfrac{\Sigma fx}{N}$, where $N = \Sigma f$.

(iv) Formulae of AM $(\bar{x})$ by short-cut method: (a) $\bar{x} = A + \dfrac{\Sigma d}{n}$; (b) $\bar{x} = A + \dfrac{\Sigma fd}{N}$; (c) $\bar{x} = A + \dfrac{\Sigma fd'}{N} \times i$, where A = Assumed Mean, $d = x - A$ = deviation, and $d' = \dfrac{x-A}{i}$ = step deviation, and i = HCF of the widths of the class intervals.

(v) Simple Geometric Mean $= G = (x_1 \times x_2 \times x_3 \times \cdots \times x_n)^{\frac{1}{n}}$ or, $\log G = \dfrac{\Sigma \log x}{n}$, or, $G = \text{antilog}\left(\dfrac{\Sigma \log x}{n}\right)$.

Weighted Geometric Mean $= G = \left[x_1^{f_1}.x_2^{f_2}.\cdots.x_n^{f_n}\right]^{\frac{1}{N}}$, where $N = \Sigma f$, or, $\log G = \dfrac{1}{N}\Sigma f \log x$,

or, $G = \text{Antilog}\left(\dfrac{1}{N}\Sigma f \log x\right)$.

(vi) (a) Simple Harmonic Mean $= H = \dfrac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n}} = \dfrac{n}{\Sigma\frac{1}{x}}$

(b) Weighted Harmonic Mean $= H = \dfrac{N}{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\cdots+\frac{f_n}{x_n}} = \dfrac{N}{\Sigma\frac{f}{x}}$, where $N = \Sigma f$.

(vii) Relations between AM, GM and HM

(a) AM $\geq$ GM $\geq$ HM, for two observations;

(b) For two observations, AM $\times$ HM $= (\text{GM})^2$.

(viii) Computation of Median

(a) For simple series : Median = value of variable in $\dfrac{n+1}{2}$th position;

(b) For grouped frequency distribution : Median $= l_1 + \dfrac{\frac{N}{2} - F}{f_m} \times c$,

where l_1 = lower class boundary of the median class,
N = total frequency,
F = cumulative frequency of the class just preceding the median class,
f = frequency of the median class,
and c = width of the median class.

(ix) Mode $= L + \dfrac{f_m - f_1}{2f_m - f_1 - f_2} \times i$,

where L = lower class boundary of the modal class,
f_m = frequency of the modal class,
f_1 = frequency of the class just preceding the modal class,
f_2 = frequency of the class just succeeding the modal class,
and i = common width of class-intervals.

(x) For a grouped frequency distribution, first quartile (Q_1), Median (Q_2) and third quartile (Q_3) correspond to cumulative frequencies $\dfrac{N}{4}$, $\dfrac{N}{2}$ and $\dfrac{3N}{4}$, where N = total frequency.

(xi) Relation between Mean, Median and Mode : Mean − Mode = 3 (Mean − Median).

(xii) (a) $Q_1 = l_1 + \dfrac{\frac{N}{4} - F_1}{f_1} \times c_1$; (b) $Q_3 = l_3 + \dfrac{\frac{3N}{4} - F_3}{f_3} \times c_3$.

(xiii) D_3 = 3rd decile $= l_3 + \dfrac{\frac{3N}{10} - F_3}{f_3} \times c_3$ and $D_7 = l_7 + \dfrac{\frac{7N}{10} - F_7}{f_7} \times c_7$.

(xiv) P_{35} = 35th percentile $= l_{35} + \dfrac{\frac{35N}{100} - F_{35}}{f_{35}} \times c_{35}$ and $P_{58} = l_{58} + \dfrac{\frac{58N}{100} - F_{58}}{f_{58}} \times c_{58}$.

14. Measures of Dispersion

(i) Range = Maximum value − Minimum value.

(ii) Quartile Deviation (or Semi-interquartile Range) $= \dfrac{Q_3 - Q_1}{2}$.

(iii) Coefficient of Quartile Deviation $= \dfrac{\text{Quartile Deviation}}{\text{Median}} \times 100\% = \dfrac{Q_3 - Q_1}{2Q_2} \times 100\%$.

(iv) Mean Deviation $= \dfrac{\Sigma|x-\bar{x}|}{n}$ or, $\dfrac{\Sigma|x - \text{Median}|}{n} = \dfrac{\Sigma|d|}{n}$, where $d = x - \bar{x}$ or, $x - M$.

(v) Mean Deviation $= \dfrac{\Sigma f|x-\bar{x}|}{N}$, where $N = \Sigma f$.

(vi) Coefficient of Mean Deviation $= \dfrac{\text{Mean Deviation}}{\text{Mean (or Median)}} \times 100\%$.

(vii) (a) Standard Deviation (SD) $= \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2} = \sqrt{\dfrac{\Sigma d^2}{n} - \left(\dfrac{\Sigma d}{n}\right)^2}$, if $d = x - \bar{x}$.

(b) SD $= \sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{N}} = \sqrt{\dfrac{\Sigma f x^2}{N} - \left(\dfrac{\Sigma f x}{n}\right)^2}$.

(c) SD $= \sqrt{\dfrac{\Sigma f d^2}{N} - \left(\dfrac{\Sigma f d}{N}\right)^2}$, where $d = x - A$, and A = Assumed Mean.

(d) SD $= \sqrt{\dfrac{\Sigma f d'^2}{N} - \left(\dfrac{\Sigma f d'}{N}\right)^2} \times i$, where $d' = \dfrac{x-A}{i}$ = step deviation.

(viii) Variance $= (\text{SD})^2 = \dfrac{\Sigma(x-\bar{x})^2}{n}$ or, $\dfrac{\Sigma f(x-\bar{x})^2}{N}$.

(ix) Coefficient of Variation (CV) $= \dfrac{\text{SD}}{\text{Mean}} \times 100\%$.

(x) Combined SD (σ) is given by $\sigma^2 = \dfrac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}$, where

$$d_1 = \bar{x}_1 - \bar{x}, d_2 = \bar{x}_2 - \bar{x},$$

and $\bar{x}$ is the combined mean given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}.$$

15. Moment, Skewness and Kurtosis

(i) m'_r (or μ'_r) = rth moment about $A = \dfrac{1}{n}\Sigma(x-A)^r$, $r = 1, 2, 3, 4, \ldots$.

(ii) rth raw moment (or moment about origin) $= \dfrac{1}{n}\Sigma x^r$, $r = 1, 2, 3, 4, \ldots$.

(iii) m_r (or μ_r) = rth central moment $= \frac{1}{n}\Sigma(x-\bar{x})^r$, where $r = 1, 2, 3, 4, \ldots$.

(iv) First raw moment $= \bar{x}$.

(v) Second central moment $= (\text{SD})^2$.

(vi) Relation between central and non-central moments:

(a) $m_1 = 0$,

(b) $m_2 = m_2' - (m_1')^2$,

(c) $m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$,

(d) $m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$.

(vii) Karl Pearson's first measure of Skewness $= \dfrac{\text{Mean} - \text{Mode}}{\text{SD}}$.

(viii) Karl Pearson's second measure of Skewness $= \dfrac{3(\text{Mean} - \text{Median})}{\text{SD}}$.

(ix) Bowley's measure of Skewness $= \dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$.

(x) Moment measure of Skewness $= \dfrac{\text{3rd Central Moment}}{(\text{SD})^3} = \dfrac{m_3}{\sigma^3}$ or, $\dfrac{m_3}{(m_2)^{3/2}} = \dfrac{m_3}{m_2\sqrt{m_2}}$.

(xi) Measure of Kurtosis $= \dfrac{\text{4th Central Moment}}{(\text{2nd Central Moment})^2} = \dfrac{m_4}{m_2^2}$ or, $\dfrac{m_4}{\sigma^4}$.

16. Correlation and Regression

(i) Correlation Coefficient $(r) = \dfrac{\text{cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$.

(ii) $r = \dfrac{\Sigma xy}{\sqrt{(\Sigma x^2)\cdot(\Sigma y^2)}}$, where $x = X - \bar{X}, y = Y - \bar{Y}$.

(iii) $r = \dfrac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$.

(iv) $R = 1 - \dfrac{6\Sigma D^2}{N^3 - N}$ or, $1 - \dfrac{6\Sigma D^2}{N(N^2-1)}$.

(v) $R = 1 - \dfrac{6\left[\Sigma D^2 + \Sigma\frac{t^3-t}{12}\right]}{N(N^2-1)}$.

(vi) Regression equation of Y on X: $Y - \bar{Y} = b_{YX}(X - \bar{X})$.

(vii) Regression equation of X on Y: $X - \bar{X} = b_{XY}(Y - \bar{Y})$.

(viii) $b_{YX} = \dfrac{\Sigma xy}{\Sigma x^2}; b_{XY} = \dfrac{\Sigma xy}{\Sigma y^2}$, where $x = X - \bar{X}, y = Y - \bar{Y}$.

(ix) $b_{YX}=\dfrac{\operatorname{cov}(X,Y)}{\sigma_X^2}=r\cdot\dfrac{\sigma_Y}{\sigma_X}.$

(x) $b_{XY}=\dfrac{\operatorname{cov}(X,Y)}{\sigma_Y^2}=r\cdot\dfrac{\sigma_X}{\sigma_Y}.$

(xi) $b_{YX}\cdot b_{XY}=r^2.$

17. Set Theory

(i) *Commutative Laws*

(a) $A\cup B=B\cup A$ (b) $A\cap B=B\cap A$

(ii) *Associative Laws*

(a) $A\cup(B\cup C)=(A\cup B)\cup C$ (b) $A\cap(B\cap C)=(A\cap B)\cap C$

(iii) *Distributive Laws*

(a) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ (b) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(iv) *De Morgan's Laws*

(a) $(A\cup B)^c=A^c\cap B^c$ (b) $(A\cap B)^c=A^c\cup B^c$

(c) $A-(B\cup C)=(A-B)\cap(A-C)$ (d) $A-(B\cap C)=(A-B)\cup(A-C)$

(v) *Result involving Cardinal Numbers*

(a) $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

(b) $n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C).$

18. Probability Theory

(i) $P(A)$ = Probability that the event A will occur.

(ii) $P(A')$ or $P(A^c)$ = Probability that the event A will not occur.

(iii) $P(A\cup B)$ = Probability of occurrence of at least one of the events A and B.

(iv) $P(A\cap B)$ = Probability of joint occurrence of events A and B.

(v) $P(A/B)$ = Probability of conditional occurrence of the event A when event B has already occurred.

(vi) *Classical definition of probability:*

$$P(A)=\frac{m}{n}\text{ or }\frac{n(A)}{n(S)}$$
$$=\frac{\text{No. of equally likely cases favourable to the event }A}{\text{No. of all equally likely cases}}.$$

$P(S)=1$, where S is the sure event (or sample space),

$P(\phi)=0$, where ϕ is the impossible event.

(vii) $0 \leq P(A) \leq 1$ and $P(A^c) = 1 - P(A)$.

(viii) *Theorem of Total Probability (or, Additive Law of Probability):*

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n),$$

where the events $A_1, A_2, \cdots, A_n$ are pairwise mutually exclusive.

(ix) For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

(x) For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

(xi) *Theorem of Compound Probability :*

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B/A) \\ \text{and} \quad P(A \cap B \cap C) &= P(A) \cdot P(B/A) \cdot P[C/(A \cap B)]. \end{aligned}$$

(xii) If the events A and B are mutually exclusive, then

$$P(A \cap B) = 0 \text{ and } P(A \cup B) = P(A) + P(B).$$

(xiii) If the events A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B).$$

(xiv) Three events A, B and C are independent if, and only if,

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

(xv) *Axioms of Probability :*

(a) For any event $A, P(A) \geq 0$;

(b) For the sure event $S, P(S) = 1$;

(c) If $A_1, A_2, \cdots$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

19. Random Variable and Mathematical Expectation

(i) If $f(x)$ is the probability function (or, probability mass function) of a discrete random variable X, then

(a) $f(x) \geq 0$, (b) $\sum f(x) = 1$.

(ii) Mathematical expectation of a random variable X

$$= E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum x_i p_i.$$

(iii) $E(x)$ = mean of the probability distribution of the random variable = $\bar{x}$ (or m).

(iv) (a) $E(a) = a$, (b) $E(ax) = aE(x)$, (c) $E(x - \bar{x}) = 0$.

(v) $E(x + y) = E(x) + E(y)$.

(vi) $E(xy) = E(x)E(y)$, when x and y are independent random variables.

(vii) $\text{Var}(x) = E(x - m)^2 = E(x^2) - m^2 = E(x^2) - \{E(x)\}^2$.

20. Index Numbers

(i) Index Number by Simple Aggregative Method $= \dfrac{\sum p_1}{\sum p_0} \times 100$.

(ii) Index Number by Weighted Aggregative Method $= \dfrac{\sum p_1 w}{\sum p_0 w} \times 100$.

(iii) Index Number by Laspeyres' Formula $= \dfrac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$.

(iv) Index Number by Paasche's Formula $= \dfrac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$.

(v) Index Number by Marshall-Edgeworth Formula $= \dfrac{\sum p_1(q_0 + q_1)}{\sum p_0(q_0 + q_1)} \times 100$.

(vi) Fisher's Ideal Index Formula

$$= \sqrt{(\text{Laspeyres' Price Index}) \times (\text{Paasche's Price Index})}$$
$$= \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}} \times 100.$$

(vii) Bowley's Formula

$$= \frac{1}{2}[\text{Laspeyres' Price Index} + \text{Paasche's Price Index}].$$

(viii) Price relative of a commodity

$$= \frac{\text{The price of the commodity in the current year}}{\text{The price of the commodity in the base year}} \times 100 = \frac{p_1}{p_0} \times 100.$$

(ix) (a) Index number by the method of relatives using Simple Arithmetic Mean

$$= \frac{100}{n} \sum \frac{p_1}{p_0};$$

(b) Index number by the method of relatives using Weighted Arithmetic Mean

$$= \frac{100}{\sum w} \sum \frac{p_1}{p_0} \cdot w.$$

(x) General Index (from group indices) $= \dfrac{\sum Iw}{\sum w}$.

(xi) Cost of Living Index $= \dfrac{\sum Iw}{\sum w}$, where I = group index and w = group weight.

(xii) Link Index of any year

$$= \{(\text{Link Relative of the year}) \times (\text{Link Index of the previous year})\} \div 100.$$

(xiii) Real Wage $= \dfrac{\text{Actual Wage}}{\text{Cost of Living Index}} \times 100.$

(xiv) Time Reversal Test: $I_{01} \times I_{10} = 1$.

(xv) Factor Reversal Test: $P_{01} \times Q_{01} = \dfrac{\sum p_1 q_1}{\sum p_0 q_0}$ = True Value Ratio.

21. Time Series

(i) If Y_t be the time series observation, then, by *additive model*, $Y_t = T+S+C+I$ and, by multiplicative model, $Y_t = T \times S \times C \times I$, where T = Trend, S = Seasonal Variation, C = Cyclical Fluctuation, I = Irregular or Random Movement.

(ii) For a linear trend: $Y = a + bX$, the normal equations are

$$\Sigma Y = Na + b\Sigma X \text{ and } \Sigma XY = a\Sigma X + b\Sigma X^2.$$

(iii) For a parabolic trend: $Y = a + bX + cX^2$, the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2, \Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3, \Sigma X^2 Y = a\Sigma X^2 + bX^3 + c\Sigma X^4.$$

(iv) Moving averages of order N for the observations $Y_1, Y_2, Y_3, \cdots, Y_n, Y_{n+1}, \cdots$ are

$$\frac{Y_1 + Y_2 + \cdots + Y_N}{N}, \frac{Y_2 + Y_3 + \cdots + Y_{N+1}}{N}, \cdots.$$

ADVANCED BUSINESS MATHEMATICS

ALGEBRA

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Chapter 1

Matrix: Definition, Different Types and Operation on Matrices

1.1 Introduction

Let us begin with the following example:

Mr X buys 1 dozen mangoes, 3 dozen bananas and 2 dozen apples. We represent them by a row matrix or row vector $(1 \quad 3 \quad 2)$.

This row matrix has one row and three columns; we say that the row matrix contains three components 1, 3 and 2.

Next, suppose that Mrs X buys from another market 2 dozen mangoes, 1 dozen bananas and 3 dozen apples. We represent these quantities by another row matrix or vector $(2 \quad 1 \quad 3)$.

The two purchases may then be represented by the rectangular array: $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

This array has two rows and three columns. The first row refers to what Mr X purchased and the second row to what Mrs X purchased. The first column refers to the amount of mangoes, the second column to the amount of bananas and the third column to the amount of apples.

Such an array when subjected to certain rules of operation is called a *Matrix.*

Matrices are now-a-days used in almost all disciplines.

1.2 Matrix: Definition

Definition 1. *A matrix is a rectangular array of numbers subject to certain rules of operations. If $m \times n$ numbers are arranged in the form of a rectangular array of m rows and n columns like*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

we call it a matrix [provided it obeys certain rules of operations] of order m by n.

We may denote this matrix by $\left(a_{ij}\right)_{m \times n}$ where $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

If $m = n$, then we call it a *square matrix* of order n. This last notation means that it is a matrix of order m by n and that its element in the ith row and jth column is a_{ij}.

We may use capital letters A, B, C, X, Y, etc., to denote matrices where the elements need not be shown explicitly.

Illustration 1. $\begin{bmatrix} 1 & 2 & 9 \\ 2 & 5 & 8 \end{bmatrix}$ *is a matrix of order* 2 *by* 3 (2×3) *having two rows and three columns.*

Illustration 2. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 9 & 8 \end{bmatrix}$ *is a matrix of order* 3 *by* 2 (3×2) *having three rows and two columns.*

Illustration 3. $\begin{bmatrix} 2 & 4 \\ 6 & 9 \end{bmatrix}$ *is a square matrix of order* 2 *or* (2×2) *having two rows and two columns.*

Illustration 4. $(1 \quad 2 \quad 8)$ *is a row matrix or row vector. It is a matrix of order* 1 *by* 3.

Illustration 5. $\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$ *is a column matrix or column vector. It is a matrix of order* 3 *by* 1.

Illustration 6. *A contractor builds three types of houses. Quantities of raw-materials needed are given in the matrix form:*

	Cement (*in* tons)	*Wood* (*in* tons)	*Steel* (*in* tons)
Type 1	20	40	10
Type 2	40	60	12
Type 3	50	80	14

It can be represented by a square matrix of order three: $\begin{bmatrix} 20 & 40 & 10 \\ 40 & 60 & 12 \\ 50 & 80 & 14 \end{bmatrix}_{3\times 3}$

The rows refer to type of houses with raw-materials needed and the first column refers to the amount of cement needed; similarly, we interpret other columns.

1.3 Definitions of a few types of Matrices

1. **Square Matrix.** A matrix having the same number of rows and columns is called a *square matrix.* If there are n rows and n columns, we call it a *square matrix* of order n or $(n \times n)$ having n rows and n columns.

   Illustration 1. $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 3 & 5 & 0 \\ -1 & 3 & 2 \\ 2 & -1 & 9 \end{bmatrix}$, [3] *are square matrices of orders* 2, 3 *and* 1 *respectively.*

2. **Zero Matrix (or Null Matrix).** A matrix of any order whose every element is zero is called a *zero matrix* (or *a null matrix*) to be denoted by a big letter O.

Illustration 2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *are null matrices of orders* $2 \times 2, 2 \times 3$ *respectively.*

3. **Row Matrix and Column Matrix.** A matrix having only one row is called a *row matrix* (or a *row vector*). $[1, 3, 5]$ is a row matrix of order 1×3. $[2, 0, 4, 1]$ is also a row matrix of order 1×4.

   A matrix having only one column is called a *column matrix* (or a *column vector*).

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ is a column matrix of order } 3 \times 1.$$

4. **Diagonal Matrix and Scalar Matrix.** A square matrix whose all elements except those in the main (leading) diagonal are zero is called a *diagonal matrix.*

   For example, $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices of orders 2 and 3 respectively.

   A square matrix whose all elements except those in the main diagonal are zero and the diagonal elements are all equal is called a *scalar matrix.* Thus, a diagonal matrix having its all diagonal elements equal is a *scalar matrix.*

   For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of orders 2 and 3 respectively.

5. **Unit (or Identity) Matrix.** A square matrix whose main diagonal elements are all unity (i.e., 1) and the other elements are all zero is called a *unit (or identity) matrix.*

   For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of orders 2 and 3 respectively. We denote them by I_2 and I_3 (or simply by I).

6. **Transpose of a Matrix.** A matrix obtained by interchanging rows and columns of A is called the *transpose* of A and is denoted by A^T or A'. Clearly, $(A^T)^T = A$.

Illustration 3. *Let* $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}_{2\times 3}$. *Then* $A^T = \begin{bmatrix} 2 & 0 \\ -1 & 2 \\ 0 & 4 \end{bmatrix}_{3\times 2}$

Properties of Transpose Matrices: (a) $(A+B)^T = A^T + B^T$; (b) $(AB)^T = B^T A^T$ [See Section 1.5.]
More generally, let $A = \left(a_{ij}\right)_{m\times n}$; then $A^T = \left(a_{ji}\right)_{n\times m}$.

Note: The transpose of an $m \times n$ matrix is of order $n \times m$.

7. **Singular and Non-singular Matrices.** With every square matrix A of certain order, we associate a determinant $|A|$ or det A of the same order formed by the corresponding elements of A. If $|A| = 0$, the matrix A is called a *singular matrix* and if $|A| \neq 0$, the matrix A is called a *non-singular matrix.*

For example,

- If $A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$, then $|A|=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}=6-6=0$ and, therefore, A is a singular matrix of order 2.
- If $A=\begin{bmatrix}2 & 3\\ -4 & -5\end{bmatrix}$, then $|A|=\begin{bmatrix}2 & 3\\ -4 & -5\end{bmatrix}=-10+12=2\neq 0$ and, therefore, A is a non-singular matrix of order 2.

8. **Symmetric and Skew-symmetric Matrices.**

A square matrix $A=\left(a_{ij}\right)$ is said to be *symmetric* if $a_{ij}=a_{ji}$ for all i and j, i.e., if (i,j)th element = (j,i)th element, i.e., if $a_{12}=a_{21}$, $a_{23}=a_{32}$, etc. In this case, $A^T=A$.

For example, $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$, $\begin{bmatrix}1 & 6 & -5\\ 6 & 2 & 4\\ -5 & 4 & 9\end{bmatrix}$ are *symmetric matrices* of orders 2 and 3 respectively.

A square matrix $A=\left(a_{ij}\right)$ is said to be *skew-symmetric* if $a_{ij}=-a_{ji}$ for all i and j, i.e., if $a_{21}=-a_{12}$, $a_{13}=-a_{31}, a_{23}=-a_{32}$, etc., and $a_{11}=a_{22}=\cdots=0$. In this case, $A^T=-A$.

For example, $\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & 3\\ -2 & -3 & 0\end{bmatrix}$ are *skew-symmetric matrices* of orders 2 and 3 respectively.

9. **Equality of Two Matrices.** Two matrices A and B are *equal* if and only if (a) A and B have the same order and (b) the corresponding elements of A and B are equal.

Illustration 4. $\begin{bmatrix}3 & 27 & 81\\ 2 & 4 & 8\end{bmatrix}$ *and* $\begin{bmatrix}3^1 & 3^3 & 3^4\\ 2^1 & 2^2 & 2^3\end{bmatrix}$ *are two equal matrices.*

Note: $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\neq\begin{bmatrix}1 & 2 & 0\\ 3 & 4 & 0\end{bmatrix}$ ($\because$ they are of different orders).

Example 1. (i) *Find the values of x and y when it is given that*

$$\begin{bmatrix}3 & x+1\\ y & 5\end{bmatrix}=\begin{bmatrix}x-1 & 5\\ x-3y & 5\end{bmatrix}.$$

(ii) *Are the following two matrices equal?*

$$A=\begin{bmatrix}1 & 0 & 9\\ 1 & 3 & -7\\ 2 & 4 & 2\\ 3 & 0 & 7\end{bmatrix},\quad B=\begin{bmatrix}1 & 0 & 9\\ 1 & 3 & -7\\ 2 & 4 & 2\\ 3 & 0 & 7\end{bmatrix}.$$

Solution: (i) We have
$$\begin{bmatrix}3 & x+1\\ y & 5\end{bmatrix}=\begin{bmatrix}x-1 & 5\\ x-3y & 5\end{bmatrix}.$$

We can write $3=x-1$, $x+1=5$, $y=x-3y$ which give $\boldsymbol{x=4, y=1}$.

(ii) The matrices A and B have the same order 4×3 (4 by 3) and their corresponding elements are also equal. Hence, $\boldsymbol{A=B}$.

10. Coefficient and Augmented Matrix of a Set of Linear Equations

For the two linear equations:
$$\left.\begin{array}{l} 3x+2y=4 \\ 2x-3y=7 \end{array}\right\},$$
the *coefficient matrix* and the *augmented matrix* are
$$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \text{ and } \left(\begin{array}{cc:c} 3 & 2 & 4 \\ 2 & -3 & 7 \end{array}\right) \text{ respectively.}$$

For the three linear equations:
$$\left.\begin{array}{l} a_{11}x+a_{12}y+a_{13}z=b_1 \\ a_{21}x+a_{22}y+a_{23}z=b_2 \\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{array}\right\},$$
the *coefficient* and the *augmented matrices* are respectively
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } \left(\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right).$$

1.4 Operational Rules for Matrices

I. Addition of Matrices

Two matrices A and B can be added only when they are of the *same order*, i.e., each of the two matrices must have the *same number of rows and columns.* When this condition is satisfied, the process of addition consists of adding the corresponding elements of the two matrices.

Illustration 1. $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 7 & 9 & 4 \end{bmatrix}.$

General Rule. Let $A=\left(a_{ij}\right)_{m\times n}$, $B=\left(b_{ij}\right)_{m\times n}$, then their sum $C=A+B=\left(c_{ij}\right)_{m\times n}$, $c_{ij}=a_{ij}+b_{ij}$.

Example 2. *Let* $A=(1 \quad 2 \quad 3)_{1\times 3}$ *be a row matrix showing that a man bought* 1 *dozen of mangoes,* 2 *dozen of bananas and* 3 *dozen of apples. Let* $B=(2 \quad 1 \quad 3)_{1\times 3}$ *be another row matrix showing that his wife purchased* 2 *dozen mangoes,* 1 *dozen bananas and* 3 *dozen apples. The sum* $A+B=(3 \quad 3 \quad 6)$ *shows that together they bought* 3 *dozen mangoes,* 3 *dozen bananas and* 6 *dozen apples.*

If two matrices are not of same order, they are not conformable for addition, i.e., we cannot find their sum.

Properties of Matrix Addition

- Matrix Addition is *Commutative*: $A+B=B+A$.
- Matrix Addition is *Associative*: $A+(B+C)=(A+B)+C$.
- Existence of *Additive Identity*: There exists a matrix O (null matrix) such that $A+O=O+A=A$, for every matrix A.

  We then call O (null matrix of same order as that of A) *additive identity.*
- Existence of the additive inverse: Given a matrix $A=\left(a_{ij}\right)_{m\times n}$. Then the negative of A is defined by $\left(-a_{ij}\right)_{m\times n}$ and is denoted by $-A$. The matrix $-A$ is the *additive inverse* of A in the sense that $A+(-A)=(-A)+A=O$.

II. Scalar Multiplication of Matrices

Remember. For our purpose a scalar is just a real number.

Let A be a matrix and k be a scalar; multiplication of A by k (denoted by kA) is effected by multiplying each element of A by k.

Example 3. *If* $A = \overset{\text{Matrix}}{\begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix}}$ *and* $\overset{\text{Scalar}}{k} = 5$, *then find* kA.

Solution: We have
$$kA = 5 \cdot \begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{15} & \mathbf{20} \\ \mathbf{25} & \mathbf{10} & \mathbf{15} \end{bmatrix}.$$

Illustration 2. *Let* $A = (50 \quad 25 \quad 75)$ *represent the cost of one mango* (50 p), *one banana* (25 p) *and one apple* (75 p). *Then* $12A = (600 \quad 300 \quad 900)$ *represents the cost of* 1 *dozen of mangoes,* 1 *dozen of bananas and* 1 *dozen of apples in paise.*

Negative of a Matrix. If A be a matrix and (-1) be a scalar, then $(-1)A = -A$ is called the *negative of* A.

Illustration 3. *Let* $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$. *Then negative of* $A = -A = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \end{bmatrix}$.

III. Subtraction of Two Matrices

Let A and B be two matrices of same order.

Then $A - B$ is obtained by adding A to the negative of B; thus, $A - B = A + (-B)$.

Illustration 4. *If* $A = \begin{bmatrix} 2 & 4 & 9 \\ 8 & 7 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$, *then*
$$A - B = A + (-B) = \begin{bmatrix} 2 & 4 & 9 \\ 8 & 7 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -5 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-2 & 9-5 \\ 8-2 & 7-1 & 3-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 4 \\ 6 & 6 & 0 \end{bmatrix}.$$

This process amounts to the following:

Rule: Subtract the corresponding elements.

Proposition I. *A square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix.*

Proof. Let A be a square matrix of order n. Then A^T is also a square matrix of order n. Also,
$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = B + C \text{ (say).} \tag{1}$$

Now,
$$B^T = \left[\frac{1}{2}\left(A + A^T\right)\right]^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left[A^T + \left(A^T\right)^T\right] = \frac{1}{2}\left(A + A^T\right) = B.$$

$\therefore$ by definition, *B is a symmetric matrix.*

Again, $$C^T = \left[\frac{1}{2}\left(A - A^T\right)\right]^T = \frac{1}{2}\left[A^T - \left(A^T\right)^T\right] = \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right) = -C.$$

$\therefore$ by definition, *C is a skew symmetric matrix.*

From (1), we see that *A is the sum of a symmetric matrix and a shew symmetric matrix.*

Example 4. *Express* $A = \begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix}$ *as the sum of a symmetric matrix and skew symmetric matrix.*

[V.U. B.Com.(H) 2008]

Solution: Here $A = \begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 3 & 2 \\ -5 & 4 \end{bmatrix}$, $\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 6 & -3 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -3/2 \\ -3/2 & 4 \end{bmatrix}$

and $\frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -5 & 4 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -7/2 \\ 7/2 & 0 \end{bmatrix}$.

Thus,
$$A = \begin{bmatrix} 3 & -3/2 \\ -3/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -7/2 \\ 7/2 & 0 \end{bmatrix} = \text{a symmetric matrix} + \text{a skew symmetric matrix.}$$

IV. Multiplication of a Matrix by a Matrix

Motivation. Before giving the formal definition, we discuss two examples:

Example 5. *Let* $A = (1 \quad 4 \quad 3)$ *be a row matrix* (1×3 *matrix*). *Suppose, A represents that a man buys* 1 *dozen mangoes,* 4 *dozen bananas and* 3 *dozen apples.*

Now suppose, mangoes cost ₹12 per dozen, bananas ₹6 per dozen and apples ₹20 per dozen. We represent these prices by a cost matrix (column matrix) $B = \begin{bmatrix} 12 \\ 6 \\ 20 \end{bmatrix}$.

The total cost may be calculated as the product AB in this manner

$$\begin{aligned} A \cdot B &= (1 \quad 4 \quad 3)_{1\times3} \cdot \begin{bmatrix} 12 \\ 6 \\ 20 \end{bmatrix}_{3\times1} \\ &= [1 \times 12 + 4 \times 6 + 3 \times 20]_{1\times1} \\ &= [96]_{1\times1}. \end{aligned}$$

Thus, we multiply a row by a column: Multiply element-wise and then add.

Here the total cost is ₹96.

Example 6. *We take two matrices — one matrix R representing the raw-materials requirements for different types of houses and another matrix P representing the cost of materials per unit quantity and the transport charges for unit quantity of the materials.*

$$R = \begin{array}{l} \\ \\ \left\{\begin{array}{l} \textit{Type 1} \\ \textit{Type 2} \\ \textit{Type 3} \end{array}\right. \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Cement} & \textit{Wood} & \textit{Steel} \\ (\textit{in}\text{ tons}) & (\textit{in}\text{ tons}) & (\textit{in}\text{ tons}) \end{array} \\ \begin{pmatrix} 20 & 40 & 10 \\ 40 & 60 & 12 \\ 50 & 80 & 14 \end{pmatrix} \end{array}$$

$$P = \begin{array}{l} \\ \\ \\ \\ \left\{\begin{array}{l} \textit{Cement} \\ \textit{Wood} \\ \textit{Steel} \end{array}\right. \end{array} \begin{array}{cc} \textit{Price} & \textit{Transport charges} \\ \textit{per ton} & \textit{per ton} \\ (₹) & (₹) \\ \multicolumn{2}{c}{\begin{pmatrix} 250 & 50 \\ 100 & 40 \\ 800 & 60 \end{pmatrix}} \end{array}$$

We can easily check: For Type 1 houses,

total money as price of the materials $= 20 \times 250 + 40 \times 100 + 10 \times 800$
$= 17000.$

[We have taken 1st row of R and 1st column of $P \to$ Multiply element-wise and then add.]

For Type 1 Houses,

total money required for transport charges $= 20 \times 50 + 40 \times 40 + 10 \times 60$
$= 3200.$

[We have taken 1st row of R and 2nd column of $P \to$ Multiply element-wise and then add.]

Similarly, proceed with Type 2 and Type 3 houses: we thus obtain

$$R \cdot P = \begin{bmatrix} 20 & 40 & 10 \\ 40 & 60 & 12 \\ 50 & 80 & 14 \end{bmatrix}_{3\times3} \begin{bmatrix} 250 & 50 \\ 100 & 40 \\ 800 & 60 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 20 \times 250 + 40 \times 100 + 10 \times 800 & 20 \times 50 + 40 \times 40 + 10 \times 60 \\ 40 \times 250 + 60 \times 100 + 12 \times 800 & 40 \times 50 + 60 \times 40 + 12 \times 60 \\ 50 \times 250 + 80 \times 100 + 14 \times 800 & 50 \times 50 + 80 \times 40 + 14 \times 60 \end{bmatrix}$$

$$= \begin{bmatrix} 17000 & 3200 \\ 25600 & 5120 \\ 31700 & 6540 \end{bmatrix}_{3\times2}$$

= a matrix showing total cost of materials and total transport charges for different types of houses.

The two examples given above show that we must make some restrictions when we go in for multiplying two matrices.

Conditions for Matrix Multiplication. In order that two matrices, A and B, are conformable for the product AB, we must have the number of columns in A = the number of rows in B.

Definition 1. *Let* $A = \left(a_{ij}\right)_{m\times n}$ *and* $B = \left(b_{jk}\right)_{n\times p}$ *be two matrices such that the number of columns in A is equal to the number of rows in B. Then the matrix C of order* $m \times p$, *given by* $C = (c_{ik})_{m\times p}$ *such that*

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is called the product of two matrices *A and B in that order and we write* $C = AB$.

Note: Summation is w.r.t. the repeated suffix.

Rules for multiplication. Suppose,

$$A = \left(a_{ij}\right)_{m\times n} \text{ be a matrix of order } m \text{ by } n \text{ and}$$

$$B = \left(b_{jk}\right)_{n\times p} \text{ be a matrix of order } n \text{ by } p$$

so that they are conformable for the product AB.

Suppose, we wish to get the element in the ith row and kth column of AB. We take the ith row $A = (a_{i1}\ a_{i2}\ a_{i3} \cdots a_{in})$ and the kth column of B

$$= \begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

Multiply element-wise and add. The sum is the element in the ith row and kth column of AB. Denoting it by c_{ik}, we get

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Thus, if $A = \left(a_{ij}\right)_{m\times n}$ and $B = \left(b_{jk}\right)_{n\times p}$, then the product AB is a matrix of order $m \times p$, given by

$$AB = (c_{ik})_{m\times p}, \text{ when } c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Illustration 5. $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}_{2\times 2} \times \begin{bmatrix} 9 & 7 & 8 \\ 6 & 3 & 2 \end{bmatrix}_{2\times 3}$ [*See that conformability condition holds*]

$$= \begin{bmatrix} 1\times 9+2\times 6 & 1\times 7+2\times 3 & 1\times 8+2\times 2 \\ 0\times 9+4\times 6 & 0\times 7+4\times 3 & 0\times 8+4\times 2 \end{bmatrix} = \begin{bmatrix} 21 & 13 & 12 \\ 24 & 12 & 8 \end{bmatrix}_{2\times 3}$$

Illustration 6. $(a\ b\ c\ d)_{1\times 4} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1} = (a^2 + b^2 + c^2 + d^2)_{1\times 1}.$

Illustration 7. $\begin{bmatrix} 3 & 4 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \\ 4 & 1 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 35 & 25 & 20 \\ 28 & 13 & 18 \\ 20 & 17 & 9 \end{bmatrix}$: *check.*

Illustration 8. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$

In fact, assuming conformability, for any matrix A, it is true that $AI = IA = A$, where I is the unit matrix such that I and A have the same order.

1.4.1 Laws of Matrix Algebra

In ordinary addition and multiplication of real numbers, we know that the following laws hold:

	Addition	*Multiplication*
(i)	Commutative law: $a+b=b+a$	$a \cdot b = b \cdot a$.
(ii)	Associative law: $a+(b+c)=(a+b)+c$	$a \cdot (b \cdot c) = (a \cdot b).c$.

(iii) Distributive law: $a \times (b+c) = a \times b + a \times c$ and $(b+c) \times a = b \times a + c \times a$.

(iv) $a \cdot b = 0$ implies that either $a=0$ or $b=0$ (if not both).

(v) $a \cdot b = a \cdot c$ implies that $b=c$ (provided that $a \neq 0$) $[a,b,c$ are any three real numbers].

We shall see that

(i) matrix addition is commutative but matrix multiplication is not, in general, commutative.

(ii) Associative law holds for three matrices if additions and multiplications are defined

(iii) Two distributive laws hold if additions and multiplications are defined.

(iv) and (v) are not, in general, true in matrix multiplication.

Proofs or justifications of these laws of Matrix Algebra are given below:

(i) Matrix addition is commutative because $\left(a_{ij}+b_{ij}\right)_{m\times n} = \left(b_{ij}+a_{ij}\right)_{m\times n}$.

[In fact, this follows from the commutative law of addition of numbers] but matrix multiplication is not, in general, commutative as can be seen from the following example:

Example 7. $$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{2\times4} \times \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{4\times2} = \begin{bmatrix} 14 & 4 \\ 4 & 14 \end{bmatrix}_{2\times2}$$

but $$\begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{4\times2} \times \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{2\times4} = \begin{bmatrix} 9 & 6 & 3 & 0 \\ 6 & 5 & 4 & 3 \\ 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}_{4\times4}$$

The two matrices on the right side cannot be equal as they are of different orders.

Even if the product AB and BA are of the same order, they may not be equal as can be seen from the following example:

Example 8. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & -2 \end{bmatrix}$, *but* $\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -3 & 3 \end{bmatrix}$. *We see that the two right-side matrices are not equal.*

The fact is that the process of constructing $A \cdot B$ and $B \cdot A$ are different and hence they may not be equal.

But this does not mean that we never have $AB = BA$. There may be special cases where $AB = BA$; but, in general, it is not true.

Proofs of (ii) and (iii), i.e., the proofs of *Associative and Distributive Laws* are given at the Appendix.

(iv) Now see that

$$\begin{bmatrix} 3 & 4 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Null matrix.}$$

This example shows that in matrix multiplication, $AB = O$ (null matrix) does not always imply that either $A = O$ or, $B = O$.

(v) Consider $A = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 3 & 3 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$.

Now, verify that $AB = \begin{bmatrix} 7 & 10 & 5 \\ 2 & 3 & 2 \\ 1 & 0 & -5 \end{bmatrix} = AC$ (but see that $B \neq C$).

Thus, here is an example to prove that $AB = AC$ does not always imply that $B = C$ (even though $A \neq O$).

Note: We shall write: $A^2 = A \cdot A$, $A^3 = A^2 \cdot A$, $A^4 = A^3 \cdot A$, etc., $A^n = A^{n-1} \cdot A$. All the laws of Matrix Algebra as explained above are stated below.

I. Properties of Addition of Matrices

- If A and B be any two matrices of the same order, then
$$A + B = B + A. \qquad \text{[Commutative law]}$$
- If A, B, C be any three matrices of the same order, then
$$A + (B + C) = (A + B) + C. \qquad \text{[Associative law]}$$
- If k be a scalar and A, B be two matrices of the same order, then
$$k(A + B) = kA + kB.$$
- If A be an $m \times n$ matrix and O be the null matrix of the same order, then
$$\text{(a) } A + O = O + A = A \text{ and (b) } A + (-A) = (-A) + A = O.$$
- If A, B, C be any three matrices of the same order, then
$$A + C = B + C \text{ gives } A = B.$$
- If A and B be any two matrices of the same order, then
$$(A + B)^T = A^T + B^T \text{ and } (A - B)^T = A^T - B^T.$$

II. Properties of Matrix Multiplication

- The product of matrices is not, in general, commutative, i.e., if two matrices A, B are conformable for the product AB and BA, then, in general,
$$AB \neq BA.$$
- If A, B, C are three matrices such that AB = AC, then, in general,
$$B \neq C.$$
- If A, B, C be three matrices of order $m \times n$, $n \times p$, $n \times p$ respectively, then
$$A \cdot (B + C) = A \cdot B + A \cdot C. \qquad \text{[Distributive law]}$$
- If A, B, C be three matrices of order $m \times n, n \times p, p \times q$ respectively, then
$$A \cdot (BC) = (AB) \cdot C. \qquad \text{[Associative law]}$$
- If A is an $m \times n$ matrix and O an $n \times m$ null matrix, then $A \cdot O$ is an $m \times m$ null matrix and $O \cdot A$ is an $n \times n$ null matrix. If $m = n$, then
$$A \cdot O = O \cdot A = O, \text{ where } O \text{ is an } n \times m \text{ null matrix.}$$
- If A be a square matrix of order n and I be the unit matrix of the same order, then
$$AI = IA = A.$$
- If $AB = O$, where A, B are two matrices, then, in general,
$$A \neq O \text{ or } B \neq O \text{ or none of } A \text{ and } B \text{ is null matrix.}$$

- If for two matrices A and B, the product AB is defined, then

$$(AB)^T = B^T A^T.$$

1.5 Illustrative Examples

Example 9. *Given two matrices* $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$ *; find, where possible,* $A+B, A-B, AB$ *and* BA*. Verify the condition of conformability in each case.* [C.U. B.Com.(H) 1991]

Solution:

(i) $\left.\begin{array}{l} A \text{ is a matrix of order } 2\times 3 \\ B \text{ is a matrix of order } 3\times 2 \end{array}\right\}$ (they are not of the same order).

Hence, A and B are not conformable for $A+B$ or $A-B$.

(ii) The number of columns in A is equal to the number of rows in B; hence, A and B are conformable for the product AB and the product is given by

$$AB = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} (4\times 2)+(2\times -3)+(-1\times -1) & (4\times 3)+(2\times 0)+(-1\times 5) \\ (3\times 2)+(-7\times -3)+(1\times -1) & (3\times 3)+(-7\times 0)+(1\times 5) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{3} & \mathbf{7} \\ \mathbf{26} & \mathbf{14} \end{bmatrix}_{2\times 2}.$$

(iii) The number of columns in B = the number of rows in $A = 2$.

$\therefore$ A and B are conformable for the product BA.

$$BA = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}_{3\times 2} \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} (2\times 4)+(3\times 3) & (2\times 2)+(3\times -7) & (2\times -1)+(3\times 1) \\ (-3\times 4)+(0\times 3) & (-3\times 2)+(0\times -7) & (-3\times -1)+(0\times 1) \\ (-1\times 4)+(5\times 3) & (-1\times 2)+(5\times -7) & (-1\times -1)+(5\times 1) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{17} & \mathbf{-17} & \mathbf{1} \\ \mathbf{-12} & \mathbf{-6} & \mathbf{3} \\ \mathbf{11} & \mathbf{-37} & \mathbf{6} \end{bmatrix}_{3\times 3}$$

We note here that $AB \neq BA$, though both exist.

Example 10. (a) *If* $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$, *find the value of* $2A+3B$. [C.U. B.Com. 2006 Type]

(b) *If* $A = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, *find the value of* $3B - A$. [C.U. B.Com. 2010]

(c) *Find the matrices* A *and* B, *if* $A+B = \begin{bmatrix} 1 & 5 & 10 \\ 5 & 9 & 8 \end{bmatrix}$ *and* $A-B = \begin{bmatrix} -1 & -1 & -4 \\ 1 & 1 & 6 \end{bmatrix}$. [V.U. B.Com.(H) 2010]

Solution: (a) We have

$$2A=2\begin{bmatrix}0&2&3\\2&1&4\end{bmatrix}=\begin{bmatrix}0&4&6\\4&2&8\end{bmatrix}_{2\times3};\quad 3B=3\begin{bmatrix}7&6&3\\1&4&5\end{bmatrix}=\begin{bmatrix}21&18&9\\3&12&15\end{bmatrix}_{2\times3}.$$

Hence,

$$2A+3B=\begin{bmatrix}0+21&4+18&6+9\\4+3&2+12&8+15\end{bmatrix}=\begin{bmatrix}\mathbf{21}&\mathbf{22}&\mathbf{15}\\\mathbf{7}&\mathbf{14}&\mathbf{23}\end{bmatrix}.$$

We note that $2A$ and $3B$ are of same order 2×3; hence, their addition is possible.

(b) $3B-A=3\begin{bmatrix}0&2&3\\2&1&4\end{bmatrix}-\begin{bmatrix}7&6&3\\1&4&5\end{bmatrix}=\begin{bmatrix}0&6&9\\6&3&12\end{bmatrix}-\begin{bmatrix}7&6&3\\1&4&5\end{bmatrix}$

$$=\begin{bmatrix}0-7&6-6&9-3\\6-1&3-4&12-5\end{bmatrix}=\begin{bmatrix}\mathbf{-7}&\mathbf{0}&\mathbf{6}\\\mathbf{5}&\mathbf{-1}&\mathbf{7}\end{bmatrix}.$$

(c) $(A+B)+(A-B)=\begin{bmatrix}1&5&10\\5&9&8\end{bmatrix}+\begin{bmatrix}-1&-1&-4\\1&1&6\end{bmatrix},$

or, $2A=\begin{bmatrix}1-1&5-1&10-4\\5+1&9+1&8+6\end{bmatrix}=\begin{bmatrix}0&4&6\\6&10&14\end{bmatrix}.$

$\therefore\ A=\frac{1}{2}\begin{bmatrix}0&4&6\\6&10&14\end{bmatrix}=\begin{bmatrix}\mathbf{0}&\mathbf{2}&\mathbf{3}\\\mathbf{3}&\mathbf{5}&\mathbf{7}\end{bmatrix}$

and $A+B=\begin{bmatrix}1&5&10\\5&9&8\end{bmatrix}$ or, $B=\begin{bmatrix}1&5&10\\5&9&8\end{bmatrix}-A=\begin{bmatrix}1&5&10\\5&9&8\end{bmatrix}-\begin{bmatrix}0&2&3\\3&5&7\end{bmatrix}=\begin{bmatrix}\mathbf{1}&\mathbf{3}&\mathbf{7}\\\mathbf{2}&\mathbf{4}&\mathbf{1}\end{bmatrix}.$

Example 11. (i) *If* $A=\begin{bmatrix}2&5\\1&3\end{bmatrix}$ *and* $B=\begin{bmatrix}1&-1\\-3&2\end{bmatrix}$, *find, if possible, AB and BA. Is $AB=BA$?*

(ii) *If the matrix* $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$ *and* $B=\begin{bmatrix}2&1&3\\-1&0&1\\3&-1&4\end{bmatrix}$, *verify whether $AB=BA$.*

(iii) *Find AB, if possible, when* $A=(-7\quad 2\quad 1)$ *and* $B=\begin{pmatrix}1\\0\\7\end{pmatrix}$. [C.U. B.Com.(H) 2008]

Solution: (i) Since A and B are square matrices of order 2, both AB and BA exist.

$$AB=\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1&-1\\-3&2\end{bmatrix}=\begin{bmatrix}(2\times1)+(5\times-3)&(2\times-1)+(5\times2)\\(1\times1)+(3\times-3)&(1\times-1)+(3\times2)\end{bmatrix}=\begin{bmatrix}-13&8\\-8&5\end{bmatrix}_{2\times2}$$

$$\text{and } BA=\begin{bmatrix}1&-1\\-3&2\end{bmatrix}\begin{bmatrix}2&5\\1&3\end{bmatrix}=\begin{bmatrix}1\times2+(-1)\times1&1\times5+(-1\times3)\\-3\times2+2\times1&-3\times5+2\times3\end{bmatrix}=\begin{bmatrix}1&2\\-4&-9\end{bmatrix}_{2\times2}$$

We see that $AB\neq BA$ though both have the same order 2×2.

(ii) $AB=\begin{bmatrix}5&-2&9\\9&-2&17\\8&2&14\end{bmatrix}$; $BA=\begin{bmatrix}10&7&10\\3&0&-1\\11&5&7\end{bmatrix}$; here, $AB\neq BA$.

Note: Students may find the products AB and BA.

(iii) $AB = (-7 \quad 2 \quad 1) \times \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} = (-7\times 1 + 2\times 0 + 1\times 7) = (-7+0+7) = (0).$

Example 12. *A and B are two matrices, where* $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. *Find the value of* $AB - 2B$. **[V.U. B.Com.(H) 2011]**

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times 1)+(2\times -1)+(1\times 0) & (1\times 4)+(2\times 2)+(1\times 0) & (1\times 0)+(2\times 2)+(1\times 2) \\ (1\times 1)+(-1\times -1)+(1\times 0) & (1\times 4)+(-1\times 2)+(1\times 0) & (1\times 0)+(-1\times 2)+(1\times 2) \\ (2\times 1)+(3\times -1)+(-1\times 0) & (2\times 4)+(3\times 2)+(-1\times 0) & (2\times 0)+(3\times 2)+(-1\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 & 6 \\ 2 & 2 & 0 \\ -1 & 14 & 4 \end{bmatrix}.$$

Now,

$$2B = 2\begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 0 \\ -2 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\therefore\ AB - 2B = \begin{bmatrix} -1-2 & 8-8 & 6-0 \\ 2+2 & 2-4 & 0-4 \\ -1-0 & 14-0 & 4-4 \end{bmatrix} = \begin{bmatrix} \mathbf{-3} & \mathbf{0} & \mathbf{6} \\ \mathbf{4} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{14} & \mathbf{0} \end{bmatrix}.$$

We note that A and B are of the same order 3×3, so their multiplication is possible. Moreover, AB and $2B$ are also of same order 3×3, so their subtraction is also possible.

Example 13. *Using matrices A, B and C, where* $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, *verify that* $(AB)C = A(BC)$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+(-1)\times 0+1\times 1 & 1\times(-1)+(-1)\times 1+1\times 1 & 1\times 0+(-1)\times(-1)+1\times 1 \\ 0\times 1+2\times 0+1\times 1 & 0\times(-1)+2\times 1+1\times 1 & 0\times 0+2\times(-1)+1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & -1-1+1 & 0+1+1 \\ 0+0+1 & 0+2+1 & 0-2+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix}.$$

$$\therefore\ (AB)C = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+0+2 & 0-1+2 \\ 1+0-1 & 0+3-1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

Again,
$$BC=\begin{bmatrix}1&-1&0\\0&1&-1\\1&1&1\end{bmatrix}\times\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}=\begin{bmatrix}1+0+0&0-1+0\\0+0-1&0+1-1\\1+0+1&0+1+1\end{bmatrix}=\begin{bmatrix}1&-1\\-1&0\\2&2\end{bmatrix}.$$

$$\therefore A(BC)=\begin{bmatrix}1&-1&1\\0&2&1\end{bmatrix}\times\begin{bmatrix}1&-1\\-1&0\\2&2\end{bmatrix}=\begin{bmatrix}1+1+2&-1+0+2\\0-2+2&0+0+2\end{bmatrix}=\begin{bmatrix}4&1\\0&2\end{bmatrix}.$$

Hence, $(AB)C=A(BC)$ **(Proved).**

Example 14. *Prove that the matrix A given by* $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ *satisfies the relation* $A^2-A(a+d)+(ad-bc)I=O$, *where I is a unit matrix of order two.* [V.U. B.Com.(H) 2009]

Solution:
$$A^2=\begin{bmatrix}a&b\\c&d\end{bmatrix}\times\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix}$$
$$A(a+d)=\begin{bmatrix}a&b\\c&d\end{bmatrix}(a+d)=\begin{bmatrix}a^2+ad&ab+bd\\ac+cd&ad+d^2\end{bmatrix}$$
$$\text{and }(ad-bc)I=(ad-bc)\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}.$$

$$\begin{aligned}\therefore A^2-A(a+d)+(ad-bc)I\\
&=\begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix}-\begin{bmatrix}a^2+ad&ab+bd\\ac+cd&ad+d^2\end{bmatrix}+\begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}\\
&=\begin{bmatrix}a^2+bc-a^2-ad+ad-bc&ab+bd-ab-bd+0\\ac+cd-ac-cd+0&bc+d^2-ad-d^2+ad-bc\end{bmatrix}\\
&=\begin{bmatrix}0&0\\0&0\end{bmatrix}=O.\end{aligned}$$

Hence, the matrix A satisfies the relation $A^2-A(a+d)+(ad-bc)I=O$. **(Proved).**

Example 15. *Find the values of a, b, c if the matrix A given by* $A=\frac{1}{3}\begin{bmatrix}a&2&2\\2&1&b\\2&c&1\end{bmatrix}$ *obeys the law* $AA'=I$ *(A' is the transpose of A and I is the unit matrix of order 3).*

Solution: We have
$$A'=\frac{1}{3}\begin{bmatrix}a&2&2\\2&1&c\\2&b&1\end{bmatrix}$$
$$AA'=\frac{1}{3}\begin{bmatrix}a&2&2\\2&1&b\\2&c&1\end{bmatrix}\times\frac{1}{3}\begin{bmatrix}a&2&2\\2&1&c\\2&b&1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}a^2+8&2a+2b+2&2a+2c+2\\2a+2b+2&b^2+5&b+c+4\\2a+2c+2&b+c+4&c^2+5\end{bmatrix}$$
$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\text{ (given) }[\because AA'=I]$$

Hence,

$$\tfrac{1}{9}(a^2+8)=1, \quad \tfrac{1}{9}(b^2+5)=1, \quad \tfrac{1}{9}(c^2+5)=1; \qquad (1)$$

and $$2a+2b+2=0, \quad 2a+2c+2=0, \quad b+c+4=0. \qquad (2)$$

From (1),

$$a^2+8=9, \quad b^2+5=9, \quad c^2+5=9,$$

or, $$a^2=1, \quad b^2=4, \quad c^2=4.$$

$$\therefore\ a=\pm 1, \quad b=\pm 2, \quad c=\pm 2.$$

Of these values of a, b, c, only $a=1$, $b=-2$, $c=-2$ satisfy the relation (2).
Hence, $\boldsymbol{a=1, b=-2, c=-2}$.

Example 16. *For the two matrices* $A=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$, *verify that* $(AB)^T=B^T\cdot A^T$.

[B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2009 Type]

Solution:

$$A\cdot B=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\times\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}=\begin{bmatrix} (2\times 1)+(1\times -1) & (2\times -2)+(1\times 1) \\ (3\times 1)+(4\times -1) & (3\times -2)+(4\times 1) \end{bmatrix}=\begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix}.$$

$$\therefore\ (AB)^T=\begin{bmatrix} 1 & -1 \\ -3 & -2 \end{bmatrix}.$$

Since,

$$A=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix};\ \therefore\ A^T=\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

Again,

$$B=\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix};\ \therefore\ B^T=\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Now,

$$B^T\cdot A^T=\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\times\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}=\begin{bmatrix} (1\times 2)+(-1\times 1) & (1\times 3)+(-1\times 4) \\ (-2\times 2)+(1\times 1) & (-2\times 3)+(1\times 4) \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ -3 & -2 \end{bmatrix}.$$

Hence, we see that

$$(AB)^T=B^T\cdot A^T \text{ (Proved).}$$

Example 17. *If for two matrices A, B of order two,* $A+B=\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$,

$$2A+3B=\begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}, \textit{ find } A \textit{ and } B.$$

[C.U. B.Com.(H) 2001]

Solution: Multiply $A+B$ by 2.

$$2A+2B=2\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}.$$

Given,

$$2A+3B=\begin{pmatrix}5 & 4\\0 & 5\end{pmatrix}.$$

On subtraction,

$$B=\begin{pmatrix}5 & 4\\0 & 5\end{pmatrix}-\begin{pmatrix}4 & 4\\0 & 4\end{pmatrix}=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\\mathbf{0} & \mathbf{1}\end{pmatrix}.$$

Again,

$$A+B=\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}.$$

$$\therefore A=\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}-B=\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}-\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}\mathbf{1} & \mathbf{2}\\\mathbf{0} & \mathbf{1}\end{pmatrix}.$$

Example 18. *If* $\begin{bmatrix}4\\1\\3\end{bmatrix}A=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$, *then find A.* [C.U. B.Com.(H) 2004]

Solution: Since $\begin{bmatrix}4\\1\\3\end{bmatrix}$ is 3×1 matrix and $\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$ is a 3×3 matrix,

$\therefore$ A must be a 1×3 matrix for $\begin{bmatrix}4\\1\\3\end{bmatrix}A$ to be a 3×3 matrix.

Let $A=[x \quad y \quad z]$. Then

$$\begin{bmatrix}4\\1\\3\end{bmatrix}\times[x \quad y \quad z]=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}\Rightarrow\begin{bmatrix}4x & 4y & 4z\\x & y & z\\3x & 3y & 3z\end{bmatrix}=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$$

$$\Rightarrow 4x=-4, 4y=8, 4z=4; x=-1, y=2, z=1; 3x=-3, 3y=6, 3z=3.$$

We see that $x=-1$, $y=2$ and $z=1$ satisfy all the equations.
Hence,

$$\boldsymbol{A=[-1 \quad 2 \quad 1].}$$

TYPES OF MATRICES

1.6 Triangular, Diagonal and Scalar Matrices

Upper Triangular Matrix: Definition. A square matrix $A=\left(a_{ij}\right)$ is called an *upper triangular matrix* if $a_{ij}=0$, whenever $i>j$.

Thus in an upper triangular matrix, all the elements below the principal diagonal (from left-hand top corner to right-hand bottom corner) are zero.

e.g., $\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\0 & a_{22} & a_{23} & \cdots & a_{2n}\\0 & 0 & a_{33} & \cdots & \cdots\\\cdots & \cdots & \cdots & \cdots & \cdots\\0 & 0 & 0 & \cdots & a_{nn}\end{bmatrix}$ is an upper triangular matrix of order $n\times n$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \text{ or, } B = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$ are examples of upper triangular matrices.

Lower Triangular Matrix: Definition. A square matrix $A = (a_{ij})$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$.

Thus in a lower triangular matrix, all the elements above the principal diagonal are zero.

$$\text{e.g., } \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}_{2\times 2}, \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}_{3\times 3}$$

are examples of lower triangular matrices.

Diagonal Matrix: Definition. A square matrix $A = [a_{ij}]_{n\times n}$ whose all the elements except the principal diagonal elements are zero, is called a *diagonal matrix.*

$$\text{e.g., } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ or, } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ is a diagonal matrix.

Scalar Matrix: Definition. A diagonal matrix whose diagonal elements are all equal is called a *scalar matrix*

$$\text{e.g., } S = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{bmatrix}$$ is a scalar matrix (scalar is K here)

A is any 4-rowed square matrix, then $AS = SA = KA$.

When $K = 1$, we get an identity matrix.

$$\text{e.g., } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is identity matrix of order 3.

See that if A is any square matrix of order 3, then $AI_3 = I_3A = A$.

A square matrix A is called an

- **Idempotent Matrix,** if $A^2 = A$.
- **Involutory Matrix,** if $A^2 = I$.
- **Nilpotent Matrix** of index n, if n is the least positive integer such that $A^n = O$.
- **Orthogonal Matrix,** if $A \cdot A^T = A^T \cdot A = I$, where A^T is the transpose of A and I is the identity matrix of appropriate order.

A square matrix is said to be

- *Symmetric* matrix, if $A^T = A$.
- *Skew-symmetric* matrix, if $A^T = -A$.

Thus, a square matrix $A=\left(a_{ij}\right)_{n\times n}$ is *symmetric*, if its (i,j)th element is the same as (j,i)th element, i.e., if $a_{ij}=a_{ji}$ for all i,j.

Again, a square matrix $A=\left(a_{ij}\right)_{n\times n}$ is *skew-symmetric* if $a_{ij}=-a_{ji}$ for all i,j.

Thus, the diagonal elements of a skew-symmetric matrix are all zero ($\because\ a_{ij}=-a_{ij}$, i.e., $a_{ij}=0$), e.g., $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Inverse of a Matrix. Suppose, A is an $n\times n$ square matrix. If there exists a square matrix B of same order $n\times n$ such that $AB=BA=I_n$, then B is called *an inverse* of A and we write $B=A^{-1}$ (Thus $A\cdot A^{-1}=A^{-1}\cdot A=I_n$). Inverse of A, if exists, is unique (one and only one). More discussions about the existence and evaluation of inverse of a square matrix will be given in a later chapter.

For an *involutory matrix* A, $A^{-1}=A$ and for an *orthogonal matrix* A, $A^{-1}=A^T$.

MORE SOLVED EXAMPLES

Continuation of Section 1.5.

Example 19. (A square matrix A such that $A^2=A$ is called an Idempotent Matrix) *Show that the matrix*

$$A=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

is idempotent.

Solution: We have by row-by-column multiplication process

$$A^2=A\cdot A=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\times\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}=A.$$

Since $A^2=A$, therefore, the matrix A **is an idempotent matrix.**

Example 20. (A matrix A such that $A^2=I$ is called an Involutory Matrix) *In this case, $A^{-1}=A$ (why?) Show that the matrix*

$$A=\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\ \textit{is involutory.}$$

Solution: Check:

$$A^2=A\cdot A=I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 21. (i) *If B is idempotent, then prove that $A=I-B$ is also idempotent and that $AB=BA=O$ (null matrix).*

(ii) *Show that a matrix A is involutory, if and only if $(I+A)\cdot(I-A)=O$ (null matrix).*

Solution: (i) Since B is idempotent, therefore, $B^2 = B$. Now

$$\begin{aligned} A^2 = (I-B)^2 &= (I-B)\cdot(I-B) \\ &= I^2 - IB - BI + B^2 \text{ (Distributive Law)} \\ &= I - B - B + B^2 \; (\because I^2 = I \text{ and } IB = BI = B) \\ &= I - B - B + B \; (\because B^2 = B) \\ &= I - B \; (\because -B + B = 0) \\ &= A. \end{aligned}$$

Since $A^2 = A$, therefore, **A is also idempotent.**

Again, $AB = (I-B)B = IB - B^2 = B - B = O$. Similarly, $BA = B(I-B) = O$.

(ii) Let A be an involutory matrix of order n.

Then $A^2 = I$ or, $I - A^2 = O$ or, $I^2 - A^2 = O$ or, $(I-A)(I+A) = O$.

Conversely, if $(I+A)(I-A) = O$, then $I^2 - IA + AI - A^2 = O$ or, $I - A^2 = O$ or, $A^2 = I$.

Example 22. (A square matrix is called a nilpotent matrix if there exists a positive integer n such that $A^n = O$). *If n is the least positive integer for which $A^n = O$, then n is called the index of the nilpotent matrix A.*

Show that the matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ *is nilpotent of index* 2.

Solution: We see that $A^2 = A \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$, i.e., **$A$ is a nilpotent matrix of index 2.**

Example 23. *Show that the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ *is nilpotent and find its index.*

Solution: Check.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}.$$

Again

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, 3 is the least positive integer such that $A^3 = O$. Hence, the matrix is **nilpotent of index 3.**

Example 24. [A square matrix A is said to be orthogonal if $AA^T = A^TA = I$. In this case, $A^T = A^{-1}$ (Inverse of A).]

Show that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ *is orthogonal.* **[C.U. B.Com.(H) 2003]**

Solution:

$$AA^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Verify that A^TA is also equal to I; thus, $AA^T = A^TA = I$ and hence, **A is orthogonal.**

Example 25. *If A and B are two square matrices of order n, show that*

(i) $(A+B)^2 = A^2 + AB + BA + B^2$.

(ii) $(A+B)(A-B) = A^2 - AB + BA - B^2$.

(iii) $(A-B)^2 = A^2 - AB - BA + B^2$.

What will happen, if A and B commute, i.e., if $AB = BA$?

Solution: (i) $(A+B)^2 = (A+B)\cdot(A+B)$

$= A\cdot A + A\cdot B + B\cdot A + B\cdot B$ (using Distributive Law)

$= A^2 + AB + BA + B^2$.

In case, $A = B$, then $(A+B)^2 = A^2 + 2AB + B^2$. Other cases may be dealt with in a similar manner. Left as exercises for the students.

Example 26. *If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that $(aI_2 + bC)^3 = a^3 I_2 + 3a^2 bC$.*

Solution: We have

$$aI_2 + bC = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = B \text{ (say)}.$$

Obtain

$$B^2 = B\cdot B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}.$$

$$(aI_2 + bC)^3 = B^3 = B^2\cdot B = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}.$$

$$a^3 I_2 + 3a^2bC = \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}.$$

Hence etc.

Example 27. *If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = O$. Hence, find A^{-1}.*

Solution:

$$A^2 = A\cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix};$$

$$-4A = \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix}; -5I = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

$$\therefore\ A^2-4A-5I=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O\ \textbf{(Proved).}$$

Now, let us write $A^2-4A-5I=O$ in the form

$$A^2-4AI=5I\quad [\because\ AI=A]$$
$$\text{or,}\quad A(A-4I)=5I\ [\text{Do not write } A(A-4)=5I]$$
$$\text{or,}\quad A\cdot\tfrac{1}{5}(A-4I)=I.$$

Since $A\cdot A^{-1}=I$, therefore, $A^{-1}=\frac{1}{5}(A-4I)$, i.e.,

$$A^{-1}=\frac{1}{5}\left[\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-\begin{pmatrix}4&0&0\\0&4&0\\0&0&4\end{pmatrix}\right]=\frac{1}{5}\begin{pmatrix}-3&2&2\\2&-3&2\\2&2&-3\end{pmatrix}=\begin{bmatrix}-3/5&2/5&2/5\\2/5&-3/5&2/5\\2/5&2/5&-3/5\end{bmatrix}.$$

$$\left[\text{Check: }\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix}-3&2&2\\2&-3&2\\2&2&-3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.\right]$$

Example 28. *If A and B are symmetric (or skew-symmetric), then so is also $A+B$.*

Solution: (i) Let A and B be two symmetric matrices of the same order. Then $A'=A$ and $B'=B$.
Now, $(A+B)'=A'+B'=(A+B)$.
Since $(A+B)'=A+B$, therefore, **$A+B$ is symmetric.**
(ii) Let A and B be two skew-symmetric matrices of same order. Then $A'=-A$, $B'=-B$.
Now, $(A+B)'=A'+B'=(-A)+(-B)=-(A+B)$.
Since $(A+B)'=-(A+B)$, therefore, **$(A+B)$ is a skew-symmetric matrix.**

EXERCISES ON CHAPTER 1

(Matrix Algebra)

1. (i) Find $A+B$, $A-B$, $A\cdot B$, $(A+B)\cdot(A-B)$, A^2, B^2, A^2-B^2 if

(a) $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}2&1\\4&3\end{bmatrix}$. Verify that here $(A+B)\cdot(A-B)\neq A^2-B^2$. Why?

(b) $A=\begin{bmatrix}2&1&2\\3&5&7\\1&0&1\end{bmatrix}$ and $B=\begin{bmatrix}-3&1&0\\6&2&1\\1&-1&2\end{bmatrix}$.

(ii) (a) If $A+B=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $A-B=\begin{pmatrix}1&0\\-1&-2\end{pmatrix}$, find A. [B.U. B.Com.(H) 2008]

(b) Find the matrices A and B, when $A+B=\begin{bmatrix}1&2&0\\3&5&4\end{bmatrix}$ and $A-B=\begin{bmatrix}3&0&2\\1&1&0\end{bmatrix}$.
[C.U. B.Com. 2007; V.U. B.Com.(H) 2010]

(c) If $A=\begin{bmatrix}-3 & 5\\ -9 & 11\end{bmatrix}$, $B=\begin{bmatrix}2 & -3\\ 6 & -7\end{bmatrix}$, determine $(2A+3B)$. [C.U. B.Com. 2006]

(d) If $A=\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}$ and $B=\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}$, find the value of $3B-A$. [C.U. B.Com. 2010]

2. (a) Determine the matrices A and B, when $A+2B=\begin{pmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{pmatrix}$ and

$$2A-B=\begin{pmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{pmatrix}.$$

[C.U. B.Com.(H) 2000; V.U. B.Com.(H) 2007]

(b) Let $A=\begin{bmatrix}2 & 3 & 1\\ 0 & 1 & -1\\ -1 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$ and $C=\begin{bmatrix}1\\ 1\\ -2\end{bmatrix}$; evaluate AB, A^2B, AC, A^2C.

(c) Find the values of x, y, z when the following matrices A and B are equal.

$$A=\begin{bmatrix}x+y & y-z\\ 1 & 7+x\end{bmatrix}, \quad B=\begin{bmatrix}4-x & z-4\\ z-y & x+z+4\end{bmatrix}.$$

[C.U. B.Com.(H) 2003]

(d) If $2A+B=\begin{pmatrix}4 & 7 & 16\\ 7 & -3 & 12\\ 13 & 6 & 2\end{pmatrix}$ and $3B-A=\begin{pmatrix}5 & 0 & 13\\ 7 & -2 & 8\\ 11 & 4 & -1\end{pmatrix}$, find A and B. [C.U. B.Com. 2005]

3. (a) If $A=\begin{bmatrix}2 & 4\\ 5 & 6\end{bmatrix}$ and $B=\begin{bmatrix}3 & 6\\ 5 & 9\end{bmatrix}$, find the matrix X such that $3A+4B-2X=O$. [X is a square matrix of order 2×2.] [C.U. B.Com.(H) 2004]

(b) If $A=\begin{bmatrix}-5 & -2 & -8\\ 2 & 3 & -1\\ 3 & -4 & 3\end{bmatrix}$ and $B=\begin{bmatrix}6 & 2 & 7\\ -1 & -2 & 1\\ -3 & 5 & -2\end{bmatrix}$, find A^T+B^T and also find its rank.

[For Rank see Section 3.4.] [C.U. B.Com.(H) 1996]

[**Hints:** $A^T=\begin{pmatrix}-5 & 2 & 3\\ -2 & 3 & -4\\ -8 & -1 & 3\end{pmatrix}$, $B^T=\begin{pmatrix}6 & -1 & -3\\ 2 & -2 & 5\\ 7 & 1 & -2\end{pmatrix}$ and hence their sum $=\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ -1 & 0 & 1\end{pmatrix}$.

For rank see worked-out Exs 18 and 21 in Section 3.4.]

(c) Express the following in a single matrix: $\begin{bmatrix}3 & -2\\ 2 & 4\\ 5 & 0\end{bmatrix}\begin{bmatrix}-2 & 1 & 5\\ 1 & -2 & 3\end{bmatrix}-\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 7\\ 4 & 7 & 2\end{bmatrix}$.

[C.U. B.Com. 2009]

4. Correct or justify the statement: If A and B are two matrices such that $A+B$ and AB are both defined, then A and B are both square matrices of the same order. [C.U. B.Com.(H) 2002]

[**Hints:** If $A+B$ is defined, then A and B have the same order, say $m\times n$. If AB is defined, the no. of columns in A = the no. of rows in B, i.e., $m=n$. Thus, A and B have the same order $n\times n$, i.e., they are both square matrices of the same order.]

5. Given $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}^T$. [$T$ stands for Transpose.]

Verify that for these three matrices the following laws hold:

(a) The associative law: $A\cdot(B\cdot C)=(A\cdot B)\cdot C$.

(b) The distributive laws: $A\cdot(B+C)=A\cdot B+A\cdot C$; $(A+B)\cdot C=A\cdot C+B\cdot C$.

6. Let $A=\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\ 2 & 0\\ -1 & 1\end{bmatrix}$.

Evaluate AB and then $(AB)^T$. Now, obtain $A^T, B^T, (BA)^T$.

Hence verify: $(AB)^T = B^T A^T$.

[See Appendix for a general proof.]

7. (a) Let $A=\begin{bmatrix}3 & 6 & 9\\ 1 & 0 & 1\\ 2 & 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 5 & 6\\ 2 & 3 & 1\\ 1 & 0 & 2\end{bmatrix}$. Obtain $(A+B)^T$. Verify that

$$(A+B)^T=A^T+B^T.$$

(b) If $A-2B=\begin{bmatrix}-7 & 7\\ 4 & -8\end{bmatrix}$ and $A-3B=\begin{bmatrix}-11 & 9\\ 4 & -13\end{bmatrix}$, find the matrices A and B. Hence, obtain adjugate of B. [C.U. B.Com.(H) 1997]

[Hints: See hints given in Q. 1(b) set in 1997 at the end of the book.]

(c) Find the matrices A and B, given that $2A+B=\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}$, $3B-2A=\begin{bmatrix}10 & 1\\ 3 & 5\end{bmatrix}$. [C.U. B.Com.(H) 2005]

(d) If $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, prove that $(xA+yB)(xA-yB)=(x^2+y^2)A$. Is $AB=BA$ true in this problem?

8. (a) Let $A=\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}$. Obtain A^2 and then A^3.

(b) If $A=\begin{pmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{pmatrix}$, show that $A^2=A$. [B.U. B.Com.(H) 2007]

9. When are two matrices A and B said to be conformable for the product AB? If $A=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{pmatrix}$ and $B=\begin{pmatrix}6 & -2 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$, evaluate AB and also BA; check that $AB=BA$. Is it, in general, true for matrix multiplication? Give an example to justify your answer.

10. (a) If $A=\begin{pmatrix}2 & 0\\ -3 & 4\end{pmatrix}$, $B=\begin{pmatrix}3 & 1\\ 4 & 3\end{pmatrix}$, show that $(AB)^T=B^TA^T$. [B.U. B.Com.(H) 2008]

(b) Verify that for $A=\begin{pmatrix}-2 & 1 & 3\\ 0 & 4 & -1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1\\ -3 & 0\\ 4 & -5\end{pmatrix}$, $(AB)^T=B^TA^T$, where A^T is the transpose of A.

(c) Verify that $(AB)^T = B^T A^T$, where $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$.

[C.U. B.Com. 2005; V.U. B.Com.(H) 2009]

(d) If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and I be the 2×2 unit matrix, find $(A-2I)(A-3I)$. [V.U. B.Com.(H) 2011]

[Hints: $A-2I = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ and $A-3I = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$.

$\therefore (A-2I)(A-3I) = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

11. (a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, show that $A^2 - (a+d)A = (bc-ad)I$. [V.U. B.Com.(H) 2009]

(b) Prove that $A^2 - 10A - I = O$ for $A = \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$. [C.U. B.Com. 2005].

(c) If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, show that $A^2 - 5A + 7I = O$. [B.U. B.Com.(H) 2005]

(d) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then show that $A^2 - 4A + 3I = O$, where I is the identity matrix of order 2. Find also A^{-1}. [C.U. B.Com. 2010; V.U. B.Com.(H) 2008]

[Hints: 2nd part: $A^{-1}(A^2-4A+3I) = A^{-1}O$ or, $(A^{-1}A)A - 4(A^{-1}A) + 3A^{-1}I = O$ or, $IA - 4I + 3A^{-1} = O$ or, $3A^{-1} = -A + 4I = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Hence, $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$.]

12. (a) Let the matrix $A = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$, verify that $A^2 + 3A + 4I = O$ (I is the unit matrix of order two; O is 2×2 null matrix).

(b) If the matrix A be given by $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, then show that $A^2 - 4A + 3I = O$.

[C.U. B.Com.(H) 1994]

(c) Show that $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ satisfies the matrix equation $3A^3 - 4A^2 - A + 2I = O$.

13. A man buys 8 dozen of mangoes, 10 dozen of apples and 4 dozen of bananas. Mangoes cost ₹18 per dozen, apples ₹9 per dozen and bananas ₹6 per dozen. Represent the quantities by a row matrix, and prices by column matrix, and hence obtain the total cost.

14. Find the product AB of the two matrices $A = (1 \quad 2 \quad 3 \quad 4)$, $B = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

15. If $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $B = \begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix}$ and $i = \sqrt{-1}$ and $i^2 = -1$, determine AB, compute also BA.

16. Find the values of a, b, c, if the matrix A is given by $A = \frac{1}{3}\begin{pmatrix} a & 2 & 2 \\ 2 & 1 & b \\ 2 & c & 1 \end{pmatrix}$ obeys the law $A \cdot A' = I$ (A' is the transpose of A, and I is the unit matrix of order 3).

17. (a) If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, prove that $(aI + bE)^3 = a^3 I + 3a^2 bE$.

(b) Express the following in a single matrix $\begin{bmatrix} 3 & 2 & 5 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix}$.

[**Hints:** $\begin{bmatrix} 3 & 2 & 5 \\ 2 & -4 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 3\times1+2\times2+5\times3 & 3\times2+2\times(-1)+5\times5 \\ 2\times1+(-4)\times2+0\times3 & 2\times2+(-4)\times(-1)+0\times5 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 29 \\ -6 & 8 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 15 & 37 \\ -11 & -1 \end{bmatrix}$$]

18. Verify: $B'A' = (AB)'$, where $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$ (dashes denote transposes). Now, prove that $(A'B')' = BA$.

19. A company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are 5 divisions, 30 districts and 200 talukas in that state.

Each office has one head clerk, one cashier, one clerk and one peon. A divisional office has, in addition, an office superintendent, 2 clerks, 1 typist and 1 peon.

A district office has in addition one clerk and one peon.

The basic monthly salaries are as follows:

Office superintendent ₹200, Head clerk ₹200, Cashier ₹175, Clerks and Typists ₹150 and Peons ₹100. Using matrix notation find:

(a) the total number of posts of each kind in all the offices taken together;

(b) the total basic monthly salary bill of each kind of office; and

(c) the total basic monthly salary bill of all the offices taken together.

[**Hints:**

$$\begin{array}{l} \\ \text{Taluka office} \\ \text{Divisional office} \\ \text{District office} \end{array}
\begin{array}{c} \begin{array}{cccccc} \text{Office Supt.} & \text{Head clerk} & \text{Cashier} & \text{Clerk} & \text{Peon} & \text{Typist} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \end{pmatrix} \end{array} = A \text{ (say)}.$$

$$\begin{array}{l} \\ \text{Office Supt.} \\ \text{Head clerk} \\ \text{Cashier} \\ \text{Clerk} \\ \text{Peon} \\ \text{Typist} \end{array}
\begin{array}{c} \text{Salary} \\ \begin{pmatrix} 200 \\ 200 \\ 175 \\ 150 \\ 100 \\ 150 \end{pmatrix} \end{array} = B \text{ (say)}.$$

Multiply the first row vector [0 1 1 1 1 0] by 200, the second row vector [1 1 1 3 2 1] by 5 and the third row vector [0 1 1 2 2 0] by 30.

Thus, the number of posts at each level is given by

$$\begin{aligned}&[0\ 200\ 200\ 200\ 200\ 0]+[5\ 5\ 5\ 15\ 10\ 5]+[0\ 30\ 30\ 60\ 60\ 0]\\&=[5\ 235\ 235\ 275\ 270\ 5]\\&=\text{total number of posts of different kinds.}\end{aligned}$$

For each office at different levels the basic monthly salary bill is given by

$$AB=\begin{bmatrix}0&1&1&1&1&0\\1&1&1&3&2&1\\0&1&1&2&2&0\end{bmatrix}\begin{bmatrix}200\\200\\175\\150\\100\\150\end{bmatrix}=\begin{bmatrix}625\\1375\\875\end{bmatrix}.$$

The total basic monthly salary bill $[200\times 625+5\times 1375+30\times 875]=[125000+6875+26250]$
$=[158125]$].

20. If $A=\begin{pmatrix}0&1&-2\\1&2&3\\2&3&4\end{pmatrix}$ and $B=\begin{pmatrix}1&-2\\-1&0\\2&-1\end{pmatrix}$, then is BA defined? Justify your answer. Also, find A^{-1}.

[C.U. B.Com.(H) 1998]

21. If $A=\begin{bmatrix}2&5&3\\3&1&2\\1&2&-1\end{bmatrix}$, verify $AA^{-1}=I$, where $A^{-1}=\frac{1}{30}\begin{bmatrix}-5&11&7\\5&-5&5\\5&1&-13\end{bmatrix}$. [C.U. B.Com.(H) 1998]

22. Let $A=\begin{bmatrix}0&0&1\\0&0&1\\0&0&0\end{bmatrix}$, $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Show that $A^2B+B^2A=A$.

23. Show that the matrix $A=\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}$ is idempotent.

24. A square matrix A is called involutory if $A^2=I$. Prove that if A is involutory, then $(I+A)(I-A)=O$. Further, prove that if $(I+A)(I-A)=O$, then A is involutory.

25. Show that the matrices $A=\begin{bmatrix}ab&b^2\\-a^2&-ab\end{bmatrix}$, $B=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$ are both nilpotent — A is of index 2 and B is of index 3.

[Hints: Show that $A^2=O$ and $B^3=O$.]

26. Show that the square matrix $A=\begin{bmatrix}-5&-8&0\\3&5&0\\1&2&-1\end{bmatrix}$ is involutory. What is A^{-1}?

27. Examine, if $AB=BA$, where $A=\begin{bmatrix}-2&3&-1\\-1&2&-1\\-6&9&-4\end{bmatrix}$ and $B=\begin{bmatrix}1&3&-1\\2&2&-1\\3&0&-1\end{bmatrix}$.

28. For the matrices A, B, C, where

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}, B=\begin{bmatrix}0&-i\\i&0\end{bmatrix}, C=\begin{bmatrix}1&0\\0&-1\end{bmatrix},$$

verify the following relations: $A^2=B^2=C^2=I$
$AB=-BA,\ AC=-CA,\ BC=-CB.$

29. Show that $A=\begin{bmatrix}1&-3&-4\\-1&3&4\\1&-3&-4\end{bmatrix}$ is nilpotent of index 2.

30. If $AB=A$ and $BA=B$, then prove that $A^2=A$, $B^2=B$.

31. If A and B are symmetric matrices, then prove that AB is symmetric if and only if $AB=BA$.

ANSWERS

1. (i) (a) $\begin{bmatrix}3&3\\7&7\end{bmatrix}; \begin{bmatrix}-1&1\\-1&1\end{bmatrix}; \begin{bmatrix}10&7\\22&15\end{bmatrix};$
$\begin{bmatrix}-6&6\\-14&14\end{bmatrix}; \begin{bmatrix}7&10\\15&22\end{bmatrix};$
$\begin{bmatrix}8&5\\20&13\end{bmatrix}; \begin{bmatrix}-1&5\\-5&9\end{bmatrix};$
since $AB\neq BA$.

(b) $\begin{bmatrix}-1&2&2\\9&7&8\\2&-1&3\end{bmatrix}; \begin{bmatrix}5&0&2\\-3&3&6\\0&1&-1\end{bmatrix}.$
$\begin{bmatrix}2&2&5\\28&6&19\\-2&0&2\end{bmatrix}; \begin{bmatrix}-11&8&8\\24&29&52\\13&0&-5\end{bmatrix},$
$\begin{bmatrix}9&7&13\\28&28&48\\3&1&3\end{bmatrix}, \begin{bmatrix}15&-1&1\\-5&9&4\\-7&-3&3\end{bmatrix};$
$\begin{bmatrix}-6&8&12\\33&19&44\\10&4&0\end{bmatrix}$; since $AB\neq BA$.

(ii) (a) $A=\begin{pmatrix}1&1\\1&1\end{pmatrix};$

(b) $A=\begin{bmatrix}2&1&1\\2&3&2\end{bmatrix},$
$B=\begin{bmatrix}-1&1&-1\\1&2&2\end{bmatrix};$

(c) $\begin{bmatrix}0&1\\0&1\end{bmatrix};$

(d) $\begin{bmatrix}-7&0&6\\5&-1&7\end{bmatrix}.$

2. (a) $A=\begin{bmatrix}1&0&2\\2&-1&3\\-1&1&1\end{bmatrix}$ and
$B=\begin{bmatrix}0&1&-1\\2&-1&0\\-2&1&0\end{bmatrix};$

(b) $\begin{bmatrix}5\\1\\-1\end{bmatrix}; \begin{bmatrix}12\\2\\-7\end{bmatrix}; \begin{bmatrix}3\\3\\-5\end{bmatrix};$
$\begin{bmatrix}10\\8\\-13\end{bmatrix}.$

(c) $x=1, y=2, z=3;$

(d) $A=\begin{pmatrix}1&3&5\\2&-1&4\\4&2&1\end{pmatrix};$
$B=\begin{pmatrix}2&1&6\\3&-1&4\\5&2&0\end{pmatrix}.$

3. (a) $\begin{bmatrix}9&18\\35/2&27\end{bmatrix};$

(b) $\begin{bmatrix}1&1&0\\0&1&1\\-1&0&1\end{bmatrix}$; rank = 2;

(c) $\begin{bmatrix}-9&5&5\\-2&-9&15\\-14&-2&23\end{bmatrix}.$

4. A and B must be both square matrices of the same order.
$\begin{bmatrix}3&6&15&0&21\\6&3&9&27&24\end{bmatrix};$
$\begin{bmatrix}-1&-2&-5&0&-7\\-2&-1&-3&-9&-8\end{bmatrix}.$

6. $\begin{bmatrix}-3&5\\4&6\end{bmatrix}; \begin{bmatrix}-3&4\\5&6\end{bmatrix}; \begin{bmatrix}1&2\\-1&1\\2&0\end{bmatrix};$
$\begin{bmatrix}1&2&-1\\3&0&1\end{bmatrix}; \begin{bmatrix}7&2&1\\2&-2&2\\2&4&-2\end{bmatrix}.$

7. (a) $A^T+B^T=\begin{bmatrix}4&3&3\\11&3&3\\15&2&7\end{bmatrix}=(A+B)^T;$

(b) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $\begin{bmatrix} 4 & -2 \\ 0 & 5 \end{bmatrix}$; $\begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix}$;

(c) $\begin{bmatrix} -1/2 & 1 \\ 3/2 & -1/4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & 3/2 \end{bmatrix}$;

(d) yes, $AB = BA$.

8. (a) $\begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$; $\begin{bmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{bmatrix}$.

9. $AB = I_3 = BA$.

10. (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

13. $[8 \quad 10 \quad 4]$, $\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$ and ₹258.

14. [30].

15. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 2i & -2 \\ -2 & -2i \end{bmatrix}$.

16. $a = 1, b = -2, c = -2$.

20. BA is not defined;

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -1 & -10 & 7 \\ 2 & 4 & -2 \\ -1 & 2 & -1 \end{pmatrix}.$$

27. $AB = I_3 = BA$.

Chapter 2

Determinants: Definition, Properties, Minors and Cofactors, Adjoint of a Determinant, Cramer's Rule

2.1 Introduction: Definitions

1. Associated with every square matrix, there is a number called *determinant of the matrix* obtained by a rule which we shall specify below:

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix of order two, where a, b, c, d are the elements of A.

We associate the quantity

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

with the matrix A and call it the *determinant of the matrix A.*

[In the discussions that follow, we may not always refer the word matrix. We shall simply call *determinant.*]

- Since there are two rows and two columns, we call it a *determinant of second order* and its value is $ad - bc$.

  For example,

  (i) $\begin{vmatrix} 2 & 3 \\ -1 & -4 \end{vmatrix} = 2\cdot(-4) - 3\cdot(-1) = -8 + 3 = -5$, -5 is the value of the second order determinant $\begin{vmatrix} 2 & 3 \\ -1 & -4 \end{vmatrix}$.

  (ii) If $\begin{bmatrix} x & a \\ a & x \end{bmatrix} = 0$, then $x^2 - a^2 = 0$ or, $x^2 = a^2$ or, $x = \pm a$.

- If there are three rows and three columns, we call it a *determinant of third order.* Its value can be obtained by expressing it as an expression containing second order determinants: thus

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

$$\text{i.e.,}\quad D = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

Rule: Write down the elements of the first row with alternately positive and negative signs, $(a_1, -b_1, c_1)$. Each such signed element is multiplied by a second order determinant obtained by deleting the row and the column in which the element occurs; the sum of these products is the value of the determinant.

Illustration 1.

$$\begin{vmatrix} 3 & -2 & 2 \\ 6 & 1 & -3 \\ -1 & -2 & 3 \end{vmatrix} = +3\begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 6 & -3 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 6 & 1 \\ -1 & -2 \end{vmatrix}$$
$$= 3(3-6) + 2(18-3) + 2(-12+1) = -9 + 30 - 22 = -1.$$

- Similar rule can be used for determinants of order more than three. We shall, however, restrict ourselves to Second and Third order determinants only.[1]

Illustration 2. *A determinant of fourth order*

$$\begin{vmatrix} 3 & 6 & 2 & 3 \\ -2 & 1 & -2 & 2 \\ 4 & -5 & 1 & 4 \\ 1 & 3 & 4 & -2 \end{vmatrix}$$
$$= 3\begin{vmatrix} 1 & -2 & 2 \\ -5 & 1 & 4 \\ 3 & 4 & -2 \end{vmatrix} - 6\begin{vmatrix} -2 & -2 & 2 \\ 4 & 1 & 4 \\ 1 & 4 & -2 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 & 2 \\ 4 & -5 & 4 \\ 1 & 3 & -2 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 & -2 \\ 4 & -5 & 1 \\ 1 & 3 & 4 \end{vmatrix}$$
$$= 3\left\{1\begin{vmatrix} 1 & 4 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} -5 & 1 \\ 3 & 4 \end{vmatrix}\right\} - 6\left\{-2\begin{vmatrix} 1 & 4 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & 4 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix}\right\}$$
$$+ 2\left\{-2\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} 4 & 4 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & -5 \\ 1 & 3 \end{vmatrix}\right\} - 3\left\{-2\begin{vmatrix} -5 & 1 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & -5 \\ 1 & 3 \end{vmatrix}\right\}$$
$$= 3\{-2-16+20-24-40-6\} - 6\{4+32-16-8+32-2\} + 2\{-20+24+8+4+24+10\}$$
$$-3\{40+6-16+1-24-10\}$$
$$= 3\times(-68) - 6\times 42 + 2\times 50 - 3(-3) = -347.$$

2.2 Properties of Determinants

Property I. *The value of a determinant does not change if rows and columns are interchanged. In other words,* $\det A = \det A^T$*, where* A *is a square matrix of any order and* A^T *is its transpose.*

Illustration 1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ *(both have the value* $a_1b_2 - a_2b_1$*).*

Illustration 2. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

[1]Fourth order determinant is not included in the new Syllabus.

Verification.

$$\begin{aligned}
\text{LHS determinant} &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)\\
&= a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\
&= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = \text{RHS determinant.}
\end{aligned}$$

Remember. The significance of this property is that any property of a determinant established for rows must be true for columns also.

Property II. *If two rows (or columns) of a determinant are interchanged, then the numerical value remains same but the sign is altered. Thus, if*

$$D = \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} \text{ and } D_1 = \begin{vmatrix} a_2 & b_2 & c_2\\ a_1 & b_1 & c_1\\ a_3 & b_3 & c_3\end{vmatrix},$$

where in D_1 we have interchanged first and second rows of D, then it is easy to verify $D_1 = -D$.

Remember. Instead of first two rows we may interchange any two rows, say, second and third or first and third. We can instead interchange any two columns — the same result will follow.

Illustration 3. $\begin{vmatrix} 7 & 5 & 3\\ 0 & 1 & 0\\ 4 & 8 & 9\end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0\\ 7 & 5 & 3\\ 4 & 8 & 9\end{vmatrix}$ [*interchanging first and second rows*].

See that we can write the value of the determinant more easily now. The value is

$$-\left\{-1\begin{vmatrix} 7 & 3\\ 4 & 9\end{vmatrix}\right\} = 63 - 12 = 51.$$

Property III. *If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.*

Illustration 4. $\begin{vmatrix} a_1 & b_1\\ a_1 & b_1\end{vmatrix} = 0.$

Illustration 5. $\begin{vmatrix} 17 & 8 & 18\\ 9 & 3 & 4\\ 17 & 8 & 18\end{vmatrix} = 0$ (*first and third rows are identical*).

Justification. If two rows of a determinant D are interchanged, then it should be $= -D$. Let us interchange those two rows which are identical. Then structurally the determinant remains the same. Hence, $D = -D$ or, $2D = 0$, i.e., $D = 0$.

Property IV. *If all the elements of one row (or one column) are multiplied by the same number, then the value of the determinant is multiplied by that number.*

Illustration 6. *If* $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, *then clearly* $D_1 = kD$.

We can apply this artifice in the following manner:

Example 1. *Evaluate:* $\begin{vmatrix} 56 & 8 & 0 \\ 24 & 15 & 18 \\ 16 & 6 & 3 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 56 & 8 & 0 \\ 24 & 15 & 18 \\ 16 & 6 & 3 \end{vmatrix} = 8\begin{vmatrix} 7 & 1 & 0 \\ 24 & 15 & 18 \\ 16 & 6 & 3 \end{vmatrix} = 8 \times 3\begin{vmatrix} 7 & 1 & 0 \\ 8 & 5 & 6 \\ 16 & 16 & 3 \end{vmatrix}$$ [Taking 8 and 3 as common factors in Rows 1 and 2 respectively in succession]

$$= 8 \times 3 \times 3\begin{vmatrix} 7 & 1 & 0 \\ 8 & 5 & 2 \\ 16 & 16 & 1 \end{vmatrix}$$ (Taking 3 as a common factor in Col. 3)

$$= 72\left\{7\begin{vmatrix} 5 & 2 \\ 16 & 1 \end{vmatrix} - 1\begin{vmatrix} 8 & 2 \\ 16 & 1 \end{vmatrix}\right\} = 72\left\{7(5-32) - 8\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right\}$$

$$= 72\{-189 + 24\} = -72 \times 165 = \mathbf{-11{,}880}.$$

2.2.1 Minors and Cofactors

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

With each element of D we associate a second order determinant by deleting the row and column in which that element occurs. We call it the *minor of that element;* thus

$$\text{minor of } a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}; \text{ minor of } b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}; \text{ minor of } b_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ and so on.}$$

Next, we define cofactor of each element of D by multiplying its minor by $(-1)^{i+j}$, where i is the row and j is the column in which the element occurs: thus

$$\text{Cofactor of } a_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}; \text{ we denote it by } A_1, \text{ i.e., } A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$\text{Cofactor of } b_2 = (-1)^{2+2}\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}; \text{ we denote it by } B_2, \text{ i.e., } B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Cofactor of } b_3 = (-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}; \text{ we denote it by } B_3, \text{ i.e., } B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Now, we can write,

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1A_1 + b_1B_1 + c_1C_1.$$

Again,

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (Interchanging first and second rows)}$$

$$= -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_2A_2 + b_2B_2 + c_2C_2.$$

Similarly, $D = a_3A_3 + b_3B_3 + c_3C_3$.

Thus, the value of a determinant is the sum of the products of elements of a row and their corresponding cofactors.

Since a determinant is unaltered when rows and columns are interchanged. We have also

$$D = \begin{cases} a_1A_1 + a_2A_2 + a_3A_3 \\ b_1B_1 + b_2B_2 + b_3B_3 \\ c_1C_1 + c_2C_2 + c_3C_3, \end{cases}$$

where the capital letters are used as the cofactors of the corresponding small letters.

Property V. *If the elements of any row (or column) are multiplied by the cofactor of the corresponding elements of another row (or column), the sum of these products is zero.*

Take, for example, the elements of second row a_2, b_2, c_2 of the determinant

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let us multiply these elements by the cofactors of the corresponding elements of, say, third row, namely, A_3, B_3, C_3. Then according to this property $a_2A_3 + b_2B_3 + c_2C_3 = 0$.

See that

$$a_2A_3 + b_2B_3 + c_2C_3 = a_2\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_2\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ } (\because \text{two rows are identical}).$$

Similarly, one can easily check $a_1A_3 + b_1B_3 + c_1C_3 = 0$, etc.

We may write down similar results with columns, e.g., $a_1B_1 + a_2B_2 + a_3B_3 = 0$.

Property VI. (Sum of Two Determinants) *If each element of any row (or column) is expressed as the sum of two numbers, the determinant can be expressed as the sum of two determinants whose remaining rows (or columns) are not altered.*

Consider the determinant

$$D=\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Justification. Let us denote

the cofactor of $a_1+\alpha_1$ by $A_1=\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$,

the cofactor of $b_1+\beta_1$ by $B_1=-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and

the cofactor of $c_1+\gamma_1$ by $C_1=\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

Then,

$$\begin{aligned} D &= (a_1+\alpha_1)A_1+(b_1+\beta_1)B_1+(c_1+\gamma_1)C_1 \\ &= (a_1A_1+b_1B_1+c_1C_1)+(\alpha_1A_1+\beta_1B_1+\gamma_1C_1) \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

Most Useful Property for Evaluating Determinants

Property VII. *The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any of the other rows (or columns).*

This property means that, for example,

$$\text{if } D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_1=\begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

we easily see that $D_1=D$.

For

$$\begin{aligned} D_1 &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= D+p\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= D+p\times 0+q\times 0=D. \end{aligned}$$

[Two rows being identical each of the last two determinants is zero.]

Note: p and q may have any positive or negative values. Instead of row, we can apply this artifice in a column.

Property VIII. *If two rows (or columns) of a determinant become identical when $x=a$ when $x=a$, then $(x-a)$ is a factor of the determinant. In this case, if r rows (or columns) are identical, when $x=a$, then $(x-a)^{r-1}$ is a factor of the determinant.*

Applications: Evaluation of Determinants using Various Properties

Example 2. (a) *Find the value of the determinant:* $\begin{vmatrix} 1 & 1 & 1 \\ 5 & 6 & 3 \\ 12 & 7 & 11 \end{vmatrix}$.

(b) *If* $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & x \\ 1 & 0 & 2 \end{vmatrix} = 6$, *find the value of* x. [C.U. B.Com. 2007]

Solution:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 5 & 6 & 3 \\ 12 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 1-1 & 1-1 \\ 5 & 6-5 & 3-5 \\ 12 & 7-12 & 11-12 \end{vmatrix}$ [new second column = old second column +(−1) old first column, i.e., = old second column − old first column, new third column = old third column−old first column]

$= \begin{vmatrix} 1 & 0 & 0 \\ 5 & 1 & -2 \\ 12 & -5 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -5 & -1 \end{vmatrix}$ (since in the first row two elements are already zero, we can at once expand by using definition)

$= -1 - 10 = \mathbf{-11}$.

(b) $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & x \\ 1 & 0 & 2 \end{vmatrix} = 6$ or, $1 \cdot \begin{vmatrix} 2 & x \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & x \\ 1 & 2 \end{vmatrix} + 0 = 6$ or, $(4-0) - 2(0-x) = 6$ or, $4 + 2x = 6$ or, $2x = 2$; $\therefore \boldsymbol{x = 1}$.

Example 3. *Find the value of the determinant*: (i) $\begin{vmatrix} 16 & 12 & 17 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix}$; (ii) $\begin{vmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 0 \end{vmatrix}$.

Solution:

(i) $\begin{vmatrix} 16 & 12 & 17 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 16-3\times5 & 12-3\times4 & 17-3\times6 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix}$ [New first row = old first row −3× old second row]

$= \begin{vmatrix} 1 & 0 & -1 \\ 5 & 4 & 6 \\ 9 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 4 & 11 \\ 9 & 7 & 20 \end{vmatrix}$ [New third col. = old third col. + old first col.]

$= 1\begin{vmatrix} 4 & 11 \\ 7 & 20 \end{vmatrix} = 80 - 77 = \mathbf{3}$.

(ii) $\Delta = \begin{vmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 0 \end{vmatrix} = -1\times-1\times-1\begin{vmatrix} 0 & -x & y \\ x & 0 & -z \\ -y & z & 0 \end{vmatrix} = -\begin{vmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 0 \end{vmatrix}$

[interchanging rows and columns]

i.e., $\Delta = -\Delta$ or, $\Delta + \Delta = 0$ or, $2\Delta = 0$ or, $\boldsymbol{\Delta = 0}$.

Example 4. *Find the values of* x, *when* $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x+1 & -1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$. [C.U. B.Com.(H) 1998; (P) 2010]

Solution:

$$\text{LHS}=\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x+1 & -1 \\ -1 & 1 & x+1 \end{vmatrix}=\begin{vmatrix} x+1 & 1 & 1 \\ x+1 & x+1 & -1 \\ x+1 & 1 & x+1 \end{vmatrix}$$
[Adding the elements of second and third columns with that of the first column]

$$=(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x+1 & -1 \\ 1 & 1 & x+1 \end{vmatrix}=(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x & -2 \\ 0 & 0 & x \end{vmatrix}$$
[Subtracting the elements of first row from second row and the elements of first row from third row]

$$=(x+1)\cdot 1\cdot\begin{vmatrix} x & -2 \\ 0 & x \end{vmatrix}=(x+1)\cdot x^2$$
[Subtracting the elements of first row from second row and the elements of first row from third row].

$\therefore$ from given relation, we get $(x+1)\,x^2=0$; $\therefore$ $x+1=0$ or, $x^2=0$;
$\therefore$ $x=-1$, or $x=0,0$. Hence $\boldsymbol{x=0,0,-1}$.

Example 5. *Find the value of x in terms of a, b, c, given that,*

(i) $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}=0$; (ii) $\begin{vmatrix} x & a & a \\ a & x & b \\ b & b & x \end{vmatrix}=0$. [B.U. B.Com.(H) 2006; V.U. B.Com.(H) 2007]

Solution: (i) We have

$$\begin{vmatrix} x+a+b+c & b & c \\ c+x+b+a & x+b & a \\ a+b+x+c & b & x+c \end{vmatrix}=0$$
(Adding the elements of second and third columns with the elements of the first column)

or, $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}=0$$

or, $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x+b-b & a-c \\ 0 & b-b & x+c-c \end{vmatrix}=0 \quad \begin{pmatrix} R_2'=R_2-R_1 \\ R_3'=R_3-R_1 \end{pmatrix}$$

or, $(x+a+b+c)\,x^2=0$.

$\therefore$ **either** $\boldsymbol{x=0}$ **or** $\boldsymbol{x=-a-b-c}$.

(ii) $$\begin{vmatrix} x & a & a \\ a & x & b \\ b & b & x \end{vmatrix}=0 \text{ or, } \begin{vmatrix} x+a+b & x+a+b & x+a+b \\ a & x & b \\ b & b & x \end{vmatrix}=0$$
[Adding R_2 and R_3 with R_1]

or, $$(x+a+b)\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ b & b & x \end{vmatrix}=0 \text{ or, } (x+a+b)\begin{vmatrix} 1 & 0 & 1 \\ a & x-a & b \\ b & 0 & x \end{vmatrix}=0 \text{ [by } C_2-C_1]$$

or, $(x+a+b)[1\cdot\{x(x-a)-0\}+1\cdot\{0-b(x-a)\}]=0$

or, $(x+a+b)(x-a)(x-b)=0$; $\therefore$ $\boldsymbol{x=a,b,-(a+b)}$.

Example 6. *Show that* $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$ [C.U. B.Com. 2010]

Solution:

$$\text{LHS} = \begin{vmatrix} a+b+2c+a+b & a & b \\ c+b+c+2a+b & b+c+2a & b \\ c+a+c+a+2b & a & c+a+2b \end{vmatrix} \text{ (Adding 2nd and 3rd cols with 1st col.)}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \text{ (Subtracting first row from 2nd and 3rd rows)}$$

$$= 2(a+b+c)\cdot 1\cdot\begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$$= 2(a+b+c)(b+c+a)(c+a+b) = 2(a+b+c)^3 = \text{RHS } \textbf{(Proved).}$$

Example 7. *Prove that* $\begin{vmatrix} x & y & 1 \\ x^2 & y^2 & 1 \\ x^3 & y^3 & 1 \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & x^2 & x^3 \\ y & y^2 & y^3 \end{vmatrix}$. *Find its value.*

Solution: The middle (i.e., second) determinant can be obtained from the first determinant by interchanging rows into columns. The last determinant can be obtained from the second by first interchanging the second and third rows and then interchanging the first and the new second rows.

We can evaluate

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x^2 & x^3 \\ y & y^2 & y^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & x^2-x & x^3-x \\ y & y^2-y & y^3-y \end{vmatrix} \begin{bmatrix} \text{Col}_2-\text{Col}_1 \\ \text{Col}_3-\text{Col}_1 \end{bmatrix} = \begin{vmatrix} x(x-1) & x(x-1)(x+1) \\ y(y-1) & y(y-1)(y+1) \end{vmatrix}$$

$$= x(x-1)y(y-1)\begin{vmatrix} 1 & x+1 \\ 1 & y+1 \end{vmatrix} = x(x-1)y(y-1)\{y+1-(x+1)\}$$

$$= \boldsymbol{xy(y-x)(x-1)(y-1).}$$

Otherwise: See that we can also evaluate thus:

$$\begin{vmatrix} x & y & 1 \\ x^2 & y^2 & 1 \\ x^3 & y^3 & 1 \end{vmatrix} = xy\begin{vmatrix} 1 & 1 & 1 \\ x & y & 1 \\ x^2 & y^2 & 1 \end{vmatrix} = xy\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & 1-x \\ x^2 & y^2-x^2 & 1-x^2 \end{vmatrix} = xy\begin{vmatrix} y-x & 1-x \\ y^2-x^2 & 1-x^2 \end{vmatrix}$$

$$= xy(y-x)(1-x)\begin{vmatrix} 1 & 1 \\ y+x & 1+x \end{vmatrix} = xy(y-x)(1-x)(1+x-y-x)$$

$$= \boldsymbol{xy(y-x)(x-1)(y-1).}$$

Example 8. *Prove that* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x).$

[C.U. B.Com.(H) 2007; B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2011]

Solution:

$$\text{LHS} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & -z(y-x) & -y(z-x) \end{vmatrix} \quad \begin{pmatrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{pmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ -z & -y \end{vmatrix}$$

$$= (y-x)(z-x)(-y+z)$$

$$= \mathbf{(x-y)(y-z)(z-x)} = \text{RHS (Proved).}$$

Example 9. *Prove that* $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$ [C.U. B.Com.(H) 2002]

Solution:

$$\text{LHS} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c+a & c+a+b & a+b+c \end{vmatrix} \quad \text{[Adding 1st row with the third]}$$

$$= (a+b+c)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \\ 1 & 0 & 0 \end{vmatrix} \quad [C_2' = C_2 - C_1 \text{ and } C_3' = C_3 - C_1]$$

$$= (a+b+c)(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ a^2 & b+a & c+a \\ 1 & 0 & 0 \end{vmatrix} \quad \text{[Taking } (b-a) \text{ and } (c-a) \text{ common]}$$

$$= (a+b+c)(b-a)(c-a)\cdot 1\cdot\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \quad \text{[Expanding in terms of the elements of the 3rd row]}$$

$$= (a+b+c)(b-a)(c-a)\cdot 1\cdot(c+a-b-a) = (a+b+c)(b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)(a+b+c) = \text{RHS } \textbf{(Proved)}.$$

Example 10. *If* $x+y+z=0$, *show that* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = 0.$ [C.U. B.Com.(H) 1991; B.U. B.Com.(H) 2006]

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} \quad \begin{pmatrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{pmatrix}$$

$$= \begin{vmatrix} y-x & z-x \\ y^3-x^3 & z^3-x^3 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y^2+yx+x^2 & z^2+zx+x^2 \end{vmatrix}$$

$$= (y-x)(z-x)\{z^2+zx+x^2-y^2-xy-x^2\}$$

$$= (y-x)(z-x)\{z^2-y^2+x(z-y)\}$$

$$= (y-x)(z-x)(z-y)(z+y+x) = 0 \textbf{ (Proved)}. \ [\because x+y+z=0]$$

Example 11. *Prove that* $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}\right)$

$= a_1a_2a_3 + a_2a_3 + a_1a_3 + a_1a_2.$

[C.U. B.Com.(H) 2001; B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2010]

Solution: Subtract the top row from each of the others and then expand with reference to the top row:

$$\begin{vmatrix} 1+a_1 & 1 & 1 \\ -a_1 & a_2 & 0 \\ -a_1 & 0 & a_3 \end{vmatrix} = (1+a_1)\begin{vmatrix} a_2 & 0 \\ 0 & a_3 \end{vmatrix} - 1\begin{vmatrix} -a_1 & 0 \\ -a_1 & a_3 \end{vmatrix} + 1\begin{vmatrix} -a_1 & a_2 \\ -a_1 & 0 \end{vmatrix}$$

$$= (1+a_1)a_2a_3 + a_1a_3 + a_1a_2$$

$$= a_1a_2a_3 + a_2a_3 + a_1a_3 + a_1a_2 \text{ (Proved)}$$

$$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right) \text{ (Proved).}$$

Example 12. *Without actual expansion (using properties of determinants only) prove that*

$$\begin{vmatrix} x+y & 2x+y & 3x+y \\ 2x+y & 3x+y & 4x+y \\ 5x+y & 6x+y & 7x+y \end{vmatrix} = 0.$$

Solution: Here the given determinant

$$= \begin{vmatrix} x+y & 2x+y-(x+y) & 3x+y-(x+y) \\ 2x+y & 3x+y-(2x+y) & 4x+y-(2x+y) \\ 5x+y & 6x+y-(5x+y) & 7x+y-(5x+y) \end{vmatrix} \quad \begin{pmatrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{pmatrix}$$

$$= \begin{vmatrix} x+y & x & 2x \\ 2x+y & x & 2x \\ 5x+y & x & 2x \end{vmatrix} = 2\begin{vmatrix} x+y & x & x \\ 2x+y & x & x \\ 5x+y & x & x \end{vmatrix} = 0 \text{ (Proved). (Since two columns are identical.)}$$

Example 13. *Without formal expansion by using properties of determinants prove that*

$$\begin{vmatrix} x^2 & x & yz \\ y^2 & y & zx \\ z^2 & z & xy \end{vmatrix} = \begin{vmatrix} x^3 & x^2 & 1 \\ y^3 & y^2 & 1 \\ z^3 & z^2 & 1 \end{vmatrix}.$$

Solution: LHS determinant $= \begin{vmatrix} x^2 & x & yz \\ y^2 & y & zx \\ z^2 & z & xy \end{vmatrix}$.

Multiply first row by x, second row by y, third row by z, and then write

$$= \frac{1}{xyz}\begin{vmatrix} x^3 & x^2 & xyz \\ y^3 & y^2 & xyz \\ z^3 & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x^3 & x^2 & 1 \\ y^3 & y^2 & 1 \\ z^3 & z^2 & 1 \end{vmatrix} = \begin{vmatrix} x^3 & x^2 & 1 \\ y^3 & y^2 & 1 \\ z^3 & z^2 & 1 \end{vmatrix}$$

= RHS determinant **(Proved).**

Example 14. *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

Solution:

$$\text{LHS determinant} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix} \begin{pmatrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{pmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$ [Taking out $a+b+c$ from 2nd and 3rd columns]

Subtracting the sum of second and third rows from the first and taking 2 outside

$$= 2(a+b+c)^2 \begin{vmatrix} bc & -c & -b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} bc & 0 & 0 \\ b^2 & c+a & \frac{b^2}{c} \\ c^2 & \frac{c^2}{b} & a+b \end{vmatrix}$$ [Adding $1/b$ times the 1st column with the 2nd and $1/c$ times the 1st column with the 3rd]

$$= 2(a+b+c)^2 \cdot bc \begin{vmatrix} c+a & \frac{b^2}{c} \\ \frac{c^2}{b} & a+b \end{vmatrix} = 2(a+b+c)^2 \cdot bc\left\{(c+a)(a+b) - \frac{b^2}{c} \times \frac{c^2}{b}\right\}$$

$$= 2(a+b+c)^2 \cdot bc\left\{ca+bc+a^2+ab-bc\right\} = 2(a+b+c)^2 \cdot bc\{a(c+a+b)\}$$

$$= 2abc(a+b+c)^3 = \text{RHS } \textbf{(Proved)}.$$

Example 15. *Show that* $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$ **[C.U. B.Com.(H) 2008]**

Solution:

$$\text{LHS} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = \begin{vmatrix} -(a+b+c) & 0 & 2a \\ 0 & -(a+b+c) & 2b \\ (a+b+c) & a+b+c & c-a-b \end{vmatrix} \begin{pmatrix} C_1' = \text{col}_1 - \text{col}_3 \\ C_2' = \text{col}_2 - \text{col}_3 \end{pmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 1 & 1 & c-a-b \end{vmatrix} = (a+b+c)^2 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 2b \\ 1 & 1 & c+a-b \end{vmatrix}$$ [Multiply col_1 by $2a$ and add with col_3]

$$= -(a+b+c)^2 \begin{vmatrix} -1 & 2b \\ 1 & c+a-b \end{vmatrix}$$

$$= -(a+b+c)^2\{-c-a+b-2b\} = (a+b+c)^3 = \text{RHS } \textbf{(Proved)}.$$

Example 16. *Prove that* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$ *and hence show that*

$$\begin{vmatrix} y^2z^2+x^2w^2 & yz+xw & 1 \\ z^2x^2+y^2w^2 & zx+yw & 1 \\ x^2y^2+z^2w^2 & xy+zw & 1 \end{vmatrix} = (x-y)(y-z)(z-x)\times(x-w)(y-w)(z-w).$$

[B.U. B.Com.(H) 2005]

Solution: 1st part:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \quad \begin{pmatrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{pmatrix}$$

$$= \begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z+x-y-x) = (x-y)(y-z)(z-x) \text{ (Proved).} \qquad (1)$$

2nd part: The given determinant (Δ)

$$= \begin{vmatrix} (yz+xw)^2-2xyzw & yz+xw & 1 \\ (zx+yw)^2-2xyzw & zx+yw & 1 \\ (xy+zw)^2-2xyzw & xy+zw & 1 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} - 2xyzw\begin{vmatrix} 1 & \alpha & 1 \\ 1 & \beta & 1 \\ 1 & \gamma & 1 \end{vmatrix},$$

where $\alpha = yz+xw$, $\beta = zx+yw$, $\gamma = xy+zw$.

The second determinant is zero, since first and third columns are identical and hence changing rows into columns and columns into rows the given determinant becomes

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

By Prop. II: 1st and 2nd rows interchanged, 2nd and 3rd rows are interchanged

$= -(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$, using (1).

Now, $$\alpha - \beta = (yz+xw)-(zx+yw) = (yz-zx)+(xw-yw)$$
$$= z(y-x) - w(y-x) = (y-x)(z-w) - (x-y)(z-z).$$

Similarly, $\beta-\gamma = -(y-z)(x-w)$ and $\gamma-\alpha = -(z-x)(y-w)$.
Hence $\Delta = (x-y)(y-z)(z-x)(x-w)(y-w)(z-w)$ **(Proved).**

Example 17. *Without formally expanding, find the value of* $D = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

[C.U. B.Com.(H) 2002]

Solution:

$$D = \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \quad [C_1' = C_1+C_2+C_3]$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = \mathbf{0}.$$

Example 18. *Show that the value of the determinant* $D=\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$ *is independent of x and prove that its value* = −2.

[C.U. B.Com.(H) 1996]

Solution:

$$D=\begin{vmatrix} x+1 & 1 & 3 \\ x+3 & 2 & 5 \\ x+7 & 3 & 7 \end{vmatrix} \begin{pmatrix} C_2'=C_2-C_1 \\ C_3'=C_3-C_1 \end{pmatrix}$$

$$=\begin{vmatrix} x+1 & 1 & 1 \\ x+3 & 2 & 1 \\ x+7 & 3 & 1 \end{vmatrix} \left(C_3'=C_3-2C_2\right)$$

$$=\begin{vmatrix} x+1 & 1 & 1 \\ 2 & 1 & 0 \\ 6 & 2 & 0 \end{vmatrix} \begin{pmatrix} R_2'=R_2-R_1 \\ R_3'=R_3-R_1 \end{pmatrix}.$$

Expanding in terms of 3rd column

$$=\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}=4-6=-2 \text{ (independent of } x\text{) } \textbf{(Proved)}.$$

Example 19. *Evaluate* $D=\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$ *and then prove that* $D=0$, *if* $x+y+z=0$.

[C.U. B.Com.(H) 2002 (Old)]

Solution:

$$D=\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \left(R_3'=R_3+R_1\right)$$

$$=(x+y+z)\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \text{ [Taking } x+y+z \text{ common from the third row]}$$

$$=(x+y+z)\begin{vmatrix} x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \\ 1 & 0 & 0 \end{vmatrix} \begin{pmatrix} C_2'=C_2-C_1 \\ C_3'=C_3-C_1 \end{pmatrix}$$

$$=(x+y+z)\begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix} \text{ [Expanding in terms of third row]}$$

$$=(x+y+z)(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} \text{ [Taking } (y-x), (z-x) \text{ common from 1st col. and 3rd col. respectively]}$$

$$=(x+y+z)(y-x)(z-x)(z+x-y-x).$$

$$\therefore\ D=\boldsymbol{(y-z)(z-x)(x-y)(x+y+z)}.$$

Now, $D=(x-y)(y-z)(z-x)(x+y+z)=0$ **(Proved)** [$\because\ \boldsymbol{x+y+z=0}$].

EXERCISES ON CHAPTER 2(I)

(Determinants)

1. Evaluate the following second order determinants:

(a) $\begin{vmatrix} 0 & 3 \\ 2 & -5 \end{vmatrix}$; (b) $\begin{vmatrix} x+y & x+2y \\ x-y & x-2y \end{vmatrix}$.

2. Evaluate the following third order determinants:

(a) $\begin{vmatrix} 3 & -2 & 2 \\ 1 & 4 & 5 \\ 6 & -1 & 2 \end{vmatrix}$; (b) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$; (c) $\begin{vmatrix} 2 & 3 & 2 \\ 0 & -2 & 1 \\ -1 & 4 & 0 \end{vmatrix}$;

(d) $\begin{vmatrix} x-2 & y+3 & z-2 \\ -2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}$; (e) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$; (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

3. Find all real values of x in the following equations:

(a) $\begin{vmatrix} x+2 & 3 & 3 \\ 3 & x+4 & 5 \\ 3 & 5 & x+4 \end{vmatrix} = 0$; (b) $\begin{vmatrix} 2-x & 2 & 3 \\ 2 & 5-x & 6 \\ 3 & 4 & 10-x \end{vmatrix} = 0$;

(c) $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x+1 & 1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$. [C.U. B.Com.(H) 1998]

4. Using properties of determinants, evaluate the following determinants:

(a) $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$; (c) $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 6 & 9 \\ 5 & 6 & 7 \end{vmatrix}$;

(d) $\begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$; (e) $\begin{vmatrix} 7 & 11 & 13 \\ 17 & 19 & 23 \\ 29 & 31 & 37 \end{vmatrix}$; (f) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$;

(g) $\begin{vmatrix} 46 & 38 & 16 \\ 27 & 21 & 9 \\ 18 & 17 & 7 \end{vmatrix}$; (h) $\begin{vmatrix} 21 & 10 & 25 \\ 26 & 13 & 32 \\ 15 & 7 & 17 \end{vmatrix}$; (i) $\begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 9 & 4 & 5 \end{vmatrix}$. [V.U. B.Com.(H) 2008]

5. Prove that

(a) $(\beta-\gamma)$ is a factor of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$ and its value $=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

[C.U. B.Com. 2008; V.U. B.Com.(H) 2010]

(b) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$. [C.U. B.Com.(H) 2007; V.U. B.Com.(H) 2011]

[Hints: Apply $R_2' = R_2 - R_1$ and $R_3' = R_3 - R_1$. See worked-out Ex. 8.]

(c) $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx);$

(d) $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2;$

(e) $\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} = (x+a+b)(x-a)(x-b);$ [C.U. B.Com. 2004]

(f) $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2.$

[Hints: (c) Take a, b, c common from R_1, R_2 and R_3. (e) Add R_2, R_3 with R_1.]

6. Without formally expanding the determinant prove that

(a) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0;$ [V.U. B.Com.(H) 2010] (b) $\begin{vmatrix} x+y & 2x+y & 3x+y \\ 2x+y & 3x+y & 4x+y \\ 5x+y & 6x+y & 7x+y \end{vmatrix} = 0.$

[See worked-out Ex. ??] [B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2010]

7. Find the value of:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}$; (b) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$; (c) $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix}$;

(d) $\begin{vmatrix} x^2+y^2+1 & x^2+2y^2+3 & x^2+3y^2+4 \\ y^2+2 & 2y^2+6 & 3y^2+8 \\ y^2+1 & 2y^2+3 & 3y^2+4 \end{vmatrix}$. [C.U. B.Com.(H) 1994]

(e) Show that: $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$. [B.U. B.Com. 2007]

[Hints: (d)

$$\begin{vmatrix} x^2+y^2+1 & x^2+2y^2+3 & x^2+3y^2+4 \\ y^2+2 & 2y^2+6 & 3y^2+8 \\ y^2+1 & 2y^2+3 & 3y^2+4 \end{vmatrix} = \begin{vmatrix} x^2 & x^2 & x^2 \\ 1 & 3 & 4 \\ y^2+1 & 2y^2+3 & 3y^2+4 \end{vmatrix}$$ [Subtracting 3rd row from the 1st and 2nd rows]

$$= \begin{vmatrix} x^2 & 0 & 0 \\ 1 & 2 & 3 \\ y^2+1 & y^2+2 & 2y^2+3 \end{vmatrix}$$ [Subtracting 1st column from the 2nd and 3rd columns]

$$= x^2 \begin{vmatrix} 2 & 3 \\ y^2+2 & 2y^2+3 \end{vmatrix} = x^2(4y^2+6-3y^2-6) = x^2y^2.$$]

8. Prove that

(a) $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc;$

(b) $\begin{vmatrix} 2a-b-c & 3b & 3c \\ 3a & 2b-c-a & 3c \\ 3a & 3b & 2c-a-b \end{vmatrix} = 2(a+b+c)^3;$

(c) $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2;$

(d) $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc;$

(e) $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$ [B.U. B.Com.(H) 2007]

[**Hints**: (c) Take a, b, c common from R_1, R_2, R_3 and then multiply C_1, C_2, C_3 by a, b, c. Add C_2', C_3' with C_1'.

(d) Take $1/c$, $1/a$, $1/b$ common from R_1, R_2, R_3 and then apply $R_1-(R_2+R_3)$. Then apply $R_2' = R_2 - R_1$ and $R_3' = R_3 - R_1$.

(e) Apply $R_1' = R_1 + (R_2+R_3)$ and take 2 common to obtain $2\begin{vmatrix} b+c & c+a & a+b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$.

Now apply $C_1' = C_1 + (C_2+C_3)$ and take 2 common to obtain $4\begin{vmatrix} a+b+c & c+a & a+b \\ c+a & c+a & a \\ a+b & a & a+b \end{vmatrix}$.

Apply $R_1' = R_1 - R_2$ and $R_2' = R_2 - R_3$ to obtain $4\begin{vmatrix} b & 0 & b \\ c-b & c & -b \\ a+b & a & a+b \end{vmatrix}$.

Apply $R_2' = R_2 + R_1$ and $R_3' = R_3 - R_1$.]

9. Prove that

(a) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix};$ (b) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc;$

(c) $\begin{vmatrix} a^2 & bc & c^2+ca \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

[**Hints**: (a) Apply $C_1' = C_1 + C_2 - C_3$.

(b) Apply $R_1' = R_1 - R_2$ and $R_3' = R_3 - R_2$ and then take 2 common. Then apply $R_2' = R_2 + R_3$ and expand.

(c) Take a, b, c common from C_1, C_2, C_3 and then apply $C_1' = C_1 + C_2 - C_3$.]

10. Without actually expanding, by using properties of determinants prove that

(a) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix};$ [C.U. B.Com. 2009]

(b) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0;$ (c) $\begin{vmatrix} x^2 & x & yz \\ y^2 & y & zx \\ z^2 & z & xy \end{vmatrix} = \begin{vmatrix} x^3 & x^2 & 1 \\ y^3 & y^2 & 1 \\ z^3 & z^2 & 1 \end{vmatrix};$

(d) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = 0$, when $x+y+z=0$. [C.U. B.Com.(H) 1997]

[Hints: (a) RHS $= \frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix} = \frac{1}{xyz} \times xyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$.

(b) Express LHS as the difference of two determinants and use (a).

(c) See worked-out Ex. 19.]

11. (a) Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$.

[C.U. B.Com.(H) 2000]

(b) Show that $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$.

[C.U. B.Com.(H) 1996]

[Hints: (a) Apply $C_1' = C_1 - bC_3$ and $C_2' = C_2 + aC_3$.

(b) $\frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}$

$$= \begin{vmatrix} b^2-a^2 & b^3-a^3 \\ c^2-a^2 & c^3-a^3 \end{vmatrix}$$
$$= (b-a)(c-a)\left\{(b+a)\left(c^2+ca+a^2\right) - (c+a)\left(b^2+ab+a^2\right)\right\}$$
$$= (b-a)(c-a)\left\{bc^2+ac^2-b^2c-ab^2\right\}$$
$$= (b-a)(c-a)\left\{bc(c-b)+a\left(c^2-b^2\right)\right\}$$
$$= (a-b)(c-a)(b-c)(bc+ac+ab).]$$

12. Prove that $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (b-c)(c-a)(a-b)$. [C.U. B.Com. 2005]

[Hints: Apply $R_2' = R_2 - R_1$ and $R_3' = R_3 - R_2$, and then take $(a-b)$ and $(b-c)$ common.]

13. Prove that

(a) $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = (3x-2)(3x-11)^2$;

(b) $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(ab+bc+ca)^3$. [C.U. B.Com.(H) 1999]

(c) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$;

(d) $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} = -2.$ [C.U. B.Com.(H) 1996]

[**Hints:**

(b) LHS determinant $\equiv \Delta = \begin{vmatrix} \frac{b^2c^2}{a^2} & c^2 & b^2 \\ c^2 & \frac{c^2a^2}{b^2} & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix}$ Putting $ab+bc+ca=0$ or, $a(b+c)=-bc$ or, $(b+c)^2 = \frac{b^2c^2}{a^2}$; similarly, $(c+a)^2 = \frac{c^2a^2}{b^2}$ in the LHS

$$= \frac{1}{a^2b^2}\begin{vmatrix} b^2c^2 & c^2a^2 & a^2b^2 \\ b^2c^2 & c^2a^2 & a^2b^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 0 \text{ } (\because \text{1st and 2nd rows are identical}).$$

$\therefore$ $(ab+bc+ca)$ is a factor of Δ. Similarly, putting $ab+bc+ca=0$ in the 2nd and 3rd rows, 1st and 3rd rows, we get $\Delta = 0$.

$\therefore$ $(ab+bc+ca)$ is a repeated factor of Δ, it being repeated thrice;

$\therefore$ $(ab+bc+ca)^3$ is a factor of Δ. Now, expansion of Δ gives 6th degree terms and $(ab+bc+ca)^3$ also gives 6th degree terms. We can write

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} \equiv k(ab+bc+ca)^3,$$

where k is numerical. Putting $a=0$, $b=1$, $c=1$, we get

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = k(0+1+0)^3 \Rightarrow k=2.$$

Hence, $\Delta = 2(ab+bc+ca)^3$.

(d) Apply $C_2' = C_2 - C_1$, $C_3' = C_3 - C_2$ and then $C_3'' = C_3' - C_2$. Then apply $R_2' = R_2 - R_1$ and $R_3' = R_3 - R_1$.]

14. Prove that

(a) $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)\left(a^2+b^2+c^2\right).$

(b) $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = (xyz+1)(x-y)(y-z)(x-z).$

[**Hints:** (a) Apply $C_2' = C_2 - 2C_3 - 2C_1$.]

15. Evaluate: $\begin{vmatrix} 1 & x-y & (x-y)^2 \\ 1 & y-z & (y-z)^2 \\ 1 & z-x & (z-x)^2 \end{vmatrix}$. [C.U. B.Com. 2005]

[**Hints:** Taking $x-y=c$, $y-z=a$ and $z-x=b$, the given determinant becomes

$$\begin{vmatrix} 1 & c & c^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = \begin{vmatrix} 1 & c & c^2 \\ 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 1 & a+c \\ 1 & b+c \end{vmatrix}$$

$$= (a-c)(b-c)(b+c-a-c) = (b-c)(c-a)(a-b)$$

$$= (y+z-2x)(z+x-2y)(x+y-2z).]$$

16. Prove that $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3.$

[Hints:

$$\Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$ (Multiplying R_1, R_2, R_3 by a, b, c and then taking a, b, c common from C_1, C_2, C_3)

$$= \begin{vmatrix} bc+ca+ab & bc+ca+ab & bc+ca+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$ $[R_1' = R_1+R_2+R_3]$

$$= (bc+ca+ab)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} = \text{etc.}$$ (adding R_2 and R_3 with R_1).

Now, subtract C_2 from C_1 and C_3 and then expand.]

ANSWERS

1. (a) -6; (b) $-2xy$.
2. (a) -67; (b) $a^3+b^3+c^3-3abc$; (c) -15; (d) $11x+6y+z-6$; (e) -18; (f) 0.
3. (a) $x=0, 1, -11$; (b) $x=1$, other values are not real; (c) $x=0, -1$.
4. (a) 2; (b) 4; (c) -1; (d) 27; (e) 36; (f) -8; (g) -6; (h) -8; (i) 0.
7. (a) ab; (b) $a^2(a+3)$; (c) $a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$; (d) x^2y^2.
15. $(y+z-2x)(z+x-2y)(x+y-2z)$.

2.3 Product of Two Determinants

1. We shall multiply two determinants of second order by the following rule:

If $D_1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$, then the product

$$D_1 \cdot D_2 = \begin{vmatrix} ap+bq & ar+bs \\ cp+dq & cr+ds \end{vmatrix}$$ (Row-by-Row multiplication)

Rule: To obtain:

(a) the element in the first row and first column of the product D_1D_2, we multiply the elements of the first row in D_1 by the corresponding elements of the first row in D_2 and then add.

(b) the element in the first row and second column of D_1D_2, we multiply the elements of the first row in D_1 by the corresponding elements of the second row in D_2 and then add.

(c) the element in the second row and first column, we multiply the elements of the second row of D_1 and by the corresponding elements of the first row of D_2 and then add.

(d) the element in the second row and second column, we multiply the elements of second row of D_1 by the elements of the second row of D_2 and then add.

2. We have exactly similar rule for the product of two determinants of third order:

If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$, then

$$D_1 \cdot D_2 = \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{vmatrix}.$$

This is again called row-by-row multiplication rule of two determinants.

Observations. Since the value of a determinant does not alter, if rows and columns are interchanged, we may also use the rule row-by-column multiplication rule just as we multiplied two matrices.

Observe that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 \end{vmatrix}$ (Row-by-Row rule)

But the product is the same as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{vmatrix} \quad \text{(Row-by-Column rule)}$$

We may use either of the rules according to convenience.

Example 1. *Express the square of* $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ *as a second-order determinant.*

Solution:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{vmatrix}.$$

Example 2. *Express the square of* $\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$ *as a third-order determinant. What is the value of this product?*

Solution:

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = \begin{vmatrix} a_1^2 + b_1^2 & a_1a_2 + b_1b_2 & a_1a_3 + b_1b_3 \\ a_2a_1 + b_2b_1 & a_2^2 + b_2^2 & a_2a_3 + b_2b_3 \\ a_3a_1 + b_3b_1 & a_3a_2 + b_3b_2 & a_3^2 + b_3^2 \end{vmatrix}.$$

The value of this product is clearly zero since it is the product of two determinants each of which is zero.

We note that if any line (row or column) of a determinant contains elements which are all zero, then the value of the determinant is zero and thus

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0.$$

Example 3. *Find a third-order determinant which is the square of* $\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix}$. *What is the value of this product?*

Solution:
$$\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix}\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = \begin{vmatrix} a^2+b^2 & ac & bc \\ ca & c^2+a^2 & ab \\ bc & ab & c^2+b^2 \end{vmatrix}.$$

Now,
$$\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = a\begin{vmatrix} 0 & a \\ c & b \end{vmatrix} - b\begin{vmatrix} c & a \\ 0 & b \end{vmatrix} = a(0-ac)-b(bc-0) = -a^2c-b^2c$$
$$= -c\left(a^2+b^2\right).$$

$$\therefore \text{ the product } \begin{vmatrix} a^2+b^2 & ac & bc \\ ca & c^2+a^2 & ab \\ bc & ab & c^2+b^2 \end{vmatrix} = \left\{-c\left(a^2+b^2\right)\right\}\left\{-c\left(a^2+b^2\right)\right\}$$
$$= \mathbf{c^2\left(a^2+b^2\right)^2}.$$

Example 4. (i) *Evaluate* $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$; $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ *(Such determinants are called circulants).*

(ii) *Using the rule of multiplication of two determinants of the types as in* (i), *show that the product of two expressions of the form* $X^3+Y^3+Z^3-3XYZ$ *is numerically of the same form.*

Solution:

(i) $$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ c+a+b & a & b \\ b+c+a & c & a \end{vmatrix} \quad \left(C_1' = C_1+C_2+C_3\right)$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{matrix}$$

$$= (a+b+c)\begin{vmatrix} a-b & b-c \\ -(b-c) & a-c \end{vmatrix} = (a+b+c)\left\{(a-b)(a-c)+(b-c)^2\right\}$$

$$= (a+b+c)\left(a^2-ab-ac+bc+b^2+c^2-2bc\right)$$

$$= (a+b+c)\left(a^2+b^2+c^2-ab-ac-bc\right) = \mathbf{a^3+b^3+c^3-3abc}.$$

Clearly, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\left(a^3+b^3+c^3-3abc\right)$. [Here 2nd and 3rd rows have been interchanged.]

(ii) Consider the product

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \begin{vmatrix} ax+by+cz & ay+bz+cx & az+bx+cy \\ bx+cy+az & by+cz+ax & bz+cx+ay \\ cx+ay+bz & cy+az+bx & cz+ax+by \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix},$$

where $\alpha = ax+by+cz$
$\beta = cx+ay+bz$
$\gamma = bx+cy+az.$

Hence, by what we have proved in (i)

$$\left\{-\left(a^3+b^3+c^3-3abc\right)\right\}\left\{-\left(x^3+y^3+z^3-3xyz\right)\right\}=\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma.$$

This proves the assertion: The product of two expressions of the form $X^3+Y^3+Z^3-3XYZ$ is also of the same form.

Example 5. *Prove that* $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}=\left(a^3+b^3+c^3-3abc\right)^2.$

Solution: Consider the product

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}=\begin{vmatrix} -a^2+2bc & c^2 & b^2 \\ c^2 & -b^2+2ac & a^2 \\ b^2 & a^2 & -c^2+2ab \end{vmatrix}$$

[The second det has been obtained from the first by interchanging col_2 and col_3 (a minus sign will come) and also multiplying each element of first col by -1. So the value of this changed determinant is the same as the value of the first determinant.]

But we have already proved

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=-\left(a^3+b^3+c^3-3abc\right)$$

and $$\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}=-\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}=+\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ (Interchanging col_2 and col_3)

$$=-\left(a^3+b^3+c^3-3abc\right)$$

Hence, we establish

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}=\left(a^3+b^3+c^3-3abc\right)^2 \text{ (Proved).}$$

Two Useful Results

Example 6. *If $A_1, B_1, \ldots$ be the cofactors of $a_1, b_1, \ldots$ in the determinant*

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0, \text{ then}$$

(i) $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}=a_1D$ [*First consider the minor of a_1; then replace each element of this minor by their cofactors*]

(ii) $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=D^2$ [*Here, replace each element of D by their corresponding cofactors*]

Solution: To prove (i) we first observe that

$$\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Now, we multiply

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \text{ (use row-by-row rule)}$$

$$= \begin{vmatrix} a_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & D & 0 \\ a_3 & 0 & D \end{vmatrix} = a_1\begin{vmatrix} D & 0 \\ 0 & D \end{vmatrix} = a_1D^2.$$

[We have used the properties of cofactors and also Property V discussed in Section 2.2.1.]

$$\therefore\ D \times \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1D^2.$$

Since $D \neq 0$, it follows $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1D.$

To prove (ii) we multiply

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \quad \text{(row-by-row rule)}$$

$$= \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}.$$

Using the properties of cofactors and Property V of Section 2.2, we get

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix} = D^3.$$

$$\therefore\ \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = D^3/D = D^2\ (\because\ D \neq 0).$$

Adjugate (or Adjoint) of a Determinant

The determinant D' which is obtained by replacing each element of a determinant D by its cofactor is called the *adjugate* (or *adjoint*) of D.

Reciprocal of a Determinant whose Value is not Zero

Let D be a determinant, such that $D \neq 0$. We obtain another determinant $\bar{D}$ or D^{-1} by dividing each element of adjugate of D by the value of the determinant D. Thus, we construct

$$\bar{D} = \begin{vmatrix} A_1/D & B_1/D & C_1/D \\ A_2/D & B_2/D & C_2/D \\ A_3/D & B_3/D & C_3/D \end{vmatrix} = \frac{1}{D^3}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \frac{1}{D^3}D',$$

where D' is the adjugate of D.

$$\therefore\ \bar{D} = \frac{1}{D^3}D' = \frac{1}{D^3}D^2 = \frac{1}{D}\ \left(\because \text{ adjugate of } D = D^2\right).$$

Thus, $\bar{D} \cdot D = 1$. So, we call $\bar{D}$, *reciprocal* or *inverse* of D. We may write $\bar{D} = D^{-1}$.

Example 7. (i) *What is the adjugate of* $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -5 \end{vmatrix}$?

(ii) *Does the reciprocal of D exist?*

(iii) *If so, find the reciprocal* D^{-1} *of the determinant D.*

Solution: (i) Adjugate D' of the determinant D

$$\begin{vmatrix} \begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & -5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} -13 & 14 & -5 \\ 6 & -8 & 2 \\ 1 & -2 & 1 \end{vmatrix}.$$

(ii) $$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & -8 \end{vmatrix} \quad \begin{matrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 \\ -2 & -8 \end{vmatrix} = -8 + 4 = -4\ (\neq 0).$$

Since the value of D is not zero, therefore, $\bar{D}$ exists.

(iii) $$D^{-1} = \begin{vmatrix} -13/-4 & 14/-4 & -5/-4 \\ 6/-4 & -8/-4 & 2/-4 \\ 1/-4 & -2/-4 & 1/-4 \end{vmatrix} = \frac{1}{(-4)^3}\begin{vmatrix} -13 & 14 & -5 \\ 6 & -8 & 2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 13/4 & -7/2 & 5/4 \\ -3/2 & 2 & -1/2 \\ -1/4 & 1/2 & -1/4 \end{vmatrix}.$$

One can easily check $D \cdot D^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$

Symmetric and Skew-symmetric Determinants

We conclude our discussions on determinants by introducing two important types of determinants: One is called a *Symmetric determinant,* e.g.,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

where any pair of elements equidistant from the main diagonal are equal in magnitude and sign, and the other is called a *Skew-symmetric determinant,* e.g.,

$$\begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix},$$

where any pair of elements equidistant from the main diagonal are equal in magnitude but opposite in sign (each element of the main diagonal of a skew-symmetric determinant is always zero).

Example 8. *It can be easily verified that a skew-symmetric determinant of third order is zero*:

$$D=\begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix}=\begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix} \text{ (Interchanging rows and columns)},$$

$$\text{i.e., } D=\begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix}=(-1)\cdot(-1)\cdot(-1)\begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix} \text{ (Taking out } (-1) \text{ from each row)}$$

$$=-D.$$

$$\therefore\ 2D=0 \Rightarrow D=0.$$

Observations. In fact, this is a general proposition: every skew-symmetric determinant of odd order is zero.

What about the skew-symmetric determinant of second order?

$$\begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix}=(f)^2.$$

In fact, every skew-symmetric determinant of even order is a perfect square.

2.4 Solutions of Linear Equations: Cramer's Rule

1. Given two linear equations

$$a_1x+b_1y=c_1, \qquad (1)$$

$$a_2x+b_2y=c_2. \qquad (2)$$

Multiply (1) by b_2 and (2) by, b_1 and then subtract. We obtain

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1},$$

where it is to be assumed that $a_1b_2-a_2b_1 \neq 0$.

Similarly, we may obtain

$$y = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1}, \ (a_1b_2 - a_2b_1 \neq 0).$$

We may write these results in the language of determinants.

Thus

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ where } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

We can frame a rule called *Cramer's Rule* for solving two linear equations (1) and (2).

- Both in x and y, the denominator is the determinant of the coefficients of x and y, namely, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D$ (say) and the rule is applicable only when this determinant is not zero, i.e., $D \neq 0$.
- The numerator of x is the same as D with the exception that the column which contains the coefficients of x is to be replaced by the column of constants on the right side of the two given equations. We take numerator of x as D_1.

Exactly similar rule for the numerator of $y \rightarrow$ the column of the coefficients of y is to be replaced by the column of constants on the right side of the two given equations. We take numerator of y as D_2.

Thus, we can write *Cramer's Rule* as $x = \frac{D_1}{D}$ and $y = \frac{D_2}{D}$, where $D \neq 0$.

Example 9. *Solve by Cramer's Rule*:

(a) $\left.\begin{aligned} 2x + 3y &= 8 \\ x - 2y &= -3 \end{aligned}\right\}$; (b) $\left.\begin{aligned} 2x - y &= 1 \\ 3x + 2y &= 5 \end{aligned}\right\}$. [C.U. B.Com.(H) 2001]

Solution: (a) Here, the determinant of the coefficients of x and y is

$$D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7 \ (\neq 0).$$

Hence, we can apply Cramer's Rule. Accordingly,

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} 8 & 3 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}} \text{ and } y = \frac{D_2}{D} = \frac{\begin{vmatrix} 2 & 8 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}},$$

$$\text{i.e., } x = \frac{-16+9}{-7} = 1 \text{ and } y = \frac{-6-8}{-7} = 2.$$

Hence, the required solutions are $\boldsymbol{x = 1, y = 2}$.

(b) Try in a similar manner. Ans. $x = 1$, $y = 1$.

2. Cramer's Rule for solution of three linear equations in three unknowns:

Statement. Let

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \tag{A}$$

be a system of three linear equations with three unknowns. If the determinant of the coefficients, namely,

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ is not zero,}$$

then the system of equations (A) has a unique solution, namely,

$$x=\frac{D_1}{D},\quad y=\frac{D_2}{D},\quad z=\frac{D_3}{D},$$

where,

$$D_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}=\begin{cases}\text{a determinant obtained by replacing the elements of the} \\ \text{first column of } D \text{ by the elements } \{d_1, d_2, d_3\} \text{ which occur} \\ \text{as constants on RHS in (A)}\end{cases}$$

$$D_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Example 10. (i) *Solve the following equations by Cramer's rule:* $x+y+z=4$, $x-2y+z=-2$, $3x+2y+7z=14$. [C.U. B.Com. 2009]

(ii) *Solve, by Cramer's Rule, the following system of linear equations:*

$$\left.\begin{aligned} x+2y-z=-3 \\ 3x+y+z=4 \\ x-y+2z=6 \end{aligned}\right\}.$$

[C.U. B.Com.(H) 1999]

Solution: (i) The given equations are

$$\left.\begin{aligned} x+y+z=4 \\ x-2y+z=-2 \\ 3x+2y+7z=14 \end{aligned}\right\} \tag{1}$$

Here, $D=$ the determinant of the coefficients of x, y, z

$$=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 7 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & 0 \\ 3 & -1 & 4 \end{vmatrix}=1\begin{vmatrix} -3 & 0 \\ -1 & 4 \end{vmatrix}=-12\neq 0.$$

[Subtracting first column from the 2nd and 3rd columns and then expanding w.r.t. first row]

By Cramer's Rule,

$$\frac{x}{D_1}=\frac{y}{D_2}=\frac{z}{D_3}=\frac{1}{D},\text{ where } D\neq 0, \tag{2}$$

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ -2 & -2 & 1 \\ 14 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 3 \\ 12 & 2 & 5 \end{vmatrix} = 3\begin{vmatrix} -2 & 3 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 12 & 5 \end{vmatrix} = 3 \times (-16) + 36 = -12.$$

[Subtracting 2nd column from the first and 3rd columns, and then expanding w.r.t. first row]

$$D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & -2 & 1 \\ 3 & 14 & 7 \end{vmatrix} = \begin{vmatrix} 0 & 6 & 0 \\ 1 & -2 & 1 \\ 3 & 14 & 7 \end{vmatrix} = -6\begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} = -6 \times 4 = -24.$$

[Subtracting 2nd row from the 1st row and then expanding w.r.t. 1st row] and

$$D_3 = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -2 & -2 \\ 3 & 2 & 14 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 3 \\ 3 & -2 & 0 \\ 1 & 2 & 12 \end{vmatrix} = -1\begin{vmatrix} 3 & 0 \\ 1 & 12 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = -36 + 24 = -12.$$

[Subtracting 2nd column from the 1st and the 3rd columns, and then expanding w.r.t. first row]

From (2),

$$x = \frac{D_1}{D} = \frac{-12}{-12} = 1,\ y = \frac{D_2}{D} = \frac{-24}{-12} = 2,\ z = \frac{D_3}{D} = \frac{-12}{-12} = 1.$$

Hence, the required solutions: $\boldsymbol{x = 1, y = 2, z = 1}$.

(ii) Here

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{vmatrix} \quad (C_3' = C_3 + C_1)$$

$$= 1\begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} = 7 - 2 \times 5 = -3 \neq 0.$$

Thus $D \neq 0$; hence Cramer's Rule is applicable. We now find:

$$D_1 = \begin{vmatrix} -3 & 2 & -1 \\ 4 & 1 & 1 \\ 6 & -1 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 6 & -1 \end{vmatrix}$$

$$= -3(2+1) - 2(8-6) - 1(-4-6) = -3.$$

Similarly, we obtain

$$D_2 = \begin{vmatrix} 1 & -3 & 1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 3 \text{ and } D_3 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = -6.$$

$\therefore$ by Cramer's Rule, the required solution is

$$\left.\begin{aligned} x &= \frac{D_1}{D} = \frac{-3}{-3} = \mathbf{1} \\ y &= \frac{D_2}{D} = \frac{3}{-3} = \mathbf{-1} \\ z &= \frac{D_3}{D} = \frac{-6}{-3} = \mathbf{2} \end{aligned}\right\}.$$

We can, for our satisfaction, check that $x = 1$, $y = -1$, $z = 2$ satisfy each of the given equations $x+2y-z=-3$, $3x+y+z=4$, $x-y+2z=6$.

[For practice, work out Question Nos. 12, 14 and 15 of the Exercises 2(II).]

Another Use of Determinants: Condition for a Common Solution

Example 11. *What is the condition that the system of equations*

$$\left.\begin{aligned} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \\ a_3x+b_3y+c_3=0 \end{aligned}\right\} \qquad \text{(B)}$$

is satisfied by the same set of values of x and y?

Our answer is that if there exists a common solution of the system, then the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ must be zero.}$$

[In many problems when we know that a system (A) has a common solution, by eliminating x and y, we obtain $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$]

Justification of Our Assertion

Solving first two equations of the system (A), we get by Cramer's Rule

$$x = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$.

Then the values of x and y obtained above must also satisfy the last equation of (A) and hence

$$a_3\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix} + b_3\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0;$$

this is the required condition which can be put in the form

$$-a_3c_1b_2 + a_3b_1c_2 - a_1b_3c_2 + a_2b_3c_1 + a_1b_2c_3 - a_2b_1c_3 = 0$$

or, $(a_1b_2c_3 - a_1b_3c_2) - (b_1a_2c_3 - b_1a_3c_2) + (c_1a_2b_3 - c_1a_3b_2) = 0$

or, $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$

or, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Observations. 1. What happens if $a_1b_2 - a_2b_1 = 0$?

Two Cases Arise

- If at the same time we find $b_1c_2 - b_2c_1 \neq 0$ and also $a_2c_1 - a_1c_2 \neq 0$, then we say that the system is INCONSISTENT, i.e., they have no common solution.
- If, however, $b_1c_2 - b_2c_1 = 0$ and $a_2c_1 - a_1c_2 = 0$, then the system can have solutions — in fact, there can exist infinite number of solutions. In this last case, observe that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Example 12. *See that the system $2x+3y+5=0$, $4x+6y+8=0$ is inconsistent. There is no solution, but the system $2x+3y+5=0$ and $4x+6y+10=0$ is consistent and they have infinite number of solutions $(x=0, y=-5/3)$, $(x=1, y=-7/3)$, etc.*

Example 13. *What is the condition that three simultaneous linear equations*

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\}$$

have a common solution other than $x = y = z = 0$?

Solution: Observe that the system has an obvious solution $x = y = z = 0$ (this is known as *Trivial Solution*). We actually find the condition that the system can have a non-trivial solution. We write the given system in the form (assume $z \neq 0$)

$$a_1\left(\frac{x}{z}\right) + b_1\left(\frac{y}{z}\right) + c_1 = 0;$$

$$a_2\left(\frac{x}{z}\right) + b_2\left(\frac{y}{z}\right) + c_2 = 0;$$

$$a_3\left(\frac{x}{z}\right) + b_3\left(\frac{y}{z}\right) + c_3 = 0.$$

$\therefore$ the condition that they have a common solution is by what we have explained above:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

In this case, eliminating x, y, z from the given equations we get the determinant of the coefficients as zero.

EXERCISES ON CHAPTER 2(II)

(Determinants)

1. Write down the determinant $\begin{vmatrix} ap+bq & ar+bs \\ cp+dq & cr+ds \end{vmatrix}$ as the sum of four determinants and hence prove that it equals to the product of the two determinants, namely,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix}.$$

[Hints: Sum of the four determinants $= pr \times 0 + ps\begin{vmatrix} a & b \\ c & d \end{vmatrix} - qr\begin{vmatrix} a & b \\ c & d \end{vmatrix} + qs \times 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}(ps - qr)$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix}.]$$

2. Express the following as second-order determinants:

(a) $\begin{vmatrix} p & q \\ r & s \end{vmatrix}^2$; (b) $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \times \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$; (c) $\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}^2$.

3. Express the square of the determinant $\begin{vmatrix} x & 0 & u \\ y & 0 & v \\ z & 0 & w \end{vmatrix}$ as a third-order determinant and hence obtain the value of this new determinant.

4. (a) Show that $\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}^2 = \begin{vmatrix} y^2+z^2 & xy & zx \\ xy & x^2+z^2 & yz \\ zx & yz & x^2+y^2 \end{vmatrix} = 4x^2y^2z^2.$

(b) Evaluate $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$ and hence prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2.$ [C.U. B.Com. 2009]

[Hints:

$$\Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix} - c\begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b\begin{vmatrix} c & 0 \\ b & a \end{vmatrix} = 0 + abc + abc = 2abc.$$

By definition,

$$\Delta' = \text{adjugate determinant of } \Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} +\begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix} & -\begin{vmatrix} c & a \\ b & 0 \end{vmatrix} & +\begin{vmatrix} c & 0 \\ b & a \end{vmatrix} \\ -\begin{vmatrix} c & b \\ a & 0 \end{vmatrix} & +\begin{vmatrix} 0 & b \\ b & 0 \end{vmatrix} & -\begin{vmatrix} 0 & c \\ b & a \end{vmatrix} \\ +\begin{vmatrix} c & b \\ 0 & a \end{vmatrix} & -\begin{vmatrix} 0 & b \\ c & a \end{vmatrix} & +\begin{vmatrix} 0 & c \\ c & 0 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix}.$$

$\therefore \Delta^2 = \Delta' = (2abc)^2 = 4a^2b^2c^2.$]

5. Find the adjugate of $\begin{vmatrix} 1 & 2 & -1 \\ 3 & -2 & 4 \\ 0 & -3 & 5 \end{vmatrix}$.

6. Show that $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = a^3 + b^3$, and hence find the value of $\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix}$.

[Hints: $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}^2 = \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} \times -\begin{vmatrix} b & a & 0 \\ a & 0 & b \\ 0 & b & a \end{vmatrix} = \text{etc.}$]

7. Express $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$ as the product of two determinants, and hence show that its value is $2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$.

[Hints: $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix} = \text{given determinant.}$]

8. Find the adjugate of $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$ and then prove that

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2.$$

9. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2.$

[**Hints:** Left side is the adjugate of the right-side determinant.]

10. If $A_1, B_1, C_1, \ldots$ are the respective cofactors of $a_1, b_1, c_1, \ldots$ in

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

prove that $A_2B_3 - A_3B_2 = C_1D$.

What can you say about $A_2C_3 - A_3C_2$?

[**Hints:** See worked-out Ex. 6 in **Section 2.3.**]

11. Show that $\begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$

12. Solve the following system of linear equations (by Cramer's Rule):

Set A: (a) $\left.\begin{aligned} 2x-5y &= -3 \\ 3x+4y &= 7 \end{aligned}\right\}$; (b) $\left.\begin{aligned} \tfrac{1}{5}(3x-2)+\tfrac{1}{10}(7y+1) &= 10 \\ \tfrac{1}{2}(x+3)-\tfrac{1}{3}(2y-5) &= 3 \end{aligned}\right\}$;

(c) $\left.\begin{aligned} \tfrac{3}{2x}+\tfrac{8}{5y} &= 3 \\ \tfrac{4}{3y}-\tfrac{1}{x} &= 1 \end{aligned}\right\}$; (d) $\left.\begin{aligned} \tfrac{2}{5}+\tfrac{3}{y} &= 5 \\ \tfrac{1}{x}+\tfrac{2}{y} &= 4 \end{aligned}\right\}$. [B.U. B.Com.(H) 2007]

[**Hints:** Solve for $1/x$ and $1/y$, and then obtain the values of x and y.]

Set B:

(a) $x-y+z=10,\ x+2y-z=7,\ x+y-z=8$; [V.U. B.Com.(H) 2011]

(b) $2x-y+z=6,\ x+2y+3z=3,\ 3x+y-z=4$; [C.U. B.Com.(H) 1990]

(c) $5x+3y=65,\ 2y-5x=-25,\ 3x+4z=57$;

(d) $2x+y+z=1,\ x-y+2z=-1,\ 3x+2y-z=4$;

(e) $2x+y-z=5,\ 3x-2y+2z=-3,\ x-3y-3z=-2$;

(f) $3x-2y+4z=2,\ x+3y-6z=8,\ 2x-y-2z=0$;

(g) $\frac{1}{x}+\frac{2}{y}+\frac{1}{z}=\frac{1}{2},\ \frac{4}{x}+\frac{2}{y}-\frac{3}{z}=\frac{2}{3},\ \frac{3}{x}-\frac{4}{y}+\frac{4}{z}=\frac{1}{3}$;

(h) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, $\frac{2}{x}+\frac{5}{y}+\frac{3}{z}=0$, $\frac{1}{x}+\frac{2}{y}+\frac{4}{z}=3$; [C.U. B.Com.(H) 1997]

[Hints: Put $u=\frac{1}{x}$, $v=\frac{1}{y}$, $w=\frac{1}{z}$. Then by Cramer's Rule solve for u, v, w. Finally obtain the values of x, y, z.]

(i) $a+b+c=6$, $4a+2b+c=11$, $9a-3b+c=6$; [C.U. B.Com.(H) 1995]

(j) $2x+3y=-2$, $5y-2z=4$, $3z+4x=-7$;

(k) $3x+2y+4z=19$, $2x-y+z=3$, $6x+7y-z=17$;

(l) $2x+3y+z=17, x-y+z=3, 3x+2y-2z=4$. [N.B.U. B.Com. 1996]

13. Eliminate x, y, z from the following system of equations:

(a) $x+by+cz=0$, $ax+y+cz=0$, $ax+by+z=0$;

(b) $a=\frac{x}{y-z}$, $b=\frac{y}{z-x}$, $c=\frac{z}{x-y}$;

(c) $ax+y-cz=0$, $bx-cy-z=0$, $x-ay+bz=0$.

14. Solve by Cramer's Rule:

(a) $\left.\begin{aligned} x+2y+3z&=6\\ 2x+4y+z&=7\\ 3x+2y+9z&=14\end{aligned}\right\}$; (b) $\left.\begin{aligned} ax+by+cz&=1\\ a^2x+b^2y+c^2z&=k\\ a^3x+b^3y+c^3z&=k^2\end{aligned}\right\}$ $(a\neq b\neq c)$;

(c) $\left.\begin{aligned} x+2y-z&=9\\ 2x-y+3z&=-2\\ 3x+2y+3z&=9\end{aligned}\right\}$; (d) $\left.\begin{aligned} x+y+z&=1\\ ax+by+cz&=k\\ a^2x+b^2y+c^2z&=k^2\end{aligned}\right\}$;

[C.U. B.Com.(H) 1999] [C.U. B.Com.(H) 2003]

(e) $\left.\begin{aligned} 2x+3y-z&=9\\ x+y+z&=9\\ 3x+y-z&=-1\end{aligned}\right\}$. [N.B.U. B.Com.(H) 2007]

15. Solve:

(a) $x+2y-2z=5, 3x-y+z=8, x+y-z=4$. [N.B.U. B.Com. 1997]

(b) $x+y-z+3=0, 2x+3y+z=2, 8y+3z=1$. [B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2009]

ANSWERS

1. $\begin{vmatrix} ap & ar\\ cp & cr\end{vmatrix}+\begin{vmatrix} ap & bs\\ cp & ds\end{vmatrix}+\begin{vmatrix} bq & ar\\ dq & cr\end{vmatrix}+\begin{vmatrix} bq & bs\\ dq & ds\end{vmatrix}$.

2. (a) $\begin{vmatrix} p^2+q^2 & pr+qs\\ pr+qs & r^2+s^2\end{vmatrix}$; (b) $\begin{vmatrix} 0 & 5\\ -3 & 1\end{vmatrix}$;

(c) $\begin{vmatrix} 13 & -14\\ -14 & 17\end{vmatrix}$.

3. $\begin{vmatrix} x^2+u^2 & xy+uv & xz+uw\\ xy+uv & y^2+v^2 & yz+vw\\ xz+uw & yz+vw & z^2+w^2\end{vmatrix}$; 0.

5. $\begin{vmatrix} 2 & -15 & -9\\ -7 & 5 & 3\\ 6 & -7 & -8\end{vmatrix}$.

8. $\begin{vmatrix} a^2+x^2 & ab+cx & ca-bx\\ ab-cx & b^2+x^2 & bc+ax\\ ca+bx & bc-ax & c^2+x^2\end{vmatrix}$.

12. Set A:

(a) $x=1, y=1$;

(b) $x=9, y=7$;

(c) $x=3/2, y=4/5$;

(d) $x=-1/2, y=1/3$.

Set B:

(a) $x=9, y=-1, z=0$;

(b) $x=2, y=-1, z=1$;

(c) $x=41/5, y=8, z=81/10$;

(d) $x=1, y=0, z=-1$;

(e) $x=1, y=2, z=-1$;

(f) $x=2, y=3, z=1/2$;

(g) $x=6, y=8, z=12$;

(h) $x=1, y=-1, z=1$;

(i) $a=1/3, b=4/3, c=7$;

(j) $x=-4, y=2, z=3$;

(k) $x=1, y=2, z=3$;

(l) $x=2, y=3, z=4$.

13. (a) $ab+bc+ca-1=2abc$;

(b) $ab+bc+ca=1$;

(c) $a^2+b^2+c^2+1=0$.

14. (a) $x=1, y=1, z=1$;

(b) $x=\dfrac{(k-b)(c-k)}{abc(a-b)(c-a)}$,
$y=\dfrac{(a-k)(k-c)}{abc(a-b)(b-c)}$,
$z=\dfrac{(b-k)(k-a)}{abc(b-c)(c-a)}$;

(c) $x=2, y=3, z=-1$;

(d) $x=\dfrac{(k-b)(c-k)}{(a-b)(c-a)}$,
$y=\dfrac{(a-k)(k-c)}{(a-b)(b-c)}$,
$z=\dfrac{(b-k)(k-a)}{(b-c)(c-a)}$.

(e) $x=-2/3, y=24/5, z=23/5$.

15. (a) $x=3, y=t, z=t-1$, where t is a parameter.

[**Hints**: Adding 2nd and 3rd equations, $4x=12 \Rightarrow x=3$. $x=3$ makes all the equations identical. If we take $y=t$, then $z=t-1$. The given equations have infinite number of solutions.]

(b) $x=1, y=-1, z=3$.

Chapter 3

Adjoint and Inverse of a Matrix: Solutions of a System of Linear Equations by Matrix Inversion Method

3.1 Adjoint (or Adjugate) of a Square Matrix

Let $A = (a_{ij})$ be a square matrix of order n. Then the determinant $|A|$ or det A is of the same order n formed by the corresponding elements of A.

Adjoint (or Adjugate) of a Square Matrix

Let $A = (a_{ij})$ be a square matrix of order n and let $|A|$ be the determinant associated with the matrix A. If a_{ij} be the cofactors of a_{ij} in $|A|$ for all pairs of values of i and j, then the transpose of the matrix (A_{ij}) formed by the corresponding cofactors of A is called the *Adjoint* (or *Adjugate*) of the matrix A and it is denoted by adj A.

Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Let

$$B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where A_{ij} are the cofactors of a_{ij} in $|A|$. Then the adjoint of A is the transpose of B given by

$$\operatorname{adj} A = B^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Reciprocal Matrix

If A be a non-singular matrix (i.e., if A be a square matrix such that $|A| \neq 0$), then the matrix $\frac{1}{|A|} \operatorname{adj} A$ is called the *reciprocal matrix of* A. Thus, if $A = (a_{ij})_{3\times 3}$ such that $|A| \neq 0$, then the reciprocal matrix of A is given by

$$\text{reciprocal matrix of } A = \frac{1}{|A|}\,\mathbf{adj}\,A = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{32} & A_{33} \end{bmatrix}.$$

Theorem 1. *For a square matrix A of order 3, prove that $A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| \cdot I$, where I is the unit matrix of order 3.*

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

where A_{ij} are the cofactors of a_{ij} in $|A|$ for $i = 1, 2, 3; j = 1, 2, 3$.

Now,

$$A(\operatorname{adj} A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I.$$

Similarly, we can prove that $(\operatorname{adj} A)A = |A| \cdot I$.

Hence, $\boldsymbol{A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| \cdot I}$.

3.2 Inverse of a Square Matrix

If A and B are two square matrices of the same order $n(= 2 \text{ or } 3)$ such that $AB = BA = I$, where I is the unit matrix of order n, then either B is the inverse of A or A is the inverse of B. In the first case, we write $B = A^{-1}$ and in the second case, $A = B^{-1}$.

Clearly, $AA^{-1} = A^{-1}A = I$ and $B^{-1}B = BB^{-1} = I$.

Formula to Compute the Inverse of the Matrix A

We have

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| \cdot I \text{ or, } A\frac{\operatorname{adj} A}{|A|} = \frac{\operatorname{adj} A}{|A|}A = I. \tag{1}$$

The relation (1) shows that

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}, \text{ where } |A| \neq 0 \;\; [\because AA^{-1} = A^{-1}A = I].$$

Remark 1. *The inverse of A, i.e., A^{-1} exists only when $|A| \neq 0$, i.e., only when A is a non-singular matrix. If $|A| = 0$, then A^{-1} does not exist.*

Remark 2. *From the formula of the inverse of the matrix A, we see that A^{-1} = reciprocal of A.*

Properties of the Inverse of a Matrix

- The inverse of a square matrix, if it exists, is unique, i.e., there cannot be two distinct inverses for the same matrix.
- If A^{-1} exists, then (a) $(A^{-1})^{-1} = A$ and (b) $(A^T)^{-1} = (A^{-1})^T$.
- If A and B be two non-singular square matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$, where A^{-1}, B^{-1} are the inverses of A and B respectively.

Orthogonal Matrix

If A be a square matrix of order n (= 2 or 3) such that $AA^T = A^TA = I$, where A^T is the transpose of A and I is the unit matrix of order n, then A is called an *orthogonal matrix.*

If A be an orthogonal matrix, then A^T is the inverse of A, i.e., $A^T = A^{-1}$, since $AA^T = A^TA = I$. If $|A| = +1, A$ is called a *proper orthogonal matrix* and if $|A| = -1, A$ is called an *improper orthogonal matrix.*

Example 1. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Solution: We can easily check: $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\therefore$ by definition, $B = A^{-1}$ or $A = B^{-1}$.

Also, A = inverse of B = inverse of $A^{-1} = (A^{-1})^{-1}$.

Example 2. (i) *Calculate the inverse of matrix* $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

(ii) *Let* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. *We can easily verify that* $A^2 - 5A + 7I = O$, *where* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and the right side is a null matrix of order two. Hence obtain* A^{-1}. **[C.U.B.Com.(H) 2001]**

Solution: (i) Here, $|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \neq 0. \therefore A^{-1}$ exists.

Now,

$$A^{-1} = \tfrac{1}{|A|}\operatorname{adj}A = \tfrac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}^T = \tfrac{1}{2}\begin{bmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{3} \end{bmatrix}.$$

(ii) See that

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}.$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix};\ 7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Now, we can write, $A^2 - 5A = -7I$, or, $\frac{1}{7}\left(5A - A^2\right) = I$,

or, $\frac{1}{7}A(5I - A) = I \quad [\because AI = A]$

or, $\frac{1}{7}(5I - A)A = I$.

If $B = \frac{1}{7}(5I - A)$, then $AB = BA = I$,

$$\text{i.e., } B = \frac{1}{7}\left\{\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\right\} = \frac{1}{7}\begin{pmatrix} 5-3 & 0-1 \\ 0+1 & 5-2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = A^{-1}.$$

One can easily check

$$\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ i.e., } AB = I$$

$$\text{and } \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ i.e., } BA = I.$$

Given any square matrix A, does inverse always exist? If it exists, how to find this inverse?

The answer of the first question is that only when the matrix A is non-singular (i.e., determinant $|A| \neq 0$), then A^{-1} exists.

We give below a method of evaluation of inverse of a square matrix A (which can be justified).

Method of Evaluation of Inverse of a Square Matrix A

Step I. *Find* det$A$ *or* $|A|$. *If it is not zero, then* A^{-1} *exists. If, however, we find* det$A = 0$, *then we shall simply say, inverse of A does not exist.*

Step II. *If det* $A \neq 0$, *then for each element of the matrix we find a cofactor.*

To find the cofactor of any element of A, consider a determinant (also called *minor*) formed by deleting the elements of the row and column in which that element occurs. Associate a sign $(-1)^{i+j}$, where i is the row and j is the column in which that element occurs. Such a signed minor is called the *cofactor of that element.*

e.g., in the matrix $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 3 & 6 \end{bmatrix}$.

What is the cofactor of the element in the 3rd row and 2nd column (or what is the cofactor of 3)? Delete the row and column in which 3 occurs and consider the determinant

$$\begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix}.$$

This is the minor of 3. Now associate with it the sign $(-1)^{3+2}$ (i.e., associate a negative sign) because 3 occurs in the third row and second column.

Thus, $-\begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix} = -28$ is the cofactor of 3.

For quick working, one may remember that for a third order square matrix the signs to be attached to the minors are $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$.

We shall then consider the matrix of the cofactors of A, i.e., we replace each element of A by its cofactor.

Step III. *The transpose of the matrix of the cofactors is called the adjoint of A or adjugate of A, denoted by* adj$A$.

The inverse of a square matrix A,

$$\text{i.e., } A^{-1} = \frac{1}{\det A}(\operatorname{adj} A) \text{ or, } \frac{1}{|A|}(\operatorname{adj} A).$$

Example 3. *Suppose, we are to find the inverse of* $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix}$.

Solution:

Step I. *First we find*

$$\det A = \begin{vmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{vmatrix} = 8\begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} + 2\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} = 8(64-8)-4(16-4)+2(4-8)$$
$$= 448 - 48 - 8 = 392\ (\neq 0).$$

$\therefore$ *the matrix A is non-singular and hence A^{-1} exists.*

Step II. *We next find the matrix of the cofactors of A*

$$= \begin{bmatrix} +\begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} & +\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} & +\begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix} & -\begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} & -\begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 56 & -12 & -4 \\ -28 & 62 & -12 \\ 0 & -28 & 56 \end{bmatrix}.$$

Step III. $\operatorname{adj} A = \begin{bmatrix} 56 & -28 & 0 \\ -12 & 62 & -28 \\ -4 & -12 & 56 \end{bmatrix}$ (*Transposing rows and columns of matrix of cofactors of A*)

$$\therefore A^{-1} = \frac{1}{\det A}(\operatorname{adj} A) = \frac{1}{392}\begin{bmatrix} 56 & -28 & 0 \\ -12 & 62 & -28 \\ -4 & -12 & 56 \end{bmatrix} = \begin{bmatrix} 56/392 & -28/392 & 0 \\ -12/392 & 62/392 & -28/392 \\ -4/392 & -12/392 & 56/392 \end{bmatrix}.$$

Example 4. *Find A^{-1} of* $A = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. [C.U. B.Com.(H) 1995; V.U. B.Com.(H) 2009]

Solution:

Step I. $\det A = \begin{vmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 12\ (\neq 0).$

$\therefore$ *in this case, A is non-singular and hence A^{-1} exists.*

Step II. *Matrix of the cofactors of A*

$$= \begin{bmatrix} +\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ -8 & 2 & 0 \\ 8 & -2 & 6 \end{bmatrix}.$$

Step III. $\operatorname{adj} A = \begin{bmatrix} 4 & -8 & 8 \\ 2 & 2 & -2 \\ 0 & 0 & 6 \end{bmatrix}$ (*Transposing the matrix of cofactors of A*)

$$\therefore\ A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{12}\begin{bmatrix} 4 & -8 & 8 \\ 2 & 2 & -2 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$\therefore\ A^{-1} = \begin{bmatrix} 1/3 & -(2/3) & 2/3 \\ 1/6 & 1/6 & -(1/6) \\ 0 & 0 & 1/2 \end{bmatrix}.$$

We can easily check

$$AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1/3 & -(2/3) & 2/3 \\ 1/6 & 1/6 & -(1/6) \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 5. *Second Order Square Matrix* $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. *Find* A^{-1}.

Solution:

Step I. $\det A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3\ (\neq 0).$

$\therefore$ *A is non-singular and hence A^{-1} exists.*

Step II. *Matrix of the cofactors of* $A = \begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix}.$

Step III. $\operatorname{adj} A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (*Transposing the matrix of the cofactors of A*)

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

Check: $A\cdot A^{-1}=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}=\begin{bmatrix}(4/3)-(1/3) & (2/3)-(2/3)\\ -(2/3)+(2/3) & -(1/3)+(4/3)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I.$

Second Method for A^{-1} : Another Method of finding the inverse of the square matrix A is the use of a special polynomial relation in A.

We explain the method through an example.

Example 6. *If the matrix A be given by $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$, then prove that $A^2-4A+3I=O$, where I is the unit matrix of order 2 and O is a null matrix of order two. Also find A^{-1}.* [C.U. B.Com.(H) 1994]

Solution:

Step I. $\det A=\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}=4-1=3\neq 0.$ *Hence, A is non-singular and so A^{-1} exists.*

Step II. $A^2=A.A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\cdot\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}.$

$$-4A=-4\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-8 & 4\\ 4 & -8\end{pmatrix}.$$

$$3I=3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}.$$

$$\therefore\ A^2-4A+3I=\begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}+\begin{pmatrix}-8 & 4\\ 4 & -8\end{pmatrix}+\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=O.$$

First part of the question is proved.

Step III. Since $A^2-4A+3I=O$, $A(A-4I)=-3I$, *i.e.*, $A\left\{-\frac{1}{3}(A-4I)\right\}=I$;

$$\therefore\ A^{-1}=-\frac{1}{3}(A-4I)=-\frac{1}{3}\left\{\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}-\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\right\}=-\frac{1}{3}\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}=\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}.$$

Check: $A\cdot A^{-1}=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I.$

3.2.1 Theorems connected with Inverse of a Square Matrix

- Let A be a square matrix. If, at all, A^{-1} exists, then it is unique. We cannot have two distinct inverses for the same square matrix A.
- A square matrix has an inverse, if and only if its determinant is not zero (i.e., if and only if it is non-singular).
- If A^{-1} exists, then $(A^T)^{-1}=(A^{-1})^T$, where A is a given square matrix and A^T is its transpose.
- Let A and B be two square matrices of same order, say, n. If A^{-1} and B^{-1} both exist, then $(AB)^{-1}=B^{-1}A^{-1}$.

- Let A be a square matrix for which A^{-1} exists. Then $A \cdot A^{-1} = A^{-1} \cdot A = I$. From this we can also conclude: Inverse of A^{-1} is A, i.e., $(A^{-1})^{-1} = A$.

  [Proofs of all these theorems are given as Exercises in Ch. 3.]

In the following examples we verify the truths of theorems III and IV.

Example 7. (i) *Let* $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$; *verify that* $(A^T)^{-1} = (A^{-1})^T$.

(ii) *Let*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}.$$

Verify $(A^T)^{-1} = (A^{-1})^T$.

Solution: (i) $\det A = -10$ $(\neq 0)$ and hence A^{-1} exists.

Matrix of the cofactors of $A = \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix}$.

$$\operatorname{adj} A = \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix}, \therefore A^{-1} = \frac{1}{-10}\begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -(3/10) & 2/5 \\ 2/5 & -(1/5) \end{bmatrix}.$$

Hence,

$$(A^{-1})^T = \begin{bmatrix} -(3/10) & 2/5 \\ 2/5 & -(1/5) \end{bmatrix}.$$

But

$$A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \det A^T = \begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} = -10.$$

$\therefore$ $(A^T)^{-1}$ exists.

Matrix of the cofactors of $A^T = \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix}$; $\operatorname{adj} A^T = \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix}$;

$$\therefore (A^T)^{-1} = -\frac{1}{10}\begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -(3/10) & 2/5 \\ 2/5 & -(1/5) \end{bmatrix}.$$

Thus, we see that $\boldsymbol{(A^{-1})^T = (A^T)^{-1}}$.

(ii) Here, $\det A = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2$ $(\neq 0)$, therefore, A^{-1} exists.

Matrix of the cofactors of $A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$. $\therefore$ $\operatorname{adj} A = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$

and

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & -(1/2) & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -(3/2) & 1/2 \end{pmatrix}.$$

$$\therefore\ \left(A^{-1}\right)^T = \begin{pmatrix} 1/2 & -4 & 5/2 \\ -(1/2) & 3 & -(3/2) \\ 1/2 & -1 & 1/2 \end{pmatrix}.$$

Now,

$$A^T = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix};\ \det A^T = \det A = -2\ (\neq 0).$$

$\therefore\ \left(A^T\right)^{-1}$ exists.

$$\text{Matrix of cofactors of } A^T = \begin{bmatrix} +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}.$$

$$\therefore\ \operatorname{adj} A^T = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}.$$

$$\therefore\ \left(A^T\right)^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -4 & 5/2 \\ -(1/2) & 3 & -(3/2) \\ 1/2 & -1 & 1/2 \end{bmatrix}..$$

See that

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = \begin{pmatrix} 1/2 & -4 & 5/2 \\ -(1/2) & 3 & -(3/2) \\ 1/2 & -1 & 1/2 \end{pmatrix}.$$

Example 8. *Let* $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$; *verify that* $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -7 & 10 \end{bmatrix}.$$

$$\det AB = \begin{vmatrix} 3 & -4 \\ -7 & 10 \end{vmatrix} = 30 - 28 = 2\ (\neq 0);\ \therefore\ (AB)^{-1} \text{ exists.}$$

$$\text{Matrix of the cofactors of } AB = \begin{bmatrix} 10 & +7 \\ +4 & 3 \end{bmatrix}.\ \therefore\ \operatorname{adj} AB = \begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}.$$

$$\therefore\ (AB)^{-1} = \frac{1}{2}\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7/2 & 3/2 \end{bmatrix}.$$

Next we see that

$$\det B = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2 - 1 = 1\ (\neq 0) \text{ and hence } B^{-1} \text{ exists;}$$

$$\det A = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 6 - 4 = 2\ (\neq 0) \text{ and hence } A^{-1} \text{ exists.}$$

Matrix of cofactors of $B = \begin{bmatrix} 2 & +1 \\ +1 & 1 \end{bmatrix}$; adj $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\therefore\ B^{-1} = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Matrix of cofactors of $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$; adj$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$.

$$\therefore\ A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 2 & 1 \end{bmatrix}.$$

$$\therefore\ B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3/2 & 1/2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7/2 & 3/2 \end{bmatrix}.$$

See that

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 5 & 2 \\ 7/2 & 3/2 \end{bmatrix}.$$

Example 9. *Let* $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$. *Verify:* $(AB)^{-1} = B^{-1}\cdot A^{-1}$.

Solution:

$$AB = \begin{bmatrix} 17 & 9 & 0 \\ 8 & 5 & -1 \\ 10 & 1 & 7 \end{bmatrix};\ \det AB = 17(35+1) - 9(56+10) = 612 - 594 = 18\ (\neq 0).$$

$\therefore\ (AB)^{-1}$ exists.

$$\text{Matrix of the cofactors of } AB = \begin{bmatrix} +\begin{vmatrix} 5 & -1 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 8 & -1 \\ 10 & 7 \end{vmatrix} & +\begin{vmatrix} 8 & 5 \\ 10 & 1 \end{vmatrix} \\ -\begin{vmatrix} 9 & 0 \\ 1 & 7 \end{vmatrix} & +\begin{vmatrix} 17 & 0 \\ 10 & 7 \end{vmatrix} & -\begin{vmatrix} 17 & 9 \\ 10 & 1 \end{vmatrix} \\ +\begin{vmatrix} 9 & 0 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 17 & 0 \\ 8 & -1 \end{vmatrix} & +\begin{vmatrix} 17 & 9 \\ 8 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 36 & -66 & -42 \\ -63 & 119 & 73 \\ -9 & 17 & 13 \end{bmatrix};$$

$$\text{adj}AB = \begin{bmatrix} 36 & -63 & -9 \\ -66 & 119 & 17 \\ -42 & 73 & 13 \end{bmatrix}.$$

$$\therefore\ (AB)^{-1} = \frac{1}{\det AB}(\text{adj}AB) = \frac{1}{18}\begin{bmatrix} 36 & -63 & -9 \\ -66 & 119 & 17 \\ -42 & 73 & 13 \end{bmatrix} = \begin{bmatrix} 2 & -7/2 & -1/2 \\ -11/3 & 119/18 & 17/18 \\ -7/3 & 73/18 & 13/18 \end{bmatrix}.$$

Next we see that

$$\det B = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 2(1-4) - 1(-1-6) + 1(2+3) = 6\ (\neq 0);\ \therefore\ B^{-1} \text{ exists.}$$

$$\operatorname{adj} B=\begin{bmatrix} \begin{vmatrix}-1 & 2\\ 2 & -1\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix} & \begin{vmatrix}1 & 1\\ -1 & 2\end{vmatrix}\\[2ex] -\begin{vmatrix}1 & 2\\ 3 & -1\end{vmatrix} & \begin{vmatrix}2 & 1\\ 3 & -1\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}\\[2ex] \begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix} \end{bmatrix}=\begin{vmatrix}-3 & 3 & 3\\ 7 & -5 & -3\\ 5 & -1 & -3\end{vmatrix}.$$

Now,

$$B^{-1}=\frac{1}{\det B}(\operatorname{adj} B)=\frac{1}{6}\begin{bmatrix}-3 & 3 & 3\\ 7 & -5 & -3\\ 5 & -1 & -3\end{bmatrix}.$$

Now,

$$\det A=\begin{vmatrix}2 & 1 & 4\\ 1 & 0 & 2\\ 2 & 3 & 1\end{vmatrix}=2(0-6)-1(1-4)+4(3-0)=3\ (\neq 0);\ \therefore\ A^{-1} \text{ exists.}$$

$$\operatorname{adj} A=\begin{bmatrix} \begin{vmatrix}0 & 2\\ 3 & 1\end{vmatrix} & -\begin{vmatrix}1 & 4\\ 3 & 1\end{vmatrix} & \begin{vmatrix}1 & 4\\ 0 & 2\end{vmatrix}\\[2ex] -\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} & \begin{vmatrix}2 & 4\\ 2 & 1\end{vmatrix} & -\begin{vmatrix}2 & 4\\ 1 & 2\end{vmatrix}\\[2ex] \begin{vmatrix}1 & 0\\ 2 & 3\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 2 & 3\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 0\end{vmatrix} \end{bmatrix}=\begin{bmatrix}-6 & 11 & 2\\ 3 & -6 & 0\\ 3 & -4 & -1\end{bmatrix}.$$

$$A^{-1}=\frac{1}{\det A}(\operatorname{adj} A)=\frac{1}{3}\begin{bmatrix}-6 & 11 & 2\\ 3 & -6 & 0\\ 3 & -4 & -1\end{bmatrix}.$$

$$\therefore\ B^{-1}A^{-1}=\frac{1}{18}\begin{bmatrix}-3 & 3 & 3\\ 7 & -5 & -3\\ 5 & -1 & -3\end{bmatrix}\begin{bmatrix}-6 & 11 & 2\\ 3 & -6 & 0\\ 3 & -4 & -1\end{bmatrix}=\frac{1}{18}\begin{bmatrix}36 & -63 & -9\\ -66 & 119 & 17\\ -42 & 73 & 13\end{bmatrix}$$

$$=\begin{bmatrix}2 & -7/2 & -1/2\\ -11/3 & 119/18 & 17/18\\ -7/3 & 73/18 & 13/18\end{bmatrix}.$$

Thus, we find $(AB)^{-1}=B^{-1}A^{-1}$.

Example 10. *If* $A^{-1}=\begin{pmatrix}2 & 5 & 3\\ 3 & 1 & 2\\ 1 & 2 & -1\end{pmatrix}$ *find* A. [C.U. B.Com.(H) 2001]

Solution: Since $(A^{-1})^{-1}=A$, we require to find inverse of the given matrix

$$A^{-1}=\begin{pmatrix}2 & 5 & 3\\ 3 & 1 & 2\\ 1 & 2 & -1\end{pmatrix}.$$

Step I. $\det A^{-1} = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 2(-1-4)-5(-3-2)+3(6-1)$
$= -10+25+15 = 30 \neq 0.$

$\therefore$ *inverse of A^{-1} exists.*

Step II. *Matrix of the cofactors of A^{-1}*

$$= \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 5 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -5 & 5 & 5 \\ 11 & -5 & 1 \\ 7 & 5 & -13 \end{bmatrix}.$$

adjoint of $A^{-1} = \begin{bmatrix} -5 & 11 & 7 \\ 5 & -5 & 5 \\ 5 & 1 & -13 \end{bmatrix}$ (*Transposing matrix of the cofactors of* A^{-1})

$$\therefore A = \left(A^{-1}\right)^{-1} = \frac{1}{\det A^{-1}}\left(\operatorname{adj} A^{-1}\right) = \frac{1}{30}\begin{bmatrix} -5 & 11 & 7 \\ 5 & -5 & 5 \\ 5 & 1 & -13 \end{bmatrix}.$$

Example 11. *If* $AB = \begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix}$ *and* $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, *find B.* **[CA Foun. 2003]**

Solution: We first obtain A^{-1}.

$$|A| = \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} = 8-7 = 1 \neq 0.$$

$\therefore$ A^{-1} exists. Matrix of the cofactor of $A = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$.

$\therefore$ adjoint of $A = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$.

$$\therefore A^{-1} = \frac{1}{\det A}\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}.$$

Now, $AB = \begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix}$.

$$\therefore \quad A^{-1}(AB) = A^{-1}\begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix}$$

$$\text{or,} \quad (A^{-1}A)B = A^{-1}\begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix}$$

$$\text{or,} \quad IB = B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 33 & 9 \\ -110 & -30 \end{bmatrix}.$$

Hence

$$\mathbf{B} = \begin{bmatrix} \mathbf{33} & \mathbf{9} \\ \mathbf{-110} & \mathbf{-30} \end{bmatrix}.$$

3.2.2 Method of Reduction

A third method of finding the inverse of a square matrix is to use elementary row operations or transformations. We explain the technique in the following lines:

Elementary Row Operations

Given any matrix, each of the following operations O_1, O_2, O_3 is called an *Elementary Row Operation.*

O_1 Interchange two rows: e.g., the operation r_{23} is the interchange of row 2 and row 3.

O_2 Multiply any row by a non-zero quantity: e.g., the operation $r_2(5)$ means multiply row 2 by 5.

O_3 Addition of any multiple of one row with another row: e.g., $r_{23}(5) \rightarrow$ multiply the third row by 5 and add with row 2.

[Elementary Column Operations are similar to Elementary Row Operations, the word *row* being replaced by *column.*]

Method of Reduction for finding the Inverse of a Square Matrix A

We explain this method through an example:

Example 12. *Find by the method of row-reduction the inverse of the square matrix*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 5 \\ -1 & 4 & 0 \end{pmatrix}.$$

Solution: First, we see that $\det A = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 5 \\ -1 & 4 & 0 \end{vmatrix} = -5\begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} = -15 \neq 0$ (expanding in terms of elements of third column).

$\therefore$ A^{-1} exists.

Next, we form an Augmented Matrix of order 3×6, namely,

$$[\ A \ \vdots\ I_3\ ], \text{ i.e., } \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right].$$

We now apply elementary row operations, so that this augmented matrix may reduce in the form

$$[\ I_3 \ \vdots\ B\ ], \text{ i.e., } \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & \end{array}\right].$$

Then B is the inverse of A. Note that we are to make the left-hand matrix of $[\ A \ \vdots\ I_3\ ]$ to $[\ I_3 \ \vdots\ B\ ]$, i.e., make A to I_3.

We show below the steps:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note: See that in A, the element in the first row, first column is already 1. So, we wish to make 2nd row, first column zero and 3rd row, first column also zero (so that the first column becomes 1 0 0). For that we multiply first row by -2 and add with 2nd row [i.e., $r_{21}(-2)$]. Again, we multiply row one by 1 and add with the third row [$r_{31}(1)$].

$$\sim\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 5 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \to r_{21}(-2) \\ \to r_{31}(1) \end{array}$$

[Note that we shall make the element in the 2nd row, 2nd column 1 and for that we multiply 2nd row by 1/5 (i.e., $r_2(1/5)$)]

$$\sim\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2/5 & 1/5 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \end{array}\right] \to r_2(1/5)$$

[Next, we shall make the element in the first row, 2nd column zero and also the element in the third row, 2nd column zero so that the 2nd col. becomes (0 1 0). For that we apply $r_{12}(1)$ and $r_{32}(-3)$]

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/5 & 1/5 & 0 \\ 0 & 1 & 1 & -2/5 & 1/5 & 0 \\ 0 & 0 & -3 & 11/5 & -3/5 & 1 \end{array}\right] \begin{array}{l} \to r_2(1/5) \\ \\ \to r_{32}(-3) \end{array}$$

[Now, we shall make the element in the third row, third column 1. For that we multiply the elements of third row by $-1/3$ (i.e., $r_3(-1/3)$)]

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/5 & 1/5 & 0 \\ 0 & 1 & 1 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1 & -11/15 & 1/5 & -1/3 \end{array}\right] \to r_3(-1/3)$$

[Next, we shall make the third column (0 0 1). For that we shall multiply third row by -1 and add with row 2 and we shall multiply third row by -1 and add with row 1]

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4/3 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & -11/15 & 1/5 & -1/3 \end{array}\right] \begin{array}{l} \to r_{13}(-1) \\ \to r_{23}(-1) = [\ I_3 \ \vdots\ B\ ] \\ \\ \end{array}$$

$$\therefore \text{ the required } A^{-1} = B = \begin{bmatrix} 4/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 \\ -11/15 & 1/5 & -1/3 \end{bmatrix}.$$

Example 13. *Try in a similar manner (by the method of reduction) to find the inverse of each of the following matrices:*

$$\text{(a)} \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & 1 & -4 \end{bmatrix}; \quad \text{(c)} \begin{bmatrix} 1 & -3 & 2 \\ 2 & 5 & -1 \\ -3 & 1 & 4 \end{bmatrix}; \quad \text{(d)} \begin{pmatrix} 1 & 3 \\ 4 & 9 \end{pmatrix}; \quad \text{(e)} \begin{pmatrix} 5 & 6 \\ 4 & 2 \end{pmatrix}.$$

[CA 1999]

[**Hints:** Proceed as in the worked-out Ex. 12.]

In each case, check the result by taking $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

3.3 Solutions of a System of Linear Equations by the Method of Inversion of Coefficient Matrix

Consider the following set of linear equations:

$$\text{(i)} \left.\begin{aligned} ax+by &= l \\ px+qy &= m \end{aligned}\right\}; \qquad \text{(ii)} \left.\begin{aligned} ax+by+cz &= l \\ px+qy+rz &= m \\ ux+vy+wz &= n \end{aligned}\right\}.$$

In the first case, there are two unknowns and two equations and in the second case, there are three unknowns and three equations. In both cases, the unknowns are of first degree (i.e., linear).

In (i) let $A=\begin{bmatrix} a & b \\ p & q \end{bmatrix}$ (called *Coefficient Matrix*) and also suppose

$$X=\begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B=\begin{bmatrix} l \\ m \end{bmatrix}.$$

Then we can write the two equations in the matrix form

$$AX=B\left(\because AX=\begin{bmatrix} ax+by \\ px+qy \end{bmatrix} \text{ and } B=\begin{bmatrix} l \\ m \end{bmatrix}\right)$$

Suppose, now A^{-1} exists (i.e., suppose $\det A \neq 0$). Then

$$A^{-1}(AX)=A^{-1}B, \quad \text{or,} \quad (A^{-1}\cdot A)X=A^{-1}B \text{ [Associative Law]}$$
$$\text{or,} \quad IX=A^{-1}B, \text{ i.e., } X=A^{-1}B.$$

Observe that

$$\det A=\begin{vmatrix} a & b \\ p & q \end{vmatrix}=aq-bp \text{ (We assume that it is not zero).}$$

Matrix of the cofactors of $A=\begin{bmatrix} q & -p \\ -b & a \end{bmatrix}$;

$$\therefore \operatorname{adj} A=\begin{bmatrix} q & -b \\ -p & a \end{bmatrix}.$$

$$\therefore A^{-1}=\frac{1}{\det A}(\operatorname{adj} A)=\frac{1}{aq-bp}\begin{bmatrix} q & -b \\ -p & a \end{bmatrix}.$$

$\therefore X=A^{-1}B$ gives

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} q/(aq-bp) & -b/(aq-bp) \\ -p/(aq-bp) & a/(aq-bp) \end{bmatrix}\begin{bmatrix} l \\ m \end{bmatrix},$$

$$\text{i.e.,} \quad \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} (ql-bm)/(aq-bp) \\ (-pl+am)/(aq-bp) \end{bmatrix}.$$

$$\therefore x=\frac{ql-bm}{aq-bp}, \; y=\frac{-pl+am}{aq-bp}.$$

Similarly, in (ii) we assume

$$A=\underset{\text{[Coefficient Matrix]}}{\begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}}, \quad X=\underset{\text{[Matrix of unknown]}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}, \quad B=\underset{\text{[Right side constants]}}{\begin{bmatrix} l \\ m \\ n \end{bmatrix}},$$

Then the given set of equations can be written in the matrix form $AX=B$.

If A^{-1} exists, i.e., if A is a non-singular matrix ($\det A \neq 0$), then we write

$$A^{-1}(AX) = A^{-1}\cdot B \quad \text{or,} \quad (A^{-1}\cdot A)X = A^{-1}\cdot B$$
$$\text{or,} \quad IX = A^{-1}\cdot B, \text{ i.e., } X = A^{-1}B.$$

We find A^{-1} and multiply by B. We then equate with X. The values of x, y, z will be obtained. See the numerical illustrations given below:

Example 14. *Solve the following system of linear equations by Matrix Inversion Method*:

$$\left.\begin{aligned} 2x+y+4z &= 2 \\ x+4y+2z &= 3 \\ 2x+3y+z &= -6 \end{aligned}\right\}.$$ [C.U. B.Com.(H) 1998 Type]

Solution: The given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}.$$

Hence, we can write $X = A^{-1}B$, if A^{-1} exists.

We next proceed to find A^{-1}.

Here

$$\det A = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 2(4-6) - 1(1-4) + 4(3-8)$$
$$= -21 \ (\neq 0) \quad (\text{Hence, } A^{-1} \text{ exists}).$$

Matrix of the cofactors of A

$$= \begin{bmatrix} +\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 3 & -5 \\ 11 & -6 & -4 \\ -14 & 0 & 7 \end{bmatrix}.$$

$$\therefore \ \operatorname{adj} A = \begin{bmatrix} -2 & 11 & -14 \\ 3 & -6 & 0 \\ -5 & -4 & 7 \end{bmatrix}; \ A^{-1} = \frac{1}{-21}\begin{bmatrix} -2 & 11 & -14 \\ 3 & -6 & 0 \\ -5 & -4 & 7 \end{bmatrix}.$$

$$\therefore \ X = A^{-1}B = \begin{bmatrix} 2/21 & -(11/21) & 2/3 \\ -(1/7) & 2/7 & 0 \\ 5/21 & 4/21 & -(1/3) \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix},$$

$$\text{i.e.,} \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{21} - \frac{33}{21} - 4 \\ -\frac{2}{7} + \frac{6}{7} + 0 \\ +\frac{10}{21} + \frac{12}{21} + 2 \end{bmatrix} = \begin{bmatrix} -(113/21) \\ 4/7 \\ 64/21 \end{bmatrix}.$$

$$\therefore\ \boldsymbol{x = -\frac{113}{21};\ y = \frac{4}{7};\ z = \frac{64}{21}}\ \text{(Reqd. solutions).}$$

Note: The students may check the correctness of the result by substituting these values in the given equations. Each of the three equations will be satisfied by these values of x, y, z.

Example 15. (i) *Solve* $2x - y = 3, -x + 2y = -3$ *by inverting the coefficient matrix.*
(ii) *Solve by matrix:* $3x + 4y = 1, x + 3y + 3 = 0.$ [C.U. B.Com.(H) 2010]

Solution: (i) Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Then the given equations can be written in the matrix form as $AX = B$, so that $X = A^{-1}B$.
Here,

$$\det A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3\ (\neq 0);\ A^{-1} \text{ exists.}$$

Matrix of the cofactors $= \begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix}$; $\operatorname{adj} A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Hence,

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

$$\therefore\ X = A^{-1}B = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ i.e., } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, $\boldsymbol{x = 1, y = -1}$ (Reqd. solutions).

(ii) Here

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

The equations may be written as $AX = B$ which gives $X = A^{-1}B$. Also $|A| = \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0.$

Now, matrix of cofactor of $A = \begin{vmatrix} 3 & -1 \\ -4 & 3 \end{vmatrix}$ and $\operatorname{adj} A = \begin{vmatrix} 3 & -4 \\ -1 & 3 \end{vmatrix}$.

$$\therefore\ A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix} \text{ and } A^{-1}B = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

$\therefore\ X = A^{-1}B$ gives $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Hence, $\boldsymbol{x = 3}$ and $\boldsymbol{y = -2}$.

Example 16. *Find the inverse of the matrix* $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$ *and hence solve the system of equations*

$$\left.\begin{aligned} x + 2y + z &= 4 \\ x - y + z &= 5 \\ 2x + 3y - z &= 1 \end{aligned}\right\}.$$

[C.U. B.Com.(H) 1994]

Solution: Given, $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$.

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{vmatrix} = 1(1-3) - 2(-1-2) + 1(3+2) = -2+6+5 = 9 \neq 0.$$

$\therefore$ A is non-singular and so A^{-1} exists.

Matrix of the cofactors of A

$$= \begin{bmatrix} \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 5 & -3 & 1 \\ 3 & 0 & -3 \end{bmatrix}.$$

$\therefore$ adj$A$ = transpose of matrix of cofactors of A

$$= \begin{bmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{\det A}(\text{adj}\, A) = \frac{1}{9}\begin{bmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}.$$

Given system of equations can be written in the matrix form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}.$$

$$\therefore X = A^{-1}B = \frac{1}{9}\begin{bmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 20 \\ -3 \\ 22 \end{bmatrix},$$

$$\text{i.e., } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20/9 \\ -1/3 \\ 22/9 \end{bmatrix}, \text{ i.e., } \boldsymbol{x = \frac{20}{9}, y = -\frac{1}{3}, z = \frac{22}{9}}.$$

[*Check*:

$$x + 2y + z = 4\left(\frac{20}{9} - \frac{2}{3} + \frac{22}{9} = \frac{20-6+22}{9} = \frac{36}{9} = 4\right)$$

$$x - y + z = 5\left(\frac{20}{9} + \frac{1}{3} + \frac{22}{9} = \frac{20+3+22}{9} = \frac{45}{9} = 5\right)$$

$$2x + 3y - z = 1\left(\frac{40}{9} - 1 - \frac{22}{9} = \frac{40-9-22}{9} = \frac{9}{9} = 1\right).]$$

3.4 Rank of a Matrix: Elementary Ideas [C.U. B.Com. 2001]

A matrix is said to be of rank r if it has at least one minor of order r which is not zero but all minors of order more than r have the value zero. We denote the rank of a matrix A by $r(A)$.

Minor of order r : Given any matrix A (either a rectangular matrix or a square matrix), we retain any r rows and r columns and delete other rows and columns. We are thus left with a square submatrix of order r. The determinant of this submatrix of order r is called a *minor of order r*, e.g., let

$$A = \begin{bmatrix} 1 & 3 & 5 & 9 & 8 \\ 2 & 5 & 7 & 0 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix};$$

a minor of order $2 = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}$ (retaining two rows and two columns, deleting the 2nd row, and first, fourth and fifth columns).

Other minors of order 2 are $\begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 9 & 8 \\ 0 & 3 \end{vmatrix}$, etc.;

minors of order 3 are $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 5 & 7 \\ 1 & 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 5 & 9 \\ 5 & 7 & 0 \\ 2 & 3 & 4 \end{vmatrix}$, etc.

There is no minor of order 4 here.

Example 17. *Let* $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. *There is a minor of order* 2, *namely,* $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6\ (\neq 0)$, *but there is no minor of order more than two.*

$\therefore$ rank of $A = 2$.

Example 18. *Let* $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

Clearly, there are minors of order two, say, $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \neq 0$.

There is only one minor of order three, namely, $\begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$; $\therefore$ rank < 3.

Hence, in this case, $\boldsymbol{r(A) = 2}$.

Example 19. *Let* $B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

There is no minor of order 4. All minors of order 3 is zero. There is a minor of order 2, namely,

$$\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5\ (\neq 0). \ \therefore\ \boldsymbol{r(B) = 2}.$$

Example 20. *Let* $C = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Here, $\boldsymbol{r(C) = 3}$, because $\begin{vmatrix} 4 & -1 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 0 \end{vmatrix} \neq 0$ and it is the only minor of order 3.

Example 21. (i) $\begin{bmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix}$ *has the rank* 2 *because there exists a minor of order* 2, *say,* $\begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} = 6\ (\neq 0)$, *but the only minor of order* $3 = \begin{vmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{vmatrix} = 2(12-12) + 7(6-6) = 0.$

(ii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ *has rank* 2, *because there exists a minor of order* 2, *say,* $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ *whose value* $= 1 \neq 0$ *and the only minor of order* $3 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0.$

Example 22. $\begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$ *has the rank* 2, *because all minors of order* 3 *and* 4 *are zero, but at least one minor of order* 2, *say,* $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0.$

EXERCISES ON CHAPTER 3

(Matrix Inversion: Rank of a Matrix)

[$\boldsymbol{I}$ = unit matrix of appropriate order; $\boldsymbol{O}$ = null matrix]

A

1. (a) Define the inverse of a square matrix A. Verify that the matrix equation $A^2 - 4A + 3I = 0$ is satisfied by $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

   Now, use your definition of the inverse to obtain A^{-1}.

   (b) If $A = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$, show that $AA^{-1} = I_2$, where A^{-1} is the inverse of A and I_2 is the second order unit matrix. **[C.U. B.Com.(H) 1996]**

   (c) If the matrix A satisfies $A^2 - A + I = O$, when I is the unit matrix and O is the zero matrix, prove that A^{-1} exists and is equal to $I - A$. **[C.U. B.Com.(H) 2003]**

   [**Hints:** $A^2 - A + I = O$ or, $I = A - A^2 = AI - A^2 = A(I - A)$. $\therefore$ $A + I = A^{-1}A(I - A)$ or, $A^{-1} = I(I - A) = I = A$.]

(d) Compute the adjoint of the matrix $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}$ and verify that $A(\operatorname{adj}A)=|A|I$.

[B.U. B.Com. 2007]

(e) If $A=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and $AB=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, find B^{-1}. [B.U. B.Com.(H) 2008]

[Hints: $|A|=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=2-1=1\neq 0$ and $A^{-1}=\frac{1}{1}\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$.

$\therefore A^{-1}(AB)=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\times\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Rightarrow(A^{-1}A)B=\begin{pmatrix}2-0 & 0-1\\ -1+0 & 0+1\end{pmatrix}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$

$\Rightarrow IB=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\Rightarrow B=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$ and $|B|=\begin{vmatrix}2 & -1\\ -1 & 1\end{vmatrix}=2-1=1\neq 0$.

$\therefore B^{-1}=\frac{1}{1}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$.]

2. (a) Show that the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$ satisfies the equation $A^3-23A-40I=O$. Hence, obtain A^{-1}.

(b) If $A=\begin{bmatrix}5 & 4 & -2\\ 4 & 5 & -2\\ -2 & -2 & 2\end{bmatrix}$, show that $A^2-11A+10I=O$, where I and O are identity matrix and null matrix of order 3 respectively. Using the above result, find A^{-1}. [C.U. B.Com.(H) 2000]

(c) If the matrix $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$, then prove that $A^2-4A+3I=O$, where I is the unit matrix of order 2 and O is a null matrix of same order. Also find A^{-1}. [V.U. B.Com.(H) 2010]

3. (a) If $A=\begin{bmatrix}-1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, verify $A^3=A^2\cdot A=A\cdot A^2=I$ and hence, find A^{-1}.

[Hints: $A^{-1}=A^2$. Find A^2.]

(b) For the matrix $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 3 & 0\\ 18 & 2 & 10\end{bmatrix}$, prove that $A(\operatorname{adj}A)=0$.

(c) Find the inverse of the matrix $A=\begin{bmatrix}3 & -10 & -1\\ -2 & 8 & 2\\ 2 & -4 & -2\end{bmatrix}$.

4. We define a square matrix A to be an orthogonal matrix if $A.A^T=I$. For an orthogonal matrix, therefore, A^T is the inverse of A $(A^T=A^{-1})$.

Verify that the following matrices are orthogonal and find their inverses:

(a) $\frac{1}{3}\begin{bmatrix}-1 & 2 & -2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$; (b) $\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$.

5. (a) A matrix A is called self-reciprocal if $A^{-1}=A$. Verify that the matrix $\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ is self-reciprocal (i.e., verify $A^{-1}=A$).

(b) For the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, find A^{-1}, the inverse of A. [C.U. B.Com.(H) 1999]

6. (a) Show that $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is an orthogonal matrix, and then write down the inverse of A.

(b) If $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -5 \\ 0 & 7 \end{bmatrix}$, find matrix A and verify your result. [C.U. B.Com.(H) 2004]

(c) Find the matrix A such that $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

[**Hints**: Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$. Then from the given matrix equation, we get

$$BAC = I \Rightarrow (B^{-1}B)A(CC^{-1}) = B^{-1}IC^{-1} \Rightarrow IAI = B^{-1}C^{-1} \Rightarrow A = B^{-1}C^{-1}$$

[by Associative Law]

Now, find B^{-1}, C^{-1} and then $B^{-1}C^{-1}$.]

7. Find the inverse of $\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ and hence solve for $x, y, z : x + 2y + z = 4;\ x - y + z = 5;\ 2x + 3y - z = 1$. [C.U. B.Com.(H) 1994]

8. Solve $2x + y + 4z = 2$, $x + 2z = 3$, $2x + 3y + z = -6$, by the method of inversion of the coefficient matrix.

9. If $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, evaluate X^2 and X^3.

If $A = lI + mX + nX^2$ and $B = l^2I - lmX + (m^2 - ln)X^2$ (l, m, n are numerical constants), prove that $AB = l^3I$. Deduce that if $l \neq 0$, then A has an inverse and equals to $(1/l^3)B$.

10. If $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 5 & 1 & -1 \end{bmatrix}$, show that $AA^{-1} = A^{-1}A = I$, where I is the unit matrix of order 3.

[V.U. B.Com.(H) 2011]

11. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, verify that $(A^{-1})^2 = I - A$.

12. Solve (by inverting the coefficient matrix):

(a) $\left.\begin{aligned} 2x+y+z &= 1 \\ x-y+2z &= -1 \\ 3x+2y-z &= 4 \end{aligned}\right\}$; (b) $\left.\begin{aligned} x-y+z &= 1 \\ x+y+2z &= 0 \\ 2x-y+3z &= 2 \end{aligned}\right\}$; (c) $\left.\begin{aligned} 5x+6y &= 39 \\ -7x+8y &= 11 \end{aligned}\right\}$;

[C.U. B.Com.(H) 1998; V.U. B.Com.(H) 2007]

(d) $\left.\begin{aligned} 2x-y &= 3 \\ -x+2y &= -3 \end{aligned}\right\}$; (e) $\left.\begin{aligned} x+2y+3z &= 6 \\ 3x-2y+z &= 2 \\ 4x+2y+z &= 7 \end{aligned}\right\}$, (f) $\left.\begin{aligned} 2x-3y+4z &= -4 \\ y-4z+2 &= 0 \\ x+z &= 0 \end{aligned}\right\}$.

[C.U. B.Com. 2004]

(g) $\left.\begin{aligned} 3x+y+z &= 1 \\ 2x+2z &= 0 \\ 5x+y+2z &= 2 \end{aligned}\right\}$; (h) $\left.\begin{aligned} 2x+3y+z &= 11 \\ x+y+z &= 6 \\ 5x-y-10z &= 34 \end{aligned}\right\}$.

[C.U. B.Com. 2007] [V.U. B.Com.(H) 2008]

13. A square matrix A such that $A^n = 0$, for some positive integer n, is called Nilpotent matrix. The least positive integer n for which $A^n = 0$ is called the index n of the nilpotent matrix A.

Verify that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3 [i.e., verify $A^2 \neq 0$, but $A^3 = 0$].

14. A square matrix A is called Idempotent if $A^2 = A$; verify that $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

15. A square matrix A is called a Symmetric matrix if $A^T = A$; skew-symmetric if $A^T = -A$.

(a) Verify that $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 8 \\ 2 & 8 & 4 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are all symmetric matrices.

(Notice that they are all symmetrical about the main diagonal.)

(b) Verify that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$ are all skew-symmetric matrices.

16. (a) Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & 1 & -4 \end{bmatrix}$. Hence, solve the system of equations $2x - y + 3z = 7$, $x + 3y - z = 8$ and $x + y - 4z = 1$. [CA Foun. May 2001]

(b) Find the inverse of $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and using the inverse solve:

$2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$. [C.U. B.Com. 2005]

(c) Find the inverse of $A=\begin{pmatrix} 2 & 3 & 4 \\ 5 & -2 & -3 \\ 3 & 1 & 6 \end{pmatrix}$ and hence, solve the equations:

$2x+3y+4z=4,\ 5x-2y-3z=4,\ 3x+y+6z=-1.$ [C.U. B.Com. 2005]

17. (a) Solve by matrix method: $7x-y-z=0, 10x-2y+z=8, 6x+3y-2z=7.$ [B.U. B.Com.(H) 2008]

(b) $x+2y+z=4, x-y+z=5, 2x+3y-z=1.$ [C.U. B.Com. 2008]

18. Find the inverse of $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{bmatrix}$ and hence solve the system of equations: $2x-y+2z=-8$, $x+2y-3z=9,\ 3x-y-4z=3.$ [C.U. B.Com. 2010]

[**Hints**: See worked-out Ex. 16.]

B

Exs 19–22 can be treated as Theorems

19. A square matrix A of order n has an inverse, if and only if $\det A \neq 0$ (i.e., if and only if it is non-singular).

Proof. Let A be a square matrix of order n.

(a) If A has an inverse, then, by definition, there exists a square matrix B of same order n such that

$$AB = BA = I, \tag{1}$$

where I is a unit matrix of order n.

Now, it can be proved (here, we assume it without proof) that if A and B are two square matrices of order n, then $\det AB = \det A \cdot \det B$.

$\therefore$ from (1) we obtain, $\det A \cdot \det B = \det I_n = 1.$

This implies that neither $\det A$ nor $\det B$ is zero.

In particular, $\det A \neq 0$, i.e., A is non-singular.

(b) Conversely, let A be a non-singular matrix of order n (i.e., let $\det A \neq 0$).

For a square matrix A of order n,

$$A \cdot (\operatorname{adj} A) = (\operatorname{adj} A) \cdot A = \det A \cdot I$$

since $\det A \neq 0$, $A \cdot \frac{\operatorname{adj} A}{\det A} = \frac{\operatorname{adj} A}{\det A} A = I.$

From definition it follows that $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}.$

(c) Does the inverse of matrix $\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ exist? Give reason. [B.U. B.Com.(H) 2008]

[**Hints**: $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1(0+1)+0+0 = 1 \neq 0$ [expanding along 3rd row]

Hence, the *inverse of the given matrix exists.*]

20. An invertible matrix has a unique inverse, i.e., if A has an inverse, then this is unique, i.e., we cannot have two distinct inverses for the same matrix A.

Proof. Suppose that A has an inverse A^{-1}.

Then, by definition $A \cdot A^{-1} = A^{-1} \cdot A = I$.

If possible, let there be another matrix B which is also an inverse of A. Then

$$A \cdot B = B \cdot A = I.$$

Our theorem will be proved if we can show that $B = A^{-1}$.

We have
$$\begin{aligned} A \cdot \left(B - A^{-1}\right) &= A \cdot B - A \cdot A^{-1} \text{ (by Distributive Law)} \\ &= I - I = O \text{ (null matrix).} \end{aligned}$$

Hence
$$\begin{aligned} & A^{-1} \cdot \left\{A \cdot \left(B - A^{-1}\right)\right\} = A^{-1} \cdot O = O \\ \text{or,}\quad & \left(A^{-1} \cdot A\right) \cdot \left(B - A^{-1}\right) = O \text{ (by Associative Law)} \\ \text{or,}\quad & I \cdot \left(B - A^{-1}\right) = O \text{ or, } B - A^{-1} = O, \\ \text{i.e.,}\quad & B = A^{-1} \textbf{ (Proved).} \end{aligned}$$

21. If A^{-1} exists, then $\left(A^{-1}\right)^{-1} = A$, ($A$ being a given square matrix) and $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$, where A^T is the transpose of A.

Proof. (a) Since A^{-1} exists, we have $A \cdot A^{-1} = A^{-1} \cdot A = I$.

This shows that the inverse of A^{-1} exists and being unique, we may write $\left(A^{-1}\right)^{-1} = A$. This proves the first part.

(b) Now, we know that

$$\begin{array}{ll|ll} & \left(AA^{-1}\right)^T = \left(A^{-1}\right)^T \cdot A^T, & & \left(A^{-1}A\right)^T = A^T \cdot \left(A^{-1}\right)^T, \\ \text{i.e.,} & (I)^T = \left(A^{-1}\right)^T \cdot A^T, & \text{i.e.,} & I^T = A^T \cdot \left(A^{-1}\right)^T \\ \therefore & \left(A^{-1}\right)^T \cdot A^T = A^T \cdot \left(A^{-1}\right)^T = I^T = I, & & \end{array}$$

i.e., the inverse of A^T is $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$, since the inverse is unique.

22. Let A and B be two square matrices of order n, and let I be the unit matrix of order n.

If A^{-1} and B^{-1} both exist, (i.e., $AA^{-1} = A^{-1}A = I$. $BB^{-1} = B^{-1}B = I$), then

$$(AB)^{-1} = B^{-1} \cdot A^{-1}.$$

Proof. $(AB)\left(B^{-1}A^{-1}\right) = A \cdot \left(BB^{-1}\right) \cdot A^{-1} = A \cdot (I) \cdot A^{-1} \quad (\because BB^{-1} = I)$
$= (AI) \cdot A^{-1} = A.A^{-1} = I.$

Similarly, we can prove that

$$\begin{aligned} \left(B^{-1}A^{-1}\right) \cdot (AB) &= B^{-1} \cdot \left(A^{-1}A\right) \cdot B = B^{-1} \cdot I \cdot B \\ &= \left(B^{-1}I\right) \cdot B = B^{-1}B = I. \end{aligned}$$

$\therefore$ (AB) has an inverse, namely, $B^{-1}A^{-1}$ and being unique we have

$$(AB)^{-1} = B^{-1} \cdot A^{-1}.$$

23. Find the ranks of the following matrices:

(a) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$; (c) $\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$;

[C.U. B.Com. 2004]

(e) $\begin{bmatrix} 1 & -5 & 3 \\ 2 & -1 & -1 \\ 5 & -1 & 1 \end{bmatrix}$; (f) $\begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix}$; (g) $\begin{bmatrix} 2 & 5 & 6 \\ 1 & -3 & -8 \\ 3 & 1 & -4 \end{bmatrix}$; (h) $\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$.

[C.U. B.Com.(H) 2000]

24. If $P = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, show that $P^2 = P$ and hence find the matrix Q, where $3P^2 - 2P + Q = I$; I is the unit matrix. Also, find the rank of P. [C.U. B.Com.(H) 1999]

[*Note*: The rank of a matrix can be conveniently found by row reduction process but it is not included here, because it is beyond the scope of this treatise.]

25. Find the rank of the matrix $\begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & -3 \\ 5 & 1 & -7 \end{pmatrix}$ and hence state with reason whether the system of equations $2x + 3y + 4z = 0$, $x - y - 3z = 0$, $5x + y - 7z = -4$ is solvable or not. [C.U. B.Com.(H) 2002]

[**Hints:** If $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & -3 \\ 5 & 1 & -7 \end{pmatrix}$, then

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & -3 \\ 5 & 1 & -7 \end{vmatrix} = 2(7+3) - 3(-7+15) + 4(1+5) = 20 \neq 0.$$

$\therefore$ rank of the matrix A is 3.

The Augmented matrix of the given system of equation is $\left(\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & -1 & -3 & 0 \\ 5 & 1 & -7 & -4 \end{array}\right)$

The rank of this matrix is also 3, since $|A| \neq 0$. Hence, the given system is consistent.]

26. If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$, show that $AB = 6I_3$ (I_3 is the identity matrix of order 3). Utilise this result to solve $2x + y + z = 5$, $x - y = 0$, $2x + y - z = 1$. [C.U. B.Com.(H) 2002]

[**Hints:** $AB = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2+2+2 & 1-2+1 & 1+0-1 \\ 2-4+2 & 1+4+1 & 1-0-1 \\ 6+0-6 & 3-0-3 & 3+0+3 \end{pmatrix} = 6I_3.$

$\therefore (AB).B^{-1} = 6I_3.B^{-1} \Rightarrow A(BB^{-1}) = 6B^{-1} \Rightarrow 6B^{-1} = A.I_3 = A.$

$\therefore B^{-1} = \frac{1}{6}A = \frac{1}{6}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}$. If $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$, then

$$X = B^{-1}C = \frac{1}{6}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \text{ etc.}]$$

27. Find the inverse of the matrix A, when $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & -2 & -3 \\ 3 & 1 & 6 \end{bmatrix}$ and hence, solve the equations $2x+3y+4z=4$, $5x-2y-3z=4$ and $3x+y+6z=-1$. [C.U. B.Com. 2005]

ANSWERS

A

1. (a) $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; (d) $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$;
   (e) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

2. (a) $\begin{bmatrix} -(1/10) & 1/10 & 1/5 \\ 1/40 & -(11/40) & 1/5 \\ 7/20 & 3/20 & -(1/5) \end{bmatrix}$;
   (b) $\frac{1}{10}\begin{bmatrix} 6 & -4 & 2 \\ -4 & 6 & 2 \\ 2 & 2 & 9 \end{bmatrix}$;
   (c) $\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$.

3. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix}$.

4. (a) $\begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$; (b) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

5. (b) $\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$.

6. (a) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$; (b) $\frac{4}{21}\begin{bmatrix} 7 & 5 \\ 0 & 3 \end{bmatrix}$;
   (c) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

7. $x = 20/9$, $y = -(1/3)$, $z = 22/9$.

8. $x = 3$, $y = -4$, $z = 0$.

9. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $X^3 = 0$.

12. (a) $x = 1, y = 0, z = -1$;
    (b) $x = -1, y = -1, z = 1$;
    (c) $x = 3, y = 4$;
    (d) $x = 1, y = -1$;
    (e) $x = y = z = 1$;
    (f) $x = -1, y = 2, z = 1$;
    (g) $x = 1, y = -1, z = -1$;
    (h) $x = \frac{143}{21}, y = -\frac{19}{21}, z = \frac{2}{21}$.

16. (a) $-\frac{1}{31}\begin{pmatrix} -11 & -1 & -8 \\ 3 & -11 & 5 \\ -2 & -3 & 7 \end{pmatrix}$; $x = 3, y = 2, z = 1$;
    (b) $-\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$; $x = 1, y = 2, z = 3$;
    (c) $-\frac{1}{91}\begin{pmatrix} -9 & -14 & -1 \\ -39 & 0 & 26 \\ 11 & 7 & -19 \end{pmatrix}$;
    $x = 1, y = 2, z = -1$.

17. (a) $x = 1, y = 3, z = 4$;
    (b) $x = 20/9, y = -1/3, z = 22/9$.

18. $-\frac{1}{31}\begin{bmatrix} -11 & -4 & 1 \\ -5 & -14 & 8 \\ -7 & -1 & 5 \end{bmatrix}$; $x = -1, y = 2, z = -2$.

B

23. (a) 1; (b) 2; (c) 1; (d) 3; (e) 3; (f) 1; (g) 2; (h) 2.

24. $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$; 1.

25. Rank = 3; solvable.

26. $x = y = 1, z = 2$.

27. $-\frac{1}{91}\begin{bmatrix} -9 & -14 & -1 \\ -39 & 0 & 26 \\ 11 & 7 & -19 \end{bmatrix}$; $x = 1, y = 2, z = -1$.

Chapter 4

Quadratic Forms

4.1 Introduction

The expressions like $3x+4$, x^2+8x-2, x^3+1, x^4+3x^2+8, etc., are called *polynomials* in x of degree 1, 2, 3, 4 respectively. *In each term the power of x is to be taken either a positive integer or zero.* The highest power determines the degree. The terms *linear, quadratic, cubic, bi-quadratic* expressions are also used for polynomials of degree one, two, three, four respectively.

When two variables, say x and y are involved, the expression will contain terms like $ax^m y^n$ (a is a constant; m, n are positive integers or zero). The degree of such a term is $m+n$. The sum of all such terms gives a polynomial in x and y of degree, which is the highest value of all such $m+n$. e.g., $3x+4x^2y^3-3x^2-4y^2+8y+7$ is a polynomial in x and y of degree 5.

If each term of a polynomial has the *same degree,* then the polynomial is called a *form.*

We can easily extend these ideas into more than two variables.

e.g., $4x-9y+8z$ is a *linear form* in three variables, and $x^2-2y^2-3z^2+4xy+6yz-8zx$ is a *quadratic form* in 3 variables.

Thus one may assume that a homogeneous polynomial of second degree in any number of variables can be called a *quadratic form* in those variables.

Definition 1. *If each term of a polynomial in two or more variables has the same degree 2, then the polynomial is called a quadratic form.*

For example, (i) $2x^2+5xy+3y^2$ is a quadratic form in two variables x and y.

(ii) $x^2+2y^2+3z^2+2yz+zx+3xy$ is a quadratic form in three variables x, y and z.

(iii) $2x^2+y^2+z^2+w^2+xy+3yz+zw+2yw$ is a quadratic form in four variables x, y, z and w.

Standard Quadratic Form

$q=ax^2+2hxy+by^2$ is a *standard quadratic form in two variables x and y*, and $q=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ is a standard quadratic form in three variables x, y and z.

Example 1. $ax^2+2hxy+by^2$ *is a standard quadratic form in two variables x and y.*

We shall investigate the formation of such expressions more closely.

Instead of x and y let us take the variables as x_1 and x_2.

See that the second degree terms are (See table):

$x_1 \cdot x_1 = x_1^2$, $x_1 \cdot x_2$, $x_2 \cdot x_1$, $x_2 \cdot x_2 = x_2^2$.

	x_1	x_2
x_1	$x_1 \cdot x_1$	$x_1 \cdot x_2$
x_2	$x_2 \cdot x_1$	$x_2 \cdot x_2$

Let us take the coefficients as a_{11}, a_{12}, a_{21}, a_{22}.

We shall agree to take the coefficients of $x_1 \cdot x_2$ and $x_2 \cdot x_1$ to be equal (i.e., $a_{12} = a_{21}$) and we shall always follow this convention and take the *most general quadratic form in two variables* x_1, x_2 as

$$q \equiv a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \text{ or, } \sum_{i=1}^{2}\left\{\sum_{j=1}^{2} a_{ij}x_ix_j\right\}, \ (a_{ij} = a_{ji}).$$

[In $ax^2 + 2hxy + by^2$ we take $a = a_{11}$, $b = a_{22}$ and $h = a_{12}$ or a_{21}.]

Note that with q we have associated a symmetric matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = A$ (say).

Example 2. *With three variables x_1, x_2, x_3 we may draw up similarly the following table*:

	x_1	x_2	x_3	Coefficients		
x_1	$x_1 \cdot x_1$	$x_1 \cdot x_2$	$x_1 \cdot x_3$	a_{11}	a_{12}	a_{13}
x_2	$x_2 \cdot x_1$	$x_2 \cdot x_2$	$x_2 \cdot x_3$	a_{21}	a_{22}	a_{23}
x_3	$x_3 \cdot x_1$	$x_3 \cdot x_2$	$x_3 \cdot x_3$	a_{31}	a_{32}	a_{33}

We agree to take $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$. Thus the most general quadratic form in 3 variables x_1, x_2, x_3 can be written as

$$q \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j \ (a_{ij} = a_{ji}).$$

The associated symmetric matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = A$ (say).

Observations. In Ex. 1 if we assume $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then its transpose $X' = [x_1 \ x_2]$ and

$$AX = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

and hence $$X'AX = [x_1 \ x_2]\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \left[a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2\right]$$

$$= \left[a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2\right] \quad [\because a_{21} = a_{12}].$$

Thus the quadratic form $q = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ can be identified by the matrix $X'AX$ of order 1×1.

Similarly, check in Ex. 2 that if $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and hence $X' = [x_1, x_2, x_3]$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$, then $q = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$ can be identified with the matrix $X'AX$ of order 1×1.

In each case A is called the Matrix of the Quadratic Form.

4.2 Formal Definition of a Quadratic Form

A homogeneous polynomial in n variables $x_1, x_2, \dots, x_n$ of the type

$$q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \left(a_{ij} = a_{ji}\right)$$

is called a *quadratic form* in n variables $x_1, x_2, \dots, x_n$.

We shall associate a symmetric matrix $A = \left(a_{ij}\right)_{m\times n} \left(a_{ij} = a_{ji}\right)$ with such an expression q.

Assuming $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $X' = [x_1, x_2, \dots, x_n]$, we shall *identify* the quadratic form q with $X'AX$.

The determinant of the symmetric matrix (i.e., $\det A = |A|$) is known as the *discriminant* of the quadratic form $X'AX$.

4.3 Positive and Negative Definiteness of Quadratic Forms

I. Let $q = ax^2 + 2hxy + by^2$ be a quadratic form in two variables x and y. A question arises in problems of maximum or minimum: What restrictions, if any, must be placed on a, b, h when x and y are allowed to take any values in order to assure a *positive* q or a *negative* q?

Definition 1. *A quadratic form q is called positive definite if q is invariably positive regardless of the values of the variables in the form q; negative definite if q is invariably negative.*

See that

$$q = ax^2 + 2hxy + by^2 = a\left(x^2 + \frac{2h}{a}x\cdot y + \frac{h^2y^2}{a^2}\right) - \frac{h^2y^2}{a} + by^2 = a\left(x + \frac{hy}{a}\right)^2 + \frac{ab-h^2}{a}y^2.$$

$\therefore$ **q is positive definite if and only if $a > 0$ and $ab - h^2 > 0$; q is negative definite if and only if $a < 0$ and $ab - h^2 > 0$.**

Now in terms of the associated matrix A of q we may state that q is $\left\{\begin{matrix}\textbf{positive definite} \\ \textbf{negative definite}\end{matrix}\right\}$ **if and only if**

$$\left\{\begin{matrix} a > 0 \\ a < 0 \end{matrix}\right\} \text{ and } \det A = \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0.$$

Example 3. *Is $q = 5x^2 + 3xy + 2y^2$ either positive or negative definite?* [C.U. B.Com.(H) 1994]

Solution: Here $a = 5$, $2h = 3$, $b = 2$; $\therefore$ $\det A = \begin{vmatrix} 5 & 3/2 \\ 3/2 & 2 \end{vmatrix} = 10 - \frac{9}{4} = \frac{31}{4} > 0$ and coefficient of $x^2 = 5 > 0$, i.e., $a = 5 > 0$, $\det A > 0$.

Hence q is positive definite.

II. Let $q = ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz$.

With three-variable quadratic forms, we also consider the associated symmetric matrix

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

It can be proved, but we assume here without proof that in this case three principal minors are considered as $D_1 = |a|$, $D_2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$ and $D_3 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and.

I. q is positive definite if (i) $D_1 > 0$; (ii) $D_2 > 0$; (iii) $D_3 > 0$,

II. q is negative definite if (i) $D_1 < 0$; (ii) $D_2 > 0$; (iii) $D_3 < 0$.

(i.e., *alternately negative and positive*)

Conclusion

I. (i) A quadratic form $q \equiv ax^2 + 2hxy + by^2$ in two variables x, y is said to be **positive definite** if and only if $a > 0$ and $ab - h^2 > 0$, i.e., if and only if $a > 0$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.

(ii) A quadratic form $q \equiv ax^2 + 2hxy + by^2$ is said to be **negative definite** if and only if $a < 0$ and $ab - h^2 > 0$, i.e., if and only if $a < 0$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.

II. A quadratic form $q \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ in three variables x, y and z is said to be

(i) **positive definite** if and only if

$$a > 0, \quad \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} > 0,$$

(ii) **negative definite** if and only if

$$a < 0, \quad \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} < 0.$$

Example 4. *Determine whether $q = x^2 + 6y^2 + 3z^2 - 2xy - 4yz$ is positive or negative definite.*

[C.U. B.Com.(H) 1998]

Solution: The matrix A of q is

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

The three principal minors are

$$D_1 = 1,\ D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 6 - 1 = 5 \text{ and } D_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 1\begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 \\ 0 & 3 \end{vmatrix} = 14 - 3 = 11,$$

i.e., **D_1, D_2, D_3 are all positive.**

∴ the quadratic form is positive definite.

Example 5. *Show that the quadratic form $-2x^2 - 3y^2 - 4z^2 + 2xy + 5yz - 4zx$ is negative definite.*

[V.U. B.Com.(H) 2011]

Solution: $q = -2x^2 - 3y^2 - 4z^2 + 2xy + 5yz - 4zx$; $q = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$.
Here $a = -2$, $b = -3$, $c = -4$; $2h = 2$, $2f = 2$, $2g = -4$ or, $h = 1$, $f = 1$, $g = -2$.

$$\therefore a = -2 < 0, \begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 6 - 1 = 5 > 0$$

$$\text{and} \begin{vmatrix} -2 & 1 & -2 \\ 1 & -3 & 1 \\ -2 & 1 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -2 \\ 1 & -3 & 1 \\ 0 & 0 & -2 \end{vmatrix} = (-2)\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = -2 \times 5 = -10 < 0.$$

Hence $q = -2x^2 - 3y^2 - 4z^2 + 2xy + 5yz - 4zx$ is negative definite.

EXERCISES ON CHAPTER 4

(Quadratic Forms)

1. By direct matrix multiplication express each of the following as a quadratic form:

   (a) $[x \quad y]\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; (b) $[x \quad y]\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$;

   (c) $[x \quad y \quad z]\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; (d) $[x \quad y \quad z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

2. Express each of the following quadratic forms as a matrix product involving a *symmetric coefficient matrix*:

   (a) $q = 3x^2 - 4xy + 7y^2$; (b) $q = x_1^2 + 5x_1x_2 + 2x_2^2$;
   (c) $q = 10xy - x^2 - 31y^2$; (d) $q = 3x^2 - 2xy + 4xz + 5y^2 + 4z^2 - 2yz$.

   In each case find the *discriminant* of the quadratic form.

3. (a) Prove that the quadratic form $6x^2 + 3y^2 + 14z^2 + 4yz + 18zx + 4xy$ is positive definite.

   (b) Prove that the quadratic form $2x^2 + 5y^2 + z^2 + 2xy - 4yz$ is positive definite.

   [V.U. B.Com.(H) 2010]

   (c) Determine the nature of the quadratic form: $5x^2 + y^2 + 5z^2 + 4xy - 8xz - 4yz$.

   [C.U. B.Com.(H) 2000]

4. Show that $6xy - 4x^2 - 3y^2$ is negative definite and $x^2 + y^2 + z^2 - yz$ positive definite.

   [V.U. B.Com.(H) 2009]

5. Define a quadratic form and show that $p = 7x^2 + 6xy + 2y^2$ is positive definite.

6. Let $q = ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz$ be a quadratic form in three variables x, y, z. Write down the conditions for which q is (a) positive definite and (b) negative definite. Prove that $x^2 + 2y^2 + z^2 + yz + 2xy$ is positive definite. [C.U. B.Com.(H) 2001]

[*Remember*: $q = ax^2 + 2hxy + by^2$ is positive for all x, y (excepting $x = y = 0$) subject to the condition $lx + my = 0$, if and only if

$$\begin{vmatrix} 0 & l & m \\ l & a & h \\ m & h & b \end{vmatrix} < 0.$$

Discriminant $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is bordered by $[0 \quad l \quad m]$ and $\begin{bmatrix} 0 \\ l \\ m \end{bmatrix}$ and q is negative for all x, y (excepting $x = y = 0$) if this determinant is > 0. Use this to show that $x^2 + y^2 + 3xy$ subject to $2x + y = 0$ is negative definite.]

7. (a) Prove that the quadratic form $4xy - 2x^2 - 3y^2$ is a negative definite.
   (b) Determine whether $2xy - x^2 - 4y^2$ is positive or negative definite. [C.U. B.Com.(H) 1997]
8. Define a quadratic form. Determine whether $2x^2 + 6xy + 7y^2$ is positive or negative definite. [C.U. B.Com.(H) 1991]
9. Check definiteness of the following forms:
   (a) $3x_1^2 + 5x_2^2, 2x_1^2 + 3x_2^2 + x_3^2;\ x_1^2;$
   (b) $-2x_1^2 - x_2^2;\ -x_1^2 - x_2^2;\ -x_1^2;$
   (c) $x^2 + 2y^2 + z^2 + 2xy + yz - 4zx.$ [C.U. B.Com.(H) 1999]
   (d) $2x^2 + 5y^2 + z^2 + 2xy - 4yz.$ [C.U. B.Com.(H) 1995]

ANSWERS

1. (a) $4x^2 + 4xy + 3y^2$;
   (b) $5x^2 + 6xy$;
   (c) $x^2 + 2y^2 + 7z^2 - 4xy - 8xz$;
   (d) $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz$.
2. (a) $[x \quad y]\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$;
   (b) $[x_1 \quad x_2]\begin{bmatrix} 1 & 5/2 \\ 5/2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$;
   (c) $[x \quad y]\begin{bmatrix} -1 & 5 \\ 5 & -31 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$;
   (d) $[x \quad y \quad z]\begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
3. (c) Positive definite.
6. (a) Positive definite if and only if $a > 0$, $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$ and $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} > 0$;
   (b) Negative definite if and only if $a < 0$, $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$ and $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} < 0$.
7. (b) Negative definite.
8. Positive definite.
9. (a) Positive definite, Positive definite, Positive definite;
   (b) Negative definite; Negative definite; Negative definite;
   (c) Neither positive definite nor negative definite;
   (d) Positive definite.

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ADVANCED BUSINESS MATHEMATICS

CALCULUS

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Chapter 0

Basic Concepts: Number System

0.1 Introduction[1]

In order to study elements of Calculus, one must be acquainted with the basic concepts of Real Numbers. The starting point is the set N of natural numbers:

$$N = \{1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \cdots\}$$

The elements of N are called *natural numbers* or *positive integers* or *signless integers.*

Next, we introduce the set of negative integers:

$$\{-1, \quad -2, \quad -3, \quad -4, \cdots\}$$

Under the heading integers we have positive and negative integers together with a signless integer 0 (read as zero).

Thus, the set Z of all integers consists of

$$Z = \{0, \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 4, \cdots\}.$$

Next, we have a more general set, known as set Q of rational numbers. A rational number is defined as the ratio of two integers p and q, written as $\frac{p}{q}$ or p/q, with $q \neq 0$.

e.g., $\frac{2}{3}$, $-\frac{3}{4}$, $\frac{0}{1} = 0$, $3 = \frac{3}{1}$, $-4 = -\frac{4}{1}$, etc., are all examples of rational numbers, but $\frac{1}{0}$ is not a rational number; in fact, $\frac{1}{0}$ is not defined.

Every rational number can be expressed as a *decimal* which is either a terminating decimal (e.g., $0.25 = \frac{1}{4}$) or a recurring decimal (e.g., $\frac{1}{3} = 0.333\cdots = 0.\dot{3}$).

We shall call a non-terminating and non-recurring decimal as an *irrational number,*

e.g., $0.1010010001000010000001\cdots$ is an irrational number.

Most common examples of irrational numbers are surds (like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., $\sqrt[3]{3}$, $\sqrt[5]{7}$, etc.) and numbers π, e which shall occur in our future discussions.

The totality of rational and irrational numbers forms what we call *Real Number System* (R).

0.2 Properties of Rational Numbers

Before we introduce the important properties of rational numbers, we shall discuss about the important concept of Inequalities ($<$, less than and $>$, greater than).

A Short Note on Inequalities of Real Numbers

- A real number a is said to be greater than a real number b (written as $a > b$), if $a - b$ is positive.

[1]For the terms Sets, Subsets, etc., see Ch. 1: Elements of Set Theory.

e.g., $10 > 3$, because $10 - 3 = 7$ which is positive.

- A real number a is said to be less than a real number b (written as $a < b$), if $a - b$ is negative.
  e.g., $3 < 10$, because $3 - 10 = -7$ (negative).
- A real number a is equal to another real number b (written as $a = b$), if $a - b = 0$.

Note: $a > b$ implies $b < a$.
$a \geq b$ signifies either $a > b$ or $a = b$.
$a \leq b$ signifies either $a < b$ or $a = b$.
$a > 0$ means a is a positive real number and
$a < 0$ means a is a negative real number.

Example 1. $\frac{2}{3} < \frac{3}{4}$, *because* $\frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$ (*negative*). *We can also write* $\frac{3}{4} > \frac{2}{3}$. (*Compare:* $a < b \Rightarrow b > a$)

Consequences.

- $a > b$ and $c > d$ together imply $a + c > b + d$, and $ac > bd$, if a, b, c, d are all positive real numbers.
- If $a > b$, then $\begin{cases} ac > bc, \text{ if } c \text{ is positive} \\ \text{but } ac < bc, \text{ if } c \text{ is negative} \end{cases}$

  In particular, if $a > b$, then $-a < -b$.
- If $a > b$, then $\frac{1}{a} < \frac{1}{b}$ (a, b are both positive).
- If a, b are positive and $a < b$, then $a < \frac{a+b}{2} < b$ and $\frac{a+b}{2} \geq \sqrt{ab}$

  (Arithmetic Mean of $a, b \geq$ Geometric Mean of a, b)

  [$(\sqrt{a} - \sqrt{b})^2 \geq 0$ (where a and b are two positive real numbers) or, $a + b - 2\sqrt{ab} \geq 0$, i.e., $\frac{a+b}{2} \geq \sqrt{ab}$; equality holds, if $a = b$.]

Important Properties of Rational Numbers

- Two rational numbers a and b can be added, subtracted, multiplied and divided (except when $b = 0$). In each case, the result is also a rational number. This property is called the *closure property* of rational numbers w.r.t. addition, subtraction, multiplication and division; in the last case we can divide only when $b \neq 0$; in fact, $\frac{a}{0}$ is not defined (do not call it ∞ infinity).
- Given two real numbers a, b, we always have

  either $a > b$ or $a < b$ or $a = b$ (known as *Law of Trichotomy*).
- If $a < b$ and $b < c$, then $a < c$ (*Law of Transitivity*).
- Between any two rational numbers there exist infinitely many rational numbers (This is known as the *Density Property of Rational Numbers*).

  [See that $\frac{a+b}{2}$ lies between a and b (assume $a < b$). Then between a and $\frac{a+b}{2}$ lies $\frac{a + \frac{a+b}{2}}{2}$. Also between $\frac{a+b}{2}$ and b lies $\frac{\frac{a+b}{2} + b}{2}$, and so on.]
- *Geometrical Representation of Rational Numbers*

  Take a line L of indefinite length (Fig. 0.1). We attribute one sense of L positive and the other sense of L negative. The positive sense of L is indicated by an arrowhead. Now, choose two points O and U on the line L. O is taken as origin and U, the unit point on the right of O (i.e., towards the arrowhead).

  The origin O represents the number zero and the point U represents the number one.

  Any positive integer, say $+3$, is represented by a point A, where the length of OA = three times the length OU and the point A is on the right of O (i.e., in the positive sense).

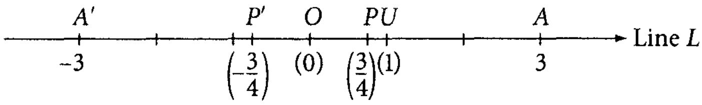


Fig. 0.1

Any fraction, like 3/4, may be represented by a point P, where length of OP = one-fourth of OA and the point P is also on the right of O. Symmetric points A' and P' about O will represent -3 and $-3/4$ respectively. Thus given any rational number p/q, we can represent it by a point on the directed line L.

0.3 Irrational Numbers

In order to prove the existence of numbers other than rational numbers, we first consider the following examples:

Example 2. *To prove that $\sqrt{2}$ is not a rational number.*

Indirect Proof. If possible, let $\sqrt{2}$ be a rational number. Suppose, then $\sqrt{2} = p/q$, where p and q are integers (prime to each other) and $q \neq 0$. We note that p and q are taken in such a way that p and q have no common factor.

Since $\sqrt{2} = p/q$, it follows $p^2 = 2q^2$, i.e., p^2 is an even integer.

Consequently, p must be an even integer $= 2k$ (say).

[We remark here that square of an odd integer is always odd. Since p^2 is even, p cannot be odd, because then its square would have been odd.]

Hence, $p^2 = 4k^2 = 2q^2$ or, $q^2 = 2k^2$, i.e., q^2 is an even integer and consequently, q must be an even integer.

This shows that p and q must have at least one common factor 2, which is against our assumption. Hence, the conclusion is that our assumption that $\sqrt{2} = p/q$ must be wrong, i.e., we cannot assume that $\sqrt{2}$ is a rational number. It must be a number other than a rational number. We shall call it an irrational number.

Example 3. *To prove that $\sqrt{3}$ is not a rational number.*

Proof. If possible, let $\sqrt{3} = p/q$, when p and q are integers (prime to each other) and $q \neq 0$. We note that p and q have no common factor or p/q is in its lowest terms.

Since $\sqrt{3} = p/q$, it follows $p^2 = 3q^2$.

We note that there are two consecutive integers 1 and 2 such that

$$1^2 < 3 < 2^2 \quad \text{or,} \quad 1^2 < p^2/q^2 < 2^2,$$
$$\text{i.e.,} \quad q^2 < p^2 < 4q^2 \quad \text{or,} \quad q < p < 2q.$$

Clearly, then $p - q < q$ ($\because$ $p < 2q$).

We next consider,

$$(3q - p)^2 = 9q^2 + p^2 - 6pq = 3p^2 + 3q^2 - 6pq \quad (\because\ p^2 = 3q^2)$$
$$= 3(p - q)^2$$

Hence,
$$\left(\frac{3q-p}{p-q}\right)^2=3,$$

i.e., there is a fraction $(3q-p)/(p-q)$, whose denominator $p-q<q$ but whose square is equal to 3. This means that the fraction p/q is not in its lowest terms for which $(p/q)^2=3$. This contradicts our assumption.

Hence, **$\sqrt{3}$ is not a rational number.**

Example 4. *Show that no positive integer m other than a square number has a square root within the system of rational numbers.*

Proof. If possible, let $\sqrt{m}=p/q$, where p/q is a fraction in its lowest terms. Then $p^2=mq^2$.

We can find two consecutive integers n and $(n+1)$ such that

$$n^2<m<(n+1)^2 \text{ or, } n^2<\frac{p^2}{q^2}<(n+1)^2 \text{ or, } n<\frac{p}{q}<(n+1).$$

Clearly, then $p-nq<q$.

We now consider,

$$(mq-np)^2=m^2q^2+n^2p^2-2mnpq=m\left(p^2\right)+n^2\left(mq^2\right)-2mnpq$$
$$=m\left[p^2+n^2q^2-2pnq\right]=m(p-nq)^2 \text{ or, } \left(\frac{mq-np}{p-nq}\right)^2=m.$$

Rule: Write m/n and p/q. Crosswise multiply and subtract: Then square it. Thus, obtain $(mq-np)^2$, i.e., there is a fraction $(mq-np)/(p-nq)$ whose denominator is $<q$ but whose square is equal to m, i.e., the fraction p/q is not in its lowest terms for which $(p/q)^2=m$. This contradicts our assumption, i.e., **$\sqrt{m}$ is not a rational number.**

Note: The arguments given in Ex. 4 are perfectly general. One can similarly prove that $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$, $\sqrt{13}$, $\sqrt{19}$, $\sqrt{23}$, $\sqrt{41}$, etc., are not rational numbers.

Example 5. *If $\sqrt{2}$ is not a rational number, then $\sqrt{2}+\sqrt{3}$ cannot be a rational number.*

Proof. If possible, let $\sqrt{3}+\sqrt{2}$ be a rational number.

Then
$$\sqrt{3}-\sqrt{2}=\frac{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}{\sqrt{3}+\sqrt{2}}=\frac{1}{\sqrt{3}+\sqrt{2}}$$

is also a rational number (since then it is the quotient of two rational numbers 1 and $\sqrt{3}+\sqrt{2}$).

$\therefore\ \sqrt{2}=\frac{1}{2}\left\{\left(\sqrt{3}+\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)\right\}$ is rational.

(Difference of two rational numbers is rational and division by 2 also makes it rational.)

But certainly $\sqrt{2}$ is not a rational number.

Thus, we arrive at a contradiction, i.e., our supposition $\sqrt{3}+\sqrt{2}$, a rational number, is wrong, i.e., **$\sqrt{3}+\sqrt{2}$ is not a rational number.**

Example 6. *To prove that* $\log_{10} 5$ *cannot be a rational number.*

Proof. If possible, let $\log_{10} 5 = \frac{p}{q}$, where p and q are integers (prime to each other) and $q \neq 0$ but positive.

Then $10^{p/q} = 5$ or, $10^p = 5^q$, i.e., $5^{q-p} = 2^p$.

This equation is not compatible, because LHS cannot be even whereas RHS is an even integer. This contradiction proves that our supposition is wrong, i.e., **$\log_{10} 5$ cannot be a rational number.**

Conclusion. The examples given above show that there must exist numbers other than rational numbers. We have observed before that rational numbers are dense (i.e., between any two of them, however close they may be, there exist many other rational numbers), but they are not dense enough to include a particular kind of decimals, namely, the decimals which are neither terminating nor recurring (e.g., 0.101001 0001 00001··· is not included in the rational number system). These numbers are called *Irrational Numbers*:

Definition 1. *A non-terminating, non-recurring decimal is an irrational number.*

Example 7. $\sqrt{2} = 1.4142134\cdots$; $\sqrt{3} = 1.7320507\cdots$; 1.03003 0003 00003,..., *etc. are irrational numbers.*

Note: There are two very important irrational numbers — one is π (approx. value 3.14159265···) and another is $e(\simeq 2.71828182\cdots)$. π and e are not solutions of any algebraic equations; and hence they are called *transcendental numbers.*

0.4 Real Numbers

The totality of the systems of rational and irrational numbers is said to form *the system of real numbers denoted by R.*

In Fig. 0.1 we have represented rational numbers on a directed line (O, origin and U, unit point). Even when all rational points are represented on such a line, there remain infinitely many gaps. These gaps correspond to irrational numbers. The existence of such an irrational point can be easily seen from Fig. 0.2 (B represents $\sqrt{2}$).

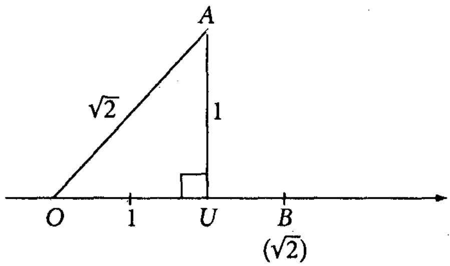


Fig. 0.2

Remember. Axiom: There is a one-to-one correspondence between the points on a directed line and the system of real numbers [Cantor-Dedekind Axiom].

This line is called the *Axis of Reals.*

The totality of points on the line is called the *Straight-line Continuum* or *Geometric Continuum* or *Linear Continuum.*

The totality of real numbers forms what we call the *Arithmetic Continuum.* What is stated in the theorem is the following:

Cantor-Dedekind Axiom

There is a one-to-one correspondence between Arithmetic Continuum and Linear Continuum.

Properties of Real Numbers: Real numbers are closed under the four fundamental operations — addition, subtraction, multiplication and division (except by zero).

- Addition and Multiplication are commutative and associative:
  $a + b = b + a$; $a \cdot b = b \cdot a$ (Commutative Property)
  $a + (b + c) = (a + b) + c$; $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associative Property)
  (a, b, c being any three real numbers)

- Addition and Multiplication are connected by Distributive Laws:
$$\left.\begin{aligned}\text{Left: } a\cdot(b+c)&=a\cdot b+a\cdot c\\ \text{Right: } (b+c)\cdot a&=b\cdot a+c\cdot a\end{aligned}\right\}.$$

Order Properties:

- Given two real numbers a and b, we must have either $a > b$ or $a = b$ or $a < b$ (Law of Trichotomy).
- If $a > b$ and $b > c$, then $a > c$ (Law of Transitivity).
- The system of real numbers is dense (i.e., between any two real numbers there exist infinitely many real numbers) (Density Property).
- Geometrically every point on a directed line is a real number and conversely, given any real number there exists a unique point on the directed line.
- Every real number is a decimal, either recurring or terminating or neither recurring nor terminating.

0.5 Intervals (Open or Closed): Absolute Values

Definitions: Intervals.

The set of all real numbers (or points) included between two given real numbers, say a and b, is called an *open interval*; a and b are called the *ends of the interval*. We write an open interval either as $a < x < b$ or, (a,b).

If, however, the ends a and b are included in the set, then we call it a *closed interval* $[a,b]$ or, $a \le x \le b$.

Half-open (or, may be called *Half-closed*) intervals are the following:
$$a \le x < b, \text{ or, } [a,b);\ a < x \le b, \text{ or, } (a,b].$$

These intervals are called *finite intervals* in the sense that the ends a, b are finite numbers.

We also have infinite intervals as given below:

- The set of all real numbers less than (or, not greater than) a given real number, say a:
$$-\infty < x < a \text{ (or, } -\infty < x \le a).$$
- The set of all real numbers greater than (or, not less than) a given real number, say a:
$$a < x < +\infty \text{ (or, } a \le x < +\infty).$$
- The set of all real numbers (Real Continuum):
$$-\infty < x < +\infty.$$

Note: $-\infty$ and $+\infty$ are merely symbols to denote that the set includes all possible real numbers of one side or other. *They are not numbers.*

0.6 Absolute Values (or Numerical Values)

Definition 1. *The absolute value (or modulus) $|a|$ of a real number a is defined in the following manner:*

$$|a| = \begin{cases} a, \text{ if } a \text{ is positive (i.e., if } a > 0) \\ -a, \text{ if } a \text{ is negative (i.e., if } a < 0) \\ 0, \text{ if } a = 0 \end{cases}$$

e.g., $|3| = 3$; $|-3| = -(-3) = 3$; $|0| = 0$.

Note: $|a|$ is always a non-negative number (i.e., it is either positive or zero).

Properties of Absolute Values

1. $|a| = |-a|$ [e.g., $|3-5| = |-2| = |2| = |5-3|$]

2. $|ab| = |a||b|$ [e.g., $|3 \times -4| = |3||-4| = 3 \times 4$]

3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $\left[\text{e.g., } \left|\frac{3}{-4}\right| = \frac{|3|}{|-4|} = \frac{3}{4}\right]$

   [1, 2, 3 follow easily from definitions]

4. $|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$.

   Proof. We shall prove:

   - if a, b are of the same sign, then $|a+b| = |a| + |b|$;
   - if a, b are of opposite signs, then $|a+b| < |a| + |b|$.

   Hence, combining the two results we write $|a+b| \le |a| + |b|$.

   - Let a, b be both positive. Then $|a| = a$, $|b| = b$. $a+b$ is certainly positive.
     $\therefore$ $|a+b| = a+b = |a| + |b|$.
     Let a, b be both negative. Then let us write $a = -m$, $b = -n$, where m, n are positive. $|a| = -a = m$, $|b| = -b = n$, $a+b = -(m+n)$, $|a+b| = m+n$; $\therefore$ $|a+b| = m+n = |a| + |b|$.
   - Let $a > 0$, $b < 0$. Then we may have $a+b > 0$ or, $a+b < 0$.
     If $a+b > 0$, then writing $b = -n(n > 0)$ we have $|a| = a$, $|b| = n$ and
     $|a+b| = a+b = a-n < a+n$, i.e., $< |a| + |b|$.
     If $a+b < 0$, then $|a+b| = -(a+b) = -a-b = n-a < n+a$, i.e., $< |b| + |a|$.
     In any case, when $a > 0$, $b < 0$, $|a+b| < |a| + |b|$.
     We interchange the roles of a and b if $a < 0$ and $b > 0$ and obtain $|a+b| < |a| + |b|$.
     $\therefore$ under all circumstances the following is true: $|a+b| \le |a| + |b|$.

5. $|\boldsymbol{a}-\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$ [$\because$ $|a-b| = |a+(-b)| \le |a| + |-b|$, i.e., $|a| + |b|$].

6. (a) $|\boldsymbol{a}+\boldsymbol{b}| \ge |\boldsymbol{a}| - |\boldsymbol{b}|$, if $|\boldsymbol{a}| \ge |\boldsymbol{b}|$
   [$\because$ $|a| = |a+b-b| \le |a+b| + |b|$, i.e., $0 \le |a| - |b| \le |a+b|$, hence, etc.]

   (b) $|\boldsymbol{a}+\boldsymbol{b}| \ge |\boldsymbol{b}| - |\boldsymbol{a}|$, if $|\boldsymbol{b}| \ge |\boldsymbol{a}|$.

7. Finally, combining 4, 5, 6, we may write, $\big||\boldsymbol{a}| - |\boldsymbol{b}|\big| \le |\boldsymbol{a} \pm \boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$.

8. $|\boldsymbol{a}| < \boldsymbol{b}$ **(**$\boldsymbol{b} > \mathbf{0}$**)** means $-\boldsymbol{b} < \boldsymbol{a} < +\boldsymbol{b}$. [For if $a > 0$, $|a| = a$, then $|a| < b$ means $a < b$, if $a < 0$, $|a| = -a$, then $|a| < b$ means $-a < b$, i.e., $a > -b$ or, $-b < a$; hence, combining $-b < a < b$.]

9. $|\boldsymbol{a}| > \boldsymbol{b}(\boldsymbol{b} > \mathbf{0})$ means either $\boldsymbol{a} > \boldsymbol{b}$, or, $\boldsymbol{a} < -\boldsymbol{b}$.

   [e.g., $|x| > 3$ means $x > 3$ or, $x < -3$, i.e., x does not take any value between -3 and 3.]

10. $\mathbf{0} < |\boldsymbol{x}-\boldsymbol{a}| \le \boldsymbol{\delta}$ means $\boldsymbol{a}-\boldsymbol{\delta} \le \boldsymbol{x} \le \boldsymbol{a}+\boldsymbol{\delta}$ (but $\boldsymbol{x} \ne \boldsymbol{a}$).

    [$|x-a| \le \delta$ means $-\delta \le x-a \le \delta$ or, $a-\delta \le x \le a+\delta$. But $0 < |x-a|$ means $x \ne a$, otherwise $x-a$ would be zero; hence, etc.]

0.7 Variables: Constants

Real Constants and Real Variables: We may think of a real number individually or we may consider an aggregate of real numbers or a collection of real numbers.

When we think of a real number individually, we say that it is a real constant, e.g., 1, 2, 3, $\sqrt{2}$, -7, $2/3$, π or e are individually real constants.

In our discussions that follow we may use symbols like a, b, c, l, m, n, etc., as real constants.

Now, sometimes we take an aggregate or a collection or a set S of real numbers. If x denotes any unspecified real number of the set S and if x may, in turn, represent any member of the set S, then we say that x is a real variable over the domain S.

Two important examples are:

(i) Closed interval: $a \leq x \leq b$ or, $[a, b]$;

(ii) Open interval: $a < x < b$ or, (a, b).

In (i), x is a real variable over the domain $[a, b]$.

In (ii), x is a real variable over the domain (a, b).

In (i), x can assume all real numbers that lie between two given real numbers a and b, including a and b.

In (ii), x can assume all real numbers that lie between two given real numbers a and b, but not a or b.

Note: The terms variables and constants are used in common day-to-day problems.

In the ordinary sense, a variable is a quantity which can assume different values. Variables frequently occur in many of our business problems: Profit, Sales, Revenue, National Income, Consumption, Investments, Exports, Imports, etc., are all examples of variables. They assume values which are non-negative real numbers, i.e., their domains are restricted to positive real numbers (including zero) only.

Constants, however, do not change their values. In problems of Economics when a variable, say the price P, takes a certain specified value (for instance, $P = 10$), we say, that the variable freezes at that value (thus, we say, P freezes at the constant value 10).

Arbitrary Constants and Fixed Constants: We call specified numbers (e.g., 1, 2, 3, -3, $\sqrt{5}$, etc.) as fixed constants — they do not change their values under any set of mathematical investigations. Sometimes we use symbols like a, b, c, d, l, m, n, etc., to represent some real numbers which remain fixed for a particular investigation but may take up different values on different investigations. Such constants are called *arbitrary constants* (or *parameters*).

In coordinate geometry, we say, *$y = mx$ represents a particular straight line passing through the origin when m has a fixed numerical value* but it will represent a different straight line passing through the origin when m has a different numerical value; here m is a parameter or an arbitrary constant.

So, when we are asked: *What does $y = mx$ represent?* the correct answer should be: *$y = mx$ represents a family of lines passing through the origin.*

Remember. For our purpose we shall specify a certain set X of real numbers over which a variable will assume values; X is the domain of the variable. X may contain discrete numbers like $X = \{100, 50, 9\}$ or X may contain continuous set of values like $[a, b]$ or (a, b). Accordingly, the variable is said to be *discrete* or *continuous*. When X contains *all real numbers*, we write, $X = (-\infty, \infty)$ = *Real Continuum.*

0.8 Illustrative Examples

Example 8. *For what values of x will the expression $\dfrac{x^2}{(x+1)(x+2)(x+3)}$ be undefined?*

Solution: We see that for $x = -1, x = -2, x = -3$, the expression involves division by zero (which is an undefined operation in mathematics).

$\therefore$ **the expression is undefined for $x = -1, -2, -3$.**

Example 9. *Prove that $\sqrt{11}$ is not a rational number.*

Solution: If possible, let $\sqrt{11} = p/q$ (or, $11q^2 = p^2$), where p and q are integers and $q \neq 0$. We assume that p and q are taken in such a way that p/q is in its lowest terms.

We see that $3^2 = 9, 4^2 = 16$; hence, $3^2 < 11 < 4^2$ or, $3^2 < \frac{p^2}{q^2} < 4^2$

or, $(3q)^2 < p^2 < (4q)^2$ or, $3q < p < 4q$, whence $p - 3q < q$.

We consider,

$$(11q - 3p)^2 = 11^2q^2 + 3^2p^2 - 2 \cdot 11q \cdot 3p = 11 \cdot p^2 + 3^2 \cdot 11q^2 - 2 \cdot 11q \cdot 3p$$
$$= 11\left(p^2 + 3^2q^2 - 2p3q\right) = 11(p - 3q)^2 \text{ or, } \left(\frac{11q - 3p}{p - 3q}\right)^2 = 11.$$

But we have proved that $p - 3q < q$; i.e., there is a fraction whose denominator is less than q but its square is 11. In other words, p/q is not in its lowest terms satisfying $(p/q)^2 = 11$. This contradiction proves that our assumption of $\sqrt{11}$ a rational number must be wrong, i.e., **$\sqrt{11}$ is not a rational number.**

Example 10. *Prove that $-7 \leq x \leq -3$ and $|x + 5| \leq 2$ are equivalent statements.*

Solution: (i) $-7 \leq x \leq -3$, or, $-7 - (-5) \leq x - (-5) \leq -3 - (-5)$,

or, $-2 \leq x + 5 \leq 2$,

or, $|x + 5| \leq 2$.

[See that the sum of the two ends $= -7 - 3 = -10$; half of this sum $= -5$]

(ii) Again, if it is given $|x + 5| \leq 2$, then we can write,

$$-2 \leq x + 5 \leq +2 \quad \text{or,} \quad -5 - 2 \leq x + 5 - 5 \leq 2 - 5$$
$$\text{or,} \quad -7 \leq x \leq -3.$$

$\therefore$ **the two relations $-7 \leq x \leq -3$ and $|x + 5| \leq 2$ are equivalent statements, i.e., one follows from the other.**

EXERCISES ON CHAPTER 0

(Number System)

1. Define a rational number. Which of the following expressions are rational/irrational/not defined?

$$\frac{0}{2}, \frac{2}{0}, \frac{0}{0}, \frac{3}{2}, -\frac{4}{5}, \frac{1}{0}, \frac{7}{3}, -\frac{3}{2}, \sqrt{2}, \sqrt[3]{5}, \infty.$$

2. Prove that $\sqrt{2}$ is not a rational number and hence prove that $\sqrt{3} + \sqrt{2}$ is also not rational.

3. Show that the sum, difference, product and quotient of two non-zero rational numbers are themselves rationals (you can assume here that the set of integers is closed under addition, subtraction and multiplication).

[Hints: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$; since a, b, c, d are integers and $b, d \neq 0$ we see that $ad+bc$ and bd are integers and $bd \neq 0$; hence the sum $(ad+bc)/bd$ is a rational number; similarly others.]

4. Prove that $a+b$ cannot be a rational number if a is rational and b is irrational.

   [If possible, let $a+b=$ a rational number k, then $b=k-a$.

   Now, $k-a$ must be a rational number, since it is the difference of two rational numbers. But b is not rational; hence a contradiction; etc.]

5. Prove that ab and $a/b (b \neq 0)$ are not rational numbers, if a is rational and b irrational.

6. What is the density property of real numbers? Are the rational numbers dense?

7. If a and b are two real numbers, prove that $|a \pm b| \leq |a|+|b|$ and $|a+b| \geq |a|-|b|$.

8. Give equivalent statements in terms of absolute value sign:

   (a) $-3 \leq x \leq 9$; (b) $a-0.01 < x < a+0.01$ $(x \neq a)$; (c) $-13 < x < 3$ or, $|x+5| < 8$.

9. Give equivalent statements in terms of inequalities:

   (a) $|x+3| \leq 2$; (b) $0 < |x-a| < \delta$; (c) $0 < |x-3| \leq 5$.

10. If $|a-x| \leq l$ and $|x-b| \leq m$, prove that $|a-b| \leq l+m$.

    [Write: $|a-b|=|a-x+x-b| \leq |a-x|+|x-b|$]

11. If $y=1+|x|$, what are the values of y when $x=0,-1,+1,-2,+2$?

12. Justify:

    (a) $x^2 > 1$ and $x > 1$ or $x < -1$ are equivalent.

    (b) $x^2 < 1$ and $-1 < x < +1$ are equivalent.

    (c) $|x| \geq 5$ and $x \geq 5$ or $x \leq -5$ are equivalent.

    (d) $-3 \leq x-8 \leq 3$ and $|x-8| \leq 3$ are equivalent.

13. State important properties of real numbers. What is Cantor-Dedekind axiom?

ANSWERS

1. Rational: 0/2, 3/2, −4/5, 7/3, −3/2;
   Irrational: $\sqrt{2}$, $\sqrt[3]{5}$;
   Not defined: $\frac{2}{0}$, $\frac{0}{0}$, $\frac{1}{0}$; ∞ is a concept (not a number).

8. (a) $(9-3)\div 2=3$, $-3-3 \leq x-3 \leq 9-3$, i.e., $|x-3| \leq 6$;

   (b) $0 < |x-a| < 0.01$;

   (c) $|x+5| < 8$.

9. (a) $-5 \leq x \leq -1$;

   (b) $a-\delta < x < a+\delta (x \neq a)$;

   (c) $-2 \leq x \leq 8 (x \neq 3)$.

11. 1, 2, 2, 3, 3.

Chapter 1

Function: Concept and Graphical Representation

1.1 Concept of Function

We shall, in our discussions, consider *real-valued functions of real variables* only.

I. Function. *Let x and y be two real variables.* Suppose, their respective domains are X and Y. Both X and Y are subsets of real numbers. Assume that there exists a rule f such that for every real number $x \in X$, we can find, by that rule, *a unique real number* $y \in Y$. The value of y, so obtained, is denoted by $f(x)$.

We then say: *y is a function of x over the domain X or f maps X into Y and write* $f: X \to Y$.

More accurately: y is a real-valued function of a real variable x; the domain of definition of the function is X.

In practice, we write: $y = f(x), x \in X$.

y may be called the *image* of x under the rule f. Often, y is called the *dependent variable* and x is called the *independent variable*:

y depends on x (y is a function of x).

What is important to note is that

- For every $x \in X$, there is one and only one $y \in Y$.
- It is possible that there may exist some $y \in Y$ which does not correspond to any $x \in X$, i.e., some values of Y may be left out. But every value of X must have an image in Y. So, we collect the images only and form a subset of Y. Such a subset is called the range of the function f. X is the domain of $f, D(f)$, and the subset of Y containing all the images is the Range of f, $R(f)$.

Illustrative Examples

Example 1. *The rule f may be given by a mathematical formula. Suppose, the rule is: In order to get y, multiply the value of x by 3 and then add 8.*

Solution: Mathematically, we write

$$y = 3x + 8$$
$$\text{or, } \boldsymbol{f(x) = 3x + 8}.$$

When $x = 0$, $y = 0 \cdot 1133 + 8 = 8$, i.e., $f(0) = 8$. It is called the functional value at $x = 0$.

Similarly,

$$f(1) = 1 \cdot 3 + 8 = \mathbf{11}$$
$$f(-2) = -2 \cdot 3 + 8 = \mathbf{2}, \text{ etc.}$$

Example 2. *The rule f may not be given by a mathematical formula. It may be given by a descriptive statement or some process by which the value of the dependent variable can be found out.*

Solution: Let x denote the dates of the month of January, 1991. So the domain of x is

$$X = \{1, 2, 3, 4, \ldots, 30, 31\}.$$

Let y be the number of babies born in Kolkata on a given date x.

Suppose, from the official records we find that on 1st January, 1991, the number of babies born $= 1231$, i.e., when $x = 1$, $y = 1231$.

In this way we may obtain datewise records, i.e., when x is known we can find y.

Here, also y depends on x or y is a function of x. There is no mathematical formula connecting x and y but here also there is a process (namely to find from official records of the corporation) by which when x is known, y can be calculated, i.e., $y = f(x)$ over the set $X = \{1, 2, 3, \ldots, 31\}$, i.e., here X is the *domain of definition of the function and the range of f is the set of those values of y which correspond to* $x = 1$, $x = 2$, $x = 3, \ldots, x = 30$, $x = 31$.

II. Constant function. If for every x of the domain X of a function f the image (or functional value) is always a particular real number c (say), then the function f is called a *constant function.* The range of a constant function contains only the element c.

We then write,

$$y = c, \text{ for every } x \in X$$
$$\text{or, } f(x) = c, \text{ for every } x \in X.$$

III. *A function f may have different rules for images corresponding to different parts of the domain.*

Example 3. $f(x) = \begin{cases} 1, & \textit{when } x \textit{ is a rational number} \\ 0, & \textit{when } x \textit{ is an irrational number.} \end{cases}$

Solution: The domain of f is the whole real number system $R(-\infty, \infty)$.

Observe that here

$$f(\sqrt{2}) = 0, \quad f(2) = 1, \quad f(\pi) = 0,$$
$$f(2/3) = 1, \quad f(-3/2) = 1, \quad f(-\sqrt{5}) = 0, \quad \text{etc.}$$

The range of f consists of two elements, namely $\{1, 0\}$.

Such a function has a special name — *Dirichlet's Function.*

Example 4. *A function f is defined as follows:*

$$f(x) = \begin{cases} 1, & \textit{when } x \textit{ is positive } (x > 0) \\ 0, & \textit{when } x = 0 \\ -1, & \textit{when } x \textit{ is negative } (x < 0). \end{cases}$$

Find $f(1), f(-2), f(0), f(\sqrt{2}), f(-\sqrt{5})$. What is the range and domain of f?

Solution: From the given function, we see that $f(1) = \mathbf{1}, f(-2) = \mathbf{-1}, f(0) = \mathbf{0}, f(\sqrt{2}) = \mathbf{1}, f(-\sqrt{5}) = \mathbf{-1}$.

2nd Part. Range consists of three elements $1, 0, -1$, i.e., **range $= \{1, 0, -1\}$.**

Clearly, *domain of f is the set R of all real numbers.*

Remarks. This function also has a special name — *Signum Function.*

Example 5. $f(x) = |x|$, $x \in R\ (-\infty, \infty)$. *What is the range of f? Find $f(1), f(-1)$ and $f(0)$?*

Solution: Here, we can write

$$f(x)=\begin{cases} x, & \text{when } x>0 \\ 0, & \text{when } x=0 \\ -x, & \text{when } x<0. \end{cases}$$

The range consists of all real numbers, positive, negative and zero.

See that here

$$\left.\begin{aligned} f(1)&=|1|=1 \\ f(-1)&=|-1|=1 \\ f(0)&=|0|=0 \end{aligned}\right\}.$$

Remarks. In view of this, we can write, Signum function of Ex. 4 as

$$f(x)=\begin{cases} |x|/x, & \text{when } x\neq 0 \\ 0, & \text{when } x=0. \end{cases}$$

IV. Greatest Integer Function.

Example 6. (a) $f(x)=[x]$, *where* $[x]$ *denotes the greatest integer which does not exceed* x. *Find the range of* f. (b) *If* $f(x)=|x|-3x$, *find* $f(-2)$. [C.U. B.Com. 2007]

Solution: (a) Here, $f(2)=[2]=2$; $f(2.01)=[2.01]=2$; $f(1.99)=[1.99]=1$; $f(-1)=-1$; $f(-3)=-3$; $f(-1.29)=[-1.29]=[-2+0.71]=-2$ (See that here it is not equal to -1, because then -1 exceeds -1.29.)

$$f(-3.04)=f(-4+0.96)=[-4+0.96]=\mathbf{-4}.$$

(Here also it is not equal to -3, because then it will exceed the value -3.04.)

We may write $f(x)$ as follows:

$$f(x)=\begin{cases} 0, & \text{if } 0\leq x<1 \\ 1, & \text{if } 1\leq x<2 \\ 2, & \text{if } 2\leq x<3, \text{ etc., etc.} \end{cases}$$

Also

$$f(x)=\begin{cases} -1, & \text{if } -1\leq x<0 \\ -2, & \text{if } -2\leq x<-1 \\ -3, & \text{if } -3\leq x<-2 \text{ and so on.} \end{cases}$$

The domain of this function is whole real number system $R(-\infty,\infty)$. *The range consists of integers — positive, negative and zero.*

Range of $f=\{0,\pm1,\pm2,\pm3,\ldots\}$.

This function has the special name — *Step Function.*

A practical example of this kind of step function.

(b) $f(x)=|x|-3x$ gives $f(-2)=|-2|-3\times -2=2+6=\mathbf{8}$.

Example 7. *Let* $x=$ *weight of a letter enclosed in an envelope (in grams)*
$y=$ *amount of postage (in paise) to be pasted on the envelope.*

Express y *as a function of* x.

Solution: For every 20 gm weight or part thereof we are to affix a stamp of ₹5 and then for every additional 20 gm or part thereof we are to put Re. 1 stamp more. Thus,

$$y = \begin{cases} ₹5.00, & \text{if } 0 < x \le 20 \\ ₹6.00, & \text{if } 20 < x \le 40 \\ ₹7.00, & \text{if } 40 < x \le 60 \\ ₹8.00, & \text{if } 60 < x \le 80 \text{ and so on.} \end{cases}$$

V. Domain and Range of a Function

Example 8. *The total cost C of a firm per day is a function of its daily output Q given by* $C = 300 + 7Q$.

The firm has a capacity limit of 100 *units of output per day. What are the domain and range of this cost function?*

Solution: Output Q can vary between 0 and 100.

$\therefore$ the *domain of the cost function* = $\{Q/0 \le Q \le 100\}$.

Here Q varies continuously from 0 to 100.

$$\therefore \quad \text{Minimum value of } C = 300, \text{ when } Q = 0$$
$$\text{Maximum value of } C = 1000, \text{ when } Q = 100.$$

$\therefore$ *Range of the cost function* is $\{C/300 \le C \le 1000\}$.

Remember. *In determining the functional values from a given rule, we shall take into consideration the following facts:*

- *Division by zero is not defined, we call it Undefined.*
- *Quantities like* $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, *etc., are not real and hence not defined for our purpose as we are concerned with real numbers only.*
- *Logarithms are defined only for positive real values, i.e.,* $\log x$ *is defined for* $x > 0$.

VI. Classification of Functions

1. **Algebraic Functions.** An expression containing fixed number of terms involving a variable, say x, formed by the operations of addition, subtraction, multiplication, division, involution (powers) and evolution (roots) is called an *algebraic function* of x.

Illustration 1.

$$3x^2 + 2, \ \frac{x^2 + 2x}{x^2 - 3x + 4}, \ \sqrt{x^2 - 5}, \ \frac{x}{\sqrt{x^2 + 6x + 5}}, \ \textit{etc.}$$

are algebraic functions.

An algebraic function may be either a *polynomial* or *rational* or an *irrational function.*

(a) **Polynomial Function.** An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and n is a positive integer, is called a *polynomial function* or a *rational integral function of x of degree n* and it is denoted by $P(x)$. Clearly to each real number a, we get the number $P(a)$.

Illustration 2. $x+2$, x^2+3x+5, x^3+2, *etc. are polynomial functions of* x *of degree* 1, 2, 3, *etc. respectively.*

(b) **Rational Function.** A rational function is the ratio of two polynomial functions of the same variable. If $P(x)$ and $Q(x)$ be two polynomial functions of x, then $R(x)=\frac{P(x)}{Q(x)}$ is a rational function of x. Clearly, this function associates to each real number a, the value $R(a)$, except those a for which $Q(a)=0$.

Illustration 3.

$$\frac{2x+3}{x^2+3x-4}, \frac{x^2+5x+6}{x^2-9x+20}, \textit{ etc.}$$

are rational functions of x.

(c) **Irrational Function.** Algebraic functions which are not rational, i.e., functions like

$$\sqrt{2x+3},\ \sqrt{x^2-5x+6},\ \frac{x}{\sqrt{x^2+3x}}, \text{ etc.}$$

are called *irrational functions.*

2. **Non-algebraic Functions.** Functions which are not algebraic are called *Non-algebraic functions.* Logarithmic and exponential functions are non-algebraic functions.

(a) **Exponential Function.** The function e^x, where $2<e<3$ ($e=2.71828$), and a^x, where $a>0, a\neq 1$, are called exponential functions. Thus, $2^x, 3^{2x}, x^x$, etc. are exponential functions.

(b) **Logarithmic Function.** $\log_e x$, $\log_a x$ (where $a>0$, $a\neq 1$) are called the logarithmic functions of x. Thus, $\log_2 x (x>0)$, $\log_e(1+x)(x>-1)$, $\log_e \sin x$, $(0<x<\pi)$ etc., are logarithmic functions.

Example 9. $f(x)=\dfrac{x^2-9}{x-3}$. *What is the value of* $f(3)$?
What is the domain of definition of this function?

Solution: $f(3)=\dfrac{3^2-9}{3-3}=$ not defined.

$\therefore$ $x=3$ does not belong to the domain of definition of f.

The *domain of* f *is the whole real number system excepting the value 3, i.e., the domain is* $R-\{3\}$ *or,* $(-\infty,\infty)-\{3\}$ *or,* $(-\infty,3)\cup(3,\infty)$.

We can, therefore, write:

$$f(x)=\frac{x^2-9}{x-3}=\frac{(x+3)(x-3)}{x-3}=x+3, \text{ when } x\neq 3$$

and is undefined when $x=3$.

Example 10. *Find the domain of definition of the function* $f(x)=\sqrt{x^2+x-12}$. [C.U. B.Com. 2009]

Solution: Here $f(x)=\sqrt{x^2+x-12}=\sqrt{(x+4)(x-3)}$.

[$\because$ $x^2+x-12=x^2+4x-3x-12=x(x+4)-3(x+4)=(x+4)(x-3)$]

We see that $f(x)$ will be real when $(x+4)(x-3)\geq 0$, i.e., when $x\geq 3$ or $x\leq -4$, and non-real when $-4<x<3$. Because, if $x=3$, then $x-3=0$; $\therefore$ $f(x)=0$ which is *real;* if $x>3$, then $(x+4)$ and $(x-3)$ are both positive so that their product $(x+4)(x-3)$ is positive and $f(x)$ is *real*, i.e., $f(x)$ is defined. Again, if $x=-4, f(x)=0$ which is *real;* if $x<-4$, then both $(x+4)$ and $(x-3)$ are negative so that $(x+4)(x-3)$ is *positive* and as such $f(x)$ is again *real*, i.e., $f(x)$ is *defined*. If $-4<x<3$, then $(x+4)$ is positive and $(x-3)$ is negative so that $(x+4)(x-3)$ is *negative* and as such $f(x)$ *is non-real*, i.e., $f(x)$ is *undefined*, if $-4<x<3$.

From above we see that *the domain of* $f(x)$ *is* $\{x : x \geq 3$ or, $x \leq -4\}$, i.e., $\boldsymbol{R-\{-4<x<3\}}$.

Example 11. *What is the domain of definition of* f, *where* $f(x)=\dfrac{x+2}{\sqrt{x^2-x-2}}$? [C.U. B.Com.(H) 1994]

Solution: We see that $x^2-x-2=0$, if $(x-2)(x+1)=0$, i.e., if $x=-1$ or $x=2$.

Moreover, $x^2-x-2<0$, if $(x-2)(x+1)<0$, i.e., if x lies between -1 and 2, because when $x>2$, both $(x-2)$ and $(x+1)$ are positive and as such $(x-2)(x+1)$ is positive and also when $x<-1$, both factors are negative and as such $(x-2)(x+1)$ is positive.

Now $f(x)$ is undefined, if $x^2-x-2=0$ or if x^2-x-2 is negative.

Therefore, when $-1 \leq x \leq 2, f(x)$ is undefined.

Thus, *the domain of* f *is* $\boldsymbol{R-\{-1 \leq x \leq 2\}}$, i.e., *all real numbers except those that are included in* $-1 \leq x \leq 2$.

Example 12. *Find the domain of definition of the function* $\log(x^2-5x+6)$. [C.U. B.Com.(H) 2008]

Solution: $x^2-5x+6=(x-3)(x-2)$ is negative or zero, when x lies between 2 and 3 or when $x=2$ or 3. For such values of x, $f(x)=\log(x^2-5x+6)$ is not defined.

$\therefore$ *the domain of definition of* $f(x)$ *is* $-\infty<x<2$ *and* $3<x<\infty$ *or,* $\boldsymbol{R-\{x \mid 2 \leq x \leq 3\}}$.

Example 13. *Find the domain of definition of the function* $f(x)=\dfrac{4x-5}{\sqrt{x^2-7x+12}}$. [C.U. B.Com.(H) 1998]

Solution: $x^2-7x+12=(x-3)(x-4)$ is negative or zero, when x lies between 3 and 4 or when $x=3$ or 4. For such values of x, $f(x)$ is not defined.

$\therefore$ the domain of definition of $f(x)$ is the system of all real numbers except the closed interval [3, 4], i.e., domain is $\boldsymbol{R-\{3 \leq x \leq 4\}}$.

Example 14. *Find the domain of definition of the function* $f(x)=\dfrac{x+2}{\sqrt{x^2-x-2}}$.

Solution: $x^2-x-2=(x-2)(x+1)$.

$\therefore$ x^2-x-2 is negative when x lies between -1 and 2, and the expression is zero when $x=2$ or -1. For such values of x, $f(x)$ is *not defined*.

$\therefore$ *the domain of definition of this function is* $\boldsymbol{R-\{-1 \leq x \leq 2\}}$.

Example 15. *Given that a function* f *is having the values shown in the following table:*

x	1	3	4
$f(x)$	0	2	3

What is the domain of f *and what is its range? Find* $f(3)$, $f(1)$, $f(4)$, $f(9)$, $f(0)$.

Solution: Here x is a discrete variable having the values $\{1,3,4\}$. This set is the domain of definition of f because for each member of the set there exists a unique value of $f(x)$.

Range of f = corresponding values of $f(x)=\{0,2,3\}$.

Here $f(1)=\mathbf{0}$, $f(3)=\mathbf{2}$, $f(4)=\mathbf{3}$; *but* $f(9)$, $f(0)$ *are not defined;* 0, 9 *do not belong to the domain of definition of* f.

Example 16. *The total cost C of a Factory per week is a function of its weekly output Q given by the equation* $C=500+12Q$. *The factory has a capacity limit of* 600 *units of output per week. Find the domain of definition and range of the cost function.* [C.U. B.Com. 2009]

Solution: The relation between the total cost C per week and weekly output Q is given by

$$C=500+12Q. \tag{1}$$

Since the capacity limit of the factory is 600 units, Q varies from 0 to 600, i.e. $0\le Q\le 600$ and for each of these values of Q, the value of C is definite finite, i.e., defined.

Hence, the required domain of definition of the cost function is $0\le Q\le 600$, where Q is a positive integer, if Q is not fractional.

Now, if $Q=0$, then $C=500$; if $Q=600$, then $C=500+12\times 600=7700$.

Hence *the range of the cost function is* $\mathbf{500\le C\le 7700}$.

Example 17. $f(x)=\dfrac{\sqrt{x}-2x}{3}$. *Obtain, wherever possible, the values of* $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(-1)$, $f(-2)$.

Solution: Clearly, $\sqrt{x}$ does not give a real value when x is negative.

Therefore, $f(-1)$, $f(-2)$ *are not defined.*

But
$$f(0)=\frac{\sqrt{0}-2\cdot 0}{3}=\mathbf{0};\ f(1)=\frac{\sqrt{1}-2\cdot 1}{3}=\frac{\mathbf{-1}}{\mathbf{3}};\ f(2)=\frac{\sqrt{\mathbf{2}}-\mathbf{4}}{\mathbf{3}};\ f(3)=\frac{\sqrt{\mathbf{3}}-\mathbf{6}}{\mathbf{3}}.$$

The domain of f *is* $0\le x<\infty$.

Example 18. (a) *If* $f(x)=|x|-2x$, *find* $f(-1)$, $f(1)$. [C.U. B.Com.(H) 2007 Type]

(b) *If* $f(x)=|x|+x$, *find* $f(3)$ *and* $f(-3)$. [B.U. B.Com.(H) 2005]

Solution: We have
$$|x|=\begin{cases} x, & \text{if } x\ge 0\\ -x, & \text{if } x<0.\end{cases}$$

If $x>0$, then $f(x)=|x|-2x=x-2x=-x$. $\therefore$ $f(1)=\mathbf{-1}$.

But, if $x<0$, then $f(x)=|x|-2x$; so

$$f(-1)=|-1|-2(-1)=1+2=\mathbf{3}$$
$$\text{and}\quad f(1)=|1|-2(1)=1-2=\mathbf{-1}.$$

(b) $f(3)=|3|+3=3+3=\mathbf{6}$ and $f(-3)=|-3|+(-3)=3-3=\mathbf{0}$.

Example 19. *If* $f(x)=|x|-[x]$, *where* $[x]$ *is the greatest integer not exceeding* x, *find the value of* $f(3.5)$ *and* $f(-3.5)$. [C.U. B.Com.(H) 2008]

Solution: If $x > 0$, $|x| = x$ and if $x < 0$, $|x| = -x$; $[3.5] = 3$ and $[-3.5] = -4$.

$$\therefore\ f(3.5) = |3.5| - [3.5] = 3.5 - 3 = \mathbf{0.5}$$
$$\text{and } f(-3.5) = |-3.5| - [-3.5] = 3.5 + 4 = \mathbf{7.5}.$$

Example 20. (a) *A function f defined by* (i) $f(x) = \frac{1}{x}$; (ii) $f(x) = 4x^2 + 2x - 3$; *find, in each case,* $\frac{f(2+h) - f(2)}{h}$, *where* $h \neq 0$.

(b) *Find* $\frac{f(x+h) - f(x)}{h}$, *when* $f(x) = \frac{1-x}{1+x}$. [C.U. B.Com. 1998]

Solution: (a) In (i) $f(2) = \frac{1}{2}$ and $f(2+h) = \frac{1}{2+h}$.

$$\therefore\ \frac{f(2+h) - f(2)}{h} = \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \frac{-h}{2(2+h)\cdot h} = \frac{\mathbf{-1}}{\mathbf{4+2h}}, \text{ where } h \neq 0.$$

Note: This function is defined over $R - \{0\}$.

In (ii), $f(x) = 4x^2 + 2x - 3$;

$$\begin{aligned}\therefore\ \frac{f(2+h) - f(2)}{h} &= \frac{\{4(2+h)^2 + 2(2+h) - 3\} - \{4\cdot 2^2 + 2\cdot 2 - 3\}}{h} \\ &= \frac{4\{(2+h)^2 - 2^2\} + 2\{(2+h) - 2\}}{h} \\ &= \frac{4(4h + h^2) + 2h}{h} = \frac{4h^2 + 18h}{h} \\ &= \mathbf{4h + 18}\ (\because\ \mathbf{h \neq 0}).\end{aligned}$$

Note: This function is defined over the entire real number system.

(b) $f(x) = \frac{1-x}{1+x}$ and $f(x+h) = \frac{1-(x+h)}{1+(x+h)} = \frac{1-x-h}{1+x+h}$.

$$\begin{aligned}\therefore\ \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left[\frac{1-x-h}{1+x+h} - \frac{1-x}{1+x}\right] \\ &= \frac{1 - x - h + x - x^2 - hx - (1 + x + h - x - x^2 - hx)}{h(1+x+h)(1+x)} \\ &= \frac{1 - h - x^2 - hx - 1 - h + x^2 + hx}{h(1+x+h)(1+x)} \\ &= \frac{-2h}{h(1+x+h)(1+x)} \\ &= -\frac{\mathbf{2}}{\mathbf{(1+x+h)(1+x)}}.\end{aligned}$$

Example 21. *A function g is given by* (i) $g(x) = \frac{1}{x-3}$; (ii) $g(x) = \sqrt{x^2 - 4}$. *Find the domain of definition of the function in each case.*

Solution: (i) $g(x)$ is not defined at $x=3$ ($\because$ division by zero is an undefined operation).

$\therefore$ the domain of g is $\boldsymbol{R}-\{3\}$.

(ii) $x^2-4<0$, if $(x+2)(x-2)<0$, i.e., if x lies between -2 and 2.

$\therefore$ $g(x)=\sqrt{x^2-4}$ will not give real values if $-2<x<2$ and for other values $g(x)$ is defined.

$\therefore$ the domain of g is $\boldsymbol{R}-\{-2<x<2\}$, i.e., all real numbers ≥ 2 or ≤ -2.

Example 22. *Which of the following define (s) y as a function of x? Give reasons where necessary:*

(i) $\begin{array}{c|c|c} x & 0 & 0 \\ \hline y & 0 & 1 \end{array}$; (ii) $\begin{array}{c|c|c} x & 0 & 1 \\ \hline y & 0 & 0 \end{array}$; (iii) $\begin{array}{c|c|c} x & 1 & 1 \\ \hline y & 1 & 2 \end{array}$.

Solution: (i) Does not define a function.

Our definition of function states that for a given x, there must exist one and only one value of y. Here for $x=0$, we find two distinct values of y (namely, 0 and 1).

For the same reason (iii) does not define a function. But (ii) defines y as a function of x; for $x=0$, $y=0$ and for $x=1$, y is again zero. For a function same image may correspond to two distinct values of the independent variable.

1.2 Different Types of Functions

VII. Bounded Functions. Let $f(x)$ be a function defined in the domain A. If two finite numbers k and K can be found such that $k\leq f(x)\leq K$ for every $x\in A$, then $f(x)$ is said to be bounded both below and above in the domain A and we say that $f(x)$ is bounded in A.

For example, the function $f(x)=x^2+2x+3$ in the domain $1\leq x\leq 3$ is bounded below by 6 and above by 18. Here 6 is the lower bound and 18 is the upper bound of the function in $1\leq x\leq 3$.

VIII. Increasing and Decreasing Functions. Let x_1, x_2 be any two values of x in the domain of definition $[a,b]$ of $f(x)$ and let $x_2>x_1$. Then $f(x)$ is said to be

- monotone increasing in $[a,b]$, if $f(x_2)\geq f(x_1)$
- monotone decreasing in $[a,b]$, if $f(x_2)\leq f(x_1)$
- strictly increasing (or steadily increasing) in $[a,b]$, if $f(x_2)>f(x_1)$ and
- strictly decreasing in $[a,b]$, if $f(x_2)<f(x_1)$.

Illustration 1. (i) *$f(x)=x+2$ is strictly increasing in any interval.*

(ii) *$f(x)=1/x$ is strictly decreasing in $1\leq x\leq 3$.*

IX. Even and Odd Functions.

- A function f is called an even function, if for any point x of the domain of f, we find $f(x)=f(-x)$. e.g., $f(x)=x^2$ is an even function, because $f(-x)=(-x)^2=x^2=f(x)$.
- A function f is called an odd function, if for any point x of the domain of f, we find $f(-x)=-f(x)$. e.g., $f(x)=x^3$ is an odd function, because $f(-x)=(-x)^3=-x^3=-f(x)$.
- Now prove that any function f of x, defined for all real x, is the sum of an even and an odd functions of x.

  Proof. Suppose, $\phi(x)=f(x)+f(-x)$.

  Then $\phi(-x)=f(-x)+f(x)$, i.e., $\phi(x)$ is an even function.

  Next, let $g(x)=f(x)-f(-x)$.

$\therefore\ g(-x)=f(-x)-f(x)=-[f(x)-f(-x)]=-g(x)$, i.e., $g(x)$ is an odd function.

See that $f(x)=\frac{1}{2}[\phi(x)+g(x)]=\frac{1}{2}\,[\{f(x)+f(-x)\}+\{f(x)-f(-x)\}]$

= sum of an even function and an odd function.

Remember. *A constant function is an even function.*

$\because\ f(x)=c \Rightarrow f(-x)=c$, i.e., $f(x)=f(-x)$.

X. Parametric Functions. If both the dependent and independent variables are expressed as functions of a third variable, then we say that the function has been represented *parametrically;* this third variable is called a *parameter,* e.g., $x=t^2$, $y=4t$ is a parametric function with parameter t.

XI. Explicit and Implicit Functions. We may express the dependent variable (y) in terms of the independent variable (x), e.g., $y=x+3$.

We then say that y has been explicitly expressed in terms of x, or y is an **explicit function** of x.

A relation between two variables (say, x and y) which is *not solved* for either of them is an **implicit relation.** If from an implicit relation it is possible to solve for one variable, say y, in terms of the other variable x, we say that y is *an explicit function of* x, e.g., $y-x-3=0$ is an implicit relation. From this we can write, $y=x+3$ or we can write, $x=y-3$. Thus, y is an explicit function of x or x is an explicit function of y. *Note that an implicit relation may not always give an explicit function or an implicit relation may give more than one explicit function,* e.g., $x^2+y^2+1=0$ does not give y as a real-valued function of x. $x^2+y^2=4 \Rightarrow y=\sqrt{4-x^2}$ and $y=-\sqrt{4-x^2}$ (two functions). $2x^2+3xy+5y^2=4$ and $e^{xy}+3\log x=4$ define y as an implicit function of x. Thus, *y is an explicit function of x, if we can write $y=f(x)$ and y is an implicit function of x, if it is given in the form $f(x,y)=0$.*

XII. Composite Function — Function of a Function. If y is a function of z, and z is a function of x, then y is called a function of a function of x (or composite function of x). Thus, if $y=f(z)$ and $z=g(x)$, then $y=f\{g(x)\}$.

Illustration 2. (i) *If $y=z^4$ and $z=x^2+3x-5$, then y is a function of z, and z is a function of x. Clearly, $y=(x^2+3x-5)^4$ is a function of a function of x.*

(ii) *If $y=\log z$ and $z=\sin x$, then $y=\log(\sin x)$ is a function of a function of x. In this case, values of x are such that* $\sin x>0$.

(iii) *If $y=e^z$ and $z=x^2-7x+11$, then $y=e^{x^2-7x+11}$ is a function of a function of x.*

XIII. Evaluation of Functional Values

Example 23. *Let $\phi(x)=a\dfrac{x-b}{a-b}+b\dfrac{x-a}{b-a}$ $(a\neq b)$. Verify that, $\phi(a+b)=\phi(a)+\phi(b)$.*

Solution:

$$\phi(a)=a\frac{a-b}{a-b}+b\frac{a-a}{b-a}=a+0\ \left(\because\ a\neq b, \frac{a-b}{a-b}=1\right), \text{ i.e., } \phi(a)=a$$

$$\phi(b)=a\frac{b-b}{a-b}+b\frac{b-a}{b-a}=0+b\ \left(\because\ a\neq b, \frac{b-a}{b-a}=1\right), \text{ i.e., } \phi(b)=b.$$

$$\phi(a+b)=a\frac{a+b-b}{a-b}+b\frac{a+b-a}{b-a}=\frac{a^2}{a-b}+\frac{b^2}{b-a}=\frac{a^2-b^2}{a-b}=\frac{(a+b)(a-b)}{a-b}$$

$$=a+b\left(\because\ a\neq b, \frac{a-b}{a-b}=1\right)=\phi(a)+\phi(b),$$

i.e., $\boldsymbol{\phi(a+b)=\phi(a)+\phi(b)}$.

Example 24. *If* $f(x)=\dfrac{ax+b}{bx+a}$, *prove that* $f(x)\cdot f(1/x)=1$. [C.U. B.Com.(H) 2010]

Solution:

$$f(x)=\frac{ax+b}{bx+a};\ \text{hence } f\left(\frac{1}{x}\right)=\frac{a\cdot\frac{1}{x}+b}{b\cdot\frac{1}{x}+a}=\frac{a+bx}{b+ax}$$

$$\therefore\ f(x)\cdot f\left(\frac{1}{x}\right)=\frac{ax+b}{bx+a}\cdot\frac{a+bx}{b+ax}=1\ \textbf{(Proved).}$$

Example 25. *If* $f(x)=\dfrac{x-1}{x+1}$, *then show that* $\dfrac{f(a)-f(b)}{1+f(a)f(b)}=\dfrac{a-b}{1+ab}$.

[C.U. B.Com.(H) 1994; V.U. B.Com.(H) 2010]

Solution:

$$f(a)=\frac{a-1}{a+1},\quad f(b)=\frac{b-1}{b+1}.$$

$$f(a)-f(b)=\frac{(a-1)(b+1)-(b-1)(a+1)}{(a+1)(b+1)}=\frac{(ab-b+a-1)-(ab-a+b-1)}{(a+1)(b+1)}$$
$$=\frac{2(a-b)}{(a+1)(b+1)}.$$

$$f(a)\cdot f(b)=\frac{(a-1)(b-1)}{(a+1)(b+1)}=\frac{ab-a-b+1}{(a+1)(b+1)}.$$

$$1+f(a)\cdot f(b)=1+\frac{ab-a-b+1}{(a+1)(b+1)}=\frac{ab+a+b+1+ab-a-b+1}{(a+1)(b+1)}=\frac{2(ab+1)}{(a+1)(b+1)}$$

$$\therefore\ \frac{f(a)-f(b)}{1+f(a)\cdot f(b)}=\frac{2(a-b)}{(a+1)(b+1)}\cdot\frac{(a+1)(b+1)}{2(ab+1)}=\frac{a-b}{1+ab}\ \textbf{(Proved).}$$

Example 26. *If* $y=f(x)=\dfrac{x+1}{x+2}$, *find* $f(y)$ *and* $f\left\{f\left(\dfrac{1}{x}\right)\right\}$, $x\neq 0$.

[C.U. B.Com.(H) 1992; V.U. B.Com.(H) 2007]

Solution:

$$f(y)=\frac{y+1}{y+2},\ f\left(\frac{1}{x}\right)=\frac{\frac{1}{x}+1}{\frac{1}{x}+2}=\frac{\mathbf{x+1}}{\mathbf{2x+1}}.$$

$$f\{f(1/x)\}=f\left\{\frac{x+1}{2x+1}\right\}=\frac{\frac{x+1}{2x+1}+1}{\frac{x+1}{2x+1}+2}=\frac{(x+1)+(2x+1)}{(x+1)+2(2x+1)}=\frac{\mathbf{3x+2}}{\mathbf{5x+3}}.$$

Example 27. *If* $f(x)=\dfrac{x^2-5x+6}{x^2-8x+12}$, *find, if possible,* $f(2)$ *and* $f(-5)$.

Solution:

$$f(2)=\frac{2^2-5\cdot2+6}{2^2-8\cdot2+12}=\textbf{not defined}$$ ($\because$ denominator is zero and division by zero is an undefined operation)

$$f(-5)=\frac{(-5)^2-5\cdot(-5)+6}{(-5)^2-8(-5)+12}=\frac{25+25+6}{25+40+12}=\frac{\mathbf{56}}{\mathbf{77}}.$$

Example 28. (a) *If $f(x)=x^2-x$, then prove that $f(h+1)=f(-h)$.* [C.U. B.Com.(H) 2007; (P) 2010]
(b) *If $f(x+1)=7x-5$, find $f(x)$.* [V.U. B.Com.(H) 2009]

Solution: (a) $f(x)=x^2-x$.

$$\therefore\ f(-h)=(-h)^2-(-h)=h^2+h$$

and $$f(h+1)=(h+1)^2-(h+1)=\left(h^2+2h+1\right)-(h+1)=h^2+h.$$

$$\therefore\ f(h+1)=f(-h)\ \textbf{(Proved).}$$

(b) We have
$$f(x+1)=7x-5. \qquad (1)$$

Replacing x by $x-1$, we get $f(x-1+1)=7(x-1)-5$ [$\because$ $(x-1)+1=x$]
or, $f(x)=7x-7-5=\mathbf{7x-12}$.

Example 29. (a) *If $f(x+3)=3x^2-2x+5$, find $f(x-1)$.* [C.U. B.Com.(H) 1996]
(b) *If $f(x)=|x-3|+3$, find $f(3)$ and $f(-3)$.* [B.U. B.Com.(H) 2008]

Solution: (a) We have
$$f(x+3)=3x^2-2x+5. \qquad (1)$$

Replacing x by $x-4$, we get

$$f(x-4+3)=3(x-4)^2-2(x-4)+5=3\left(x^2-8x+16\right)-2x+8+5$$

or, $$f(x-1)=3x^2-24x+48-2x+13=\mathbf{3x^2-26x+61}.$$

(b) $f(x)=|x-3|+3$; $\therefore f(3)=|3-3|+3=0+3=3$ and $f(-3)=|-3-3|+3=|-6|+3=6+3=\mathbf{9}$.

Example 30. *If $f(x)=\dfrac{2x+1}{2x^2+1}$ and $\phi(x)=2f(2x)$, then find $\phi(2.5)$.* [C.U. B.Com.(H) 1995]

Solution: We have
$$f(x)=\frac{2x+1}{2x^2+1} \text{ and } \phi(x)=2f(2x).$$

$$\therefore\ \phi(x)=2f(2x)=2\cdot\frac{2(2x)+1}{2(2x)^2+1}=2\left(\frac{4x+1}{8x^2+1}\right).$$

$$\therefore\ \phi(2.5)=2\left\{\frac{4\times 2.5+1}{8(2.5)^2+1}\right\}=2\left(\frac{10+1}{50+1}\right)=\mathbf{\frac{22}{51}}.$$

Example 31. *Find the range of the function $\dfrac{2x}{4+x^2}$, where x is real.* [C.U. B.Com. 2003]

Solution: Let $y=\dfrac{2x}{4+x^2}$. Then $y\left(4+x^2\right)=2x$ or, $yx^2-2x+4y=0$.

$$\therefore\ x=\frac{2\pm\sqrt{4-16y^2}}{2y}, \quad y\neq 0.$$

Since x is real; $\therefore$ $4-16y^2\geq 0$ or, $1-4y^2\geq 0$ or, $4y^2\leq 1$ or, $y^2\leq\frac{1}{4}$;
$\therefore -\frac{1}{2}\leq y\leq\frac{1}{2}$ and $y\neq 0$.

Hence *the required range of the given function is* $-\frac{1}{2} \leq y \leq \frac{1}{2}$, $y \neq 0$.

XIV. Break-even Analysis. We shall denote

$C(x)$ = Cost function (x is the number of units produced)

$R(x)$ = Revenue function (revenue obtained by selling x units of commodity)

$P(x)$ = Profit function.

We know that, Profit = Revenue − Cost, i.e., $P(x) = R(x) - C(x)$.

If $R(x) = C(x)$, then $P(x) = 0$, i.e., when the revenue equals cost, there is no profit or loss — this value is called the *break-even point.*

Example 32. *If the cost function for x units is given by* $C = ₹(400 + 16x - x^2)$, *obtain the* (i) *average cost,* (ii) *average variable cost.* [C.U. B.Com.(H) 2001]

Solution: Cost function $f(x)$ for x units is $C = ₹(400 + 16x - x^2)$.

$\therefore$ average cost $= \dfrac{C}{x} = ₹\,\dfrac{(400 + 16x - x^2)}{x} = ₹\left(\dfrac{400}{x} + 16 - x\right)$.

Variable cost of x units is $V = ₹(16x - x^2)$.

$\therefore$ average variable cost $= \dfrac{₹(16x - x^2)x}{=}₹(16 - x)$.

Example 33. *The daily cost of production of x units of a commodity is* ₹$C(x)$, *where* $C(x) = ₹2.05x + ₹550$.

(i) *If each unit is sold for* ₹3, *what is the minimum number that must be produced and sold daily for ensuring no loss?*

(ii) *If this selling price is increased by* 30 paise *per unit, what would be the break-even point?*

(iii) *If it is given that at least* 500 *units can be sold daily, what price should be charged per unit to guarantee no loss?*

Solution: (i) $R(x)$ = Revenue from sales of x units = ₹$3x$.

For the break-even point, we have

$$C(x) = R(x) \quad \text{or,} \quad 2.05x + 550 = 3x$$

$$\text{or,} \quad 550 = 3x - 2.05x = 0.95x$$

$$\text{or,} \quad x = \frac{550}{0.95} = \frac{550 \times 100}{95} = \frac{11000}{19} = 578.95.$$

Thus, *at least* **579** *units are to be produced daily to ensure no loss.*

(ii) In the case, $R(x)$ = ₹$3.30x$.

For the break-even point, we have $R(x) = C(x)$.

$$3.30x = 2.05x + 550 \quad \text{or,} \quad (3.30 - 2.05)\,x = 550$$

$$\text{or,} \quad x = \frac{550}{1.25} = \frac{550}{125} \times 100 = 440.$$

$\therefore$ when the selling price is increased by 30 paise per unit, the break-even point is **440**.

(iii) If 500 units are sold daily, the price p per unit needed to ensure no loss is given by

$500p = 2.05 \times 500 + 550 \Rightarrow p = ₹3.15$.

Example 34. *A company decides to set up a small production plant for manufacturing electronic clocks. The total cost for initial set-up (Fixed Cost) is ₹9 lac. The additional cost (Variable Cost) for producing each clock is ₹300. Each clock is sold at ₹750. During the first month 1500 clocks are produced and sold.*

(i) *Determine the cost function $C(x)$ for the total cost of producing x clocks.*

(ii) *Determine the revenue function $R(x)$ for the total revenue from the sale of x clocks.*

(iii) *Determine the profit function $P(x)$ for the profit from the sale of x clocks.*

(iv) *What profit or loss the company incurs during the first month when all the 1500 clocks are sold?*

(v) *Determine the break-even point.* [CA May 1992]

Solution: (i) $\boldsymbol{C(x)}$ = Fixed cost + Variable cost for x clocks
= **₹9,00,000 + ₹300*x*.**

(ii) $\boldsymbol{R(x)=750x}$ = Revenue from the sale of x clocks.

(iii) **Profit function** $\boldsymbol{P(x)}=R(x)-C(x)=750x-(9{,}00{,}000+300x)=\mathbf{450x-9{,}00{,}000}$.

(iv) $450 \times 1500 - 9{,}00{,}000 = 6{,}75{,}000 - 9{,}00{,}000 = -2{,}25{,}000$.

There is a loss of ₹2,25,000 on the first month.

(v) *Break-even point* is given by $P(x)=0$, i.e., when $450x=9{,}00{,}000$, i.e., when $\boldsymbol{x=2{,}000}$, i.e., **production of 2000 electronic clocks.**

1.3 Graphical Representation of Functions

Introduction. The concept of function has now been introduced. We wish to represent (wherever possible) functions by graphs. They will be of great help in the study of characteristics of different functions.

When we say that y is a function of x we have, in mind, two sets of values — one, the domain of x; another, the range of y. Any x-value of the domain corresponds to a unique y-value of the range.

The set of x-values of the domain can be represented by a line-segment and the set of y-values of the range can also be represented by another line-segment [See Fig. 1.1(a)].

Instead, we can take two rectangular axes OX and OY. The line-segment of the domain can be considered as a part of the X-axis and that of the range, a part of the Y-axis. We then plot the corresponding values of x and y by points on the plane determined by OX and OY.

Suppose, when $x=x_1$, then $y=y_1$. We plot the point (x_1, y_1) in the usual manner. In this way corresponding to all x-values of the domain we plot points in the plane of OX and OY. The totality of all such points forms the *graph of the function* (not always — we shall get an unbroken line when we plot the graph) [See Fig. 1.1(b)].

Example 35. *Suppose, the total cost C of a firm per day is a function of its output Q given by $C=150+7Q$. Let us assume that the firm has a capacity limit of 100 units per day.*

Solution: Here the domain of Q is $0 \le Q \le 100$, the range of C is $150 \le C \le 850$.

We take two lines Q-axis (OQ) and C-axis (OC).

Our domain is the line-segment from $Q=0$ to $Q=100$ and the range is the line-segment from $C=150$ to $C=850$.

We plot points like (Q_1, C_1), where if $Q=Q_1$, then $C=C_1=150+7Q_1$.

The totality of all such points will give the graphical representation of the cost functions [See Fig. 1.1(c)].

We note that in drawing a graph of a function it is not always necessary to know the exact mathematical relationship between the variables. One may have arbitrary set of discrete points.

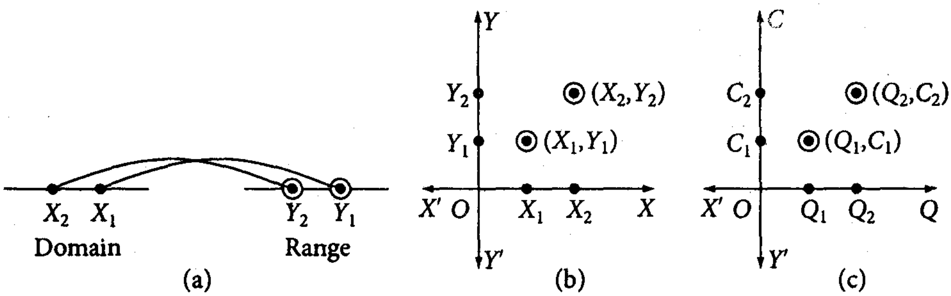


Fig. 1.1

What is essential is that corresponding to each value of the domain we must have one and only one value of the range. Then we can draw the set of points (called the graph).

1.4 Graphs of Polynomials: Everywhere Continuous Graphs

Linear Functions

Example 36. (i) $x = 3$; (ii) $y = 5$; (iii) $y = x$; (iv) $\frac{x}{3} + \frac{y}{4} = 1$; (v) $y = 2x + 3$.

Solution: [The domain of definition, in each case, is the whole system of real numbers.]

(i) Here for all values of y (i.e., $-\infty < y < \infty$), $x = 3$, e.g., $y = 0$, $x = 3$; $y = -1$, $x = 3$, etc., etc. The corresponding points on the plane determined by two rectangular axes OX and OY give an unbroken line parallel to Y-axis at a distance 3 units from it [Fig. 1.2(a)].

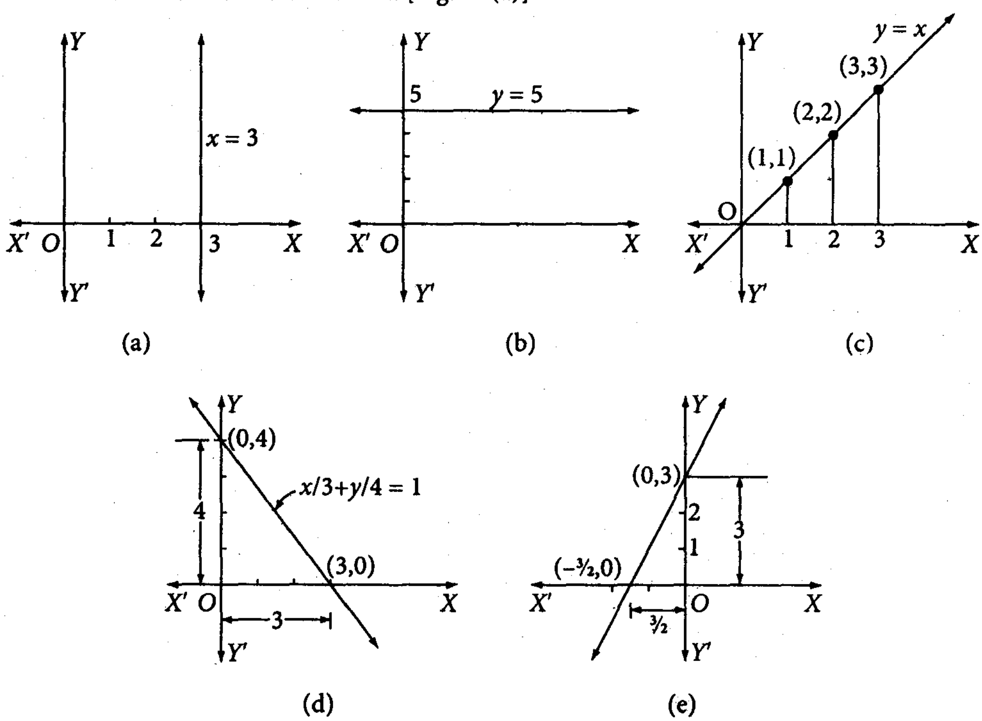


Fig. 1.2

(ii) $y = 5$ for all values of x (i.e., $-\infty < x < \infty$). The assemblage of points satisfying this relation gives the graph [some typical points are $(0,5)$, $(1,5)$, $(2,5)$, $(-1,5)$, $(-2,5)$, etc.]. The graph is a straight line parallel to the X-axis at a height of 5 units above it [See Fig. 1.2(b)].

(iii) Here also the domain of x is $-\infty < x < \infty$. See that when $x = 0, y = 0$, i.e., the origin $(0,0)$ is a point on this graph. Now $(1,1),(2,2),(3,3)$, etc., are all points of the graph. $y = x$, therefore, gives a graph of an unbroken line passing through the origin. This line bisects the angle between OX and OY [Fig. 1.2(c)].

(iv) The domain is, as before, $-\infty < x < \infty$. The given equation is the intercept form of a straight line whose x- and y-intercepts are respectively 3 and 4. One may first take $x = 0$; in that case, $y = 4$. Then take $y = 0$ so that $x = 3$, i.e., $(0,4)$ and $(3,0)$ are points of the graph. Other points of the graph are points on the line joining these two points [See Fig. 1.2(d)].

(v) Given: $y = 2x + 3$ (domain is $-\infty < x < \infty$). One may write, $y - 2x = 3$ or, $y/3 - 2x/3 = 1$ or, $\frac{y}{3} + \frac{x}{-3/2} = 1$.

Now take x-intercept $= -3/2$ and y-intercept $= 3$, i.e., draw the line joining $(-3/2, 0)$ and $(0,3)$. This line is the required graph [Fig. 1.2(e)].

Remember. *In general, the graph of any linear function like* $y = ax + b$ *is always a straight line* (x*-intercept* $= -b/a$, y*-intercept* $= b$, *if* $a, b \neq 0$).

Quadratic Functions

Example 37. (i) $y = x^2$; (ii) $y = x^2 - \frac{1}{2}$; (iii) $y = 2x^2 + 4x + 3$.

[*The domain of definition is, in each case,* $-\infty < x < \infty$.]

Discuss the shapes of graphs of $y = x^4$, $y = x^6$, $y = x^8$, *etc., along with* (i).

Solution: (i) In order to plot a fair number of (say, at least ten) points we make a table of values comprising x and the corresponding values of y.

Function $y = x^2$ in $-\infty < x < \infty$

x	0	1/4	1/2	3/4	1	−1/4	−1/2	−3/4	−1	2	−2	3	−3	etc.
y	0	1/16	1/4	9/16	1	1/16	1/4	9/16	1	4	4	9	9	etc.

On a suitable scale plot these points. The graph will be a parabola. We observe the following points:

- The curve passes through $(0,0),(1,1),(-1,1)$.

  Any function like $y = x^4$, $y = x^6$, $y = x^8$, etc., (with even powers of x) will have the same property.
- For all non-zero x-values (positive or negative), the values of y are always positive. This means that the curve lies only in the first and second quadrants. This statement is also true for functions like $y = x^4$, $y = x^6$, etc.
- In $0 < x < \infty$, as x increases, y also increases, [while in $0 < x < 1$, the increase is not so fast, in $1 < x < \infty$, y increases very fast, e.g., $x = 2$, $y = 4$ but $x = 3$, $y = 9$, etc.] i.e., in the first quadrant the curve is rising. In $-\infty < x < 0$, as x increases, y decreases, the curve is a falling curve in the second quadrant as we proceed from $-\infty$ to 0 values of x.

  This statement is also true for functions which are even power of x (x^4, x^6, x^8, etc.).

  See that the curve $y = x^4$ is similar in appearance to $y = x^2$ but the former is more flat near $(0,0)$ and steeper beyond $(1,1)$, and $(-1,1)$ and as the powers become higher (x^6, x^8, etc.), the flatness and steepness become more and more pronounced.

  Ultimately, when the power of x becomes infinitely large (but even), then the curve is practically indistinguishable from the U-shaped line [the line $x = -1$, part of X-axis from -1 to $+1$ and the line

$x = +1]$. In Fig. 1.3(a) the graph of $y = x^2$ has been shown with a solid line and the graph of $y = x^4$ has been shown with dotted line.

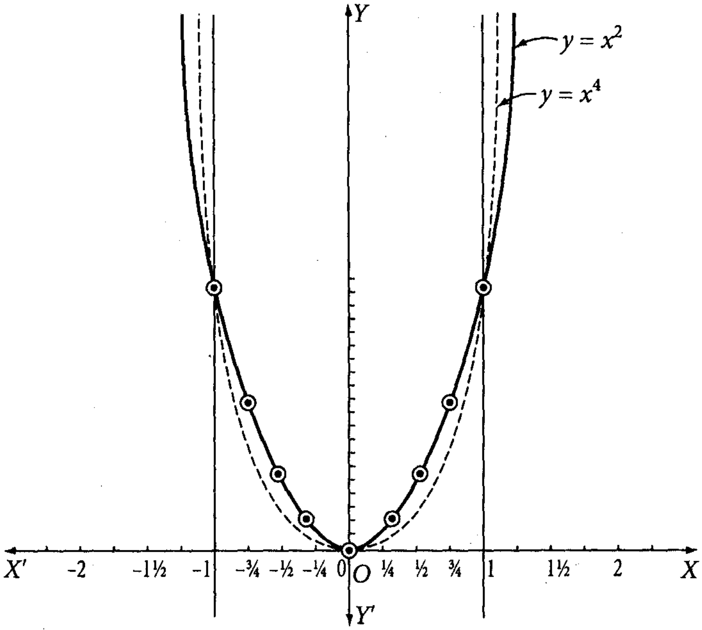


Fig. 1.3(a)

- In the curve $y = x^2$ (and also for $y = x^4$, x^6, etc.) the lowest point is $(x = 0, y = 0)$, called the vertex.
- The concavity of all these curves is upwards.

(ii) The graph of $y = x^2 - 1/2$. We first write, $y + 1/2 = x^2$ or, $Y = x^2$, where $Y = y + 1/2$. In this case, $Y = 0$ means $y = -1/2$, i.e., in this case the lowest point is $A(0, -1/2)$ and the curve has the shape of a parabola. Plot a fair number of points, the shape will be apparent [Fig. 1.3(b)].

(iii) We write, $y = 2x^2 + 4x + 3$ as $y = 2(x^2 + 2x + 1) + 1 = 2(x+1)^2 + 1$

or, $y - 1 = 2(x+1)^2$, i.e., $Y = 2X^2$, where $Y = y - 1$ and $X = x + 1$.

The lowest point here is $X = 0$, $Y = 0$, i.e., $x = -1$, $y = 1$. This curve also is a parabola opening upwards, vertex being at $(-1, 1)$, Fig. 1.3(c). If the coefficient of X^2 were negative, then the curve would be concave downwards.

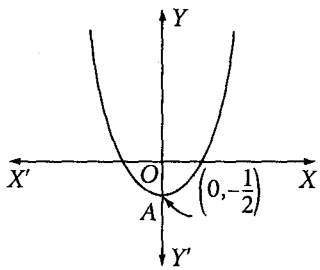


Fig. 1.3(b)

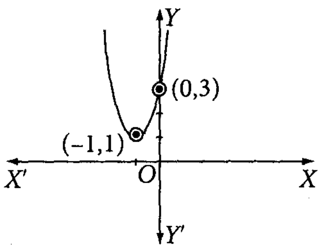


Fig. 1.3(c)

Remember. *Every function of the type $y = ax^2 + bx + c(a \neq 0)$ gives the curve of a parabola. The concavity will be upwards if $a > 0$, downwards if $a < 0$. To find the lowest point, i.e., vertex, write:*

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$\text{or,}\quad y - \left(c - \frac{b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

$$\text{or,}\quad Y = aX^2,\ \text{where } Y = y - c + \frac{b^2}{4a}, X = x + \frac{b}{2a}.$$

The lowest point is $X = 0$, $Y = 0$, i.e., $x = -b/2a, y = c - b^2/4a$.

Cubic Functions

Example 38. $y = x^3$. *Discuss the appearance of curve of odd powers of* x.

Solution: We take the function $y = x^3$.

- The graph of this function contains the points $(0,0)$, $(1,1)$, $(-1,-1)$. All odd-power functions have the same characteristic.
- When x is positive, y is positive and also when x is negative, y is negative.
  $\therefore$ the curve lies in the first and the third quadrants.
  This is true for all curves like $y = x^3, y = x^5$, etc.
- In $-\infty < x \leq 0$ as x increases from very large negative values to zero, y also increases from very large negative values to zero. Similar is the behaviour of y as x increases from 0 to ∞. The curve thus rises both in the third and in the first quadrants.
- We can tabulate a few values of (x, y) thus:

$x=0$	1/2	1	−1/2	−1	2	−2	3	−3	etc.
$y=0$	1/8	1	−1/8	−1	8	−8	27	−27	etc.

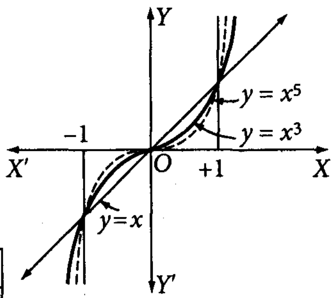


Fig. 1.4

  Plot these points, taking a suitable scale.
  The shape of the curve $y = x^3$ is shown in Fig. 1.4.
- For the curves of higher powers of x (odd powers), say $y = x^5$ (shown by dotted line in the figure), we observe that it is more flat near origin and steeper beyond $(1,1)$ and $(-1,-1)$. For still higher odd powers of x, the flatness and steepness become more and more pronounced and ultimately when the powers become extremely large odd numbers, the curve approaches a shape like the line $x = -1$, a part of X-axis between -1 to 1 and the line $x = 1$. See that $y = x$ belongs to these types of curves.

Polynomials

A function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n(a_0 \neq 0),$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are all constants and n is a positive integer is known as a *polynomial* in x of degree n.

- Every polynomial is defined in the entire real domain (i.e., defined in $-\infty < x < \infty$).
- They always give curves which are continuous (i.e., no break anywhere in the curve).
- The simplest polynomials are $y = x$, x^2, x^3, x^4, x^5, etc. We call $y = ax + b$, $y = ax^2 + bx + c$, $y = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ respectively linear, quadratic and cubic functions, whose graphs are discussed above. They are all very useful in ordinary applications.

  Note: A function $f(x)$ is even if $f(-x) = f(x)$; it is odd if $f(-x) = -f(x)$. So even powers of x are even functions of x and odd powers of x are odd functions of x.

- $f(x) = c$ (constant) for all values of x, is a polynomial of degree zero [Constant Function].

1.5 Graphs of Rational Functions: Not Everywhere Continuous Graphs

Rational functions. A function such as

$$y = f(x) = \frac{x-1}{x^2+5x+6}$$

in which y is expressed as a ratio of two polynomials in x is known as a rational function (mark the word rational). Accordingly polynomials are also rational functions with the denominator = constant function 1.

A particular rational function has many interesting applications: $f(x) = 1/x$ or $f(x) = a/x$.

Example 39. *Draw the graph of $y = 1/x$ or, $xy = 1$.*

Solution:

First notice that y is undefined, when $x = 0$.

The graph passes through $(1, 1)$ and $(-1, -1)$.

The curve lies wholly in the first and third quadrants.

($\because$ when x is positive, y is positive, and when x is negative, y is also negative.)

The curve is a falling curve (in $0 \le x < \infty$, y decreases from very large values to extremely small values and in $-\infty, < x \le 0$ also y decreases from very small negative values to very large negative values).

The shape of the curve is shown in Fig. 1.5. The curve is known as a *rectangular hyperbola*.

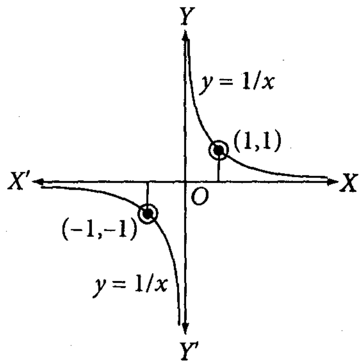


Fig. 1.5

Note that since at $x = 0$, there exists no corresponding y, there is a gap there. We say that the function is not continuous (or it is discontinuous) at the origin. We have thus two branches of the curve — one in the first quadrant, another in the third quadrant. As we take values of x nearer and nearer to zero (but positive) the values of y become infinitely large.

But at $x = 0$, there is no value of y. Again, when x takes values nearer and nearer to zero (but negative) the values of y become numerically infinitely large but negative.

The graph of $y = a/x$ can now easily be drawn. This curve will also be a rectangular hyperbola passing through (a, a) and $(-a, -a)$.

Applications. The function $xy = a$ may be used to represent the special demand curve — with price P and quantity Q on the two axes for which the total expenditure PQ is constant at all levels of price.

Another application is to the average fixed cost (AFC) curve. With AFC on one axis and output Q on the other, the AFC curve is a rectangular hyperbola because (AFC)$\times Q$ (= total fixed cost) is a fixed constant.

The rectangular hyperbola $xy = 1$ never meets either X-axis or Y-axis, even if extended indefinitely upward or to the right. We say in this case that the curve approaches the axes asymptotically; as y becomes very large, the curve comes closer to the Y-axis but never actually reach it, and similarly for the X-axis. The axes constitute what we call the ASYMPTOTES of the rectangular hyperbola.

1.6 Algebraic and Non-algebraic Functions

A polynomial (such as x^3, x^2+x, etc.) or root of a polynomial (such as $\sqrt{x}$, $\sqrt[3]{x}$, etc.) is an *algebraic function.*

Note: $y = \sqrt{x^2+8}$ is not a rational function but it is an algebraic function.

Exponential functions (such as $y = a^x$, $a > 1$, $y = e^x$ in which the independent variable appears in the exponent) and logarithmic functions $y = \log_a x$ or $y = \log_e x$ are examples of non-algebraic functions. Another important type of non-algebraic function is trigonometric function whose discussions will not be included in the present text. Non-algebraic functions are also known as *Transcendental Function.*

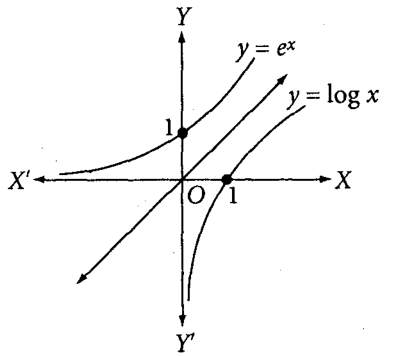


Fig. 1.6 Shapes of the Curves : $y = e^x$, $y = \log x$.

We recall here that the logarithm of a non-zero positive real number x is the power or exponent y to which a constant $a > 0$ and $a \neq 1$, called the base of the logarithm, must be raised in order to give the number x, i.e., $\log_a x = y$ so that $x = a^y$. (x must not be zero or negative).

When the base a is taken as 10, the logarithm is called *common logarithm.* But when the base is taken as a special number e (defined by $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$), the logarithm is called *natural logarithm.* In this case, $y = \log_e x$ or simply $y = \log x$ or $y = \ln x$.

In our discussions we shall mostly take logarithm to the base e. We shall consider logarithmic functions as $y = \log x$ $(x > 0)$; $x \neq 0$, x is not negative. We now show the shapes of the curve $y = \log x$ and $y = e^x$. They are inverse to each other. About the line $y = x$ one is the reflection of the other.

1.7 Further Illustrations on Graphs

Example 40. *Draw the graphs of $y = |x|$ or what is same as to write:*

$$f(x) = \begin{cases} x, \text{ when } x > 0 \\ -x, \text{ when } x < 0 \\ 0, \text{ when } x = 0. \end{cases}$$

Solution: We tabulate some values of x and y

x	0	1	2	3	−1	−2	−3
y	0	1	2	3	1	2	3

On the axes of x and y we take a certain scale (say 2 small squares on a graph paper may be taken as the number 1)

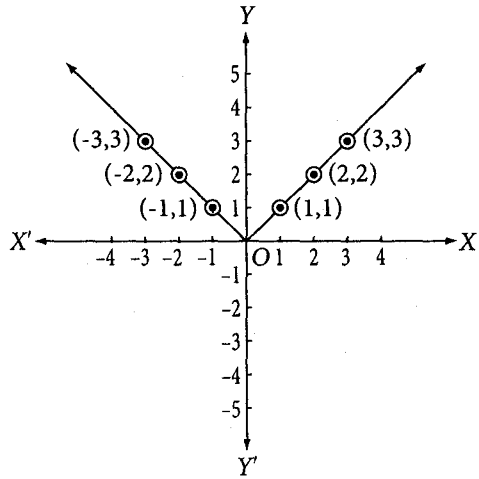


Fig. 1.7

Here the graph consists of two half-lines, one in the first quadrant, bisecting the angle between the axes and the other in the second quadrant, bisecting the angle between the axes, the origin is included in the graph. (See Fig. 1.7). There is no break, i.e., the graph is continuous.

Example 41. *Draw the graph of the function* $y = \frac{x^2}{x}$ *or which is the same as* $f(x) = x$, *when* $x \neq 0$ *and is undefined when* $x = 0$. [B.U. B.Com.(H) 2007]

Remember. *This function is not the same as* $y = x$ *or* $f(x) = x$ *because for the function* $f(x) = x$, *the domain is* $(-\infty, \infty)$ *including the point* $x = 0$, *but* $f(x) = x^2/x$ *has the domain* $(-\infty, 0) \cup (0, \infty)$, *i.e., whole real number system excluding the point* $x = 0$.

Some of the values of $y = x^2/x$ can be plotted as follows:

x	0	1	2	3	−1	−2	−3
y	Undefined	1	2	3	−1	−2	−3

Thus, there corresponds no point on the graph corresponding to $x = 0$. The scale on the two axes can be conveniently chosen (say 2 units to represent 1). The graph in Fig. 1.8 here is a straight line (except origin) lying in the first and third quadrants.

The graph has a break at the origin. We say that the function is not continuous at $(0, 0)$.

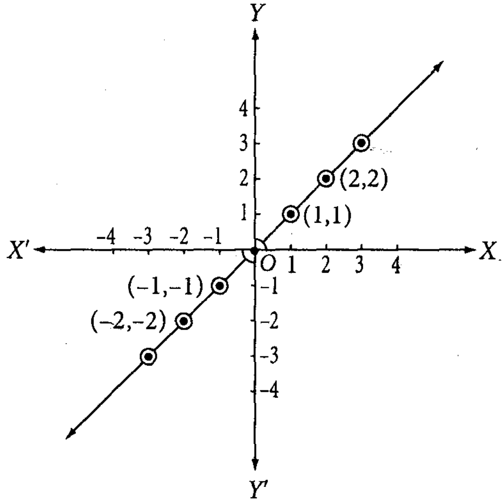


Fig. 1.8

Example 42. (Greatest Integer Function) $y = [x]$, *where* $[x]$ *denotes the greatest integer in x but not greater than x.* $y = f(x) = [x]$ *can be written as*

$$y = f(x) = \begin{cases} 0, \text{ when } & 0 \leq x < 1 \\ 1, \text{ when } & 1 \leq x < 2 \\ 2, \text{ when } & 2 \leq x < 3 \\ -1, \text{ when } & -1 \leq x < 0 \\ -2, \text{ when } & -2 \leq x < -1, \quad \text{etc.} \end{cases}$$

On the convenient scale we draw the graph of each segment: $-1 \leq x < 0$, $-2 \leq x < -1$, $0 \leq x < 1$, $1 \leq x < 2$, etc. In each segment y has a constant value.

e.g., for $-1 \leq x < 0$, y has the constant value -1
$-2 \leq x < -1$, y has the constant value -2
$0 \leq x < 1$, y has the constant value 0, etc.

The graph in Fig. 1.9 thus consists of a number of parallel line-segments, all parallel to the X-axis. These line-segments (drawn in bold) look like steps, hence the function is called a *step function*. There is a break at every integral value of x, i.e., the function is not continuous at $x = 0, \pm 1, \pm 2$, etc.

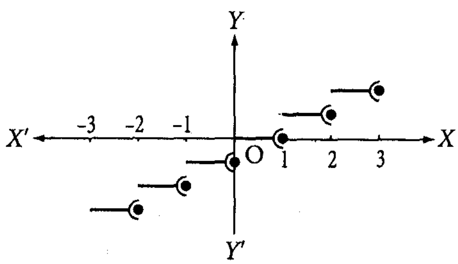


Fig. 1.9

Example 43. (Signum Function) *Draw the graph of the following function:*

$$y = f(x) = \begin{cases} 1, \text{ when } x > 0 \\ -1, \text{ when } x < 0 \\ 0, \text{ when } x = 0. \end{cases}$$

Solution: This function is same as $f(x) = \begin{cases} |x|/x, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0. \end{cases}$

x	0	1	$\frac{1}{2}$	$\frac{3}{4}$	2	-1	$-\frac{1}{2}$	$-\frac{3}{4}$	-2
y	0	1	1	1	1	-1	-1	-1	-1

Here, when $x = 0$, $y = 0$. For every positive x, y is always 1 and for every negative x, y is always -1. Thus, the graph in Fig. 1.10 consists of (a) the point $(0,0)$; (b) a line parallel to X-axis in the first quadrant at a height 1 above the X-axis, the point $(0,1)$ is not to be included; (c) a line parallel to the X-axis in the third quadrant at a depth 1 below the X-axis, the point $(0,-1)$ is not to be included.

The graph has a discontinuity or a break at $x = 0$.

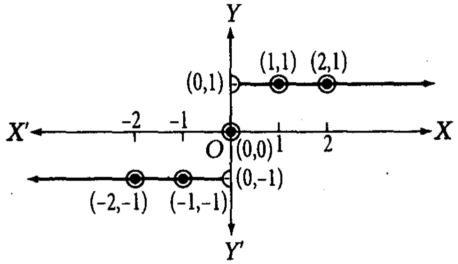


Fig. 1.10

Example 44. $y = f(x) = \begin{cases} 1, \text{ for } x \geq 0 \\ -1, \text{ for } x < -3. \end{cases}$

The graph in Fig. 1.11 consists of (a) a line parallel to X-axis in the first quadrant at a height 1 above the X-axis, the point $(0,1)$ being included; (b) another line parallel to X-axis in the third quadrant at a depth 1 below the X-axis, the point $(-3,1)$ is not included; (c) the function has not been defined for $-3 \leq x < 0$ and hence we do not get the corresponding ys, i.e., there is no graph of the function in this interval.

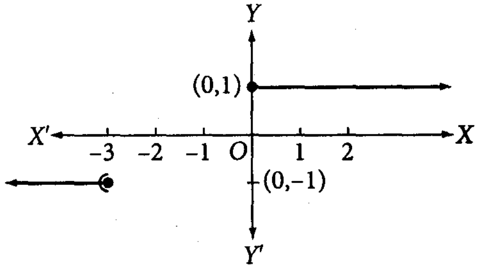


Fig. 1.11

Example 45. *Draw the graph of the following function:*

$$y = \begin{cases} 2x+1, \text{ when } x \geq 1 \\ 2x-1, \text{ when } x < 1. \end{cases}$$

Solution:

x	0	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	3	-1	-2
y	-1	0	$\frac{1}{2}$	3	4	5	7	-3	-5

(1)

Thus, in (1) when $x \geq 1$ we use the formula $y = 2x + 1$ and when $x < 1$, we use $y = 2x - 1$.

The graph in Fig. 1.12 consists of two half-lines, one in the first quadrant intersecting the line $x = 1$ at the point $(1,3)$ and the other lies in first, third and fourth quadrants intersecting the X-axis at the point $(1/2, 0)$ and Y-axis at the point $(0,-1)$. The origin is not included here. The graph has a break at $x = 1$. So we say that the function is not continuous at $x = 1$.

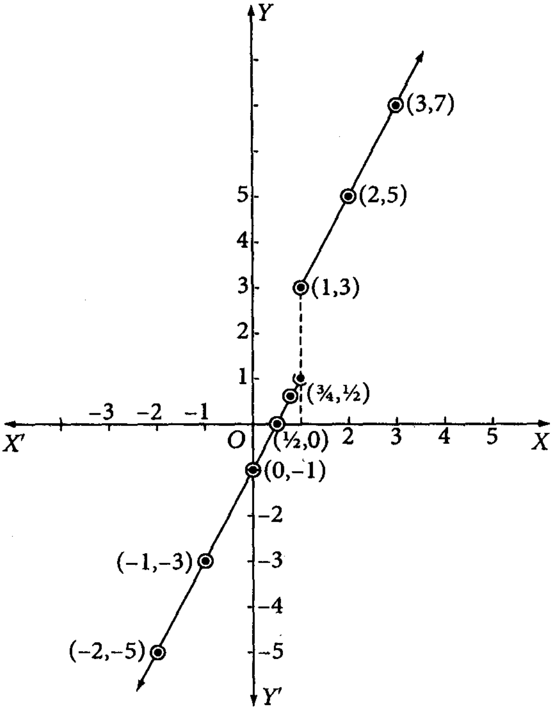


Fig. 1.12

Example 46. *Sketch the graph of the function defined by*

$$f(x)=\begin{cases} x-1, & when\ x>0 \\ 1/2, & when\ x=0 \\ x+1, & when\ x<0. \end{cases}$$

[C.U. B.Com.(H) 1992]

Solution: We tabulate some values of x and y as follows:

x	1	−1	0	2	−2	3	−3
y	0	0	$\frac{1}{2}$	1	−1	2	−2

Thus, when $x>0$, we shall use the formula $y=x-1$ and when $x<0$ we use $y=x+1$.

On the axes of x and y we take a certain scale (say 2 small squares on graph paper may be taken as number 1)

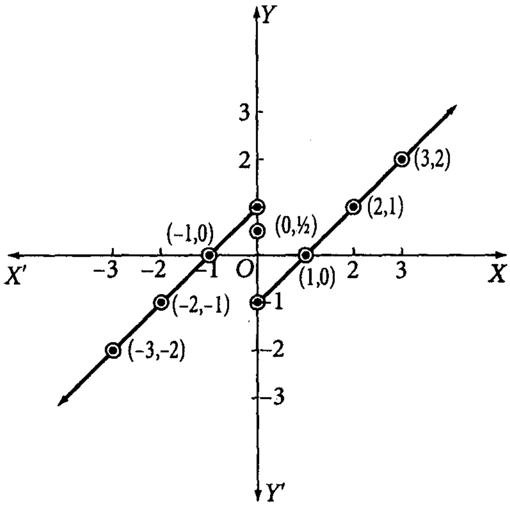


Fig. 1.13

Hence, the graph in Fig. 1.13 consists of two half-lines and a point $\left(0,\frac{1}{2}\right)$. One half-line lies in the first and fourth quadrants cutting the X-axis at the point $(1,0)$ and the other lies in the second and third quadrants cutting the X-axis at the point $(-1,0)$; the point $\left(0,\frac{1}{2}\right)$ is included in the graph, it is an isolated point. Here the graph has a break at $x=0$. So the function $f(x)$ is not continuous there.

Example 47. *A function $f(x)$ is defined by $f(x) = |x-2|+1$ over all real values of x. Show that $f(x)$ is continuous at $x=2$.*

Solution: Now, $f(x)=|x-2|+1$ over all real values of x can be written as

$$f(x)=\begin{cases} x-2+1=x-1, & \text{when } x>2 \\ 1, & \text{when } x=2 \\ -x+2+1=3-x, & \text{when } x<2. \end{cases}$$

We tabulate some values of x and y as under:

x	0	1	2	3	4	−1	−2	−3
y	3	2	1	2	3	4	5	6

Thus, when $x \geq 2$ we shall use the formula $y = x-1$ and when $x \leq 2$ we use $y = 3-x$.

The graph in Fig. 1.14 consists of two half-lines, one in the left side of the line $x = 2$ and other in the right side of the line $x = 2$, both intersect the line $x = 2$ at the point $(2, 1)$. The point $(2, 1)$ is included in the graph (Fig. 1.14). There is no break at $x = 2$, i.e., the graph is continuous at $x = 2$.

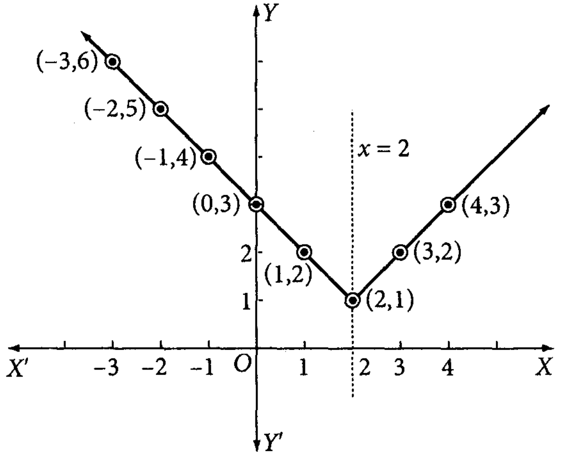


Fig. 1.14

Example 48. ***Dirichlet's Function***: *This function is defined by*

$$y = f(x) = \begin{cases} 1, & \textit{when } x \textit{ is rational} \\ 0, & \textit{when } x \textit{ is irrational.} \end{cases}$$

This function is well-defined for all x in $-\infty < x < \infty$. But we cannot draw the graph of this function. Within any interval of x we have infinite number of rational values for which $y = 1$, and also there exist infinite number of irrational values of x for which $y = 0$. So, we cannot plot them on a graph paper.

EXERCISES ON CHAPTER 1

A
(Concept of Function)

1. x and y are variables connected by $y = 6x + 3$. Find the value of y when $x = 2$. Can y be regarded as a function of x? If so, find its domain of definition.

2. Write the functional relation between the area A and radius r of a circle. Find A, when $r = 3$ cm.

3. Express the volume of a sphere as a function of its radius. Find the volume of a sphere of radius 3 cm, if $\pi = 3.14$.

4. An aeroplane flies at the constant speed of 400 km per hour; write a functional relation between the distance $\{s$ (in km)$\}$ covered and time $\{t$ (in hours)$\}$ of flight.

5. A manufacturer observes that there is a functional relationship between the machine operating cost C and the age of the machine t; given by a mathematical formula: $C = t^2 + 3$. Complete the following Table:

Age t years	1	2	3	4	5
Cost C thousand of ₹	—	—	—	—	28

6. A firm has found that sales are related to advertising effort, as given in the following data. Find a possible functional relation between s and a.

Advertising effort a, in thousand of ₹	10	12	14	16
Sales s, in thousand of ₹	100	120	140	160

7. A departmental store has found that sales volume (S) of women's dresses is inversely related to its sale price (P) given by the formula: $S = 100{,}000/(10 + P)$. Find S, when $P = 10, 20, 30, 40$.

8. (a) If $f(x) = (x - a)(x - b)(x - c)$, find $f(a)$, $f(b)$, $f(c)$, $f(0)$.

   (b) If $f(x) = |x| - 3x$, find $f(-2)$. [C.U. B.Com. 2007]

   (c) If $f(x) = |x| - [x]$, where $[x]$ = greatest integer not exceeding x, then find the value of $f(2.5)$ and $f(-2.5)$. [C.U. B.Com. 2008]

9. (a) **Remember**: $|x| = x$, if $x \geq 0$, $|x| = -x$, if $x < 0$. Given: $f(x) = 3x + |x|$, find $f(3)$ and $f(-3)$.

   (b) If $f(x) = x - |x|$, find $f(-4)$. [B.U. B.Com.(H) 2007]

10. (a) If $f(x) = \dfrac{1-x}{1+x}$, find $f(1/x)$ and $f\left\{f\left(\frac{1}{x}\right)\right\}$ while $x \neq 0$. [C.U. B.Com. 2009]

    (b) $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{x}{1-x}$, obtain $\dfrac{f(x+h) - f(x)}{h}$ and $\dfrac{g(x+h) - g(x)}{h}$.

11. (a) $f(x) = x^2 + x^4$, verify $f(x) = f(-x)$.

    (b) $g(x) = 3x^3 + x$, verify $g(-x) = -g(x)$.

    (c) If $f(x) = e^{ax+b}$, prove that $e^b \cdot f(x + y) = f(x).f(y)$. [C.U. B.Com.(H) 1993]

    (d) If $f(x) = \dfrac{1+e^x}{1-e^x}$, then show that $f(x)$ is an odd function. [C.U. B.Com. 2006]

(e) Show that $f(x)=\log\left(x+\sqrt{x^2+1}\right)$ is an odd function of x.

[B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2008]

[Hints: $f(-x)=\log\left(-x+\sqrt{x^2+1}\right)=\log\left\{\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right\}=\log\left(\frac{x^2+1-x^2}{x+\sqrt{x^2+1}}\right)=\log\left(\frac{1}{x+\sqrt{x^2+1}}\right)$
$=-\log\left(x+\sqrt{x^2+1}\right)=-f(x)$]

(f) Show that $f(x)=\log\left(\sqrt{x^2-1}-x\right)$ is an odd function of x.

12. If $f(x)=[x]$, where $[x]$ denotes the greatest integer in x, but not greater than x; find $f(0)$, $f(1)$, $f(1/2)$, $f(7/5)$, $f(5.02)$, $f(-3)$, $f(-2.3)$, $f(-7.9)$.

13. If $f(x)=\frac{1}{2}(x+|x|)$, find $f(0)$, $f(1)$, $f(-1)$, $f(2)$, $f(-2)$, $f(1.3)$, $f(-1.3)$.

14. (a) If $\phi(x)=a\dfrac{x-b}{a-b}+b\dfrac{x-a}{b-a}(a\neq b)$, obtain $\phi(a)$, $\phi(b)$ and $\phi(a+b)$ and then verify: $\phi(a)+\phi(b)=\phi(a+b)$.

(b) If $f(x)=|x|-[x]$, find the values of $f(3.5)$ and $f(-3.5)$. [C.U. B.Com.(H) 1997]

(c) If $f(x)=\log_3 x$ and $\phi(x)=x^2$, find the value of $f\{\phi(3)\}$. [N.B.U. B.Com. 2006]

15. Let $g(x)=\dfrac{x-1}{x+1}$. Verify $\dfrac{g(x)-g(y)}{1+g(x)\cdot g(y)}=\dfrac{x-y}{1+xy}$. [C.U. B.Com.(H) 1994; V.U. B.Com.(H) 2010]

16. (a) Let $y=f(x)=\dfrac{ax+b}{cx-a}$, verify $f(y)=x$. [B.U. B.Com.(H) 2008; C.U. B.Com. 2008]

(b) If $y=f(x)=\dfrac{px+q}{rx-p}$, then show that $f(y)=x$. [V.U. B.Com.(H) 2008]

[Hints: $y=f(x)=\frac{px+q}{rx-p}$ or, $f(y)=\frac{py+q}{ry-p}=\frac{p\left(\frac{px+q}{rx-p}\right)+q}{r\left(\frac{px+q}{rx-p}\right)-p}$, i.e., $f(y)=\frac{(p^2+rq)x+pq-pq}{prx+rq-prx+p^2}=\frac{(p^2+rq)x}{p^2+rq}=x$.]

17. (a) If $f(x)=ax^2+bx+c$, find a, b, c, if $f(1)=3$, $f(2)=7$ and $f(3)=13$. What is the value of $f(0)$?

(b) If $f(x+3)=3x^2-2x+5$, find $f(x-1)$.

(c) If $f(x)=\log x(x>0)$, show that $f(p)+f(q)+f(r)=f(pqr)$. [B.U. B.Com.(H) 2002]

18. A function f is defined over the whole real number system R defined by

$$f(x)=\begin{cases} 2x^2+1, & \text{for } x\leq 2 \\ 1/(x-2), & \text{for } 2<x\leq 3 \\ 2x-5, & \text{for } x>3. \end{cases}$$

Find $f(-1)$, $f(0)$, $f\left(\sqrt{2}\right)$, $f(-2)$, $f(4)$, $f(2.5)$, $f(7)$.

19. Find for what values of x the following functions are not defined:

(a) $f(x)=\dfrac{1}{x-3}$; (b) $\sqrt{x}$; (c) $\sqrt{x-2}$; (d) $\sqrt{(x-2)(x-3)}$; (e) $\dfrac{x^2-9}{x-3}$; (f) $\sqrt{1-x^2}$.

Write down the domain of definition in each case.

Remember. Non-real values are regarded, in this book, undefined. Also $\log x$ is defined, only when $x>0$.

20. Which of the following tables define y as a function of x?

(a) $\begin{array}{c|c|c} x & 2 & 2 \\ \hline y & 4 & 1 \end{array}$; (b) $\begin{array}{c|c|c} x & 1 & 3 \\ \hline y & -1 & -1 \end{array}$; (c) $\begin{array}{c|c|c} x & 1 & 1 \\ \hline y & 1 & 2 \end{array}$; (d) $\begin{array}{c|c|c} x & 0 & 0 \\ \hline y & 1 & 0 \end{array}$; (e) $\begin{array}{c|c|c} x & 1 & 2 \\ \hline y & 1 & 1 \end{array}$.

21. Find the value of $\{f(a+h)-f(a)\}/h$, $(h \neq 0)$, where

(a) $f(x)=x$;

(b) $f(x)=x^2$;

(c) $f(x)=1/x$;

(d) $f(x)=$ a constant, for all real values of x;

(e) $f(x)=4x^2+2x-3$.

22. Definition. Two functions f and ϕ are said to be equal $(f=g)$, if

(a) the domain of definition of f and g are same,

(b) $f(a)=g(a)$, for any member a of their domains.

Are the following functions equal?

(a) $f(x)=|x|$ and $g(x)=+\sqrt{x^2}$.

(b) i. $f(x)=x^2/x$ and $g(x)=x$;

ii. $f(x)=x+1$ and $\phi(x)=\dfrac{x^2-1}{x-1}$.

(c) $f(x)=x^2/x$ and

$$g(x)=\begin{cases} x, & \text{when } x \neq 0 \\ 0, & \text{when } x=0. \end{cases}$$ [B.U. B.Com.(H) 2007]

(d) $f(x)=x^2/x$ and

$$g(x)=\begin{cases} x, & \text{when } x \neq 0 \\ 1, & \text{when } x=0. \end{cases}$$

23. Find the range of each of the following functions, where x is real:

(a) $\sqrt{9-x^2}$; (b) $\dfrac{x^2}{1+x^2}$; (c) $\dfrac{2x}{1+x^2}$.

24. Find $\dfrac{f(x+h)-f(x)}{h}$ when $f(x)=\dfrac{1-x}{1+x}$. [C.U. B.Com.(H) 1998]

[Hints: $f(x)=\frac{1-x}{1+x}$ and $f(x+h)=\frac{1-(x+h)}{1+(x+h)}=\frac{1-x-h}{1+x+h}$.

$\therefore\ \frac{f(x+h)-f(x)}{h}=\frac{\frac{1-x-h}{1+x+h}-\frac{1-x}{1+x}}{h}=\frac{1-x^2-h-hx-(1-x^2+h-hx)}{h(1+x+h)(1+x)}=\frac{-2h}{h(1+x+h)(1+x)}=-\frac{2}{(1+x+h)(1+x)}$]

25. If $f(x)=\dfrac{ax-b}{bx-a}$, show that $f(a)\cdot f\left(\dfrac{1}{a}\right)-f(b)\cdot f\left(\dfrac{1}{b}\right)=0$. [C.U. B.Com.(H) 2007]

[Hints: $f(x)=\dfrac{ax-b}{bx-a}$; $\therefore\ f(a)=\dfrac{a\cdot a-b}{b\cdot a-a}=\dfrac{a^2-b}{ab-a}$,

$f\left(\frac{1}{a}\right)=\frac{a\cdot\frac{1}{a}-b}{b\cdot\frac{1}{a}-a}=\frac{1-b}{\frac{b}{a}-a}=\frac{1-b}{\frac{b-a^2}{a}}=\frac{a(b-1)}{a^2-b}$, $f(b)=\frac{ab-b}{b^2-a}=\frac{b(a-1)}{b^2-a}$

and $f\left(\frac{1}{b}\right)=\frac{a\cdot\frac{1}{b}-b}{b\cdot\frac{1}{b}-a}=\frac{a-b^2}{b-ab}=\frac{b^2-a}{b(a-1)}$. Put these values in the LHS of the equation to be proved.]

26. (a) Find the range of the function $\dfrac{x}{1+x^2}$. [C.U. B.Com.(H) 2006]

(b) Find for what values of x, the function $f(x)=\sqrt{(x-1)(x-2)}$ is not defined.

[**Hints:** If $1<x<2$, then $(x-1)$ is positive and $(x-2)$ is negative so that $(x-1)(x-2)$ is *negative* and $f(x)$ is *non-real*, i.e., not defined. If $x<1,(x-1)$ and $(x-2)$ are both negative and their product is positive, i.e., $f(x)$ is *defined*. If $x>2$, then both $(x-1)$ and $(x-2)$ are positive and $f(x)$ is defined.

Hence $f(x)$ is not defined, only when $1<x<2$.]

27. The taxi-fare is ₹3.50 per 1 km or less from start and ₹1.20 per km or any fraction thereof for additional distance. If the fare be ₹y for a distance of x km, express y as a function of x.

[C.U. B.Com.(H) 2003]

[**Hints:** $y=3.50$ if $0<x\leq 1$, $y=3.50+1.20$ if $1<x\leq 2$, $y=3.50+2\times 1.20$ if $2<x\leq 3$, and so on and in general, $y=3.50+1.20p$ when $p<x\leq p+1$, where $p=0$ or a positive integer.]

28. If $f(x)=\dfrac{e^x-1}{e^x+1}$ and $\phi(x)=\dfrac{1+f(x)}{1-f(x)}$, then show that $\phi(x+y)=\phi(x)\phi(y)$. [C.U. B.Com. 2005]

29. Find the domain of definition of the function: (a) $\dfrac{5}{\sqrt{(x+1)(x-3)}}$. [C.U. B.Com. 2004]

(b) $f(x)=\dfrac{x-2}{\sqrt{x^2-x-2}}$. [C.U. B.Com.(H) 1994; B.U. B.Com.(H) 2003]

[**Hints:** (a) Let $f(x)=\dfrac{t}{\sqrt{(x+1)(x-3)}}$. (1)

We see that $f(x)$ is undefined at the values of x given by $(x+1)(x-3)\leq 0$ and is defined when $(x+1)(x-3)>0$. (2)

If $-1<x<3$, then $(x+1)$ is positive but $(x-3)$ is negative so that their product is negative which contradicts eq. (2). If $x<-1$, then both $(x+1)$ and $(x-3)$ are both negative so that their product is positive and (2) is satisfied. Again, $x>3$, then both $(x+1)$ and $(x-3)$ are positive and in eq. (1) is again satisfied. Hence, the domain of definition of $f(x)$ is $\{x:\infty<x<-1\}\cup\{x:3<x<\infty\}$, i.e., $R-\{x:-1\leq x\leq 3\}$.]

30. Find the domain of definition of the function: $y=\log(x^2-7x+12)$. [C.U. B.Com. 2008]

[**Hints:** $y=\log(x^2-7x+12)$ is defined only when $x^2-7x+12>0$, i.e., when

$$(x-3)(x-4)>0. \tag{1}$$

From (1), we see that if $3<x<4$, then $(x-3)$ is positive and $(x-4)$ is negative, i.e., (1) is not satisfied. If $x<3$ or, $x>4$, then (1) is satisfied.

Hence, the required domain is $\{x:x<3 \text{ or } x>4\}$.]

B
(Graphical Representation of Functions)

1. Draw the graphs of the following linear functions:

   (a) $y=2x$; (b) $y=5+3x$; (c) $y=5-3x$; (d) $x=8$; (e) $y=5$; (f) $y=3x+7$.

   [In each case, consider the domain of x consisting of all non-negative real numbers only.]

   What difference is reflected in the graphs of (b), (c) and (f)?

2. Graph the functions:

   (a) $y=-x^2+4x-2$; (b) $y=x^2+4x-2$

   with the set of values $-5\leq x\leq 5$ as the domain. What difference do you notice in the two graphs?

3. Graph the function $y = 24/x$ assuming that x and y can take positive values only.

   Next, suppose that x and y can take negative values as well; how much the graph is to be modified to reflect this change?

4. Draw the graph of $y = 2x^3(-\infty < x < \infty)$.

5. Draw the graphs of

   (a) $y = x^2 + x$; (b) $y = x^3 + x^2$; (c) $y = x^5$; (d) $y = x^4$; (e) $y = x^3 + x^2 - 5$; (f) $y = 1/x^2$.

6. Sketch the graph of $f(x) = 3x + 1$, when $x \geq 1$; $f(x) = 3x - 1$, when $x < 1$.

7. A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 3 + 2x, & \text{for } -\frac{3}{2} \leq x < 0 \\ 3 - 2x, & \text{for } 0 \leq x < \frac{3}{2} \\ -3 - 2x, & \text{for } x \geq \frac{3}{2}. \end{cases}$$

   Sketch the graph.

8. Sketch the graph and examine whether $f(x)$ is continuous at $x = 1$ or not.

   The function is given as $f(x) = 3x + 1$, when $x \geq 1$; $f(x) = 3x - 1$ when $x < 1$.

9. Sketch the graph of the functions

   (a) $f(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ -1, & \text{for } x < 0 \end{cases}$

   (b) $f(x) = \begin{cases} -x, & \text{when } x \leq 0 \\ x, & \text{when } 0 < x \end{cases}$

   From the graph examine the continuity of $f(x)$ at $x = 0$. [C.U. B.Com.(H) 1990]

   (c) $f(x) = \dfrac{|x|}{x}$ and examine its continuity at $x = 0$. [C.U. B.Com.(H) 2001]

   (d) $f(x) = \dfrac{x}{|x|}$ and discuss its continuity at $x = 0$. [C.U. B.Com.(H) 1995]

10. Sketch the graph of the function

$$f(x) = \begin{cases} 3 + 2x, & \text{when } x \leq 0 \\ 3 - 2x, & \text{when } x > 0. \end{cases}$$

   From the graph examine the continuity of $f(x)$ at $x = 0$. [C.U. B.Com.(H) 1991]

11. A function $f(x)$ is defined by $f(x) = |x - 2| + 1$ over all real values of x. Show that $f(x)$ is continuous at $x = 2$.

12. Draw the graph of the following function:

$$f(x) = \begin{cases} 2x - 1, & \text{if } 0 \leq x < 4 \\ 2 - x^2, & \text{if } -4 < x < 0. \end{cases}$$

   [C.U. B.Com.(H) 1993]

13. Draw the graph (rough sketch) of $g(x) = \dfrac{|x-1|}{x-1}$ and examine the continuity at $x = 1$.

[C.U. B.Com. 2005]

Note: The set R represents the entire real numbers $(-\infty, \infty)$.

ANSWERS

A

1. 15; Yes, R.
2. $A = xr^2$; 9π sq cm.
3. $V = \frac{4}{3}\pi r^2$; 113.04 cu cm.
4. $s = 400t$.
5. 4, 7, 12, 19.
6. $s = 10a$.
7. 5000, 3333, 2500, 2000.
8. (a) 0, 0, 0, $-abc$; (b) 8; (c) 0.5, 5.5.
9. (a) 12, -6; (b) -8.
10. (a) $\dfrac{x-1}{x+1}; -\dfrac{1}{x}$;
    (b) $-1/x(x+h)$;
    $1/(1-x)(1-x-h)$.
11. (a) Even function;
    (b) Odd function.
12. 0, 1, 0, 1, 5, -3, -3, -8.
13. 0, 1, 0, 2, 0, 1.3, 0.
14. (b) 0.5; 7.5; (c) 2.
17. (a) $a, b, c, f(0) = 1$ in each case.
    (b) $3x^2 - 26x + 61$.
18. 3, 1, 5, 9, 3, 2, 9.
19. (a) $x = 3$;
    (b) $x < 0$;
    (c) $x < 2$;
    (d) $2 < x < 3$;
    (e) $x = 3$;
    (f) $x > 1$ or $x < -1$,
    [Domain for (a) $R - \{3\}$; (b) $R^+ + \{0\}$, etc.]
20. (a), (c), (d) are not functions; for others y is a function of x.
21. (a) 1;
    (b) $2a + h$;
    (c) $-1/a(a+h)$;
    (d) 0;
    (e) $8a + 2 + 4h$.
22. Only in (a) functions are equal; others not equal. Explain why.
23. (a) $0 \le y \le 3$;
    (b) $0 \le y < 1$;
    (c) $-1 \le y \le 1, y \ne 0$.
24. $-\dfrac{2}{(1+x+h)(1+x)}$
26. $-\frac{1}{2} \le y \le \frac{1}{2}$ $(y \ne 0)$.
27. $y = 3.50 + 1.20p$ where $p < x \le p + 1$ where $p = 0$ or a positive integer.
29. (a) $R - \{-1 \le x \le 3\}$;
    (b) $R - \{-1 \le x \le 2\}$.
30. $\{x : x < 3 \text{ or } x > 4\}$.

B

8. $f(x)$ is not continuous at $x = 1$.
9. (b) $f(x)$ is continuous at $x = 0$;
   (c) $f(x)$ is not continuous at $x = 0$;
   (d) $f(x)$ is not continuous at $x = 0$.
10. $f(x)$ is continuous at $x = 0$.

Chapter 2

Limit and Continuity

2.1 Introduction

The concept of limit is the most basic concept in Calculus. A particular case of limit leads to the notion of Continuity. In the present chapter we shall explain these two ideas, mainly from graphical standpoint. We have not sacrificed the underlying rigour though every attempt has been made to make the ideas clear from intuitive approach.

2.2 Limit of an Independent Variable x

Let x be a real variable and a be a finite real number. We often say $x \to a$ (x tends to a) or $\lim x = a$ or Lt $x = a$. What does this statement mean?

$x \to a$ and $x = a$ have different meanings. $x = a$ means that we put a for x; $x \to a$ is associated with the approach of the successive values of the variable x towards a, given a number a (here we are not interested for the value $x = a$; we are interested for values of x close to a). To make the idea clear we explain from a geometric standpoint:

On the number axis OX we plot the fixed number a; let the point corresponding to a be A.

We know that x is a real variable and hence it can assume different real numbers; we shall plot these moving points on the number axis.

Suppose, the successive values of x be so taken that one can discriminate the values that precede and the values that follow. Remember, not at any stage, shall we say that this is the last value of x.

We first begin with some value of x, greater than a, and plot it on the right of A, say, the point is P (Fig. 2.1).

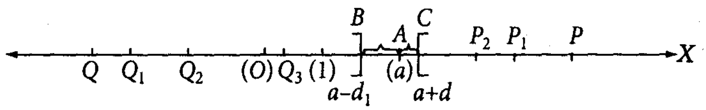


Fig. 2.1

We next take the value of x so that the corresponding point P_1 lies between A and P (this means that the value of x is nearer to a than what we had taken before).

Next point P_2 is taken between A and P_1 (i.e., the value of x corresponding to P_2 is still nearer to A than P_1 or P). We continue this process of taking x nearer and nearer to A (but remaining always to the right of A). In this case, we say that the moving point P is approaching A from the right side or we say that the successive values of x are approaching the fixed number a from the right. If the process is unending, we can come closer and closer to A without ever reaching it.

How close to A? To measure this closeness we take an arbitrary point C on the right of A (C is different from A but may be as close to A as we please). Let the number corresponding to C be $a+\delta$(δ is thus a positive number, no matter how small). If the successive values of x be such that the corresponding points *ultimately cross C and remain within* (A, C), then we say that the moving point $P \rightarrow A$ from the right or we say that x tends to a from the right; we then write ($x \rightarrow a^{+0}$ or $x \rightarrow a^{+}$).

Similarly, one may start from a moving point Q on the left of A (i.e., the starting value of x may be less than a). Then take the point Q_1 between Q and A (i.e., the corresponding x is taken nearer to a than the previous value of x). The process is continued. This is what we say 'Q is approaching A from the left' or 'x is moving towards a from the left'. Now, if B be an arbitrary point on left of A [suppose, the corresponding number is $a-\delta_1$; δ_1 is thus any positive number, no matter how small] and if the successive positions of the moving point ultimately cross B and remain within (B, A) (i.e., if the successive values of x ultimately remain within $a-\delta_1$ and a), then we say that the moving point $Q \rightarrow A$ from the left (or x tends to a from the left, written as $x \rightarrow a^{-0}$ or $x \rightarrow a^{-}$).

Remember $x \rightarrow a$ means $x \rightarrow a^{+0}$ as well as $x \rightarrow a^{-0}$

When $x \rightarrow a$, the successive values of x ultimately remain within an interval $(a-\delta, a+\delta)$, but $x = a$ is excluded. We write in this case, $0 < |x-a| < \delta$, where δ is any arbitrary positive number, no matter how small.

Formal Rigorous Definitions:

- $x \rightarrow a$ has the following definition: *Given any positive number δ, no matter how small, the successive values of x ultimately satisfy the relation $0 < |x-a| < \delta$ or, $a-\delta < x < a+\delta (x \neq a)$*;
- $x \rightarrow a^{+0} \Rightarrow a < x < a+\delta$;
- $x \rightarrow a^{-0} \Rightarrow a-\delta < x < a$.

[In every case δ is a preassigned positive number, no matter how small.]

Meaning of $x \rightarrow \infty$ (x tends to infinity). Suppose, the successive values of x ultimately become and remain greater than any arbitrary positive number G, then x is said to have an infinite limit. We write, $x \rightarrow \infty$.

Meaning of $x \rightarrow -\infty$ (x tends to minus infinity): Suppose, the successive values of x ultimately become and remain less than any given number G', usually we take $G' = -G$, where G is any arbitrary positive number, then x is said to have a limit $-\infty$. We write, $x \rightarrow -\infty$.

Note: $x = \infty$ or $x = -\infty$ have no meanings.

2.3 Limiting Value of a Function (or Limit of a Dependent Variable)

[B.U. B.Com.(H) 2008]

Our next problem: What value does a dependent variable y approach when the independent variable x approaches a specific finite quantity a? We assume that y is a function of x, say $y = f(x)$. Thus our problem is: To what value $f(x)$ approaches when x approaches a? Such a value may or may not exist — we are not certain. In case such a value exists (let us call it l), we say that $f(x)$ has a limit l as $x \rightarrow a$. We describe this situation by writing

$$\lim_{x \rightarrow a} f(x) = l \text{ or, } \underset{x \rightarrow a}{\mathrm{Lt}} f(x) = l.$$

Read. Limiting value of $f(x)$ when x tends to a is l.

We make the following two observations:

- The equality sign ($= l$) is to be understood as an *approach to equality*, not actually an equality. Actually we wish to emphasize $f(x) \to l$, when $x \to a$.
- When we put $x = a$ in $f(x)$, we get the value $f(a)$, called the *functional value* at $x = a$. It has, in general, nothing to do with the limiting value l. In fact, $f(a)$ may be undefined, but we may still obtain some limiting value of $f(x)$ when $x \to a$ and vice versa. A special case arises for a class of functions (called continuous functions) where the limiting value and functional value are identical. We shall discuss them later.

We start with an example:

Example 1. *What does the statement* $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} = 4$, *mean?*

Solution: *Geometric Approach.* Let us draw the graph of $y = f(x) = \dfrac{x^2-4}{x-2}$ and from the graph we shall find the limit of y (if it exists) as $x \to 2$.

Clearly, $y = f(x) = x + 2$, when $x \neq 2$, but y is undefined, when $x = 2$.

Then we draw the graph of $f(x)$. Such a graph is a straight line $y = x + 2$; the point $P(2, 4)$ of this line is, however, excluded (Fig. 2.2).

On the X-axis the point A corresponds to $x = 2$ and on the Y-axis the point C corresponds to $y = 4$.

Let $x \to 2^+$. The moving point x takes successive positions (on the right of A),

$A_1, A_2, A_3, A_4, \ldots$ coming nearer and nearer to A from the right.

Let $P_1, P_2, P_3, P_4, \ldots$ be the corresponding points on the graph of $y = f(x) = (x^2 - 4)/(x - 2)$ and let C_1, $C_2, C_3, C_4, \ldots$ be corresponding points on the Y-axis.

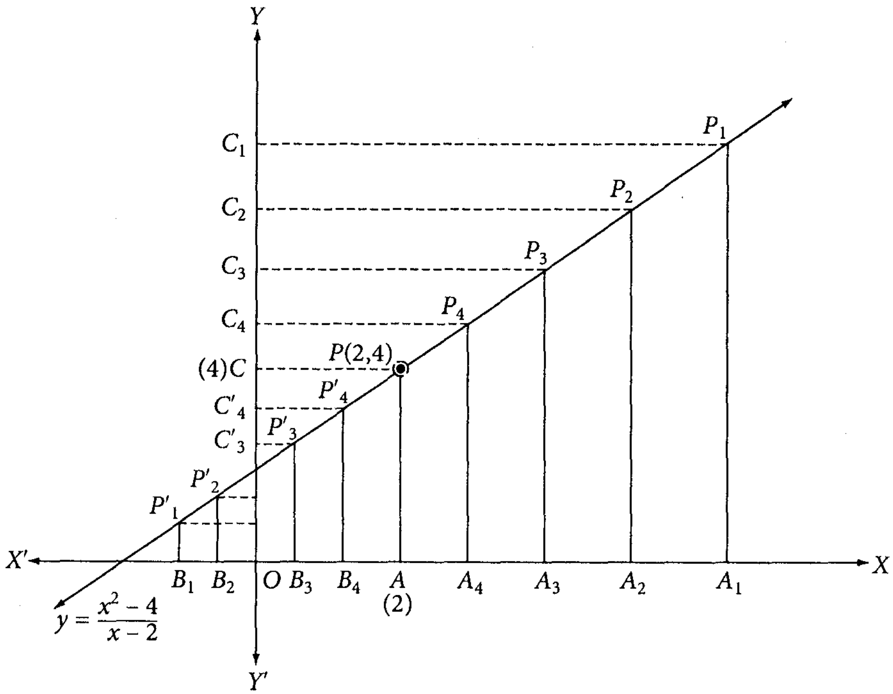


Fig. 2.2

Then suppose $x \to 2^-$. The moving point x takes now successive positions (on the left of A), B_1, B_2, B_3, B_4, ... coming closer and closer to A from the left. Let P_1', P_2', P_3', P_4', ... be corresponding points on the graph of $y = f(x) = \frac{x^2-4}{x-2}$ and C_1', C_2', C_3', C_4', $\cdots$ be the corresponding points on the Y-axis.

Now, we see that when the moving point on the X-axis approaches the point A from the right (i.e., when $x \to 2^+$), the corresponding moving point of the graph approaches the point P; consequently, the corresponding moving point on the Y-axis approaches the point C from above. Similarly, when the moving point on the X-axis approaches A from the left, the corresponding points on the graph approach P and the corresponding points on Y-axis approach C from below.

Since the moving point on Y-axis approaches $C(y = 4)$ both from below and from above, we say that $y \to 4^+$ and $y \to 4^-$ as the point $x \to 2$ from the right and from the left respectively.

In other words, $y \to 4$ as $x \to 2$ or, $\lim\limits_{x \to 2} y = 4$ or, $\lim\limits_{x \to 2} \frac{x^2-4}{x-2} = 4$.

Notations.

- $f(x)$ or $y \to 4$ as $x \to 2^+$ is denoted by $\lim\limits_{x \to 2^+} f(x) = 4 = f(2+0)$.

  [This is the right-hand limit of $f(x)$ when x approaches 2 from the right.]
- $f(x)$ or $y \to 4$ as $x \to 2^-$ is denoted by $\lim\limits_{x \to 2^-} f(x) = 4 = f(2-0)$.

  [This is the left-hand limit of $f(x)$ when x approaches 2 from the left.]
- For the existence of limit of $f(x)$, both right-hand and left-hand limits must exist and they must be equal.

More generally, let a be a specified point where $f(x)$ may or may not be defined but $f(x)$ must be defined near $x = a$ (both on its left and on its right). We denote the right-hand limit of $f(x)$ as $x \to a^+$ by $\lim\limits_{x \to a^+} f(x)$ or as $f(a+0)$. Similarly, the left-hand limit is denoted by $\lim\limits_{x \to a^-} f(x)$ or $f(a-0)$.

For the existence of $\lim\limits_{x \to 0} f(x)$ we must have:

- $f(a+0)$ must exist;
- $f(a-0)$ must exist;
- $f(a+0)$ must be equal to $f(a-0)$; and
- this equal value need not be equal to $f(a)$.

Only when the point a is a point at one end of the domain of definition of $f(x)$, we do not insist on the equality of $f(a+0)$ and $f(a-0)$. Thus, suppose $f(x)$ is defined in the closed interval $[\alpha, \beta]$. Then $\lim\limits_{x \to \alpha^+} f(x) = f(\alpha+0)$ and $\lim\limits_{x \to \beta^-} f(x) = f(\beta-0)$ can only be investigated. If $f(\alpha+0)$ exists, we take that limit as the limit of $f(x)$ when x approaches α. Similar is the case at $x = \beta$.

A More Precise Definition of Limit

A function $f(x)$ is said to have a limit l as x approaches a if, for any given positive number ϵ (no matter how small), we can find a positive number δ such that for all values of x in $a - \delta < x < a + \delta$ $(x \neq a)$, the corresponding values of $f(x)$ will satisfy $l - \epsilon < f(x) < l + \epsilon$, i.e., for all x in $0 < |x - a| < \delta$, we have

$$\left|f(x) - l\right| < \epsilon. \tag{I}$$

We then write,

$$\lim_{x \to a} f(x) = l.$$

Graphically it means that: When any positive number ϵ is given, we draw three lines parallel to X-axis: $y = l - \epsilon$, $y = l$, $y = l + \epsilon$.

To establish $\lim_{x \to a} y = l$ we are to obtain a positive number δ (for each given ϵ) such that the values of $f(x)$ for x lying between $a - \delta$ and $a + \delta$ (except for $x = a$) will fall within the rectangle formed by $y = l - \epsilon$, $y = l + \epsilon$; $x = a - \delta$, $x = a + \delta$.

Observations. The definition demands a suitable δ for each positive number ϵ, given in advance, so that $\forall x$ in $(a - \delta, a + \delta)$ excepting for $x = a$, we find $l - \epsilon < f(x) < l + \epsilon$. If no δ exists for some given ϵ, we shall say that the limit does not exist.

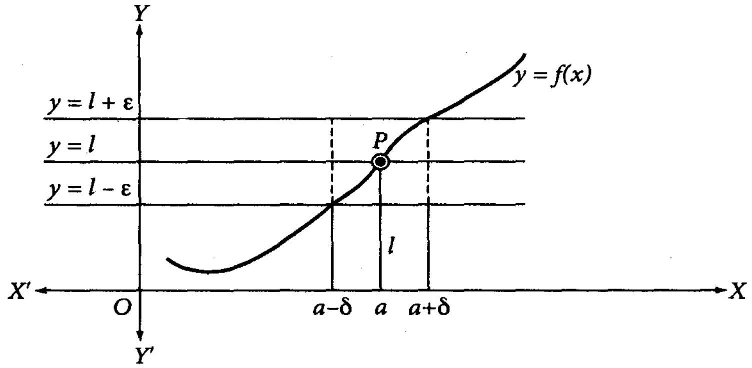


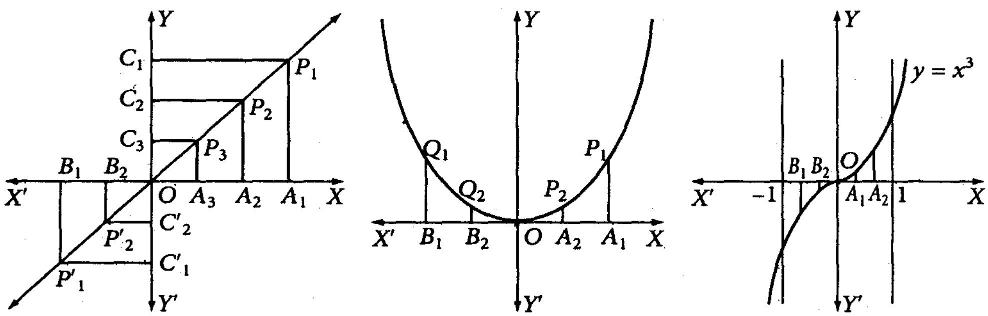
Fig. 2.3

- **Right-hand limit:** $\lim_{x \to a^+} f(x) = l_1 \Rightarrow$ given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that for all x in $a < x < a + \delta$ we have $|f(x) - l_1| < \epsilon$, (i.e., $l_1 - \epsilon < f(x) < l_1 + \epsilon$).
- **Left-hand limit:** $\lim_{x \to a^-} f(x) = l_2 \Rightarrow$ given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that for all x in $a - \delta < x < a$ we have $|f(x) - l_2| < \epsilon$, (i.e., $l_2 - \epsilon < f(x) < l_2 + \epsilon$).

In order that $\lim_{x \to a} f(x)$ may exist, l_1 and l_2 both must exist and they must be equal. This common value is the limit of $f(x)$ when $x \to a$.

2.4 Illustrative Examples

Example 2. $y = x$, $y = x^2$, $y = x^3$. *In each case we draw the graphs of the functions.*



$\lim_{x \to 0} x = 0$ (a)

$\lim_{x \to 0} x^2 = 0$ (b)

$\lim_{x \to 0} x^3 = 0$ (c)

Fig. 2.4

We can then easily check from the graphs: $\lim_{x\to 0} y = 0$ in each case. See *Figs 2.4(a), (b), (c).* As $x \to 0$ from either side, the corresponding points on the Y-axis approach the origin $\lim_{x\to 0} x = 0$, $\lim_{x\to 0} x^2 = 0$, $\lim_{x\to 0} x^3 = 0$.

Example 3. *To establish*: $\lim_{x\to 0} \frac{x^2}{x} = 0$.

Solution: We first draw the graph of $y = x^2/x$. Here $y = x$, when $x \neq 0$ and y is undefined at $x = 0$.

The graph is the line $y = x$, the point $(0, 0)$, however, is excluded.

But as $x \to 0$ from either side the points on the Y-axis $C_1, C_2, \ldots$ approach O (from above) and C_1', $C_2', \cdots$ (from below) also approach O, i.e., $\lim_{x\to 0} y = 0$.

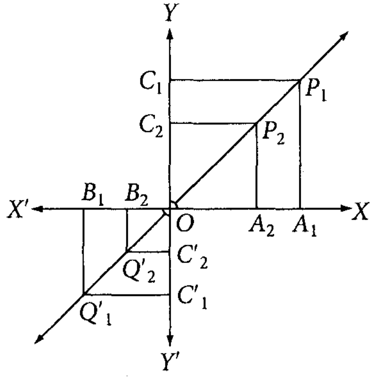


Fig. 2.5

Another Approach:

R.H. limit $= \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0.$

When $x \to 0^+$, we are not concerned with the particular value $x = 0$; we are interested with neighbouring values; for such neighbouring values $x^2/x = x$.

L.H. limit $= \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0$ (we consider only neighbouring values of 0 on the left side and hence for $x \neq 0$, we write $x^2/x = x$).

$\therefore \lim_{x\to 0} \frac{x^2}{x} = 0$ ($\because$ R.H. limit = L.H. limit $= 0$).

Example 4. *Let* $f(x) = |x|$. *To prove* $\lim_{x\to 0} f(x) = 0$.

Solution: The graph of $y = |x|$ (i.e., $y = x$, $x > 0$; $y = 0$, $x = 0$; $y = -x$, $x < 0$) is shown in Fig. 2.6.

As $x \to 0$ from either side {through $A_1, A_2, \ldots$ from the right; $B_1, B_2, \ldots$ from the left towards 0}, the corresponding $y \to 0$ (through the points C_1, C_2, $\ldots$, C_1', C_2', $\cdots$, i.e., for all approach). Here $\lim_{x\to 0} f(x) = 0.$

Another Approach:

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$

[When $x \to 0^+$, the successive values of x are > 0 and hence $|x| = x$]

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

[Here values of x are < 0. So $|x| = -x$]

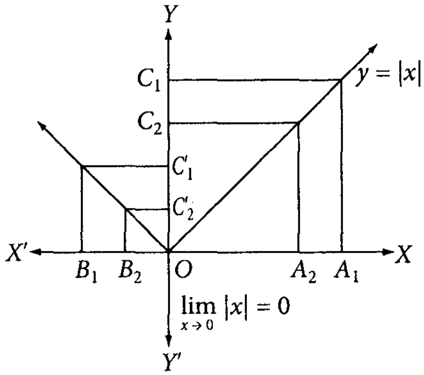


Fig. 2.6

$\therefore \lim_{x\to 0} f(x) = 0$ ($\because$ R.H. limit = L.H. limit).

Example 5.

$$f(x)=\begin{cases} 1, & x\geq 0 \\ -1, & x<0. \end{cases} \quad \textit{To discuss } \lim_{x\to 0} f(x).$$

Solution: The graph in Fig. 2.7 consists of the part of the line $y=1$ in the first quadrant and part of the line $y=-1$ [excluding $(0,-1)$] in the third quadrant.

As $x\to 0$ from the right, $y=f(x)\to 1$

$\{$when $A_1, A_2, \ldots \to 0$, then $P_1, P_2, \ldots \to P(0,1)\}$.

As $x\to 0$ from the left, $y=f(x)\to -1$

$\{$when $B_1, B_2, \ldots \to 0$, then $P_1', P_2', \ldots \to Q(0,-1)\}$,

i.e., R.H. limit $\neq$ L.H. limit,

i.e., $\lim_{x\to 0} f(x)$ does not exist.

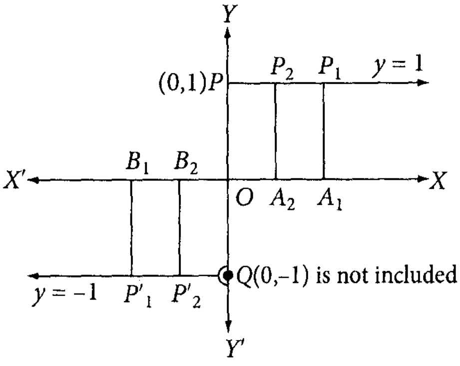


Fig. 2.7

Example 6.

$$f(x)=\begin{cases} 2x+5, & x>2 \\ 3x+9, & x\leq 2. \end{cases} \quad \textit{Does } \lim_{x\to 2} f(x) \textit{ exist?}$$

Solution: Here
$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+5)=\lim_{h\to 0^+}\{2(2+h)+5\}=\lim_{h\to 0^+}(9+2h)=9$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(3x+9)=\lim_{h\to 0^+}\{3(2-h)+9\}=\lim_{h\to 0^+}(15-3h)=15.$$

$\therefore$ R.H. limit $\neq$ L.H. limit, i.e., **$\lim_{x\to 2} f(x)$ does not exist.**

Example 7. *How to evaluate limits in very simple cases? Obtain* $\lim_{x\to 2}\left(x^2+3x+4\right)$ *(assume that the limit exists).*

[**Hints:** When $x\to a$ $(a\neq 0)$, write $x=a+h$ and make $h\to 0$. Here $x=2+h$; as $x\to 2$, $h\to 0$.]

Thus,
$$\lim_{x\to 2}\left(x^2+3x+4\right)=\lim_{h\to 0}\left\{(2+h)^2+3(2+h)+4\right\}=\lim_{h\to 0}\left(h^2+7h+14\right)$$
$$=\lim_{h\to 0}h^2+\lim_{h\to 0}7h+14=0+0+14=14.$$

Note: We have used *limit of a sum = sum of the limits.* This will be explained in the next article.

Example 8. *Use ϵ-δ definition to establish* $\lim_{x\to 1}(5x+3)=8$.

Solution: The statement $\lim_{x\to 1}(5x+3)=8$ is true, if given any positive number ϵ, we can determine a positive number δ such that for all x in $|x-1|<\delta$, we have $|5x+3-8|<\epsilon$.

Our object is to make $|5x+3-8|<\epsilon$.

This will happen, if $|5x-5|<\epsilon$, i.e., if $|x-1|<\epsilon/5$.

So if δ is taken as $\epsilon/5$, then for x is $|x-1|<\delta$, we have $|5x+3-8|<\epsilon$, i.e., the definition requirement is met.

$\therefore$ $\lim_{x\to 1}(5x+3)=8$ *is a true statement.*

Example 9. *Use ϵ-δ definition to establish* $\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = 8.$

Solution: Let ϵ be a given positive number. We require to find $\delta > 0$ s.t. for all x in $|x - 4| < \delta$ (except $x = 4$), the inequality $\left|\frac{x^2 - 16}{x - 4} - 8\right| < \epsilon$ holds.

Our object is to make $\left|\frac{x^2 - 16}{x - 4} - 8\right| < \epsilon.$

This will occur if $|x + 4 - 8| < \epsilon$ provided we exclude $x = 4$, i.e., if $0 < |x - 4| < \epsilon$.

So choosing $\delta = \epsilon$ our requirement for the definition is met.

$\therefore \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = 8$ *is true.*

2.5 Other Types of Limits

$\lim_{x \to a} f(x) = +\infty$ (read: limit of $f(x)$ as $x \to a$ is plus infinity).

Intuitively, when x approaches a from either side, the corresponding values of $f(x)$ grow larger and larger and ultimately exceed and remain more than any number we name, then, we say $f(x) \to +\infty$ as $x \to a$.

More precisely, $\lim_{x \to a} f(x) = +\infty$ means:

Given any number $G(> 0)$, no matter how large, we can determine $\delta > 0$ such that for x in $a - \delta < x < a + \delta (x \neq a)$ the inequality $f(x) > G$ always holds.

Example 10. $\lim_{x \to 0} \frac{1}{x^2} = \infty.$

Solution:

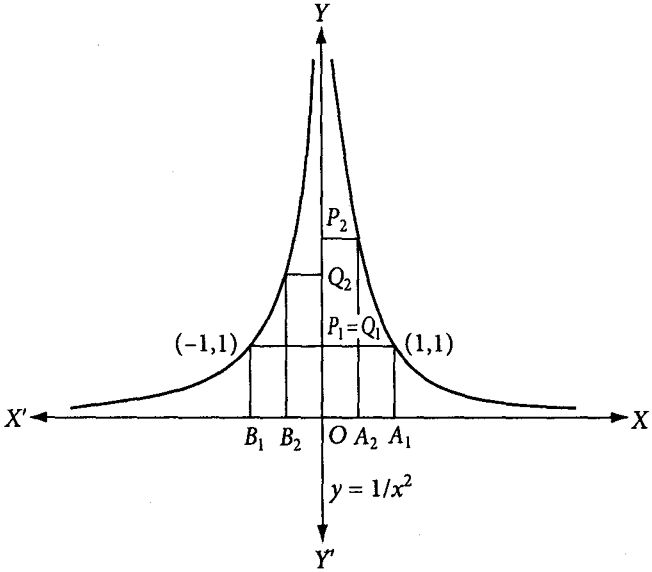


Fig. 2.8

We draw the graph of $y = \frac{1}{x^2}$.

At $x = 0$, y is undefined. As $x \to 0$ from the right or from the left, the corresponding y becomes larger and larger. y can exceed any number $G(> 0)$ and remain $> G$, if x is suitably chosen near 0.

Precise Definition. Given $G > 0$, we wish to make $\frac{1}{x^2} > G$, i.e., we wish to make $x^2 < \frac{1}{G}$, i.e., $|x| < \frac{1}{\sqrt{G}}$.

This will happen if $-\frac{1}{\sqrt{G}} < x < \frac{1}{\sqrt{G}}$.

$\therefore\ \delta = \frac{1}{\sqrt{G}}$ is a suitable choice, so that $-\delta < x < \delta$, i.e., $0 < |x| < \delta$ holds.

The inequality $\frac{1}{x^2} > G$ holds.

$\therefore\ \lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$ is a true statement.

$\lim\limits_{x \to +\infty} f(x) = l \Rightarrow$ given any positive number ϵ, no matter how small, we can find $G > 0$ such that for all $x > G$, the inequality $|f(x) - l| < \epsilon$ holds.

Intuitively, when x grows larger and larger, the values of $f(x)$ become closer and closer to l.

$\lim\limits_{x \to +\infty} f(x) = +\infty \Rightarrow$ given any $G > 0$, we can determine $N > 0$ such that $f(x) > G$ for all $x > N$, i.e., when x grows larger and larger, $f(x)$ also grows larger and larger.

$\lim\limits_{x \to a} f(x) = -\infty$ means $\lim\limits_{x \to a} \{-f(x)\} = +\infty$.

$$\lim_{x \to -\infty} f(x) = l \text{ means } \lim_{y \to +\infty} f(-y) = l \text{(put } x = -y)$$

Example 11. *Does* $\lim\limits_{x \to 0} \frac{1}{x}$ *exist?*

Solution:

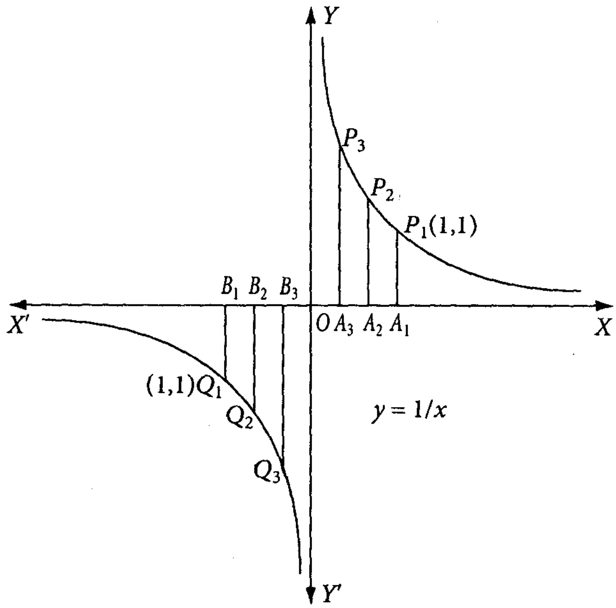


Fig. 2.9

We draw the graph of $y = 1/x$. It has two branches — one in the first quadrant and the other in the third quadrant (Fig. 2.9). As $x \to 0$ from the right, y goes on increasing without any bound. As $x \to 0$ from the left, y goes on decreasing in value (but increases numerically).

From the figure it is clear that $y \to +\infty$ as $x \to 0^+$ and $y \to -\infty$ as $x \to 0^-$.

$\therefore$ the limit of $y = 1/x$ as $x \to 0$ does not exist.

We express in the following way: $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$; $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$. $\therefore$ $\lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist.

Observations.

Example 12. $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$. *What about* $\lim\limits_{x\to\infty} \dfrac{1}{x^2}$?

Solution: We can establish $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$; moreover, $\lim\limits_{x\to+\infty} \frac{1}{x} = 0$.

See that when $x \to 0$, $\frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^8}, \ldots$ tend to $+\infty$; but $\frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^5}, \ldots$ do not tend to any limit.

Example 13. *To establish* $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$ *in a precise manner.*

Solution: Let $\epsilon > 0$ be any given number. Our statement will be true, if we can find $N > 0$ such that for all values of $x > N$, $|\frac{1}{x^2} - 0| < \epsilon$, i.e., $\frac{1}{x^2} < \epsilon$.

Now to make $\frac{1}{x^2} < \epsilon$ we require $x^2 > 1/\epsilon$ or we require $x > \dfrac{1}{\sqrt{\epsilon}}$.

So take $N = 1/\sqrt{\epsilon}$, then our definition requirement is met and the statement $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$ *is true.*

2.6 Theorems of Limits

In evaluation of limits of functions, the following theorems will be useful. We shall apply them whenever necessary.

Suppose, $f(x)$, $\phi(x)$ and $\psi(x)$ are three functions of a real variable x. Suppose that

$$\lim_{x\to a} f(x) = l, \quad \lim_{x\to a} \phi(x) = m \quad \text{and} \quad \lim_{x\to a} \psi(x) = n.$$

Then the following results hold:

Theorem 1.

(i) $\lim\limits_{x\to a}\{f(x) \pm \phi(x)\} = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} \phi(x) = l \pm m.$

(ii) $\lim\limits_{x\to a}\{f(x) + \phi(x) - \psi(x)\} = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} \phi(x) - \lim\limits_{x\to a} \psi(x) = l + m - n.$

(iii) $\lim\limits_{x\to a}\{f(x) \times \phi(x)\} = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} \phi(x) = l \times m.$

(iv) $\lim\limits_{x\to a}\{f(x) \times \phi(x) \times \psi(x)\} = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} \phi(x) \times \lim\limits_{x\to a} \psi(x) = l\,m\,n.$

(v) $\lim\limits_{x\to a} \dfrac{f(x)}{\phi(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} \phi(x)} = \dfrac{l}{m}$, *provided* $\lim\limits_{x\to a} \phi(x) = m \neq 0.$

Wrong Step:

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \frac{\lim\limits_{x\to 3} \left(x^2 - 9\right)}{\lim\limits_{x\to 3}(x - 3)}$$

is *not a correct procedure, because the limit in the denominator is zero.*

Correct Procedure for Such Problems.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{h\to 0}\frac{(3+h)^2-9}{3+h-3}=\lim_{h\to 0}\frac{h(6+h)}{h},\text{ where } x=3+h$$
$$=\lim_{h\to 0}(6+h)=6.$$

We can divide both numerator and denominator by h, since *when $h\to 0$, we need not take $h=0$ (We take h close to zero but not zero).*

Theorem 2.

(i) $\lim_{x\to a} c=c$. *Here $f(x)=c$ for all x and when the successive values of x approach a from either side, $f(x)$ remains always c which is a constant.*

(ii) $\lim_{x\to a} cf(x)=c\lim_{x\to a} f(x)=cl$, *if* $\lim_{x\to a} f(x)=l$.

Theorem 3. *Limit of a function of a function.*

If $\lim_{x\to a}\phi(x)=b$ and $\lim_{y\to b} f(y)=f(b)$, then $\lim_{x\to a} f\{\phi(x)\}=f\left\{\lim_{x\to a}\cdot\phi(x)\right\}=f(b)$.

Theorem 4. *Suppose, in some interval around $x=a$, $\phi(x)\le f(x)\le\psi(x)$ and $\lim_{x\to a}\phi(x)=l$ and $\lim_{x\to a}\psi(x)=l$. Then $\lim_{x\to a} f(x)=l$.* [Sandwich Theorem]

SOME IMPORTANT LIMITS

1. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1.$ 2. $\displaystyle\lim_{x\to 0}\frac{1}{x}\log(1+x)=1.$

3. $\displaystyle\lim_{x\to\infty} x^n=0\ (-1<x<1).$ 4. $\displaystyle\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a\ (a>0).$

5. $\displaystyle\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, where $a>0$ and n is any rational number.

2.7 Evaluation of Limits: Illustrative Examples

In case of evaluation, we shall assume (without the rigorous proof) that the limit exists.

Example 14. *Evaluate*:

(i) $\displaystyle\lim_{x\to 1}\frac{x^2+4x-5}{x-1}$, (*assuming that the limit exists;*) [C.U. B.Com.(H) 1994]

(ii) $\displaystyle\lim_{x\to 2}\frac{2x^2-7x+6}{3x^2-7x+2}$. [C.U. B.Com.(H) 2000]

(iii) $\displaystyle\lim_{x\to 0}\frac{e^{3x}-1}{x}$. [B.U. B.Com.(H) 2008]

(iv) $\displaystyle\lim_{x\to 2}\left(\frac{x^5-32}{x-2}\right)$.

Solution: (i) $\displaystyle\lim_{x\to 1}\frac{x^2+4x-5}{x-1}=\lim_{h\to 0}\frac{(1+h)^2+4(1+h)-5}{1+h-1}$ | Put $x=1+h$; as $x\to 1$, $h\to 0$.

$$=\lim_{h\to 0}\frac{h^2+6h}{h}=\lim_{h\to 0}(h+6)$$ (here $h\to 0$ and so we need not consider $h=0$)

$$=\lim_{h\to 0} h+\lim_{h\to 0} 6=0+6=6.$$

[$\because$ when $h \to 0$, we are not interested for the value $h = 0$. So we assume $h \neq 0$. We, therefore, divide both numerator and denominator by h]

(ii) Exactly in the same way, obtain $\lim\limits_{x\to 2} \dfrac{2x^2-7x+6}{3x^2-7x+2} = \dfrac{1}{5}$.

(iii) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x} = \lim\limits_{x\to 0} \dfrac{e^{3x}-1}{3x} \times 3 = 3\lim\limits_{z\to 0} \dfrac{e^z-1}{z}$ (where $z = 3x$) $= 3 \times 1 = 3$.

(iv) $\lim\limits_{x\to 2}\left(\dfrac{x^5-32}{x-2}\right) = \lim\limits_{x\to 2} \dfrac{x^5-2^5}{x-2}$, which is of the form $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a}$

$$= 5\cdot(2)^{5-1} = 5 \times 2^4 = \mathbf{80}.$$

Example 15. *Evaluate*:

(i) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$;

(ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{a+x}-\sqrt{a}}{2x}$; [C.U. B.Com.(H) 2001]

(iii) $\lim\limits_{x\to 3} \dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$. [C.U. B.Com. 1994]

Solution:

(i) $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim_{x\to 0} \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$ (See the explanations given in Ex. 1, above)

$$= \frac{2}{\lim\limits_{x\to 0}\sqrt{1+x}+\lim\limits_{x\to 0}\sqrt{1-x}} = \frac{2}{\sqrt{\lim\limits_{x\to 0}(1+x)}+\sqrt{\lim\limits_{x\to 0}(1-x)}} = \frac{2}{1+1} = \mathbf{1}.$$

(ii) $$\lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{2x} = \lim_{x\to 0} \frac{a+x-a}{2x\left(\sqrt{a+x}+\sqrt{a}\right)} = \lim_{x\to 0} \frac{1}{2\left(\sqrt{a+x}+\sqrt{a}\right)} = \frac{\mathbf{1}}{\mathbf{4\sqrt{a}}}.$$

(iii) $$\lim_{x\to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} = \lim_{x\to 3} \frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}\right)^2-\left(\sqrt{4-x}\right)^2} = \lim_{x\to 3} \frac{(x-3)\left[\sqrt{x-2}+\sqrt{4-x}\right]}{(x-2-4+x)}$$

$$= \lim_{x\to 3} \frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{2(x-3)}$$

$$= \lim_{x\to 3} \frac{\sqrt{x-2}+\sqrt{4-x}}{2}$$ [When $x \to 3$, we may consider values of x near $x = 3$ but not necessarily at $x = 3$. Here we take $x \neq 3$.]

$$= \lim_{h\to 0} \frac{\left(\sqrt{3+h-2}+\sqrt{4-3-h}\right)}{2}, \text{ where } x = 3+h$$

$$= \lim_{h\to 0} \frac{\sqrt{h+1}+\sqrt{1-h}}{2} = \frac{2}{2} = \mathbf{1}.$$

Example 16. *Evaluate*:

(i) $\lim\limits_{x\to\infty}\dfrac{5x^{12}+7x^9+12}{14x^{16}+9x^2-3}$; (iii) $\lim\limits_{x\to\infty}\dfrac{4x^2-3x+2}{5x^4+2x^2+3}$. [C.U. B.Com. 1994]

(ii) $\lim\limits_{x\to\infty}\dfrac{15x^7+12x+17}{5x^7+9x^2+12}$; [C.U. B.Com.(H) 2000] (iv) $\lim\limits_{x\to\infty}\dfrac{3x^3+2x-1}{4x^3+3x-2}$. [C.U. B.Com.(H) 2011]

Solution: (i) See that the highest power of x occurred in the denominator is x^{16}. Divide both numerator and denominator by x^{16}.

$$\lim_{x\to\infty}\frac{5x^{12}+7x^9+12}{14x^{16}+9x^2-3}=\lim_{x\to\infty}\frac{5/x^4+7/x^7+12/x^{16}}{14+9/x^{14}-3/x^{16}}$$

$$=\lim_{y\to0}\frac{5y^4+7y^7+12y^{16}}{14+9y^{14}-3y^{16}}$$ [Use this idea. When it is given $x\to\infty$, put $x=1/y$ and make $y\to0$]

$$=\frac{\lim\limits_{y\to0}\left(5y^4+7y^7+12y^{16}\right)}{\lim\limits_{y\to0}\left(14+9y^{14}-3y^{16}\right)}$$ [See that the limit in the denominator $\neq 0$. So we apply Theorem 1(v)]

$$=\frac{5\lim\limits_{y\to0}y^4+7\lim\limits_{y\to0}y^7+12\lim\limits_{y\to0}y^{16}}{14+9\lim\limits_{y\to0}y^{14}-3\lim\limits_{y\to0}y^{16}}=\frac{0+0+0}{14+0-0}=\mathbf{0}.$$

(ii) Given limit $=\lim\limits_{x\to\infty}\dfrac{15+\frac{12}{x^6}+\frac{17}{x^7}}{5+\frac{9}{x^5}+\frac{12}{x^7}}$ [Dividing both numerator and denominator by x^7]

$$=\lim_{y\to0}\frac{15+12y^6+17y^7}{5+9y^5+12y^7}$$ [Putting $1/x=y$, so that when $x\to\infty$, $y\to0$]

$$=\frac{\lim\limits_{y\to0}\left(15+12y^6+17y^7\right)}{\lim\limits_{y\to0}\left(5+9y^5+12y^7\right)}=\frac{15+\lim\limits_{y\to0}12y^6+\lim\limits_{y\to0}17y^7}{5+\lim\limits_{y\to0}9y^5+\lim\limits_{y\to0}12y^7}=\frac{15}{5}=\mathbf{3}.$$

(iii) Required to find

$$\lim_{x\to\infty}\frac{4x^2-3x+2}{5x^4+2x^2+3}=\lim_{x\to0}\frac{\frac{4}{x^2}-\frac{3}{x^3}+\frac{2}{x^4}}{5+\frac{2}{x^2}+\frac{3}{x^4}}$$ [Dividing both numerator and denominator by x^4; now put $\frac{1}{x}=y$]

$$=\lim_{y\to0}\frac{4y^2-3y^3+2y^4}{5+2y^2+3y^4}=\frac{0+0+0}{5+0+0}=\frac{0}{5}=\mathbf{0}.$$

(iv) Try in a similar way: $\lim\limits_{x\to\infty}\dfrac{3x^3+2x-1}{4x^3+3x^2-2}=\dfrac{\mathbf{3}}{\mathbf{4}}$. [C.U. B.Com.(H) 1995]

Example 17. *Evaluate*: $\lim\limits_{x\to0}\dfrac{(1+x)^2-1}{x}$. [V.U. B.Com.(H) 2010]

Solution: $$\lim_{x\to0}\frac{(1+x)^2-1}{x}=\lim_{x\to0}\frac{2x+x^2}{x}=\lim_{x\to0}(2+x)=\lim_{x\to0}2+\lim_{x\to0}x=2+0=\mathbf{2}.$$

Example 18. *Evaluate*: $\lim\limits_{x\to\infty}\dfrac{5-2x^2}{3x+5x^2}$.

Solution:

$$\lim_{x\to\infty}\frac{5-2x^2}{3x+5x^2}=\lim_{x\to\infty}\frac{5/x^2-2}{3/x+5}=\lim_{y\to 0}\frac{5y^2-2}{3y+5} \text{ [See Ex. 16 above]}$$

$$=\frac{\lim\limits_{y\to 0}\left(5y^2-2\right)}{\lim\limits_{y\to 0}(3y+5)}=\frac{5\lim\limits_{y\to 0}y^2-2}{3\lim\limits_{y\to 0}y+5}=-\mathbf{\frac{2}{5}}.$$

Example 19. *Evaluate*: $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ $(x>0)$. [N.B.U. B.Com.(H) 2007]

Solution:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}(x>0)=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \text{ [cf. Ex. 15]}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \text{ [Since } h \text{ approaches to zero, but not equal to zero]}$$

$$=\frac{1}{\lim\limits_{h\to 0}\sqrt{x+h}+\lim\limits_{h\to 0}\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\mathbf{\frac{1}{2\sqrt{x}}}.$$

Example 20. *Evaluate*:

(i) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+a}-\sqrt{a-x^2}}{x^2}$; (ii) $\lim\limits_{x\to\infty}\dfrac{1-\sqrt{x}}{1+\sqrt{x}}$.

[C.U. B.Com.(H) 1998; V.U. B.Com.(H) 2010]

Solution:

$$\text{(i) }\lim_{x\to 0}\frac{\sqrt{x^2+a}-\sqrt{a-x^2}}{x^2}=\lim_{x\to 0}\frac{\left(\sqrt{x^2+a}-\sqrt{a-x^2}\right)\left(\sqrt{x^2+a}+\sqrt{a-x^2}\right)}{x^2\left(\sqrt{x^2+a}+\sqrt{a-x^2}\right)}$$

$$=\lim_{x\to 0}\frac{(x^2+a)-(a-x^2)}{x^2\left(\sqrt{x^2+a}+\sqrt{a-x^2}\right)}=\lim_{x\to 0}\frac{2x^2}{x^2\left(\sqrt{x^2+a}+\sqrt{a-x^2}\right)}$$

$$=\lim_{x\to 0}\frac{2}{\sqrt{x^2+a}+\sqrt{a-x^2}}=\frac{2}{\sqrt{0+a}+\sqrt{a-0}}=\frac{2}{2\sqrt{a}}=\mathbf{\frac{1}{\sqrt{a}}}.$$

$$\text{(ii) }\lim_{x\to\infty}\frac{1-\sqrt{x}}{1+\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}-1}{\frac{1}{\sqrt{x}}+1}=\lim_{h\to 0^+}\frac{h-1}{h+1}$$

$$=\frac{0-1}{0+1}=\mathbf{-1}. \text{ [Put } x=\tfrac{1}{h^2}\text{; then } h=\tfrac{1}{\sqrt{x}}\text{. As } x\to\infty,\ h\to 0^+\text{]}$$

Example 21. *Evaluate:*

(i) $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}+2x}{x}$; [C.U. B.Com.(H) 2001]

(ii) $\lim\limits_{x\to 0}\dfrac{10^x-5^x-2^x+1}{x^2}$;

(iii) $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{e^x-1}$;

(iv) $\lim\limits_{x\to 5}\dfrac{x^3-125}{x^4-625}$; [C.U. B.Com.(H) 1997; V.U. B.Com.(H) 2007]

(v) $\lim\limits_{x\to 0}\dfrac{14^x-7^x-2^x+1}{x^2}$. [C.U. B.Com.(H) 2002]

Solution:

(i) $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}+2x}{x}=\lim\limits_{x\to 0}\dfrac{(e^{3x}-1)-(e^{2x}-1)+2x}{x}$

$=\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x}-\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}+\lim\limits_{x\to 0}2$ [Put $3x=y$ and $2x=z$; then as $x\to 0$, $y\to 0$ and $z\to 0$ and $x=y/3$ and $x=z/2$]

$=\lim\limits_{y\to 0}\dfrac{e^y-1}{y/3}-\lim\limits_{z\to 0}\dfrac{e^z-1}{z/2}+2=3\lim\limits_{y\to 0}\dfrac{e^y-1}{y}-2\lim\limits_{z\to 0}\dfrac{e^z-1}{z}+2$

$=3\times 1-2\times 1+2=\mathbf{3}.$

(ii) $\lim\limits_{x\to 0}\dfrac{10^x-5^x-2^x+1}{x^2}=\lim\limits_{x\to 0}\dfrac{5^x\cdot 2^x-5^x-2^x+1}{x^2}=\lim\limits_{x\to 0}\dfrac{5^x(2^x-1)-1(2^x-1)}{x^2}=\lim\limits_{x\to 0}\dfrac{(2^x-1)(5^x-1)}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{2^x-1}{x}\times\lim\limits_{x\to 0}\dfrac{5^x-1}{x}=\mathbf{\log_e 2\cdot\log_e 5}\ \left(\because\ \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a\right)$

(iii) $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{e^x-1}=\lim\limits_{x\to 0}\dfrac{\frac{1}{x}\log(1+x)}{\frac{e^x-1}{x}}=\dfrac{\lim_{x\to 0}\frac{1}{x}\log(1+x)}{\lim_{x\to 0}\frac{e^x-1}{x}}=\dfrac{1}{1}=\mathbf{1}.$

(iv) $\lim\limits_{x\to 5}\dfrac{x^3-125}{x^4-625}=\lim\limits_{x\to 5}\dfrac{\frac{x^3-125}{x-5}}{\frac{x^4-625}{x-5}}=\dfrac{\lim_{x\to 5}\frac{x^3-5^3}{x-5}}{\lim_{x\to 5}\frac{x^4-5^4}{x-5}}=\dfrac{3\cdot 5^{3-1}}{4\cdot 5^{4-1}}=\dfrac{3\cdot 5^2}{4\cdot 5^3}=\dfrac{3}{4}\times\dfrac{1}{5}=\mathbf{\dfrac{3}{20}}.$

(v) Similar to (ii).

$$\lim_{x\to 0}\frac{14^x-7^x-2^x+1}{x^2}=\lim_{x\to 0}\frac{(2^x-1)(7^x-1)}{x^2}=\lim_{x\to 0}\frac{2^x-1}{x}\times\lim_{x\to 0}\frac{7^x-1}{x}=\mathbf{\log_e 2\cdot\log_e 7.}$$

Example 22. *If* $\lim\limits_{x\to 2}\dfrac{ax^2-b}{x-2}=4$, *find the values of* a *and* b. [C.U. B.Com. 2009]

Solution: Given,
$$\lim_{x\to 2}\frac{ax^2-b}{x-2}=\mathbf{4}. \tag{1}$$

Since the limit is 4 which is definite and finite, (ax^2-b) must be divisible by $(x-2)$, i.e., remainder $=0$, i.e., $4a-b=0$. [See division]

$$\therefore\ b=4a \tag{2}$$

From (1), we get

$$\lim_{x\to 2}\left(\frac{ax^2-4a}{x-2}\right)=4 \text{ or, } \lim_{x\to 2}\frac{a\,(x^2-4)}{x-2}=4$$

$$\text{or, } \lim_{x\to 2}\frac{a\cdot(x+2)(x-2)}{x-2}=4 \text{ or, } \lim_{x\to 2}(x+2)=4$$

or, $a(2+2)=4$ or, $4a=4$ or, $a=1$.

$$\begin{array}{r|l} x-2) & ax^2-b(ax+2a \\ & \underline{ax^2-2ax} \\ & 2ax-b \\ & \underline{2ax-4a} \\ & 4a-b \end{array}$$

From (2), $b=4a=4\times 1=4$.

Hence $\boldsymbol{a=1}$ **and** $\boldsymbol{b=4}$.

2.8 Concept of Continuity of a Function

[B.U. B.Com.(H) 2008]

We know that the word continuity is used to indicate the absence of any gap. See Fig. 2.3; there is a break of the curve at the point P (corresponding to $x=a$). We say that $f(x)$ is not continuous (or discontinuous) at $x=a$. See Fig. 2.4; there is no break anywhere in these curves. We say that the functions representing these curves are everywhere continuous.

Perhaps, we may say that a function is continuous, if small changes (however, small it may be) in the values of the independent variable x cause small changes in the values of the dependent variable y and the function is discontinuous, if small changes in x can make a sudden change (or jump) in the value of y. The concept of continuity can be best described in the language of limit:

Definition 1. *Suppose, $f(x)$ is defined in a certain domain and let c be a point in this domain, where $f(c)$ is defined.*

- If $\lim_{x\to c} f(x)$ exists and equals $f(c)$, then $f(x)$ is said to be continuous at $x=c$.
  In other words, if $\lim_{x\to c^+} f(x)=f(c+0)$, $\lim_{x\to c^-} f(x)=f(c-0)$ and $f(c)$ all exist and are equal to one another, then $f(x)$ is continuous at $x=c$.
  This leads to a more rigorous definition:
- $f(x)$ is continuous at $x=c$, if for any given $\epsilon>0$, we can find $\delta>0$ such that for all x in $|x-c|<\delta$ (including $x=c$), the result $|f(x)-f(c)|<\epsilon$ holds.
- If $f(x)$ is continuous at every point c of an open interval $a<x<b$ in the sense given above, then $f(x)$ is said to be continuous over that interval (a,b).
  If, further, at $x=a$, $\lim_{x\to a^+} f(x)=f(a+0)=f(a)$ and at $x=b$, $\lim_{x\to b^-} f(x)=f(b-0)=f(b)$, then $f(x)$ is said to be continuous in the closed interval $[a,b]$.

Observations. If at a point α of the domain, $f(x)$ fails to be continuous there, it is said to be discontinuous at $x=\alpha$; the point α is the point of discontinuity.

Graphically, for a continuous function in a certain interval there are no breaks.

Note that for continuous functions, limits can be at once found by just substitution process.

Theorem 1. *The sum, difference, product of a finite number of functions which are continuous at a point (or continuous in an interval) are continuous at that point (or continuous in that interval). The same statement is true for the quotient of two functions, provided that the denominator function does not vanish at that point (or at any point of the interval).*

Polynomials. $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_nx^n$.

Here, $\lim_{x\to c} a_0 = a_0$

$\lim_{x\to c} a_1 x = a_1 \lim_{x\to c} x = a_1 c,$

$\lim_{x\to c} a_2 x^2 = a_2 \lim_{x\to c} x \times \lim_{x\to c} x = a_2 c^2$ and so on;

$$\lim_{x\to c} a_n x^n = a_n c^n.$$

Now using extensions of Theorem 1(i) of Section 2.6, we can easily see that

$$\lim_{x\to c} f(x) = a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n = f(c),$$

i.e., $f(x)$ is continuous at $x = c$.

Conclusion: Polynomials are always continuous.

Rational functions: $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are two polynomials. Every rational function is also always continuous except at those points where the denominator function vanishes; the function is not defined there, e.g., $\frac{x^3+x^2-4x-4}{x^2-4}$ is everywhere continuous except at $x = 2$ and at $x = -2$.

2.9 Illustrative Examples

Example 23. *Prove that $f(x) = 3x + 2$ is continuous at $x = 4$.*

Solution: Let us draw the graph of $y = 3x+2$ or, $y/2 + x/(-2/3) = 1$. Clearly, the line has no break anywhere.

However, at $x = 4$, $y = 14$; the corresponding point on the graph is $P(4, 14)$. As $x \to 4$ from the right the corresponding points P_1, etc., approach P. Also as $x \to 4$ from the left the corresponding points Q_1, etc., also approach P. [See Fig. 2.10.]

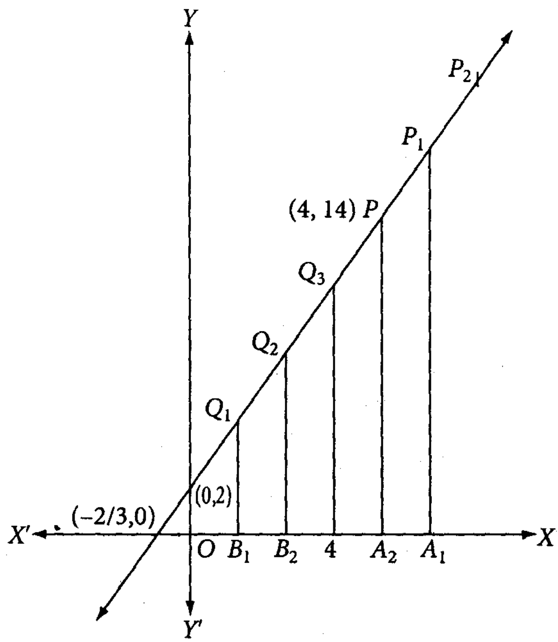


Fig. 2.10

$\therefore$ the curve has no break at P or **$f(x)$ is continuous at $x = 4$.**

Another Approach

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (3x+2) = \lim_{h\to 0^+} (3(4+h)+2) = \lim_{h\to 0^+} (14+3h) = 14.$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (3x+2) = \lim_{h\to 0^+} (3(4-h)+2) = \lim_{h\to 0^+} (14-3h) = 14.$$

Moreover, $f(4) = 3\cdot 4+2 = 14$, i.e., $f(4+0) = f(4-0) = f(4) = 14$ and hence **$f(x)$ is continuous at $x = 4$.**

Example 24.

$$f(x) = \begin{cases} -x, \ when\ x \le 0 \\ x, \ when\ 0 < x < 1 \\ 2-x, \ when\ x \ge 1. \end{cases}$$

Discuss the continuity of $f(x)$ at $x = 0$, $x = 1$. [V.U. B.Com.(H) 2008]

Solution: At $x = 0$,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0+0).$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0 = f(0-0) \text{ and } f(0) = 0.$$

$\therefore$ **$f(x)$ is continuous at $x = 0$.**

At $x = 1$,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = \lim_{h\to 0^+} \{2-(1+h)\} = \lim_{h\to 0^+} (1-h) = 1 = f(1+0).$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = \lim_{h\to 0^+} (1-h) = 1 = f(1-0)$$

and $f(1) = 2-1 = 1$.

$\therefore$ **$f(x)$ is also continuous at $x = 1$.**

Note: Draw the graph of the function and see that there are no breaks at these points.

Example 25. (***The concept of continuous or discontinuous function is important in many business cost and pricing problems.***)

A firm may offer discounts on large order of quantities in order to attract large orders. Suppose that the total cost of purchasing x gallons is ₹C and is proportional to x within each order of interval, as given by

$$C = \begin{cases} 20x, \ 0 \le x \le 1000 \\ 18x, \ 1000 < x \le 2000 \\ 16x, \ x > 2000 \end{cases}$$

The cost function does not have the same limiting value as $x \to 1000$ from right and from left.

$$\lim_{x\to 1000^-} C = \lim_{x\to 1000^-} 20x = \lim_{h\to 0^+} 20(1000-h) = 20000.$$

$$\lim_{x\to 1000^+} C = \lim_{x\to 1000^+} 18x = \lim_{h\to 0^+} 18(1000+h) = 18000,$$

i.e., there is a break or a jump at $x = 1000$. [See Fig. 2.11.]

Similarly, there is a break at $x = 2000$.

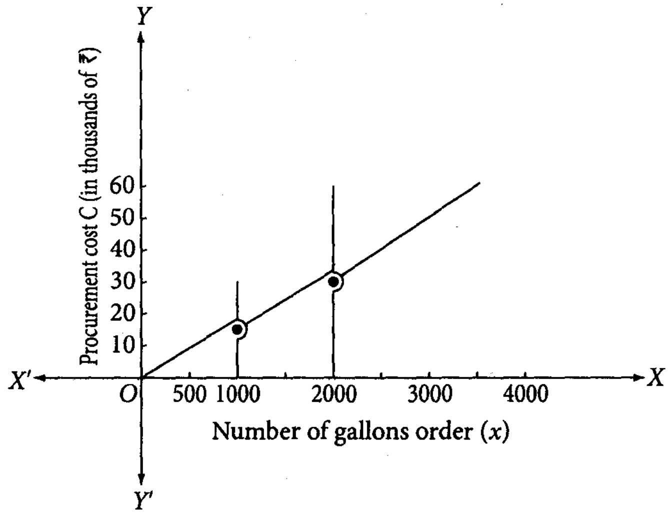


Fig. 2.11

Example 26. *Examine the continuity of the function*

$$f(x)=\begin{cases} 2-3x, & when\ x>0 \\ 2, & when\ x=0 \\ 2+3x, & when\ x<0 \end{cases}$$

at $x=0$. [C.U. B.Com.(H) 2001; V.U. B.Com. 2010]

Solution: At $x=0$, $f(x)=2$; $\therefore$ $f(0)=2$.

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(2-3x)=2-3\times 0=2$$

and $$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(2+3x)=2+3\times 0=2.$$

Thus, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0)=2$.

Hence, **$f(x)$ is continuous at $x=2$.**

Example 27. *A function $f(x)$ is defined as follows:*

$$f(x)=\begin{cases} x+1, & for\ x\le 1 \\ 3-ax^2, & for\ x>1. \end{cases}$$

If $f(x)$ is continuous at $x=1$, find the value of a. [V.U. B.Com.(H) 2011]

Solution: Since $f(x)$ is continuous at $x=1$,

we have $\lim_{x\to 1+} f(x)=\lim_{x\to 1-} f(x)=f(a)$. (1)

Now, $$\lim_{x\to 1+} f(x)=\lim_{x\to 1+}\left(3-ax^2\right)=3-a\cdot 1^2=3-a$$

$$\lim_{x\to 1-} f(x)=\lim_{x\to 1-}(x+1)=1+1=2 \text{ and } f(1)=1+1=2.$$

$\therefore$ from (1), $3-a=2$ or, $a=3-2=1$. Hence **$a=1$.**

Example 28. *Discuss the continuity of $f(x)$ at $x=-2$, where*

$$f(x)=\begin{cases} x+\dfrac{x+2}{|x+2|}, & \text{for } x\neq -2 \\ -1, & \text{for } x=-2. \end{cases}$$

[C.U. B.Com.(H) 2000; B.U. B.Com.(H) 2007]

Solution: At $x=-2$, $f(x)=-1$; $\therefore$ $f(-2)=-1$.

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+}\left(x+\frac{x+2}{|x+2|}\right) \qquad \left[|x+2|=\begin{cases} x+2, \text{ if } x>-2 \\ -(x+2), \text{ if } x<-2 \end{cases}\right]$$

$$=\lim_{x\to -2^+}\left(x+\frac{x+2}{x+2}\right)=\lim_{x\to -2^+}(x+1)=-2+1=-1$$

and

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}\left(x+\frac{x+2}{|x+2|}\right)=\lim_{x\to -2^-}\left[x+\frac{x+2}{-(x+2)}\right]$$

$$=\lim_{x\to -2^-}(x-1)=-2-1=-3.$$

Thus,

$$\lim_{x\to -2^+} f(x)\neq \lim_{x\to -2^-} f(x).$$

Hence, $\boldsymbol{f(x)}$ **is not continuous at $\boldsymbol{x=-2}$.**

Example 29. *Given $f(x)=\dfrac{x^2-4}{x-2}$, when $x\neq 2$; find the value of $f(2)$ so that $f(x)$ is continuous at $x=2$.*

[C.U. B.Com.(H) 2005; B.U. B.Com.(H) 2005]

Solution: For $f(x)$ to be continuous at $x=2$, we must have

$$\lim_{x\to 2} f(x)=f(2).$$

Now

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-4}{x-2} \quad \Bigg| \; \text{Put } x=2+h; \text{ as } x\to 2,\ h\to 0$$

$$=\lim_{h\to 0}\frac{(2+h)^2-4}{2+h-2}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$$

$$=\lim_{h\to 0}\frac{h(4+h)}{h}=\lim_{h\to 0}(4+h) \quad [\because h\neq 0]$$

$$=4+0=4.$$

Hence, required value of $\boldsymbol{f(2)}$ **is 4.**

EXERCISES ON CHAPTER 2

(Limit and Continuity)

A (Limit)

1. (a) Explain the statements: $x \to 5$, $x \to +\infty$, $x \to -\infty$.

   (b) Define the limit of a function $f(x)$ as $x \to$ a finite quantity and also as $x \to \infty$.

   (c) Explain clearly the distinction between $\lim\limits_{x \to a} f(x)$ and $f(a)$.

   (d) Give ϵ-δ definition of $\lim\limits_{x \to a} f(x) = l$. Fit in this definition to establish

$$\lim_{x \to 1}(3x+15) = 18.$$

2. Obtain R.H. limit and L.H. limit (both from graphical considerations and also from analytic considerations) and then discuss the existence of limits of the following functions as $x \to a$.

$f(x)$	a	
(a) $5x+3$	$a=0$	
(b) x^2	$a=1$	
(c) $\dfrac{x^2-16}{x-4}$	$a=4$	
(d) $\dfrac{\lvert x\rvert}{x}$	$a=0$	$[x]$ = greatest integer in x
(e) $[x]$	$a=1,2$	but not greater than x.

3. $f(x) = \begin{cases} x, & \text{when } x \leq 1 \\ 2-x, & \text{when } x > 1. \end{cases}$

   Discuss the existence of $\lim\limits_{x \to 1} f(x)$.

4. (a) $f(x) = \begin{cases} x^2, & \text{when } x > 1 \\ 2, & \text{when } x = 1 \\ x, & \text{when } x < 1. \end{cases}$

   Does $\lim\limits_{x \to 1} f(x)$ exist?

   (b) $f(x) = \begin{cases} 3x-2, & \text{when } x \geq 0 \\ 2-3x, & \text{when } x < 0. \end{cases}$

   Does $\lim\limits_{x \to 5} f(x)$ exist?

   [C.U. B.Com. 2009]

5. $f(x) = \begin{cases} 5, & \text{when } x \text{ is any integer} \\ 0, & \text{when } x \text{ is not an integer.} \end{cases}$

   Does $\lim\limits_{x \to 5} f(x)$ exist?

6. Using definitions (graphical and analytical) establish:

(a) $\lim_{x\to 2} 3x = 6$;

(b) $\lim_{x\to 1} \sqrt{x} = 1$;

(c) $\lim_{x\to\infty} \frac{1}{x^2} = 0$;

(d) $\lim_{x\to\infty} \left(\frac{1}{x^2}+2\right) = 2$;

(e) $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$;

(f) $\lim_{x\to 0} \frac{1}{x^4} = \infty$;

(g) $\lim_{x\to\infty} x^2 = \infty$;

(h) $\lim_{x\to 0} \frac{1}{x^8} = \infty$;

(i) $\lim_{x\to 0} \frac{1}{x^3}$ does not exist.

7. Use ϵ-δ definition, to establish $\lim_{x\to 1}(2x+7) = 9$. [C.U. B.Com. 2004]

[Hints: $|(2x+7)-9| < \epsilon$, when $|2x-2| < \epsilon$, i.e., when $2|x-1| < \epsilon$, i.e., when $|x-1| < \frac{\epsilon}{2}$. Taking $\delta = \frac{\epsilon}{2}$, we see that $|(2x+7)-9| < \epsilon$, when $|x-1| < \delta$. Hence by ϵ-δ definition, $\lim_{x\to 1}(2x+7) = 9$.]

8. (a) Does $\lim_{x\to 2}[x]$ exist, where $[x]$ denotes greatest integer not exceeding the values of x. [B.U. B.Com.(H) 2005]

[Hints: $\lim_{x\to 2+}[x] = 2$, since x is greater than 2, and $\lim_{x\to 2-}[x] = 1$, since x is less than 2, x always remaining close to 2. Thus $\lim_{x\to 2+}[x] \neq \lim_{x\to 2^-}[x]$. Hence, the limit does not exist.]

(b) Evaluate $\lim_{x\to 2+}[x+2]$ and hence show that $\lim_{x\to 2}[x+2]$ does not exist.

[B.U. B.Com.(H) 2008]

B

(Limit)

1. Evaluate the following limits (using limit theorems, where necessary):

[*Rules to be used*: (a) When $x \to a$, where $a \neq 0$, write $a+h$ for x and make $h \to 0$.

(b) When $x \to \infty$, write $1/y$ for x and make $y \to 0$]

(a) $\lim_{x\to 1} \frac{x^2-x+3}{x+4}$.

(b) $\lim_{x\to 2} \frac{x^2-3x+7}{4x^3+x+1}$;

(c) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$;

(d) $\lim_{x\to 1} \frac{4x^5+9x+7}{3x^6+x^3+1}$;

(e) $\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2}$;

(f) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2}$; [V.U. B.Com.(H) 2009]

(g) $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-3x+2}$;

(h) $\lim_{x\to -1} \frac{2x^2-x-3}{x^2-2x-3}$; [C.U. B.Com.(H) 1990]

(i) $\lim_{x\to 1} \frac{x^2-3x+2}{x^3-4x+3}$; [C.U. B.Com.(H) 1991]

(j) $\lim_{x\to 0} \frac{(1+x)^2-(1-x)^2}{2x}$;

(k) $\lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{x+h}-\frac{1}{x}\right\} (x>0)$;

(l) $\lim_{x\to 0} \frac{\frac{1}{x-2}-\frac{1}{x}}{\frac{1}{x-3}+\frac{1}{x}}$;

(m) $\lim_{x\to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x}$;

(n) $\lim_{x\to 2} \frac{\sqrt{x+1}-\sqrt{3}}{x-2}$;

(o) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$; [C.U. B.Com.(H) 1996]

(p) $\lim_{x\to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$; [C.U. B.Com.(H) 1994]

(q) $\lim_{x\to 0} \frac{x^3}{3-\sqrt{9-x^2}}$;

(r) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1-x^2}-\sqrt{1-x}}$.

(Here rationalise both Numr. and Denr.)

(s) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$;

(t) $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$;

(u) $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$; [C.U. B.Com.(H) 1994]

(v) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+3x}-\sqrt{2-5x}}{4x}$;

(w) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{x^2}$;

(x) $\lim\limits_{x\to\infty}\dfrac{x^2+3x+2}{x^3+x-4}$;

(y) $\lim\limits_{x\to\infty}\dfrac{2x^3+5x+9}{x^3+7x-3}$;

(z) $\lim\limits_{x\to\infty}\dfrac{x^2+3x+2}{x^3+x-4}$;

(aa) $\lim\limits_{x\to\infty}\dfrac{1-\sqrt{x}}{1+\sqrt{x}}$;

(ab) $\lim\limits_{x\to\infty}\dfrac{2x^3+5x+9}{x^2+7x-3}$;

(ac) $\lim\limits_{x\to 25}\dfrac{(\sqrt{x}-5)(x+1)}{x^2-24x-25}$; [C.U. B.Com.(H) 1996]

(ad) $\lim\limits_{x\to\infty}\dfrac{3x^2-4x+6}{x^2+6x-7}$. [C.U. B.Com.(H) 2001]

[**Hints**: Rules to be used:

(a) When $x\to a$, where $a\neq 0$, write $a+h$ for x and make $h\to 0$.

(b) When $x\to\infty$, write $1/y$ for x and make $y\to 0$.

$$(\text{m})=\lim_{x\to 0}\frac{\{\sqrt{a+x}-\sqrt{a}\}\{\sqrt{a+x}+\sqrt{a}\}}{x\{\sqrt{a+x}+\sqrt{a}\}}=\lim_{x\to 0}\frac{(a+x)-a}{x\{\sqrt{a+x}+\sqrt{a}\}}=\lim_{x\to 0}\frac{x}{x\{\sqrt{a+x}+\sqrt{a}\}}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{a+x}+\sqrt{a}}\ \text{(We assume } x\neq 0 \text{ as we are interested only for } x\to 0)=\frac{1}{2\sqrt{a}}.$$

$$(\text{r})=\lim_{x\to 0}\frac{\{\sqrt{1+x}-\sqrt{1+x^2}\}\{\sqrt{1+x}+\sqrt{1+x^2}\}\{\sqrt{1-x^2}+\sqrt{1-x}\}}{\{\sqrt{1-x^2}-\sqrt{1-x}\}\{\sqrt{1-x^2}+\sqrt{1-x}\}\{\sqrt{1+x}+\sqrt{1+x^2}\}}$$

$$=\lim_{x\to 0}\frac{(x-x^2)\left(\sqrt{1-x^2}+\sqrt{1-x}\right)}{(x-x^2)\left(\sqrt{1+x}+\sqrt{1+x^2}\right)}$$

[We can cancel $x-x^2$ as we may take x near zero but not $x=0$]

$$=\tfrac{2}{2}=1.$$

$$(\text{x})\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{3}{x^2}+\frac{2}{x^3}}{1+\frac{1}{x^2}-\frac{4}{x^3}}=\lim_{y\to 0}\frac{y+3y^2+3y^3}{1+y^2-4y^3}$$ [Dividing numr. and denr. by x^3 and then putting $x=\frac{1}{y}$. Since $x\to\infty$, $\therefore\ y\to 0$]

$$=\tfrac{0+0+0}{1+0-0}=0.]$$

2. Evaluate:

(a) $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x^2+2x-15}$; [C.U. B.Com.(H) 1998]

(b) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+4x}-\sqrt{1-3x}}{x}$; [C.U. B.Com.(H) 1999; V.U. B.Com.(H) 2008]

(c) $\lim\limits_{x\to -4}\left[\dfrac{1}{x+4}+\dfrac{8}{x^2-16}\right]$; [C.U. B.Com.(H) 1999]

(d) $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$ where $f(x)=2x^2-7x+1$. [C.U. B.Com.(H) 2003]

3. Evaluate each of the following limits:

(a) $\lim_{x\to\infty} \frac{3x^4-2x^2+1}{x^4-2x^2+3}$; [C.U. B.Com.(H) 1996]

(b) $\lim_{x\to\infty} \frac{5x^2-3x+7}{3x^2+x+4}$; [C.U. B.Com.(H) 1999]

(c) $\lim_{x\to\infty} \left[\frac{3x^3+2x-1}{4x^3+3x^2-2}\right]$; [C.U. B.Com.(H) 1995]

(d) $\lim_{x\to\infty} \frac{4x^2-3x+2}{5x^4+2x^2+3}$. [C.U. B.Com.(H) 1994]

4. Evaluate:

(a) $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$; [C.U. B.Com.(H) 1995]

(b) $\lim_{h\to\infty} \frac{f(x+h)-f(x)}{h}$, where $f(x)=\frac{1}{\sqrt{x}}(x>0)$. [C.U. B.Com.(H) 2000]

(c) $\lim_{x\to a} \frac{\sqrt{2x+a}-\sqrt{x+2a}}{x-a}$. [C.U. B.Com.(H) 2005]

(d) $\lim_{x\to 0} \frac{x^2}{a-\sqrt{a^2-x^2}}$.

5. Evaluate:

(a) $\lim_{x\to 0} \frac{e^{2x}+e^x-2}{x}$;

(b) $\lim_{x\to 0} \frac{6^x-3^x-2^x+1}{x}$;

(c) $\lim_{x\to 0} \frac{5x+|x|}{2x+|x|}$, if the limit exists;

(d) $\lim_{x\to 3} \frac{x^3-27}{x^4-81}$;

(e) $\lim_{x\to 0} \frac{e^x-1}{\log(1+x)}$;

(f) $\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x^2-9}$. [C.U. B.Com. 2006]

6. Verify:

(a) $\lim_{x\to\infty} \sqrt{x}\left\{\sqrt{x+a}-\sqrt{x}\right\}=\frac{a}{2}$;

(b) $\lim_{x\to\infty} \left(\sqrt[3]{x+1}-\sqrt[3]{x}\right)=0$;

(c) $\lim_{x\to 0} \frac{a_0x^m+a_1x^{m-1}+a_2x^{m-2}+\cdots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_{n-1}x+b_n}=\frac{a_m}{b_n}$ $(b_n\neq 0)$.
(m and n are positive integers).

7. (a) If $\lim_{x\to 3} \frac{px^2-q}{x-3}=6$, find the values of p and q.

(b) If $\lim_{x\to 5} \frac{px^2-q}{x-5}=20$, find the values of p and q.

8. Evaluate: $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$. [C.U. B.Com.(H) 1997]

9. Evaluate: $\lim_{x\to\infty} \left[x-\sqrt{(x-a)(x-b)}\right]$. [C.U. B.Com. 2004]

[Hints: Given limit $= \lim_{x\to\infty} \frac{\left\{x-\sqrt{(x-a)(x-b)}\right\}\left\{x+\sqrt{(x-a)(x-b)}\right\}}{x+\sqrt{(x-a)(x-b)}}$

$= \lim_{x\to\infty} \frac{x^2-(x-a)(x-b)}{x+\sqrt{(x-a)(x-b)}} = \lim_{x\to\infty} \frac{(a+b)x-ab}{x+x\sqrt{\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)}}$

$$= \lim_{x \to \infty} \frac{(a+b) - \frac{ab}{x}}{1+\sqrt{\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)}} = \frac{(a+b)-0}{1+\sqrt{(1-0)(1-0)}} = \frac{a+b}{2}.] \qquad [\because x \to \infty, \tfrac{1}{x} \to 0.]$$

C

(Continuity)

1. (a) Define continuity of $f(x)$ at a point $x = b$.
   (b) What is meant by $f(x)$ is continuous in $2 \le x \le 3$?
   (c) What do you mean by $y = f(x)$ is continuous at $x = 2$? [B.U. B.Com.(H) 2008]

2. Show, by definition, that $f(x) = 7x + 3$ is continuous at $x = 1$.

3. Draw the graphs of the following functions and discuss from each graph the continuity of the function at $x = 0$:

   (a) $f(x) = 3x + 5$;
   (b) $f(x) = |x|$;
   (c) $f(x) = |x|/x$. [C.U. B.Com.(H) 2001; B.U. B.Com.(H) 2007]

4. Discuss the continuity of the following functions at the indicated points: (See the Answers)

   (a) (i) $f(x) = \dfrac{x^2-9}{x-3}$ at $x = 3$; (ii) $f(x) = \frac{1}{x-1}$ at $x = 1$. [N.B.U. B.Com.(H) 2007]

   (b) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 6, & \text{when } x = 3 \end{cases}$ at $x = 3$;

   (c) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 1, & \text{when } x = 3 \end{cases}$ at $x = 3$;

   (d) $f(x) = \begin{cases} x+1, & x \ge 1 \\ 2x+1, & x < 1 \end{cases}$ at $x = 1$;

   (e) $f(x) = \begin{cases} x^2, & x > 2 \\ 4, & x = 2 \\ 2x, & x < 2 \end{cases}$ at $x = 2$;

   (f) $f(x) = \begin{cases} 4x+3, & x \neq 4 \\ 3x+7, & x = 4 \end{cases}$ at $x = 4$;

   (g) $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \le x < 2 \\ \frac{1}{4}x^3, & 2 \le x < 3 \end{cases}$ at $x = 1, x = 2$;

(h) $f(x)=\begin{cases} 0, & x=0 \\ \frac{1}{2}-x, & 0<x<\frac{1}{2} \\ \frac{1}{2}, & x=\frac{1}{2} \\ \frac{3}{2}-x, & \frac{1}{2}<x<1 \\ 1, & x=1 \end{cases}$ at $x=0, \frac{1}{2}, 1$;

(i) $f(x)=\begin{cases} x, & x<1 \\ 1+x, & x>1 \\ \frac{3}{2}, & x=1 \end{cases}$ at $x=\frac{1}{2}, x=1$.

5. (a) Prove that every polynomial function is always continuous.

   (b) Prove that $\dfrac{x^3+3x+7}{x^2-6x+8}$ is not continuous at $x=2, x=4$.

   (c) Prove that $\dfrac{1}{1+1/x}$ is discontinuous at $x=0, x=-1$.

6. A function $f(x)$ is defined as

$$f(x)=\begin{cases} |x-3|, & \text{if } x\neq 3 \\ 1, & \text{if } x=3. \end{cases}$$

   Discuss the continuity of $f(x)$ at $x=3$.

D

(Continuity)

1. Indicate the points of discontinuity of the function

$$\frac{2x^2+6x-5}{12x^2+x-20}.$$ [N.B.U. B.Com.(H) 2006]

2. Examine the continuity of the function defined by

$$f(x)=\begin{cases} x-1, & \text{when } x>0 \\ 1/2, & \text{when } x=0 \\ x+1, & \text{when } x<0. \end{cases} \quad \text{at } x=0.$$ [C.U. B.Com.(H) 1996]

3. Sketch the graph of the function:

$$f(x)=\begin{cases} 3+2x, & \text{when } x\leq 0 \\ 3-2x, & \text{when } x>0. \end{cases}$$

   From the graph examine the continuity of $f(x)$ at $x=0$. [C.U. B.Com.(H) 1991; B.U. B.Com.(H) 2008]

4. Examine continuity of the function

$$g(x)=\begin{cases} 2-3x, & \text{when } x>0 \\ 2, & \text{when } x=0 \\ 2+3x, & \text{when } x<0. \end{cases} \quad \text{at } x=0.$$ [C.U. B.Com.(H) 2001; V.U. B.Com.(H) 2010]

5. A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}\dfrac{|x-3|}{x-3}, & \text{if } x\neq 0\\ 1, & \text{if } x=3.\end{cases}$$

Discuss the continuity of $f(x)$ at $x=3$. [C.U. B.Com.(H) 1997]

6. If the function $f(x)$ is defined by

$$f(x)=\begin{cases}\dfrac{x^2-16}{x-4}, & \text{for } x\neq 4\\ 10, & \text{for } x=4,\end{cases}$$

say whether it is continuous at $x=4$. If not, why?

7. The function $f(x)=\dfrac{2x^2-8}{x-2}$ is undefined at $x=2$. What should be assigned to $f(2)$ so that $f(x)$ is continuous at $x=2$?

8. A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}x+1, & \text{when } x\leq 0\\ 3-ax, & \text{when } x>1.\end{cases}$$

For what value of a will $f(x)$ be continuous at $x=1$? With this value of a draw the graph of $f(x)$. [B.U. B.Com.(H) 2006]

9. For what value of $f(3), f(x)=\frac{x^2-9}{x-3}$ will be continuous at $x=3$? [C.U. B.Com. 2006]

10. For what value of $f(5), f(x)=\frac{x^2-25}{x-5}$ will be continuous at $x=5$? [C.U. B.Com. 2008]

E

(Miscellaneous)

1. Prove:

(a) $\lim\limits_{x\to\infty}\dfrac{4x+5}{2x+3}=2;$

(b) $\lim\limits_{t\to 0}\dfrac{4t^2+3t+2}{t^3+2t-6}=-\dfrac{1}{3};$

(c) $\lim\limits_{h\to 0}\dfrac{x^2h+3xh^2+h^3}{2xh+5h^2}=\dfrac{x}{2};$

(d) $\lim\limits_{h\to\infty}\dfrac{3h+2xh^2+x^2h^3}{4-3xh-2x^3h^3}=-\dfrac{1}{2x};$

(e) $\lim\limits_{y\to\infty}\dfrac{4y^3-3}{2y^3+3y^2}=2;$

(f) $\lim\limits_{k\to 0}\dfrac{(2z+3k)^3-4k^2z}{2z(2z-k)^2}=1;$

(g) $\lim\limits_{x\to\infty}\dfrac{6x^3-5x^2+3}{2x^3+4x-7}=3;$

(h) $\lim\limits_{h\to 0}\dfrac{(x+h)^n-x^n}{h}=nx^{n-1},$ (n is a positive integer);

(i) $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x^2-4}=\dfrac{5}{4};$

(j) $\lim\limits_{s\to a}\dfrac{s^4-a^4}{s^2-a^2}=2a^2;$

(k) $\lim\limits_{x\to 5}\dfrac{x^3-125}{x^4-625}=\dfrac{3}{20};$ [C.U. B.Com.(H) 1997]

(l) $\lim\limits_{x\to 0}\dfrac{e^{px}-e^{qx}}{x}=p-q.$ [B.U. B.Com.(H) 2006]

2. If $f(x) = ax^2 + bx + c$, show that $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = 2ax + b$.

3. Given $q = \frac{(v+2)^2-4}{v}$, find $\lim\limits_{v \to 0} q$, $\lim\limits_{v \to 2} q$, $\lim\limits_{v \to a} q$.

4. A function $y = f(x)$ is discontinuous at $x = a$ when any one of the three requirements:

   (a) $f(a+0)$ exists,

   (b) $f(a-0)$ exists,

   (c) $f(a) = f(a+0) = f(a-0)$,

   for continuity is violated. Construct three graphs to illustrate the violation of each of these requirements.

5. Given $y = f(x) = \dfrac{x^2 - x - 12}{x - 4}$.

   (a) Is it possible to apply the limit theorem on quotient of two functions to find the limit of this function, when $x \to 4$?

   (b) Is this function continuous at $x = 4$? Why?

   (c) Find a function that, for $x \neq 4$, is equivalent to the above function and obtain from the equivalent function the limit of y as $x \to 4$.

6. The total cost of ₹C of producing x items is given by

$$C = \begin{cases} 1000 + 5x, & 0 \leq x \leq 500 \\ 2000 + 4x, & 500 < x \leq 2000. \end{cases}$$

   (a) Graph C for different values of x and show that the function is discontinuous at $x = 500$.

   (b) Suppose, further, that we are given the relationship between the number of items sold (x) and the selling price per item (p): $x = 3000 - 25p$.

   Express p as a function of x. Find the relationship between net profit P and sales x. Plot the graph of P against x for $200 \leq x \leq 1500$.

[For 6. (b), note that p = S.P. per item
x = number of items sold.

We can express p as a function of x, i.e., S.P. per item depends on the number of items sold by the expression $p = \frac{3000-x}{25} = 120 - \frac{x}{25}$.

Now, the net profit P = S.P. - C.P.

$$\begin{aligned} &= \left(120 - \tfrac{x}{25}\right)x - (1000 + 5x), && \text{if } 0 \leq x \leq 500 \\ &= \left(120 - \tfrac{x}{25}\right)x - (2000 + 4x), && \text{if } 500 < x \leq 2000 \\ \text{or, } P &= -\tfrac{x^2}{25} + 115x - 1000, && \text{if } 0 \leq x \leq 500 \\ &= -\tfrac{x^2}{25} + 116x - 2000, && \text{if } 500 < x \leq 2000 \text{]} \end{aligned}$$

ANSWERS

A

2. (a) 3;
   (b) 1;
   (c) 8;
   (d) L.H. limit $= -1$,
   R.H. limit $= 1$
   and thus limit does not exist;
   (e) L.H. limit $= 0$,
   R.H. limit $= 1$,
   when $a = 1$;
   L.H. limit $= 1$,
   R.H. limit $= 2$,
   when $a = 2$.
   In both cases limit does not exist.
3. 1.
4. (a) 1; (b) No.
5. 0.
8. (a) No; (b) 4.

B

1. (a) $\frac{3}{5}$;
   (b) $\frac{1}{7}$;
   (c) 7;
   (d) 4;
   (e) 1;
   (f) $\frac{5}{3}$;
   (g) -1;
   (h) $\frac{1}{2}$;
   (i) 1;
   (j) 2;
   (k) $-\frac{1}{x^2}$;
   (l) -1;
   (m) $\frac{1}{2\sqrt{3}}$;
   (n) $\frac{1}{2\sqrt{3}}$;
   (o) $\frac{5}{2}$;
   (p) 1;
   (q) 0;
   (r) 1;
   (s) $\frac{7}{12}$;
   (t) $\frac{1}{6}$;
   (u) $\frac{1}{2\sqrt{a}}$;
   (v) $\frac{1}{\sqrt{2}}$;
   (w) $\frac{1}{2}$;
   (x) 0;
   (y) 2;
   (z) 0;
   (aa) -1;
   (ab) ∞;
   (ac) $\frac{1}{10}$;
   (ad) 3.
2. (a) $\frac{7}{8}$;
   (b) $\frac{5}{2}$;
   (c) $-\frac{1}{8}$;
   (d) 1.
3. (a) 3;
   (b) $\frac{5}{3}$;
   (c) $\frac{3}{4}$;
   (d) 0.
4. (a) 1;
   (b) $-\frac{1}{2x\sqrt{x}}$;
   (c) $\frac{1}{2\sqrt{3a}}$;
   (d) $2a$.
5. (a) 3;
   (b) $\log 2.\log_e 3$;
   (c) The limit does not exist;
   (d) $\frac{1}{4}$;
   (e) 1;
   (f) $\frac{1}{12\sqrt{3}}$.
7. (a) $p=1$, $q=9$;
   (b) $p=2$, $q=50$.
8. 1.
9. $\frac{1}{2}(a+b)$.

C

3. (a) Continuous;
   (b) Continuous;
   (c) Discontinuous.
4. (a) (i) $\lim_{x\to 3} f(x) = 6$ but function is not defined at $x = 3$; hence the function is discontinuous at $x = 3$.
   (ii) Discontinuous at $x = 1$.
   (b) $\lim_{x\to 3} f(x) = 6 = f(3)$; hence continuous at $x = 3$.
   (c) $\lim_{x\to 3} f(x) = 6 \neq f(3) = 1$; hence discontinuous at $x = 3$.
   (d) $\lim_{x\to 1^{-0}} f(x) = \lim_{x\to 1^{-0}} (2x+1) = \lim_{h\to 0^-} \{2(1+h)+1\}$
   $= 3$;
   $\lim_{x\to 1^{+0}} f(x) = \lim_{x\to 1^{+0}} (x+1) = \lim_{h\to 0^+} \{(1+h)+1\}$
   $= 2$.
   $\therefore$ R.H. limit $\neq$ L.H. limit. Hence limit does not exist and as such the function cannot be continuous at $x = 1$.
   (e) Continuous at $x = 2$;
   (f) Continuous at $x = 4$.
   (g) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$;

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1,\ f(1) = 1;$
hence continuous at $x = 1$.
Similarly, see that it is continuous at $x = 2$.

(h) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\frac{1}{2} - x) = \frac{1}{2};$
$\lim_{x \to 0^-} f(x)$: there $f(x)$ is not defined, when x has values less than zero; so we cannot determine the left-hand limit. But $f(0) = 0$. Thus, even the right-hand limit $\neq$ functional value, there $f(x)$ is discontinuous at $x = 0$. Discuss the other cases: $f(x)$ is discontinuous at $x = \frac{1}{2}$ or $x = 1$.

(i) See that $f(x)$ is continuous at $x = \frac{1}{2}$ but discontinuous at $x = 1$.

6. Discontinuous at $x = 3$.

D

1. $x = -\frac{4}{3}, \frac{5}{4}$.
2. Discontinuous at $x = 0$.
3. Continuous at $x = 0$.
4. Continuous at $x = 0$.
5. $f(x)$ is not continuous at $x = 3$.
6. Not continuous at $x = 4$, since $\lim_{x \to 4} f(x) = 8$ and $f(4) = 10$.
7. $f(2) = 8$.
8. $a = 1$.
9. $f(3) = 6$.
10. $f(5) = 10$.

E

3. $\lim_{v \to 0} q = \lim_{v \to 0} \frac{(v+2)^2 - 4}{v} = \lim_{v \to 0} \frac{v^2 + 4v}{v}$
$= \lim_{v \to 0}(v + 4) = 4.$

(We can cancel v because in determining limit as $v \to 0$ we need not consider the value $v = 0$; we consider value of v near 0.)

$\lim_{v \to 2} q = \lim_{v \to 2} \frac{(v+2)^2 - 4}{v} = \lim_{h \to 0} \frac{(2+h+2)^2 - 4}{2+h}$
$= 6,$
$\lim_{v \to a} q = a + 4.$

5. (a) No; because here the limit in the denominator is zero. But the limit of this function can be determined in the following way:

$$\lim_{x \to 4} f(x) = \lim_{h \to 0} \frac{(4+h)^2 - (4+h) - 12}{4+h-4}$$
$$= \lim_{h \to 0} \frac{h(7+h)}{h} = \lim_{h \to 0}(7+h)$$
$$= 7.$$

(b) The function is not continuous at $x = 4$ because though limit exists as $x \to 4$, the function does not exist at $x = 4$.

(c)
$$f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x - 4}, & \text{when } x \neq 4 \\ \text{undefined}, & \text{at } x = 4 \end{cases}$$

Limit is 7.

Chapter 3

Differentiation: Its Meaning and Rules of Differentiation

3.1 Introduction

In Differential Calculus we investigate the rate of change of a dependent variable w.r.t. an independent variable. We call this rate as *derivative* of a dependent variable w.r.t. its independent variable. Geometrically, this derivative gives the measure of slope of a curve.

In the present chapter we shall give a precise definition of derivative and then deduce the standard rules of finding derivatives. The proofs of these rules are given at the Appendix to this Chapter.

We shall be concerned only with Algebraic, Exponential and Logarithmic functions. Trigonometric functions have not been included in the present treatise.

3.2 Derivative: Definition

Let x be an independent variable and y, the dependent variable, depending on x. Let $y = f(x)$.

Let $x = c$ be a given value of x. The corresponding value of y is $f(c)$. We make a change in the value of x, say, $x = c + h$, where h may be positive or negative. The corresponding value of y is, say, $y + k = f(c + h)$. Then we call h, the increment of x, and k — the increment of y. Clearly, $k = f(c + h) - f(c)$.

Then we consider the ratio:

$$\frac{k}{h} = \frac{f(c+h)-f(c)}{h},$$

called *increment ratio;* now make $h \to 0$. Then, if

$$\lim_{h\to 0} \frac{k}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists,}$$

we say that $f(x)$ is *derivable* at $x = c$ and the limit thus obtained is the *derivative* of $f(x)$ w.r.t. x at $x = c$ and is denoted by $f'(c)$ or $\frac{d}{dx}\{f(x)\}$ at $x = c$ or $\frac{dy}{dx}$ at $x = c$.

A Formal Precise Definition

Let $y = f(x)$ be a function of x.

Suppose, the function is defined at a given value of x, say, at $x = c$ and it is also defined in some neighbourhood of c.

If

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists,}$$

then $f(x)$ is said to be derivable at $x = c$ and the limit is called the **derivative of $f(x)$ w.r.t. x at $x = c$ and is denoted by $f'(c)$.**

Observation. The derivative has been defined at a particular point $x = c$. If the point is not specified, then usually we find the derivative at any point x and write,

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \quad \textit{if this limit exists.}$$

Now for a particular value of x, say, $x = c$, we can easily get the derivative by putting that value of $x = c$ in $f'(x)$, i.e., find $f'(x)$ and put $x = c$ to get $f'(c)$.

The method of finding the derivative of a function using this definition is called *first principle.*

3.3 Finding Derivative from First Principle (i.e., Using Definition)

Example 1. (a) *Let $f(x) = x^3$. Find, from first principle, the derivative of $f(x)$ w.r.t. x at $x = 2$.*
(b) *If $f(x) = x^2$, find $f'(x)$ from the definition.* [B.U. B.Com.(H) 2008]

Solution: (a) By definition (or first principle), the derivative of $f(x)$ w.r.t. x at $x = 2$ is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}, \quad \text{if this limit exists.}$$

Now,
$$\frac{f(2+h)-f(2)}{h} = \frac{(2+h)^3-2^3}{h} = \frac{12h+6h^2+h^3}{h}$$
$$= 12+6h+h^2, \text{ if } h \neq 0.$$

$$\therefore \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\left(12+6h+h^2\right) \text{ (when } h\neq 0\text{, we need not take } h=0)$$
$$= 12+6\lim_{h\to 0} h + \lim_{h\to 0} h^2 = 12.$$

$\therefore$ **$f'(2) = 12$.**

(b) By definition,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x+0 = \mathbf{2x}.$$

Example 2. (i) *If $y = 3x^2-2x+9$, find, from definition, the value of $\frac{dy}{dx}$ at any point x.* (ii) $f(x) = 2x^2-x+1$. *Is $f(x)$ derivable at $x = 2$? If so, what is the derivative at $x = 2$? Is the function continuous at $x = 2$?*

Solution: (i) We write, $y = f(x) = 3x^2-2x+9$.

By definition,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{[3\left\{(x+h)^2\right\}-2(x+h)+9]-[3x^2-2x+9]}{h}$$
$$= \lim_{h\to 0} \frac{3\left\{(x+h)^2-x^2\right\}-2h}{h} = \lim_{h\to 0} \frac{6xh+3h^2-2h}{h}$$
$$= \lim_{h\to 0}(6x+3h-2) \; (\because \text{ we assume } h\neq 0),$$

i.e., **$f'(x) = 6x-2$.**

(ii) $f(x) = 2x^2-x+1$.

This function is derivable at $x=2$, if $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

exists and the limiting value is called the derivative of $f(x)$ at $x=2$.

Here,
$$\frac{f(2+h)-f(2)}{h}=\frac{[2(2+h)^2-(2+h)+1]-[2\cdot 2^2-2+1]}{h}$$
$$=\frac{8+2h^2+8h-2-h+1-7}{h}.$$
$$\therefore\ \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{2h^2+7h}{h}$$
$$=\lim_{h\to 0}(2h+7)\ (\because\ h\to 0,\text{ we take } h\neq 0)$$
$$=\lim_{h\to 0}2h+\lim_{h\to 0}7=0+7=7.$$

Since, $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$ exists,

$f(x)$ is derivable at $x=2$ and since the limit = 7, this is the derivative at $x=2$.

The *function $f(x)$, being a polynomial in x, is* **continuous at $x=2$.**

Conclusion. Existence of derivative at a point ⇒ function is continuous at that point.

Example 3. (i) $f(x)=1/x$. *Find, from first principles, the derivative of $f(x)$ at any point $x\neq 0$.* (ii) *Find, from first principle, the derivative of $f(x)=1/x^2(x\neq 0)$.* **[B.U. B.Com.(H) 2008]**

Solution: (i) By definition,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},\ \text{if this limit exists}\quad \Big|\ f(x)=\tfrac{1}{x}\text{ and } f(x+h)=\tfrac{1}{x+h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{-h}{hx(x+h)}$$
$$=-\lim_{h\to 0}\frac{1}{x(x+h)}\ (\text{taking } h\neq 0),$$

i.e., $\boldsymbol{f'(x)=-\frac{1}{x^2}}$, *where $x\neq 0$.*

(ii) We have
$$f(x)=\frac{1}{x^2};\ \therefore\ f(x+h)=\frac{1}{(x+h)^2}.$$

By definition,
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$
$$=\lim_{h\to 0}\frac{x^2-(x+h)^2}{h\cdot(x+h)^2\cdot x^2}=\lim_{h\to 0}\frac{-2xh-h^2}{h\cdot(x+h)^2\cdot x^2}$$
$$=-\lim_{h\to 0}\frac{h(2x+h)}{h\cdot(x+h)^2\cdot x^2}=-\lim_{h\to 0}\frac{2x+h}{(x+h)^2\cdot x^2}\ [\because\ h\neq 0]$$
$$=-\frac{2x+0}{(x+0)^2\cdot x^2}=-\frac{2}{x^3},\ \text{where } x\neq 0.$$

Note: Both in (i) and (ii) $f(x)$ is not defined at $x=0$ and as such, derivative of $f(x)$ at $x=0$ does not exist.

Example 4. *Find, from definition, the derivative of* $3x^3+7$ *w.r.t.* x.

Solution: By definition,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\{3(x+h)^3+7\}-\{3x^3+7\}}{h}$$

$$=\lim_{h\to 0}\frac{9x^2h+9xh^2+3h^3}{h}=\lim_{h\to 0}\{9x^2+9xh+3h^2\}$$

$$=9x^2+\lim_{h\to 0}9xh+\lim_{h\to 0}3h^2=9x^2, \quad \text{i.e., } \boldsymbol{f'(x)=9x^2}.$$

Note: We can now write down the derivative of $f(x)$ at any given value of x, say, for $x=2$, as $f'(2)=36$.

Example 5. *Using definition, find the derivative of* $f(x)=\sqrt{x}$ $(x>0)$. [N.B.U. B.Com.(H) 2007]

Solution: Using definition, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \qquad (\because h\neq 0)$$

$$=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}, \text{ where } x>0.$$

Thus,

$$\boldsymbol{f'(x)=\frac{1}{2\sqrt{x}}}.$$

Note: $f'(x)$ is not defined for $x\le 0$ and hence $f'(x)$ does not exist for any $x\le 0$.

Example 6. $f(x)=\sqrt[3]{x}$. *Find* $f'(x)$ *from first principle.* [C.U. B.Com.(H) 2004; B.U. B.Com.(H) 2007]

Solution: From *first principle*, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}{h\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

[Remember: $(a-b)(a^2+ab+b^2)=a^3-b^3$. Here take: $a=\sqrt[3]{x+h}$, $b=\sqrt[3]{x}$.]

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}}$$

$$=\frac{1}{3\sqrt[3]{x^2}}=\boldsymbol{\frac{1}{3}x^{-(2/3)}\ (x\neq 0).}$$

Example 7. *Differentiate from first principle:* $f(x)=(x-1)^3$. [C.U. B.Com.(H) 1993]

Solution: We have $f(x)=(x-1)^3$; $\therefore$ $f(x+h)=(x+h-1)^3$.

From *first principle,* we get

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{(x+h-1)^3-(x-1)^3}{h},$$

$$\text{i.e., } f'(x)=\lim_{h\to0}\frac{(x-1)^3+3(x-1)^2h+3(x-1)h^2+h^3-(x-1)^3}{h}$$

$$=\lim_{h\to0}\frac{h\left\{3(x-1)^2+3(x-1)h+h^2\right\}}{h}$$

$$=\lim_{h\to0}\left\{3(x-1)^2+3(x-1)h+h^2\right\}, \text{ since } h\neq0$$

$$=3(x-1)^2+3(x-1)\cdot0+0=\mathbf{3(x-1)^2}.$$

3.3.1 Exponential and Logarithmic Functions

We shall assume the following limits without proof:

$$\text{(A) } \lim_{x\to0}\frac{e^x-1}{x}=1; \quad \text{(B) } \lim_{x\to0}\frac{\log_e(1+x)}{x}=1; \quad \text{(C) } \lim_{x\to0}\frac{a^x-1}{x}=\log_e a.$$

[(B) follows from (A) by putting $\log_e(1+x)=h$ so that $1+x=e^h$, or, $x=e^h-1$ and as $x\to0$, $h\to0$.

$$\therefore \lim_{x\to0}\frac{\log_e(1+x)}{x}=\lim_{h\to0}\frac{h}{e^h-1}=\lim_{h\to0}\frac{1}{\frac{e^h-1}{h}}=\frac{1}{1}=1.]$$

Using these limits we can find the derivatives of e^x, a^x and $\log x$ w.r.t. x.

Note: We shall take the base of the logarithm always as e, unless otherwise mentioned.

Example 8. *Let* $f(x)=e^x$. *Find, from first principle, the derivative of* $f(x)$ *w.r.t.* x *at any given point* x. [V.U. B.Com.(H) 2008]

Solution: From first principle, we get

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{e^{x+h}-e^x}{h}=e^x\lim_{h\to0}\frac{e^h-1}{h}=e^x\cdot1=\boldsymbol{e^x}.$$

Example 9. (i) *Derivative of* $\log x$ $(x>0)$; [N.B.U. B.Com.(H) 2006; V.U. B.Com.(H) 2007]

(ii) *Derivative of* $a^x (a>0)$.

Solution: (i) Let $f(x)=\log x (x>0)$ [If $x<0$ or, $x=0, f(x)$ is not defined and there is no question of finding the derivative then.]

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\log(x+h)-\log x}{h}=\lim_{h\to0}\frac{1}{h}\log\left(\frac{x+h}{x}\right)$$

$$=\lim_{h\to0}\frac{1}{h}\log\left(1+\frac{h}{x}\right) \quad \left[\text{Put } \tfrac{h}{x}=z, \text{ then } h=xz \text{ and when } h\to0, z\to0\ (x \text{ has a fixed value})\right]$$

$$=\lim_{z\to0}\frac{1}{xz}\log(1+z)=\frac{1}{x}\lim_{z\to0}\frac{\log(1+z)}{z}$$

$$=\frac{1}{x}\cdot1=\mathbf{\frac{1}{x}} \quad \text{[See limit (B) above].}$$

(ii) Let $f(x)=a^x$; then $f(x+h)=a^{x+h}$.

$$\therefore\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}=\boldsymbol{a^x\log_e a}\ \text{[See limit (C) page 177].}$$

Example 10. *Find by using definition, the derivative of* $f(x)=e^{\sqrt{x}}$.

Solution:
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h}.$$

Put $\sqrt{x}=y$ and $\sqrt{x+h}=y+k$; then $k=\sqrt{x+h}-y=\sqrt{x+h}-\sqrt{x}$.
Now, when $h\to 0$, $k\to 0$ and $h=(y+k)^2-x=(y+k)^2-y^2$.

$$\therefore\ f'(x)=\lim_{k\to 0}\frac{e^{y+k}-e^y}{k}\cdot\frac{k}{(y+k)^2-y^2}=e^y\lim_{k\to 0}\frac{e^k-1}{k}\times\lim_{k\to 0}\frac{k}{2yk+k^2}$$

$$=e^y\cdot 1\cdot\lim_{k\to 0}\frac{1}{2y+k}=e^y\cdot\frac{1}{2y}=\boldsymbol{e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}.$$

Example 11. *Find by using definition, the derivative of* $f(x)=\log_{10}x$ *w.r.t.* x *at any given point* x.

Solution:
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\log_{10}(x+h)-\log_{10}x}{h}$$

$$=\lim_{h\to 0}\frac{\log_e(x+h)\times\log_{10}e-\log_e x\times\log_{10}e}{h}$$

$$=\log_{10}e\lim_{h\to 0}\frac{\log(x+h)-\log x}{h}\qquad\text{(Now proceed as in Ex. 9.)}$$

$$=\log_{10}e\cdot\frac{1}{x}=\frac{1}{x\log_e 10}=\boldsymbol{\frac{1}{x\log 10}}.\qquad\left[\because\ \log_{10}e=\frac{1}{\log_e 10}\right]$$

Example 12. *Prove that* $f(x)=|x|$ *is continuous at* $x=0$ *but* $f(x)$ *has no derivative there.*

Solution: We first observe, $f(x)=|x|$ can be written as

$$f(x)=\begin{cases} x, & \text{when } x>0\\ -x, & \text{when } x<0\\ 0, & \text{when } x=0.\end{cases}$$

$$\therefore\quad \lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x=0\ \text{(R.H. limit = 0)}$$

and $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}(-x)=0$ (L.H. limit = 0) and $f(0)=0$.

Hence, **$f(x)$ is continuous at $x=0$.**

We shall show that $f'(0)$ does not exist, i.e., $\lim_{x\to 0}\frac{f(h)-f(0)}{h}$ does not exist.

Now, $$Rf'(0)=\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^+}\frac{h-0}{h}=1,\text{ i.e., R.H. derivative}=1$$

and $$Lf'(0)=\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^-}\frac{-h-0}{h}=-1,\text{ i.e., L.H. derivative}=-1.$$

Since R.H. derivative $\neq$ L.H. derivative at $x=0$, we say **derivative $f'(0)$ does not exist at $x=0$.**

Example 13. *Given,*

$$f(x)=\begin{cases} x, \text{ when } 0<x<1 \\ 2-x, \text{ when } 1\le x\le 2 \\ 3x-x^2, \text{ when } x>2.\end{cases}$$

Show that $f(x)$ is not continuous at $x=2$ and $f'(2)$ also does not exist finitely.

Solution: We have

$$\lim_{x\to2^+}f(x)=\lim_{x\to2^+}\left(3x-x^2\right)=\lim_{h\to0}\left\{3(2+h)-(2+h)^2\right\}=\lim_{h\to0}\left(2-h-h^2\right)=2$$

and $$\lim_{x\to2^-}f(x)=\lim_{x\to2^-}(2-x)=\lim_{h\to0^-}\{2-(2+h)\}=0\text{ and }f(2)=0.$$

Thus R.H. limit = 2, L.H. limit = 0 and L.H. limit $\neq$ R.H. limit.

$\therefore\ \lim_{x\to2}f(x)$ does not exist and hence **$f(x)$ is not continuous at $x=2$.**

$$Rf'(x)=\lim_{h\to0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to0}\frac{3(2+h)-(2+h)^2-0}{h}$$
$$=\lim_{h\to0}\frac{2-h-h^2}{h}$$ This limit does not exist (it tends to infinity).

We can at once conclude that **$f'(2)$ does not exist finitely.**

Example 14. *Prove that if the function $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$ (i.e., differentiability implies continuity).*

Solution: Since $f(x)$ is differentiable at $x=a$, $f'(a)$ exists and is finite.

$$\therefore\ f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\text{ is a definite finite quantity.}$$

Now,
$$\lim_{h\to0}\{f(a+h)-f(a)\}=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\times h\quad[\because\ h\neq0]$$
$$=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\times\lim_{h\to0}h$$
$$=f'(a)\times0=0\ [\because\ f'(a)\text{ is finite}].$$

$\therefore\ \lim_{h\to0}f(a+h)=f(a)$. This shows that **$f(x)$ is continuous at $x=a$.**

3.4 Rules for Differentiation

We first give a list of Rules which will be very helpful in the evaluation of derivatives.

- Derivative of any constant is zero (See Appendix), i.e., let $f(x)=c$ for all values of x; then $f'(x)=0$ at every x.

  e.g., $$\frac{d}{dx}(5)=0;\ \frac{d}{dx}(-3000)=0, \text{ etc. } \therefore\ \frac{d(c)}{dx}=\mathbf{0}.$$

- Derivative of $x^n = nx^{n-1}$, where n is any real number [Except, where the function becomes undefined] (See Section 3.5, Ex. 1),

  $$\text{i.e., } \frac{d}{dx}(x^n)=nx^{n-1}.$$

  Remember. $\frac{d}{dx}(x^{-1})$ *does not exist at* $x=0$.

  Note: $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$, except when $x=0$ or any negative number.

- Derivative of sum (or difference) of two derivable functions = sum of their separate derivatives, (provided both these derivatives exist), i.e., $\frac{d}{dx}(u \pm v)=\frac{du}{dx} \pm \frac{dv}{dx}(u, v$ are functions of x). [See Appendix]

  e.g., $$\frac{d}{dx}\left(x^2+3x+8\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(8).$$

- **Differentiation of Product: [Product Rule]**

  $$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}, \text{ [See Rule III given later]}$$

  where u and v are two functions, both derivable w.r.t. x.

  e.g.,

  (i) $\frac{d}{dx}\left(2^x x^5\right)=2^x\frac{d}{dx}\left(x^5\right)+x^5\frac{d}{dx}(2^x)=2^x\cdot 5x^4+x^5\cdot 2^x\log_e 2=x^4\cdot 2^x(5+x\log 2).$

  (ii) $\frac{d}{dx}\left(x^3\cdot\sqrt{x}\right)=x^3\frac{d}{dx}\left(\sqrt{x}\right)+\sqrt{x}\frac{d}{dx}\left(x^3\right).$

  As a corollary: $\frac{d}{dx}(cu)=c\frac{du}{dx}$, *where c is a constant.*

  e.g., $$\frac{d}{dx}\left(3x^2\right)=3\frac{d}{dx}\left(x^2\right).$$

- **Quotient Rule**

  $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}, \text{ [See Rule IV given later]}$$

  where u, v are derivable functions of x; more specifically remember that the derivative is to be obtained at those values of x for which $\frac{u}{v}$ is defined.

  For example,

  $$\frac{d}{dx}\left(\frac{x^2}{\sqrt{x}}\right)=\frac{\sqrt{x}\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}.$$

  Remember the Special Case:

  $$\frac{d}{dx}\left(\frac{1}{v}\right)=-\frac{1}{v^2}\cdot\frac{dv}{dx}\quad\left(\because \frac{du}{dx}=\frac{d(1)}{dx}=0\right).$$

- **Derivative of Function of a Function**

  Suppose, y is some derivable function of u, where u is itself a derivable function of x, then we say that y is not only a FUNCTION of a function of x, but also y is a derivable function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \qquad \text{[CHAIN RULE].}$$

For proof, see **Rule V**, given later

e.g., $y=\sqrt{x^2+3}$. Let us write, $y=\sqrt{u}$, where $u=x^2+3$. Then y is a function of u, where u is a function of x.

So, $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{d}{du}\left(\sqrt{u}\right)\cdot\frac{d}{dx}\left(x^2+3\right)=\frac{1}{2\sqrt{u}}\cdot 2x=\frac{x}{\sqrt{x^2+3}}.$$

Important Special Case:
$$\frac{dy}{dx}\times\frac{dx}{dy}=1 \text{ or, } \frac{dy}{dx}=1\Big/\frac{dx}{dy}\ \left(\text{if } \frac{dx}{dy}\neq 0\right).$$

Extensions of Rules III *and* IV:

- $\frac{d}{dx}(u+v-w+z)=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}+\frac{dz}{dx}$,

  where u, v, w, z are all derivable functions of x.
- $\frac{d}{dx}(uvw)=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$.

Three Important Derivatives:

$$\frac{d}{dx}(e^x)=e^x;\qquad \frac{d}{dx}(a^x)=a^x\log_e a\ (a>0);$$
$$\frac{d}{dx}(\log_e x)=\frac{1}{x}\ (x\neq 0 \text{ or negative}).$$

We have already proved these results (See Exs 8, 9; Section 3.3.1).

Proofs of General Rules of Differentiation

Rule I. *Derivative of a constant function is zero.*

Proof. Let $f(x)=c$, for all values of x. $\therefore\ f(x+h)=c$.

Hence, $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{c-c}{h}=0.\ \therefore\ \frac{d}{dx}\{f(x)\}=\frac{d}{dx}(c)=0.$

[e.g., $\frac{d}{dx}(5)=0$; $\frac{d}{dx}(-3000)=0$, etc.]

Rule II. *Derivative of the sum or difference of two derivable functions is the sum or difference of their derivatives.*

Proof. Let u, v be two derivable functions of x and let $f(x)=u\pm v$.

Let us give an increment h to the independent variable x. Let the corresponding increments of u and v be k and l respectively. Then,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(u+k)\pm(v+l)-(u\pm v)}{h}$$

$$\therefore\quad \frac{d}{dx}\{f(x)\}=\lim_{h\to 0}\frac{k}{h}\pm\lim_{h\to 0}\frac{l}{h}=\frac{du}{dx}\pm\frac{dv}{dx}.$$

[e.g., $\frac{d}{dx}(x^2+3x+8)=\frac{d}{dx}(x^2)+\frac{d}{dx}(3x)+\frac{d}{dx}(8)=2x+3\cdot 1+0=2x+3.$]

Rule III. *The derivative of the product of two derivable functions = first function × derivative of the second function + second function × derivative of the first function.*

Proof. Let u, v be two derivable functions of x and let $f(x)=uv$.

Let us give an increment h to the independent variable x. Suppose, the corresponding increments of u and v be k and l respectively. Then when $h\to 0, k\to 0$ and $l\to 0$. We have

$$\frac{d}{dx}\{f(x)\}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(u+k)(v+l)-uv}{h}=\lim_{h\to 0}\left(\frac{k}{h}v+\frac{l}{h}u+\frac{k}{h}l\right)$$

$$=v\lim_{h\to 0}\frac{k}{h}+u\lim_{h\to 0}\frac{l}{h}+\lim_{h\to 0}\frac{k}{h}\times\lim_{h\to 0}l$$

$$\text{or, } f'(x)=v\frac{du}{dx}+u\frac{dv}{dx}+\frac{du}{dx}\times 0=u\frac{dv}{dx}+v\frac{du}{dx}.$$

[e.g., (i) $\frac{d}{dx}(2^x\cdot x^5)=2^x\frac{d}{dx}(x^5)+x^5\frac{d}{dx}(2^x)=2^x.5x^4+x^5.2^x\log_e 2=x^4.2^x(5+x\log_e 2)$.
(ii) $\frac{d}{dx}(x^3.\sqrt{x})=x^3\frac{d}{dx}(\sqrt{x})+\sqrt{x}.\frac{d}{dx}(x^3)=$ etc.]

Corollary 1. $\frac{d}{dx}(cu)=c\frac{du}{dx}$. [See that: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c(u+k)-cu}{h}=c\lim_{h\to 0}\frac{k}{h}=c\frac{du}{dx}$.]

Rule IV. (Quotient Rule)

Derivative of the quotient of two derivable functions

$$=\frac{\textit{Denominator} \times \textit{deriv. of Numerator} - \textit{Numerator} \times \textit{deriv. of Denominator}}{(\textit{Denominator})^2}.$$

i.e., $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

Proof. Let $f(x)=\frac{u}{v}$, where u and v are derivable functions of x. We give an increment h to the independent variable x. Suppose, k and l are corresponding increments of u and v respectively. When $h\to 0$, both k and $l\to 0$.

$$\therefore\ \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(\frac{u+k}{v+l}-\frac{u}{v}\right)\frac{1}{h}=\lim_{h\to 0}\frac{kv-ul}{h\cdot v(v+l)}$$

$$=\lim_{h\to 0}\left(\frac{k}{h}\cdot\frac{1}{v+l}\right)-\lim_{h\to 0}\frac{l}{h}\cdot\frac{u}{v(v+l)}$$

$$=\frac{du}{dx}\cdot\frac{1}{v}-\frac{dv}{dx}\cdot\frac{u}{v^2}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

$$\therefore \frac{d}{dx}\{f(x)\} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Remember the Special Case: $\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\cdot\frac{dv}{dx}\left(\because \frac{du}{dx} = \frac{d(l)}{dx} = 0\right)$.

[e.g., $\frac{d}{dx}\left(\frac{x^2}{\sqrt{x}}\right) = \frac{\sqrt{x}\cdot\frac{d}{dx}(x^2) - x^2\cdot\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$.]

Note: In these discussions we are assuming that the function is defined at or near the point x, where we wish to find the derivative.

Rule V. *Rule for finding the derivative of function of a function.*

Let $u = \phi(x)$ be a derivable function of x in the closed interval $a \le x \le b$, the range of u being $[\alpha, \beta]$. If $y = f(u)$ be a derivable function of u in the closed interval $\alpha \le u \le \beta$, then the composite function $y = f\{\phi(x)\}$ is derivable w.r.t. x and the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \quad \text{[CHAIN RULE]}$$

Proof. Let h be an increment of x, and k be the corresponding increment of u in $u = \phi(x)$. Suppose that corresponding to the increment k of u, l is the increment of y determined from $y = f(u)$.

Case I. *Suppose, $k \neq 0$. Then*

$$\frac{l}{h} = \frac{l}{k}\cdot\frac{k}{h} \text{ or, } \lim_{h\to 0}\frac{l}{h} = \lim_{k\to 0}\frac{l}{k}\cdot\lim_{h\to 0}\frac{k}{h}\left(\because \text{ when } \begin{matrix} h\to 0 \\ k\to 0\end{matrix}\right), \text{ i.e., } \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

Case II. *Suppose, $k = 0$. Then l is also zero.*

$$\therefore \frac{dy}{dx} = \lim_{h\to 0}\frac{k}{h} = 0 \text{ and } \frac{du}{dx} = \lim_{h\to 0}\frac{k}{h} = 0.$$

$\therefore$ *here also the result* $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ *is TRUE (both sides being zero).*

Corollary 2. $\frac{dx}{dy} = 1\Big/\frac{dy}{dx}$. [*The operations being defined*]

3.5 Illustrative Examples

Example 15. $f(x) = x^n$. *To prove* $f'(x) = nx^{n-1}$, *where n is a rational number.*

[*For a particular value of x, say $x = c$, derivative is not defined, if $f(x)$ is not defined at $x = c$.*]

Case I. *Let n be a positive integer.*

Then, by definition, $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$, provided that this limit exists.

Now,
$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$
$$= \frac{\left\{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n\right\} - x^n}{h}.$$

[Expanding $(x+h)^n$ by the Binomial Theorem for positive integral index.]

i.e., $\frac{f(x+h)-f(x)}{h} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \frac{n(n-1)(n-2)}{3}x^{n-3}h^2 + \cdots + h^{n-1}$ [if $h \neq 0$].

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} nx^{n-1} + \lim_{h\to 0}\frac{n(n-1)}{2!}x^{n-2}h + \cdots + \lim_{h\to 0} h^{n-1}$$
$$= nx^{n-1} + 0 + 0 + \cdots + 0.$$
$$\therefore\ \boldsymbol{f'(x) = nx^{n-1}}.$$

Case II. *Let n be a negative integer. Suppose, we require to find $f'(x)$ for such x for which $f(x)$ is not undefined.*

We write, $n = -m$ (m is a positive integer)

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m\frac{d}{dx}(1) - 1\left[\frac{d}{dx}(x^m)\right]}{(x^m)^2} \text{ [using Quotient Rule]}$$
$$= \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}\ (\because\ -m = n).$$

Case III. *Let $n = \frac{p}{q}$, where p is either a positive or a negative integer and q is a positive integer ($q \neq 0$).*

Then we write,
$$y = x^n = x^{p/q} = \left(x^{1/q}\right)^p = z^p,$$
where $z = x^{1/q}$ or $x = z^q$.

$\therefore$ by Chain Rule, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\bigg/\frac{dx}{dz} = \frac{d}{dz}(z^p)\bigg/\frac{d}{dz}(z^q)$.

$$\therefore\ \frac{dy}{dx} = \frac{pz^{p-1}}{qz^{q-1}} = \frac{p}{q}z^{p-q} = \frac{p}{q}\left(x^{1/q}\right)^{p-q} = \frac{p}{q}x^{(p-q)/q} = \frac{p}{q}x^{(p/q)-1} = nx^{n-1}\ (\because\ n = p/q).$$

Otherwise. (using $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$, where $a > 0$ and n is rational.)

By definition,
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{(x+h)-x} = \lim_{z\to x}\frac{z^n - x^n}{z-x},$$
where $x + h = z$; as $h \to 0$, $z \to x$
$$= nx^{n-1}\ [\because\ x > 0 \text{ and } n \text{ is rational.}]$$

Example 16. *Find the derivative of* $\frac{x^4}{\sqrt[5]{x^2}} - \frac{7x}{\sqrt[3]{x^4}} + 8\sqrt[3]{x^7}$.

Solution: Let $y = \frac{x^4}{\sqrt[5]{x^2}} - \frac{7x}{\sqrt[3]{x^4}} + 8\sqrt[3]{x^7}$.

Then
$$y = \frac{x^4}{x^{2/5}} - \frac{7x}{x^{4/3}} + 8x^{7/3} = x^{4-(2/5)} - 7x^{1-(4/3)} + 8x^{7/3} = x^{18/5} - 7x^{-1/3} + 8x^{7/3}.$$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}\left(x^{18/5}\right) - 7\frac{d}{dx}\left(x^{-1/3}\right) + 8\frac{d}{dx}\left(x^{7/3}\right) = \frac{18}{5}x^{(18/5)-1} - 7\left(-\frac{1}{3}\right)x^{-(1/3)-1} + 8\cdot\frac{7}{3}x^{(7/3)-1}$$
$$= \boldsymbol{\frac{18}{5}x^{13/5} + \frac{7}{3}x^{-(4/3)} + \frac{56}{3}x^{4/3}}.$$

Example 17. (a) *Given*: $y=\dfrac{x^2}{a^2-x^2}$; *find* $\dfrac{dy}{dx}$, *when* $x=1$. [C.U. B.Com.(H) 2000]

(b) *If* $f(x)=\dfrac{1}{x-1}$, *then show that* $f'(0)=-1$. [B.U. B.Com.(H) 2008]

Solution: (a) We have $y=\dfrac{x^2}{a^2-x^2}$. [$\frac{u}{v}$ form]

$$\therefore\ \frac{dy}{dx}=\frac{(a^2-x^2)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}=\frac{(a^2-x^2)\cdot 2x-x^2(0-2x)}{(a^2-x^2)^2}$$
$$=\frac{2a^2x-2x^3+2x^3}{(a^2-x^2)^2}=\frac{2a^2x}{(a^2-x^2)^2}.$$

At $x=1$, $\frac{dy}{dx}=\frac{2a^2\cdot 1}{(a^2-1)^2}=\mathbf{\frac{2a^2}{(a^2-1)^2}}$.

(b) $f(x)=\frac{1}{x-1}$; $\therefore f'(x)=-\frac{1}{(x-1)^2}\cdot\frac{d}{dx}(x-1)=-\frac{1}{(x-1)^2}\times 1=-\frac{1}{(x-1)^2}$.

$$\therefore f'(0)=-\frac{1}{(0-1)^2}=-\frac{1}{1}=\mathbf{-1}.$$

Example 18. *Find* $\dfrac{dy}{dx}$, *when* (i) $y=(x^2+3x-5)^{3/2}$.

(ii) $y=\log\left(x-\sqrt{x^2-a^2}\right)$. [V.U. B.Com.(H) 2007]

Solution: (i) $y=(x^2+3x-5)^{3/2}=z^{3/2}$, where $z=x^2+3x-5$.

$$\therefore\ \frac{dy}{dz}=\frac{3}{2}z^{(3/2)-1}=\frac{3}{2}\sqrt{z}\text{ and }\frac{dz}{dx}=2x+3.$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{3}{2}\sqrt{z}\cdot(2x+3)=\mathbf{\frac{3}{2}(2x+3)\sqrt{x^2+3x-5}}.$$

Otherwise.

$$\frac{dy}{dx}=\frac{3}{2}\left(x^2+3x-5\right)^{(3/2)-1}\times\frac{d}{dx}\left(x^2+3x-5\right)=\frac{3}{2}\left(x^2+3x-5\right)^{1/2}\cdot(2x+3)$$
$$=\mathbf{\frac{3}{2}(2x+3)\cdot\sqrt{x^2+3x-5}}.$$

(ii) $y=\log\left(x-\sqrt{x^2-a^2}\right)=\log z$, where $z=x-\sqrt{x^2-a^2}$.

$$\therefore\ \frac{dy}{dz}=\frac{1}{z}\text{ and }\frac{dz}{dx}=1-\frac{1}{2}\left(x^2-a^2\right)^{(1/2)-1}\times\frac{d}{dx}\left(x^2-a^2\right)=1-\frac{1}{2}\left(x^2-a^2\right)^{-1/2}\times 2x$$
$$=1-\frac{x}{\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}-x}{\sqrt{x^2-a^2}}=\frac{-z}{\sqrt{x^2-a^2}}.$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{1}{z}\times\frac{-z}{\sqrt{x^2-a^2}}=\mathbf{-\frac{1}{\sqrt{x^2-a^2}}}.$$

Otherwise.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x-\sqrt{x^2-a^2}} \times \frac{d}{dx}\left(x-\sqrt{x^2-a^2}\right)\\ &= \frac{1}{x-\sqrt{x^2-a^2}} \times \left\{1-\frac{1}{2}\left(x^2-a^2\right)^{(1/2)-1} \cdot \frac{d}{dx}\left(x^2-a^2\right)\right\}\\ &= \frac{1}{x-\sqrt{x^2-a^2}} \times \left\{1-\frac{1}{2}\left(x^2-a^2\right)^{-1/2} \times 2x\right\}\\ &= \frac{1}{x-\sqrt{x^2-a^2}} \times \left\{1-\frac{x}{\sqrt{x^2-a^2}}\right\}\\ &= \frac{1}{x-\sqrt{x^2-a^2}} \times \frac{\sqrt{x^2-a^2}-x}{\sqrt{x^2-a^2}}\\ &= \frac{1}{x-\sqrt{x^2-a^2}} \times \frac{-\left(x-\sqrt{x^2-a^2}\right)}{\sqrt{x^2-a^2}} = -\frac{1}{\sqrt{x^2-a^2}}.\end{aligned}$$

Similarly, try and obtain: $\frac{d}{dx}\left[\log_e\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{\sqrt{1+x^2}}$. [C.U. B.Com.(H) 1994]

3.5.1 Differentiation of Implicit Functions

By an implicit function of x and y, we mean a relation involving x and y, where none of the variables is directly expressed in terms of the other.

$$x^2+y^2=a^2,\ y^2=4ax;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\ ax^2+2hxy+by^2=0$$

are examples of implicit functions of x and y.

Note: $x^2-4y=0$ is an implicit function but $y=x^2/4$ is an explicit relation. It is to be understood that when an implicit function will be given, we shall assume that it is possible to express y as a function of x though not explicitly shown.

To find $\frac{dy}{dx}$ from an implicit function we differentiate each term w.r.t. x and then solve for $\frac{dy}{dx}$.

Remember. *In all such cases:* $\frac{d}{dx}(y^n) = ny^{n-1}\cdot\frac{dy}{dx}$.

Example 19. *Find* $\frac{dy}{dx}$, *if* $x^3-9xy+y^3=0$.

Solution: We differentiate each term w.r.t. x (assuming that it is possible to express y as a function of x) and obtain

$$\frac{d}{dx}\left(x^3\right)-\frac{d}{dx}(9xy)+\frac{d}{dx}\left(y^3\right)=0$$

$$\text{or,}\quad 3x^2-9\left[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\right]+3y^2\frac{dy}{dx}=0$$

$$\text{or,}\quad 3x^2-9x\frac{dy}{dx}-9y+3y^2\frac{dy}{dx}=0$$

$$\text{or,}\quad 3x^2-9y=\frac{dy}{dx}\left(9x-3y^2\right).$$

$$\therefore\ \frac{dy}{dx}=\frac{3(x^2-3y)}{3(3x-y^2)}=\frac{x^2-3y}{3x-y^2}\ \left(\text{where } y^2\neq 3x\right).$$

Example 20. *Find* $\frac{dy}{dx}$, *when*

(i) $x^2+y^2=a^2$; [C.U. B.Com. 2007]

(ii) $x^2+y^2+2gx+2fy+c=0$

(g, f, c *are constants*);

(iii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; [C.U. B.Com. 2010]

(iv) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$;

(v) $3x^2+2xy-y^2=4$. [C.U. B.Com.(H) 1993]

Solution: In each case we differentiate every term w.r.t. x.

(i) $\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(a^2)$, i.e., $2x+2y\frac{dy}{dx}=0$, i.e., $\frac{dy}{dx}=-\frac{x}{y}$.

(ii) $\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)+2g\frac{d}{dx}(x)+2f\frac{d}{dx}(y)+\frac{d}{dx}(c)=0$,

i.e., $2x+2y\frac{dy}{dx}+2g\cdot 1+2f\frac{dy}{dx}+0=0$ or, $\frac{dy}{dx}=-\frac{x+g}{y+f}$.

(iii) $\frac{d}{dx}\left(\frac{x^2}{a^2}\right)+\frac{d}{dx}\left(\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$;

i.e., $\frac{1}{a^2}2x+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$, or, $\frac{dy}{dx}=-\frac{x}{a^2}\Big/\frac{y}{b^2}=-\frac{b^2}{a^2}\cdot\frac{x}{y}$.

(iv) Similarly, $\frac{dy}{dx}=\frac{b^2x}{a^2y}$.

(v) $\frac{d}{dx}(3x^2)+\frac{d}{dx}(2xy)-\frac{d}{dx}(y^2)=\frac{d}{dx}(4)$ or, $3\frac{d}{dx}(x^2)+2\frac{d}{dx}(xy)-\frac{d}{dx}(y^2)=0$

or, $3\times 2x+2\left(1.y+x\frac{dy}{dx}\right)-2y\cdot\frac{dy}{dx}=0$

or, $2\frac{dy}{dx}(x-y)=-2(3x+y)$, or, $\frac{dy}{dx}=-\frac{3x+y}{x-y}$.

3.5.2 Parametric Functions

If it is possible to express both x and y in terms of a third variable t (or θ or α, etc.), e.g., $x=f(t)$, $y=\phi(t)$, then we say that the two relations together define a parametric function of x and y; t (or θ or α, etc.) is called *parameter.*

To find $\frac{dy}{dx}$ in $x=f(t)$, $y=\phi(t)$, we first obtain $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and then write

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}.$$

Example 21. (a) *Given,* $x=at^2$, $y=2at$. *Find* $\frac{dy}{dx}$. [C.U. B.Com.(H) 2008]

(b) *If* $x=\frac{a}{t}$ *and* $y=2at$, *find* $\frac{dy}{dx}$. [C.U. B.Com. 2009]

Solution: (a) $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{d}{dt}(2at)\Big/\frac{d}{dt}(at^2)=\frac{2a}{2at}=\frac{1}{t}$.

(b) $\frac{dx}{dt}=\frac{d}{dt}\left(\frac{a}{t}\right)=a\times\left(-\frac{1}{t^2}\right)=-\frac{a}{t^2}$ and $\frac{dy}{dt}=\frac{d}{dt}(2at)=2a\frac{d}{dt}(t)=2a$.
Hence

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=2a\Big/\left(-\frac{a}{t^2}\right)=\mathbf{-2t^2}.$$

Example 22. *Let* $x=\sqrt{1-t^2}$, $y=t\sqrt{4-t^2}$; *find* $\frac{dy}{dx}$.

Solution: Here, $x=\sqrt{1-t^2}$;

$$\therefore\ \frac{dx}{dt}=\frac{d}{dt}\left(\sqrt{1-t^2}\right)=\frac{d}{dt}\left(u^{1/2}\right),\text{ where } u=1-t^2$$
$$=\frac{1}{2}u^{-(1/2)}\cdot\frac{du}{dt}=\frac{1}{2\sqrt{1-t^2}}\times -2t\ \left(\because\ \frac{du}{dt}=-2t\right)$$
$$=\frac{-t}{\sqrt{1-t^2}}$$

and $y=t\sqrt{4-t^2}$;

$$\therefore\ \frac{dy}{dt}=\frac{d}{dt}(t)\cdot\sqrt{4-t^2}+t\cdot\frac{d}{dt}\left(\sqrt{4-t^2}\right)=\sqrt{4-t^2}+t\cdot\frac{1}{2\sqrt{4-t^2}}\times -2t$$
$$=\frac{4-t^2-t^2}{\sqrt{4-t^2}}=\frac{4-2t^2}{\sqrt{4-t^2}}.$$

$$\frac{d}{dt}\left(\sqrt{4-t^2}\right)=\frac{d}{dt}u^{1/2},\text{ where } u=4-t^2$$
$$=\frac{1}{2}u^{-(1/2)}\cdot\frac{du}{dt}=\frac{1}{2}\cdot\frac{1}{\sqrt{u}}(-2t)=\frac{1}{2\cdot\sqrt{4-t^2}}\times(-2t).$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{4-2t^2}{\sqrt{4-t^2}}\Big/\frac{-t}{\sqrt{1-t^2}}=\mathbf{\frac{2(t^2-2)\sqrt{1-t^2}}{t\sqrt{4-t^2}}}.$$

Example 23. *Let* $y=\frac{a-t}{a+t}$ *and* $t=\frac{b-x}{b+x}$; *find* $\frac{dy}{dx}$.

Solution: Here, we use CHAIN RULE.

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$
$$=\frac{(a+t)\frac{d}{dt}(a-t)-(a-t)\frac{d}{dt}(a+t)}{(a+t)^2}\cdot\frac{(b+x)\frac{d}{dx}(b-x)-(b-x)\frac{d}{dx}(b+x)}{(b+x)^2}$$
$$=\frac{-(a+t)-(a-t)}{(a+t)^2}\cdot\frac{-(b+x)-(b-x)}{(b+x)^2}=\frac{-2a}{(a+t)^2}\cdot\frac{-2b}{(b+x)^2}.$$

$$\therefore\ \frac{dy}{dx}=\frac{4ab}{\{(a+t)(b+x)\}^2}.$$

Example 24. *Find* $\frac{dy}{dx}$ *if* $x=\frac{2at}{1+t^2}, y=\frac{1-t^2}{1+t^2}$. [C.U. B.Com.(H) 2001]

Solution:

$$x = \frac{2at}{1+t^2} \Rightarrow \frac{dx}{dt}=\frac{2a(1+t^2)-2at.2t}{(1+t^2)^2}=\frac{2a-2at^2}{(1+t^2)^2}=\frac{2a(1-t^2)}{(1+t^2)^2}$$

$$\text{and } y = \frac{1-t^2}{1+t^2} \Rightarrow \frac{dy}{dt}=\frac{-2t(1+t^2)-(1-t^2)2t}{(1+t^2)^2}=\frac{-4t}{(1+t^2)^2}.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{-4t}{(1+t^2)^2}\bigg/\frac{2a(1-t^2)}{(1+t^2)^2}=\frac{-4t}{2a(1-t^2)}=\boldsymbol{\frac{-2t}{a(1-t^2)}}.$$

Example 25. *Find* $\frac{dy}{dx}$ *if* $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$. [C.U. B.Com.(H) 2000]

Solution:

$$x = \frac{3at}{1+t^3} \Rightarrow \frac{dx}{dt}=\frac{3a(1+t^3)-3at\cdot 3t^2}{(1+t^3)^2}=\frac{3a-6at^3}{(1+t^3)^2}=\frac{3a(1-2t^3)}{(1+t^3)^2}$$

$$y = \frac{3at^2}{1+t^3} \Rightarrow \frac{dy}{dt}=\frac{6at(1+t^3)-3at^2\cdot 3t^2}{(1+t^3)^2}=\frac{6at-3at^4}{(1+t^3)^2}=\frac{3at(2-t^3)}{(1+t^3)^2}.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{3at(2-t^3)}{(1+t^3)^2}\div\frac{3a(1-2t^3)}{(1+t^3)^2}=\frac{3at(2-t^3)}{3a(1-2t^3)}=\boldsymbol{\frac{t(2-t^3)}{1-2t^3}}.$$

3.5.3 Exponential and Logarithmic Functions

We recall here,

$$\boldsymbol{\frac{d}{dx}(e^x)=e^x;\ \frac{d}{dx}(a^x)=a^x\log a (a>0);\text{ and } \frac{d}{dx}(\log_e x)=\frac{1}{x}.}$$

See that,

$$\frac{d}{dx}\left(e^{5x}\right)=5e^{5x};\ \frac{d}{dx}(2^x)=2^x\log 2;\ \frac{d}{dx}\log(1+x)=\frac{1}{1+x}.$$

Example 26. *Find* $\frac{dy}{dx}$, *if* $y=\log_e\sqrt{1-x^2}$.

Remember. *Logarithms w.r.t. the base e are called* Natural Logarithms. *So, one may write* $\log\sqrt{1-x^2}$ *instead of* $\log_e\sqrt{1-x^2}$.

Solution:

$$y=\log_e\sqrt{1-x^2}=\tfrac{1}{2}\log\left(1-x^2\right).$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left\{\frac{1}{2}\log\left(1-x^2\right)\right\}=\frac{1}{2}\cdot\frac{d}{dx}(\log v),\text{ where } v=1-x^2\ \left(\because \frac{dv}{dx}=-2x\right)$$

$$=\frac{1}{2}\cdot\frac{1}{v}\cdot\frac{dv}{dx}=\frac{1}{2(1-x^2)}\times(-2x)=-\boldsymbol{\frac{x}{1-x^2}}.$$

Example 27. *Differentiate w.r.t. x*: $\log\left(\sqrt{x-a}+\sqrt{x-b}\right)$. [C.U. B.Com.(H) 1998]

Solution: Let $y=\log\left(\sqrt{x-a}+\sqrt{x-b}\right)$.

$$\therefore\ \frac{dy}{dx}=\frac{1}{\sqrt{x-a}+\sqrt{x-b}}\cdot\frac{d}{dx}\left(\sqrt{x-a}+\sqrt{x-b}\right)$$
$$=\frac{1}{\sqrt{x-a}+\sqrt{x-b}}\times\left\{\frac{1}{2}(x-a)^{-1/2}\frac{d}{dx}(x-a)+\frac{1}{2}(x-b)^{-1/2}\cdot\frac{d}{dx}(x-b)\right\}$$
$$=\frac{1}{\sqrt{x-a}+\sqrt{x-b}}\times\frac{1}{2}\left\{\frac{\sqrt{x-b}+\sqrt{x-a}}{\sqrt{x-a}\cdot\sqrt{x-b}}\right\}=\mathbf{\frac{1}{2\sqrt{(x-a)(x-b)}}}.$$

Example 28. *Find* $\frac{dy}{dx}$, *when* $y=\sqrt{\frac{1+x}{1-x}}$. [C.U. B.Com. 1993; N.B.U. B.Com.(H) 2007]

Solution: We have,

$$y=\sqrt{\frac{1+x}{1-x}};\ \therefore\ \log y=\frac{1}{2}\left[\log(1+x)-\log(1-x)\right].$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)-\frac{1}{1-x}\cdot\frac{d}{dx}(1-x)\right]=\frac{1}{2}\left[\frac{1}{1+x}\times 1-\frac{1}{1-x}\times(-1)\right]$$
$$=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=\frac{1}{2}\left\{\frac{1-x+1+x}{(1+x)(1-x)}\right\}=\frac{1}{2}\times\frac{2}{1-x^2}=\frac{1}{1-x^2}.$$
$$\frac{dy}{dx}=y\left(\frac{1}{1-x^2}\right)=\sqrt{\frac{1+x}{1-x}}\times\frac{1}{1-x^2}=\mathbf{\frac{1}{(1-x)\sqrt{1-x^2}}}.$$

Example 29. *If* $y=\log\sqrt{\frac{1+x^2}{1-x^2}}$, *then find* $\frac{dy}{dx}$.

Solution: We have $y=\frac{1}{2}\left[\log(1+x^2)-\log(1-x^2)\right]$.

$$\therefore\ \frac{dy}{dx}=\frac{1}{2}\cdot\frac{d}{dx}\left\{\log\left(1+x^2\right)\right\}-\frac{1}{2}\cdot\frac{d}{dx}\left\{\log\left(1-x^2\right)\right\}=\frac{1}{2}\cdot\frac{d}{dx}(\log v)-\frac{1}{2}\cdot\frac{d}{dx}(\log w),$$
$$\text{where } v=1+x^2\left(\therefore\ \frac{dv}{dx}=2x\right)\text{ and } w=1-x^2\left(\therefore\ \frac{dw}{dx}=-2x\right)$$
$$=\frac{1}{2}\cdot\frac{1}{v}\cdot\frac{dv}{dx}-\frac{1}{2}\cdot\frac{1}{w}\cdot\frac{dw}{dx}=\frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x-\frac{1}{2\left(1-x^2\right)}\cdot(-2x)$$
$$=\frac{x}{1+x^2}+\frac{x}{1-x^2}=\frac{x\left(1-x^2\right)+x\left(1+x^2\right)}{1-x^4}=\mathbf{\frac{2x}{1-x^4}}.$$

3.5.4 Logarithmic Differentiation

In this process we first take logarithm w.r.t. the base e and then differentiate. Examples will make the idea clear.

Example 30. *Find* $\frac{dy}{dx}$ *in each of the following* 3 *cases:* (i) $y = x^x$; [B.U. B.Com.(H) 2007]

(ii) $y = \sqrt{\frac{(x-1)(x-4)}{(x-2)(x-5)}}$; (iii) $y = x^x + e^{ax^2+bx+c}$. [C.U. B.Com.(H) 2001]

Solution: (i) $y = x^x$. Take natural logarithm of both sides. Thus, $\log y = x \log x$.

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x)$$

$$\text{or,} \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x) \cdot \log x + x \cdot \frac{d}{dx}(\log x) = \log x + x \cdot \frac{1}{x} = 1 + \log x.$$

$$\therefore \frac{dy}{dx} = y(1 + \log x) = \mathbf{x^x(1 + \log x)} \; (\because y = x^x).$$

Note: $\frac{d}{dx}(\log y) = \frac{d}{dy}(\log y) \cdot \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}$.

Remember. $\frac{d}{dx}(x^a) = ax^{a-1}$; $\frac{d}{dx}(a^x) = a^x \log a$; $\frac{d}{dx}(x^x) = x^x(1 + \log x)$ [See (i)]

It, therefore, follows $\frac{d}{dx}(x^x + 2^x) = x^x(1 + \log x) + 2^x \log 2$. [C.U. B.Com.(H) 1994]

(ii) $y = \sqrt{\frac{(x-1)(x-4)}{(x-2)(x-5)}}$. Take natural logarithm of both sides.

$$\log y = \tfrac{1}{2}[\log(x-1) + \log(x-4) - \log(x-2) - \log(x-5)].$$

Differentiate both sides w.r.t. x.

$$\frac{d}{dx}(\log y) = \frac{1}{2}\left[\frac{d}{dx}\log(x-1) + \frac{d}{dx}\log(x-4) - \frac{d}{dx}\log(x-2) - \frac{d}{dx}\log(x-5)\right]$$

$$\text{i.e.,} \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-4} - \frac{1}{x-2} - \frac{1}{x-5}\right] = \frac{1}{2}\left[\frac{x-2-x+1}{(x-1)(x-2)} + \frac{x-5-x+4}{(x-4)(x-5)}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{(x-1)(x-2)} + \frac{-1}{(x-4)(x-5)}\right] = -\frac{1}{2}\left[\frac{x^2-9x+20+x^2-3x+2}{(x-1)(x-2)(x-4)(x-5)}\right]$$

$$= -\frac{1}{2}\left[\frac{2x^2-12x+22}{(x-1)(x-2)(x-4)(x-5)}\right].$$

$$\therefore \frac{dy}{dx} = -y\left[\frac{x^2-6x+11}{(x-1)(x-2)(x-4)(x-5)}\right],$$

where y *is the given function.*

(iii) Given $y = u + v$, where $u = x^x$ and $v = e^{ax^2+bx+c}$. [C.U. B.Com.(H) 2001]

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Since, $u = x^x$;

$$\frac{du}{dx} = x^x(1 + \log x) \text{ [refer to (i) above]}.$$

Again, $v = e^{ax^2+bx+c} = e^w$, where $w = ax^2+bx+c$ so that $v = e^w$

$$\text{and } \frac{dv}{dw} = e^w;\ \frac{dw}{dx} = a\frac{d}{dx}\left(x^2\right)+b\frac{d}{dx}(x)+\frac{d}{dx}(c) = 2ax+b$$

$$\text{i.e., } \frac{dv}{dx} = \frac{dv}{dw}\times\frac{dw}{dx} = e^w(2ax+b) = e^{x^2+bx+c}(2ax+b).$$

$$\therefore\ \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} = \mathbf{x^x(1+\log x)+e^{ax^2+bx+c}\cdot(2ax+b).}$$

Example 31. *Find $\frac{dy}{dx}$, when $y = (1+2x)^x$.* [C.U. B.Com.(H) 1999]

Solution: $y = (1+2x)^x$. Taking logarithm of both sides w.r.t. base e, we get

$$\log_e y = x\cdot\log_e(1+2x).$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log(1+2x)+x\cdot\frac{1}{1+2x}\cdot\frac{d}{dx}(1+2x)$$

$$= \log(1+2x)+\frac{x}{1+2x}\times 2 = \log(1+2x)+\frac{2x}{1+2x}.$$

$$\therefore\ \frac{dy}{dx} = y\left\{\log(1+2x)+\frac{2x}{1+2x}\right\} = \mathbf{(1+2x)^x\cdot\left\{\log(1+2x)+\frac{2x}{1+2x}\right\}.}$$

Example 32. *Differentiate w.r.t. x* : (i) $\left(x^2+a^2\right)^x$. [C.U. B.Com.(H) 1998]
(ii) x^x+2^x. [C.U. B.Com.(H) 1994]

Solution: (i) Let $y = \left(x^2+a^2\right)^x$. Taking logarithm of both sides w.r.t. base e, we get $\log y = x\log\left(x^2+a^2\right)$.
Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log\left(x^2+a^2\right)+x\times\frac{1}{x^2+a^2}\cdot\frac{d}{dx}\left(x^2+a^2\right)$$

$$= \log\left(x^2+a^2\right)+\frac{x}{x^2+a^2}\times 2x = \log\left(x^2+a^2\right)+\frac{2x^2}{x^2+a^2}.$$

$$\therefore\ \frac{dy}{dx} = y\left\{\log\left(x^2+a^2\right)+\frac{2x^2}{x^2+a^2}\right\} = \mathbf{\left(x^2+a^2\right)^x\left\{\log\left(x^2+a^2\right)+\frac{2x^2}{x^2+a^2}\right\}.}$$

(ii) Let

$$y = x^x+2^x = u+2^x, \text{ where } u = x^x. \tag{1}$$

$$\therefore\ \frac{dy}{dx} = \frac{du}{dx}+2^x\log_e 2. \tag{2}$$

Now, from eq. (1), $\log u = x\log x$. Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = 1\cdot\log x+x\cdot\frac{1}{x} = \log x+1 \text{ or, } \frac{du}{dx} = u(\log x+1) = x^x(\log x+1).$$

Hence, from (2), we have

$$\frac{dy}{dx} = \mathbf{x^x\left(\log_e x+1\right)+2^x\log_e 2.}$$

Example 33. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$; *find* $\dfrac{dy}{dx}$.

Solution: We use Quotient Rule,

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}.$$

Now
$$\frac{d}{dx}(e^{-x}) = \frac{d}{dx}(e^u), \text{ (where } u = -x)$$
$$= e^u \cdot \frac{du}{dx} = e^{-x} \cdot (-1) = -e^{-x}.$$

$$\therefore \frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$= \boldsymbol{\frac{4}{(e^x + e^{-x})^2}} \qquad \left[\text{Numerator} = (e^x + e^{-x})^2 - (e^x - e^{-x})^2 = 4 \cdot e^x \cdot e^{-x} = 4\right]$$

Example 34. $(x + y)^{m+n} = x^m y^n$; *find* $\dfrac{dy}{dx}$.

[N.B.U. B.Com.(H) 2006; B.U. B.Com.(H) 2008; C.U. B.Com. 2010; V.U. B.Com.(H) 2009]

Solution: Taking natural logarithm of both sides, we get

$$(m + n)\log(x + y) = m\log x + n\log y.$$

Differentiating both sides w.r.t. x, we get

$$(m+n)\frac{1}{x+y}\left[1 + \frac{dy}{dx}\right] = m \cdot \frac{1}{x} + \frac{n}{y} \cdot \frac{dy}{dx}$$

or, $\dfrac{dy}{dx}\left(\dfrac{m+n}{x+y} - \dfrac{n}{y}\right) = \dfrac{m}{x} - \dfrac{m+n}{x+y}$

or, $\dfrac{dy}{dx} \cdot \dfrac{my + ny - nx - ny}{y(x+y)} = \dfrac{mx + my - mx - nx}{x(x+y)}$

or, $\dfrac{dy}{dx} \cdot \dfrac{my - nx}{y(x+y)} = \dfrac{my - nx}{x(x+y)}$,

i.e., $\boldsymbol{\dfrac{dy}{dx} = \dfrac{y}{x}}$ (assuming $my \neq nx$), i.e., $\frac{dy}{dx}$ is independent of m, n.

Verify that: If $x^{2p}y^q = (x + y)^{2p+q}$, then $\frac{dy}{dx} = \frac{y}{x}$. [C.U. B.Com.(H) 2001]

Example 35. $y = \dfrac{1}{1 + x^{n-m} + x^{p-m}} + \dfrac{1}{1 + x^{m-n} + x^{p-n}} + \dfrac{1}{1 + x^{m-p} + x^{n-p}}$. *To find* $\dfrac{dy}{dx}$.

Solution:

$$y = \frac{x^m}{x^m + x^n + x^p} + \frac{x^n}{x^n + x^m + x^p} + \frac{x^p}{x^p + x^m + x^n} = \frac{x^m + x^n + x^p}{x^m + x^n + x^p} = 1.$$

$$\therefore \boldsymbol{\frac{dy}{dx} = 0.}$$

Example 36. *If* $f(x)=\left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$, *prove that* $f'(0)=\left(2\log\dfrac{a}{b}+\frac{b^2-a^2}{ab}\right)\left(\dfrac{a}{b}\right)^{a+b}$.

[C.U. B.Com.(H) 2006]

Solution: Note that $f'(0)$ means the derivative of $f(x)$ w.r.t. x at the value $x=0$.

We first find the derivative $f'(x)$ at any given value of x and then for x we shall substitute 0.

Here,

$$\log f(x)=(a+b+2x)[\log(a+x)-\log(b+x)].$$

$$\therefore \quad \frac{d}{dx}\{\log f(x)\}=\frac{d}{dx}(a+b+2x)\cdot\log\frac{a+x}{b+x}$$
$$+(a+b+2x)\left[\frac{d}{dx}\log(a+x)-\frac{d}{dx}\log(b+x)\right]$$

$$\text{or,}\quad \frac{1}{f(x)}f'(x)=2\log\frac{a+x}{b+x}+(a+b+2x)\left[\frac{1}{a+x}-\frac{1}{b+x}\right].$$

Put $x=0$. We obtain,

$$f'(0)=f(0)\left[2\log\frac{a}{b}+(a+b)\frac{b-a}{ab}\right]=\left(\frac{a}{b}\right)^{a+b}\left[2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right] \text{ (Proved).}$$

Example 37. (i) *If* $x^y=e^{x-y}$, *prove that* $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$. [C.U. B.Com.(H) 2008; B.U. B.Com.(H) 2008]

(ii) *If* $e^{xy}-4xy=2$, *then prove that* $\dfrac{dy}{dx}=-\dfrac{y}{x}$. [C.U. B.Com.(H) 2002; V.U. B.Com.(H) 2010]

Solution: (i) Given, $x^y=e^{x-y}$. Taking logarithms of both sides w.r.t. the base e, we get

$$y\log x=(x-y). \tag{1}$$

Now differentiating w.r.t. x, we obtain

$$\frac{d}{dx}(y\log x)=\frac{d}{dx}(x-y) \quad \text{or,} \quad \frac{dy}{dx}\log x+y\frac{d}{dx}(\log x)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\text{or,}\quad \frac{dy}{dx}\log x+\frac{y}{x}=1-\frac{dy}{dx}$$

$$\text{or,}\quad \frac{dy}{dx}\log x+\frac{dy}{dx}=1-\frac{y}{x}$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{1-y/x}{1+\log x}.$$

Now, from eq. (1), we get $y(1+\log x)=x$ or, $y=\dfrac{x}{1+\log x}$.

$$\therefore \quad \frac{dy}{dx}=\frac{1\cdot(1+\log x)-x.\left(0+\frac{1}{x}\right)}{(1+\log x)^2}=\frac{1+\log x-1}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}.$$

We can write, $\quad 1+\log x=\log e+\log x=\log(ex)$

and then $\boldsymbol{\dfrac{dy}{dx}=\dfrac{\log x}{(\log ex)^2}}$.

(ii) Given: $e^{xy}-4xy=2$.

Differentiating w.r.t. x, we get

$$e^{xy}\left[y+x\frac{dy}{dx}\right]-4y-4x\frac{dy}{dx}=0 \text{ or, } \frac{dy}{dx}(xe^{xy}-4x)=4y-ye^{xy}$$

$$\text{or, } \frac{dy}{dx}=\frac{y(4-e^{xy})}{-x(4-e^{xy})}=-\frac{y}{x}, \text{ assuming } e^{xy}\neq 4.$$

Example 38. (i) *Find the derivative of* x^4 *w.r.t.* $\sqrt{1+x}$.

(ii) *Find the derivative of* $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ *w.r.t.* $\sqrt{1-x^4}$.

Solution: Let $y=x^4$, $z=\sqrt{1+x}$. To find $\frac{dy}{dz}$. We know,

$$\frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=\frac{dy}{dx}\cdot\frac{1}{\frac{dz}{dx}}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}.$$

Here, $$\frac{dy}{dx}=4x^3,\ \frac{dz}{dx}=\frac{1}{2\sqrt{1+x}}.\quad \therefore\ \frac{dy}{dz}=\frac{4x^3}{1/\left(2\sqrt{1+x}\right)}=\mathbf{8x^3\sqrt{1+x}}.$$

(ii) Let $$y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)^2}{\left(\sqrt{1+x^2}\right)^2-\left(\sqrt{1-x^2}\right)^2}\quad \text{(rationalizing the denominator)}$$

$$\text{or, } y=\frac{1+x^2+1-x^2+2\sqrt{1-x^4}}{(1+x^2)-(1-x^2)}=\frac{1+\sqrt{1-x^4}}{x^2}, \tag{1}$$

$$\text{i.e., } y=\frac{1}{x^2}+\frac{1}{x^2}\sqrt{1-x^4}.$$

$$\therefore\ \frac{dy}{dx}=-\frac{2}{x^3}-\frac{2}{x^3}\sqrt{1-x^4}+\frac{1}{x^2}\cdot\frac{1}{2\sqrt{1-x^4}}\times\left(-4x^3\right)=-\frac{2}{x^3}\left(1+\sqrt{1-x^4}\right)-\frac{2x}{\sqrt{1-x^4}}$$

$$=-\frac{\left(2+2\sqrt{1-x^4}\right)\sqrt{1-x^4}+2x^4}{x^3\sqrt{1-x^4}}=-\frac{2\sqrt{1-x^4}+2(1-x^4)+2x^4}{x^3\sqrt{1-x^4}}$$

$$=-\frac{2\left\{1+\sqrt{1-x^4}\right\}}{x^3\sqrt{1-x^4}}.$$

Let $z=\sqrt{1-x^4}$; so that $\dfrac{dz}{dx}=\dfrac{1}{2\sqrt{1-x^4}}(-4x^3)=-\dfrac{2x^3}{\sqrt{1-x^4}}$.

Hence, $$\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=-\frac{2\left\{1+\sqrt{1-x^4}\right\}}{x^3\sqrt{1-x^4}}\times\left(-\frac{\sqrt{1-x^4}}{2x^3}\right)=\frac{1+\sqrt{1-x^4}}{x^6}$$

$$=\frac{1}{x^4}\cdot\frac{1+\sqrt{1-x^4}}{x^2}=\frac{1}{x^4}y=\frac{1}{x^4}\cdot\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\ \text{[by (1)]}.$$

Example 39. *If* $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, *show that* $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$. [C.U. B.Com. 2009]

Solution: Given: $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$.

Differentiating w.r.t. x, we get

$$\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)+\frac{1}{2\sqrt{1-y^2}}\cdot\left(-2y\frac{dy}{dx}\right)=a\left[1-\frac{dy}{dx}\right]$$

or, $$\frac{dy}{dx}\left(\frac{y}{\sqrt{1-y^2}}-a\right)=-\left[a+\frac{x}{\sqrt{1-x^2}}\right]$$

or, $$\frac{dy}{dx}\left[\frac{y}{\sqrt{1-y^2}}-\frac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}\right]=-\left[\frac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}+\frac{x}{\sqrt{1-x^2}}\right]$$

or, $$\frac{dy}{dx}\cdot\frac{y(x-y)-\sqrt{1-x^2}\sqrt{1-y^2}-(1-y^2)}{(x-y)\sqrt{1-y^2}}=-\left[\frac{(1-x^2)+\sqrt{1-y^2}\sqrt{1-x^2}+x(x-y)}{(x-y)\sqrt{1-x^2}}\right]$$

or, $$\frac{dy}{dx}\cdot\frac{xy-1-\sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}=-\frac{1-xy+\sqrt{1-y^2}\sqrt{1-x^2}}{\sqrt{1-x^2}}.$$

$$\therefore\ \boldsymbol{\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}}.$$

Example 40. *Find* $\frac{dy}{dx}$, *if* $x^y+y^x=1$. [C.U. B.Com.(H) 2007]

Solution: Let $u=x^y$ and $v=y^x$. Then $\log u=y\log x$ and $\log v=x\log y$.

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{y}{x}+\log x\frac{dy}{dx}\text{ and }\frac{1}{v}\cdot\frac{dv}{dx}=\log y+\frac{x}{y}\cdot\frac{dy}{dx}$$

or, $$\frac{du}{dx}=u\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)\text{ and }\frac{dv}{dx}=v\left(\log y+\frac{x}{y}\cdot\frac{dy}{dx}\right).$$

Given: $x^y+y^x=1$ or, $u+v=1$.

$$\therefore\quad \frac{du}{dx}+\frac{dv}{dx}=0\text{ or, }u\left\{\frac{y}{x}+\log x\frac{dy}{dx}\right\}+v\left\{\log y+\frac{x}{y}\cdot\frac{dy}{dx}\right\}=0$$

or, $$\frac{dy}{dx}\left\{u\log x+v\frac{x}{y}\right\}=-v\log y-u\frac{y}{x}$$

or, $$\frac{dy}{dx}=-\frac{v\log y+u\frac{y}{x}}{u\log x+v\frac{x}{y}}=-\frac{y^x\log y\cdot x+x^y\cdot y}{x}\Big/\frac{x^y\log x\cdot y+y^x\cdot x}{y}$$

$$\boldsymbol{=-\frac{y}{x}\cdot\frac{y^x x\log y+x^y\cdot y}{x^y y\log x+y^x\cdot x}}.$$

Note: It is wrong to write from $x^y+y^x=1$, $y\log x+x\log y=\log 1=0$. (Why?)

Example 41. *If* $y = 1 + \dfrac{a}{x-a} + \dfrac{bx}{(x-a)(x-b)}$, *prove that* $\dfrac{dy}{dx} = \dfrac{y}{x}\left\{\dfrac{a}{a-x} + \dfrac{b}{b-x}\right\}$.

Solution: Given:

$$y = 1 + \frac{a}{x-a} + \frac{bx}{(x-a)(x-b)}$$

$$\text{or,} \quad y = \frac{x}{x-a} + \frac{bx}{(x-a)(x-b)} = \frac{x}{x-a}\left(1 + \frac{b}{x-b}\right)$$

$$\text{or,} \quad y = \frac{x^2}{(x-a)(x-b)}.$$

$\therefore$ $\log y = 2\log x - [\log(x-a) + \log(x-b)]$ (taking logarithm w.r.t. the base e)

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x-a} - \frac{1}{x-b} = \left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right)$$

$$= \frac{x-a-x}{x(x-a)} + \frac{x-b-x}{x(x-b)} = \frac{1}{x}\left[\frac{-a}{x-a} - \frac{b}{x-b}\right].$$

$$\therefore \quad \boldsymbol{\frac{dy}{dx} = \frac{y}{x}\left[\frac{a}{a-x} + \frac{b}{b-x}\right]}.$$

Example 42. *Let* $S_n = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$ $(x > 1)$. *Prove that*

$$(x-1)\frac{dS_n}{dx} = (n-1)S_n - nS_{n-1}.$$

Solution:

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$$

$$\text{or,} \quad S_n = \frac{x^n - 1}{x-1} \quad \text{(assuming the GP with common ratio } x > 1) \qquad (1)$$

[For those who are not acquainted with this formula: consider $xS_n = x + x^2 + x^3 + \cdots + x^n$. Now, obtain $xS_n - S_n = x^n - 1$; hence, etc.]

See that,

$$S_{n-1} = 1 + x + x^2 + \cdots + x^{n-2};$$

$$\therefore \quad S_n - S_{n-1} = x^{n-1}.$$

From eq. (1),

$$(x-1)S_n = x^n - 1 \quad \text{or,} \quad (x-1)\frac{dS_n}{dx} + S_n = nx^{n-1} = n(S_n - S_{n-1})$$

$$\text{or,} \quad \boldsymbol{(x-1)\frac{dS_n}{dx} = (n-1)S_n - nS_{n-1}}.$$

Example 43. (i) *Obtain* $\dfrac{dy}{dx}$, *when* $y = x^{x^x} + 2$. [C.U. B.Com.(H) 2001]

(ii) *Find* $\dfrac{dy}{dx}$, *if* $y = (x^x)^x$.

Solution: (i) Let $u = x^{x^x} = (x)^{x^x}$.

Taking logarithm of both sides w.r.t. base e, we get $\log_e u = x^x . \log_e x$.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\cdot\frac{du}{dx} = x^x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x^x) = x^x \cdot \frac{1}{x} + \log x \cdot \frac{dv}{dx}, \text{ where } v = x^x. \quad (1)$$

$$\therefore \ \frac{du}{dx} = u\left(x^{x-1} + \log x \cdot \frac{dv}{dx}\right). \quad (2)$$

From $v = x^x$, we get

$$\log_e v = x \log_e x.$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = 1\cdot\log x + x \cdot \frac{1}{x} = \log_e x + 1 \text{ or, } \frac{dv}{dx} = v\left(\log_e x + 1\right) = x^x\left(1 + \log_e x\right).$$

$\therefore$ from (2),

$$\frac{du}{dx} = u\left(x^{x-1} + \log x \cdot \frac{dv}{dx}\right) = x^{x^x}\left[x^{x-1} + \log_e x\left\{x^x\left(1 + \log_e x\right)\right\}\right]$$

$$= x^{x^x}\left\{x^{x-1} + x^x \log_e x\left(1 + \log_e x\right)\right\}.$$

Now,

$$y = x^{x^x} + 2 = u + 2.$$

$$\therefore \ \frac{dy}{dx} = \frac{du}{dx} + 0 = \boldsymbol{x^{x^x}\left\{x^{x-1} + x^x \log_e x\left(1 + \log_e x\right)\right\}}.$$

(ii) $y = (x^x)^x = x^{x\cdot x} = x^{x^2}$ $[\because (a^m)^n = a^{mn}]$

Now taking logarithm, we get

$$\log y = x^2 \log x.$$

Differentiating both sides, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 2x\log x + x^2 \cdot \frac{1}{x}.$$

$$\therefore \ \frac{dy}{dx} = y\left(2x\log x + x\right) = \boldsymbol{(x^x)^x\left(2x\log x + x\right)}.$$

EXERCISES ON CHAPTER 3

(Derivative of a Function)

Remember.

- $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$, if exists,
  is called the *derivative* of $f(x)$ w.r.t. x, at $x = c$, denoted by $f'(c)$.
- $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, if exists,
  is called the *derivative* of $f(x)$ w.r.t. x, at any point x, denoted by $f'(x)$.
- $\lim\limits_{h\to 0^+}\dfrac{f(x+h)-f(x)}{h}$ and $\lim\limits_{h\to 0^-}\dfrac{f(x+h)-f(x)}{h}$
  are respectively the right-hand and left-hand derivatives at the point x.

1. Find, from first principle (i.e., using the definition), the derivative of:

(a) x^2 at $x=1$;
(b) x^2+7x+9 at any point x;
(c) $\sqrt{x}$ at $x=4$;
(d) $\frac{1}{\sqrt{x}}$, $\sqrt[3]{x}$ and $\frac{1}{x}$ at any point $x>0$; [V.U. B.Com.(H) 2011]
(e) $5x+7$ at some point x;
(f) $|x|$ at $x=0$;
(g) $\frac{1}{x^2}$, x^3, x^2-2x at any point x;
(h) $2x^3+3$, $2x^3+5$ at any point x;
(i) x^3+4 at $x=1$; [C.U. B.Com.(H) 1991; V.U. B.Com.(H) 2010]
(j) e^{mx}, $e^{\sqrt{x}}$ at any point x.
(k) $(x-1)^2$. [C.U. B.Com.(H) 2005]

Remember.

- $\frac{d}{dt}(t^n)=nt^{n-1}$ [n is any real number].
- derivative of a constant is zero and $\frac{d}{dx}\{cf(x)\}=c\frac{d}{dx}[f(x)]$.
- $\frac{d}{dx}(u+v-w)=\frac{d}{dx}(u)+\frac{d}{dx}(v)-\frac{d}{dx}(w)$.

2. Differentiate (w.r.t. the independent variable involved) [a,b,c are constants]:

(a) $5x^4$;
(b) $\sqrt{x}$;
(c) $\frac{2}{\sqrt{x}}$;
(d) $\frac{1}{x^2}$;
(e) $\frac{3}{x^5}$;
(f) $\frac{6}{x}$;
(g) $\frac{c}{v}$;
(h) $2\sqrt[3]{t^4}$;
(i) $\frac{1}{\sqrt[4]{x^5}}$;
(j) $\frac{1}{x^7}$;
(k) $3x^2-4x+3$;
(l) $3x^4-2x^2+8$;
(m) $8+7x-3x^3$;
(n) $x^{4/3}+5$;
(o) $\frac{3x^3}{\sqrt[5]{x^2}}$;
(p) at^5-5bt^3;
(q) $x^{2/3}-a^{2/3}$;
(r) $\sqrt{5x}-\frac{5}{\sqrt{5x}}$;
(s) $\frac{a+bx+cx^2}{\sqrt{x}}$;
(t) $3^{4x}+\frac{3}{\sqrt[3]{x}}$;
(u) $x^2 5^{32}$;
(v) $(3-5x)^{3/2}$;
(w) $4x^{5/4}+2x^{1/4}-\frac{x}{4\sqrt{x}}+\frac{x}{4\sqrt{x^5}}+7x+8$.

Remember. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ and $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

3. Find the derivative of each of the following functions w.r.t. the independent variable involved:

(a) $(ax+b)(cx+7)$; $10^x \cdot x^{10}$; $2^x \cdot \log_e x$;

(b) $\dfrac{(at^2+2bt+c)}{(At^2+2Bt+c)}$;

(c) $\dfrac{(x-1)(x-2)}{(x+1)(x+2)}$;

(d) $\dfrac{a-x}{a+x}$;

(e) $\dfrac{5}{5+7x}$;

(f) $\dfrac{a^2+u^2}{a^2-u^2}$; [C.U. B.Com.(H) 1990]

(g) $t^2\left(3t^{4/3}-8\right)$;

(h) $\dfrac{2-2x}{1+2x^2}$;

(i) $\dfrac{t^2+2}{2-t^2}$;

(j) $\dfrac{x^2-1}{x^2+1}$;

(k) $\dfrac{5-4x}{5+4x}$;

(l) $\dfrac{x^2-2x-3}{x-1}$;

(m) $\left(x^2+1\right)e^x$;

(n) $\dfrac{x}{e^x-1}$;

(o) $\dfrac{x^2+1}{x-1}$;

(p) $x^4.e^{3x^2}$; [C.U. B.Com.(H) 1993]

(q) $7^{x^2}+2x$.

Remember. If $y=f(v)$ and $v=\phi(x)$, then $\dfrac{dy}{dx}=\dfrac{dy}{dv}\cdot\dfrac{dv}{dx}$.

4. Find $\dfrac{dy}{dx}$ for each of the following functions:

(a) $y=\sqrt{x-1}$;

(b) $y=\sqrt{x^2+a^2}$;

(c) $y=\dfrac{1}{\sqrt{x^2+a^2}}$;

(d) $y=\left(5+\dfrac{6}{x^2}\right)^3$;

(e) $y=x\sqrt{x^2+a^2}$;

(f) $y=x^2\sqrt{a+bx}$;

(g) $y=\dfrac{x}{\sqrt{a^2-x^2}}$;

(h) $y=\sqrt[3]{\dfrac{2+3x}{2-3x}}$;

(i) $y=\left(a^{2/3}-x^{2/3}\right)^{3/2}$;

(j) $y=v^6$, $v=1+2\sqrt{x}$;

(k) $y=\sqrt{u}$, where $u=x^3-x^2$;

(l) $y=\dfrac{b-u}{b+u}$, where $u=\dfrac{a-x}{a+x}$;

(m) $y=\sqrt{ax^2+bx+c}$;

(n) $y=x^2\sqrt{ax^3+b}$;

(o) $y=e^{ax^2+bx+c}$;

(p) $y=\left(3x^2-9x+7\right)^{-(1/2)}$;

(q) $(e^x+1)y=e^x-1$. [C.U. B.Com.(H) 1995]

Remember. If y is a function of x, then $\frac{d}{dx}(y^n)=ny^{n-1}\cdot\frac{dy}{dx}$.

5. Find $\dfrac{dy}{dx}$, if

(a) $y^2=8x$;

(b) $x^2+y^2=a^2$;

(c) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; [C.U. B.Com.(H) 2010]

(d) $x^2+y^2+2gx+2fy+c=0$ (f, g, c are constants);

(e) $x^2+xy+2y^2=28$;

(f) $x^3+3x^2y+y^3=a^3$; [V.U. B.Com.(H) 2009]

(g) $x^2+7\sqrt{xy}+y^2=8$;

(h) $\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}=a^2$;

(i) $x^3-3axy+y^3=0$; [C.U. B.Com.(H) 1990]

(j) $\sqrt{x}+\sqrt{y}=\sqrt{a}$;

(k) $(x/a)^{2/3}+(y/b)^{2/3}=1$;

(l) $x^{2/3}+y^{2/3}=a^{2/3}$;

(m) $x\sqrt{y}+y\sqrt{x}=\sqrt{a}$;

(n) $x^4+x^2y^2+y^4=0$. [C.U. B.Com.(H) 1998]

(o) $x^{2p}y^q=(x+y)^{2p+q}$; [C.U. B.Com.(H) 2001]

(p) $2x^2+3xy+y^2=4$ at the point (0, 2). [C.U. B.Com.(H) 1995]

(q) $x^4+x^3y^3+y^4=0$. [C.U. B.Com.(H) 2006]

Parametric Functions: Remember. If $x=f(t)$, $y=\phi(t)$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{\phi'(t)}{f'(t)}.$$

6. Find $\frac{dy}{dx}$ for each of the following parametric functions:

(a) $x=t^2, y=2t+1$;

(b) $x=t^3, y=3t$ at $t=-1$;

(c) $x=ct, y=\frac{c}{t}$; [C.U. B.Com.(H) 1991; (P) 2010]

(d) $x=t^2, y=2-t$ at $t=1$;

(e) $x=3t, y=\frac{2}{t}$ at $t=2$;

(f) $3x=t^3, 2y=t^2$;

(g) $x=6t-t^2, y=2t+3$ at $t=0$;

(h) $x=\frac{1}{t}, y=5t$ at $t=-1$;

(i) $x=3t-t^3, y=t+1$;

(j) $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$; [C.U. B.Com.(H) 2000]

(k) $x=\sqrt{1+u}, y=\sqrt{1-u}$ at $u=\frac{1}{2}$;

(l) $x=ct^3, y=\frac{c}{t^3}$. [C.U. B.Com.(H) 1994; V.U. B.Com.(H) 2010]

(m) $x=\frac{t}{1+t}, y=\frac{t}{1-t}$. [C.U. B.Com.(H) 1999]

(n) $x=t\log t, y=\frac{\log t}{t}$. [N.B.U. B.Com.(H) 2006]

[Hints: (l) $\frac{dx}{dt}=3ct^2$; $\frac{dy}{dt}=-\frac{3c}{t^4}$ and hence $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-3c/t^4}{3ct^2}=-\frac{1}{t^6}$.

(m) $\frac{dx}{dt}=\frac{(1+t)-t}{(1+t)^2}=\frac{1}{(1+t)^2}$; $\frac{dy}{dt}=\frac{(1-t)-t(-1)}{(1-t)^2}=\frac{1}{(1-t)^2}$.

$\therefore \frac{dy}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{1}{(1-t)^2}/\frac{1}{(1+t)^2}=\left(\frac{1+t}{1-t}\right)^2$.]

Exponential and Logarithmic Functions: Remember.

$$\frac{d}{dx}(a^x)=a^x\log_e a;\quad \frac{d}{dx}(e^x)=e^x;\quad \frac{d}{dx}(a^v)=a^v\log_e a\frac{dv}{dx};$$
$$\frac{d}{dx}(e^v)=e^v\frac{dv}{dx};\quad \frac{d}{dx}(\log_e x)=\frac{1}{x};\quad \frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e.$$

7. Differentiate:

(a) $y=\log_e\left(x^2+a\right)$;

(b) $y=\log_e\frac{2x}{1+x^2}$;

(c) $y=\log_e\sqrt{1-x^2}$;

(d) $y=\log_e\sqrt{\frac{1+x^2}{1-x^2}}$;

(e) $y=\log_e(5x+7)$;

(f) $y=\log_e(ax+b)^3$;

(g) $y = (\log_e x)^3$;

(h) $y = \log_e x^3$;

(i) $y = \log_e \left(ax^2 + bx + c\right)$ assuming $u = ax^2 + bx + c$; [C.U. B.Com.(H) 1999]

(j) $y = \log_e \left(x + \sqrt{x^2 - a^2}\right)$; [C.U. B.Com.(H) 2009]

(k) $y = x \log_e x$;

(l) $y = \log_e \sqrt{\dfrac{a + bx}{a - bx}}$;

(m) $y = 10^x$; $y = 10^{5x}$;

(n) $y = e^{ax}$;

(o) $y = e^{x^2}$;

(p) $y = \dfrac{a}{e^x}$;

(q) $y = e^{\sqrt{x}}$;

(r) $z = b^{2y}$;

(s) $y = \dfrac{1}{x} \log_e x$;

(t) $y = \dfrac{e^x - 1}{e^x + 1}$;

(u) $y = x^2 e^{-x}$;

(v) $y = \log_e \dfrac{\sqrt{x^2 + a^2} - x}{\sqrt{x^2 + a^2} + x}$;

(w) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$;

(x) $y = \log_{10} x$;

(y) $y = \dfrac{1}{x^2} \log_e x^2$;

(z) $y = x^{5/2} \log x$; [C.U. B.Com.(H) 1990]

(aa) $x = y \log\left(x^2 y^2\right)$;

(ab) $x = y \log \dfrac{x}{y}$; [C.U. B.Com.(H) 1992]

(ac) $x = y \log\left(x^2 y^3\right)$;

(ad) $x = y \log(xy)$. [C.U. B.Com.(H) 1994]

Logarithmic Differentiation: Take logarithm and then differentiate and obtain the values of $\frac{dy}{dx}$:

8. (a) $y = x^x$; [C.U. B.Com.(H) 1990]

(b) $y = x^{\sqrt{x}}$;

(c) $y = \left(\dfrac{a}{x}\right)^x$;

(d) $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$;

(e) $y = \dfrac{x^3\sqrt{5x+7}}{\sqrt{2x+8}}$;

(f) $y = \dfrac{\sqrt{4 + x^2}}{x\sqrt{4 - x^2}}$;

(g) $y = e^{\sqrt{x}} \log \sqrt{x}$;

(h) $x^m y^n = (x + y)^{m+n}$; [C.U. B.Com. 2010]

(i) $x^y y^x = 1$; [C.U. B.Com.(H) 2001; V.U. B.Com.(H) 2008]

(j) $x = y \log(xy)$; [C.U. B.Com.(H) 1994]

(k) $\log xy = \dfrac{x}{y}$;

(l) $y = (x)^{x^x}$; $y = (x^x)^x$; $y = x^x$; [C.U. B.Com.(H) 1996]

(m) $y = x^2 \sqrt{\dfrac{x^2 + a^2}{x^2 - a^2}}$;

(n) $x^y + y = 1$;

(o) $x^y + xy = 8$;

(p) $y = (1 + x)^x$;

(q) $a^x + x^y = 4$; [C.U. B.Com. 2008]

(r) $x^y + y^x = 11$; [C.U. B.Com.(H) 2005]

(s) $x^y = y^x$; [C.U. B.Com.(H) 1992] [B.U. B.Com.(H) 2002]

(t) $y = x^x + \log\left(3x^2 + 4x + 5\right)$; [C.U. B.Com.(H) 2000]

(u) $(1 + x)^{1+x}$.

[**Hints:** (j) $x = y\{\log x + \log y\}$. Differentiate w.r.t. x :

$$1 = \frac{dy}{dx}(\log xy) + y\left(\frac{1}{x} + \frac{1}{y}\frac{dy}{dx}\right).$$

Hence,

$$\frac{dy}{dx}[\log xy+1]=1-\frac{y}{x},$$

$$\text{i.e., } \frac{dy}{dx}=\frac{x-y}{x(\log xy+1)}=\frac{x-y}{x\left[\frac{x}{y}+1\right]}=\frac{y(x-y)}{x(x+y)}.$$

(t) We write, $y=u+v$, where $u=x^x$ and $v=\log(3x^2+4x+5)$.

Then,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

We now separately find

$$\frac{du}{dx} \text{ and } \frac{dv}{dx}.$$

Now,

$$u=x^x \Rightarrow \log u=x\log x \Rightarrow \frac{1}{u}\cdot\frac{du}{dx}=\log x+x\cdot\frac{1}{x}=1+\log x.$$

$$\therefore\ \frac{du}{dx}=u(1+\log x)=x^x(1+\log x).$$

Again, $v=\log(3x^2+4x+5)$.

$$\frac{dv}{dx}=\frac{1}{3x^2+4x+5}\cdot\frac{d}{dx}\left(3x^2+4x+5\right)=\frac{6x+4}{3x^2+4x+5}=\frac{2(3x+2)}{3x^2+4x+5}.$$

$$\therefore\ \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}=\boldsymbol{x^x(1+\log x)+\frac{2(3x+2)}{3x^2+4x+5}}.$$

(u) Let $y=(1+x)^{1+x}$; then $\log y=(1+x)\log(1+x)$.

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=(0+1)\log(1+x)+(1+x)\cdot\frac{1}{1+x}\times 1=\log(1+x)+1$$

$$\therefore\ \frac{dy}{dx}=y\{\log(1+x)+1\}=\boldsymbol{(1+x)^{1+x}\cdot\{\log(1+x)+1\}}]$$

9. (a) If $y=2x^3-9x^2+12x+25$, find for what values of x, $\frac{dy}{dx}=0$.

(b) Prove that in each of the functions given below, $\frac{dy}{dx}=0$:

$$y=\frac{1}{1+x^{n-m}+x^{p-m}}+\frac{1}{1+x^{m-n}+x^{p-n}}+\frac{1}{1+x^{m-p}+x^{n-p}}$$

$$y=\left(x^{\frac{l+m}{l-m}}\right)^{1/(n-l)}\times\left(x^{\frac{m+n}{n-l}}\right)^{1/(l-m)}\times\left(x^{\frac{n+l}{n-l}}\right)^{1/(m-n)}$$

(c) Find the derivative of x^6 w.r.t. x^3.

(d) Find for what value of x, the value of $\frac{dy}{dx}=0$, if $y=2x^3-15x^2+36x+12$.

(e) If $y=\dfrac{x-2}{x+2}$, show that $2x\dfrac{dy}{dx}=1-y^2$. [C.U. B.Com.(H) 2003]

(f) If $f(x)=\left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$, show that $f'(0)=\left(2\log\dfrac{a}{b}+\dfrac{b^2-a^2}{ab}\right)\left(\dfrac{a}{b}\right)^{a+b}$

[C.U. B.Com.(H) 2002]

(g) Find $\frac{ds}{dt}$, if $s = t^{(1-t)} + t^2$. [C.U. B.Com.(H) 1995]

[s = sum of two functions of t, the first function $t^{1-t} = u$ (say) and the second function = $t^2 = v$ (say).

$$\therefore \frac{ds}{dt} = \frac{du}{dt} + \frac{dv}{dt} \qquad \left(\frac{dv}{dt} = 2t\right).$$

We find $\frac{du}{dt}$. Since $u = t^{1-t}$, $\log u = (1-t)\log t$ or, $\frac{1}{u}\frac{du}{dt} = -\log t + (1-t)\frac{1}{t}$.

$$\therefore \frac{du}{dt} = u\left[-\log t + (1-t)\frac{1}{t}\right] = t^{1-t}\left[-\log t + (1-t)t^{-1}\right].$$

$$\therefore \frac{ds}{dt} = \frac{du}{dt} + \frac{dv}{dt} = \boldsymbol{t^{1-t}\left[-\log t + (1-t)t^{-1}\right] + 2t}.$$

Alternatively, $u = t^{1-t} = e^{\log t^{1-t}} = e^{(1-t)\log t}$. Then proceed to find $\frac{du}{dt}$]

10. (a) Evaluate:

$$\frac{d}{dx}\left\{\frac{1}{4\sqrt{2}}\log\frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1}\right\}.$$

(b) Show that the derivative of

$$\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$$

w.r.t. $\sqrt{1-x^4}$ is

$$\frac{1}{x^4} \cdot \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}.$$

(c) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

(d) If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(\log ex)^2}$. [C.U. B.Com.(H) 1997]

(e) If we denote $S_n = 1 + r + r^2 + \cdots r^{2n-1}$, prove that

$$(r-1)\frac{d}{dr}(S_{2n}) = (2n-1)S_{2n} - 2nS_{2n-1}.$$

(f) If

$$y = 1 + \frac{a}{x-a} + \frac{bx}{(x-a)(x-b)} + \frac{cx^2}{(x-a)(x-b)(x-c)},$$

prove that

$$\frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}.$$

(g) Find $\frac{dy}{dx}$, when $\frac{x^2}{a^2} + \frac{2xy}{k} + \frac{y^2}{b^2} = 1$. [C.U. B.Com.(H) 1991]

(h) If $y = \dfrac{x}{\sqrt{1+x^2}}$, show that $x^3 \dfrac{dy}{dx} = y^3$. [C.U. B.Com.(H) 1999]

(i) If $e^{xy} = 4(1+xy)$ and $e^{xy} \neq 4$, then show that $\frac{dy}{dx} = -\frac{y}{x}$. [C.U. B.Com.(H) 1992]

(j) Find the gradient of the curve, $\log(xy) = x^2 + y^2 - 2$, at the point $(1, 1)$. [C.U. B.Com.(H) 1992]

[Hints: $\dfrac{dy}{dx} = \dfrac{2x - \frac{1}{x}}{\frac{1}{y} - 2y}$. At $x = 1$, $y = 1$, the value of $\frac{dy}{dx} = -1$.]

(k) If $y^x = e^{y-x}$, show that $\dfrac{dy}{dx} = \dfrac{(\log ey)^2}{\log y}$. [C.U. B.Com.(H) 2003]

[Hints: $x \log y = (y - x)\log e = y - x$ or, $x(\log y + 1) = y$ or, $x = \frac{y}{\log y + 1}$.

$\therefore \frac{dx}{dy} = \frac{1(\log y+1) - y \cdot \frac{1}{y}}{(\log y+1)^2} = \frac{\log y}{(\log y+1)^2}$; $\therefore \frac{dy}{dx} = 1 / \frac{dx}{dy} = \frac{(\log y + \log e)^2}{\log y} = \frac{(\log ey)^2}{\log y}$.]

11. If $f(x) = \dfrac{6-4x}{1+2x+2x^2}$, find $f'(0)$. Is the function continuous at $x = 0$? [C.U. B.Com.(H) 2003]

12. (a) If $y = \dfrac{x-2}{x+2}$, show that $2x\dfrac{dy}{dx} = 1 - y^2$. [C.U. B.Com. 2003]

(b) If $y = e^{y/x}$, show that $\dfrac{dy}{dx} = \dfrac{y^2}{x(y-x)}$. [B.U. B.Com.(H) 2008]

13. If $f(x) = \left(\dfrac{a+x}{b+x}\right)^x + 2x$, show that $f'(0) = 2 + \log\dfrac{a}{b}$. [C.U. B.Com. 2004; B.U. B.Com.(H) 2008]

14. If $y = \dfrac{x}{\sqrt{1-x^2}}$, then show that $(1-x^2)\dfrac{dy}{dx} = \dfrac{y}{x}$. [C.U. B.Com. 2005]

15. If $z = \dfrac{2x^3 + 3x^2 + 4x + 5}{\sqrt{x}}$, find $\dfrac{dz}{dx}$, when $x = 4$. [N.B.U. B.Com.(H) 2007]

[Hints: $z = \frac{2x^3+3x^2+4x+5}{\sqrt{x}} = 2x^{5/2} + 3x^{3/2} + 4x^{1/2} + 5x^{-1/2}$. Now find $\frac{dz}{dx}$ and put $x = 4$.]

ANSWERS

1. (a) 2;
   (b) $2x + 7$;
   (c) $\frac{1}{4}$;
   (d) $-\frac{1}{2\sqrt{x^3}}, \frac{1}{3\sqrt[3]{x^2}}, -\frac{1}{x^2}$;
   (e) 5;
   (f) derivative does not exist;
   (g) $\frac{-2}{x^3}, 3x^2, 2x - 2$;
   (h) $6x^2, 6x^2$;
   (i) 3;
   (j) $me^{mx}, e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$;
   (k) $2(x-1)$.
2. (a) $20x^3$;
   (b) $\frac{1}{2\sqrt{x}}$;
   (c) $-\frac{1}{\sqrt{x^3}}$;
   (d) $\frac{-2}{x^3}$;
   (e) $\frac{-15}{x^6}$;
   (f) $\frac{-6}{x^2}$;
   (g) $-\frac{c}{v^2}$; here we require to find the derivative w.r.t. v.
   (h) $\frac{8}{3}t^{1/3}$;
   (i) $\frac{-5}{\sqrt[4]{x^9}}$;
   (j) $-7x^{-8}$;
   (k) $6x - 4$;

(l) $12x^3-4x$;

(m) $7-9x^2$;

(n) $\frac{4}{3}\sqrt[3]{x}$;

(o) $\frac{39}{5}x^{8/5}$;

(p) $5at^4-15bt^2$;

(q) $\frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}}$;

(r) $\sqrt{5}\frac{1}{2\sqrt{x}}+\sqrt{5}\frac{1}{2\sqrt{x^3}}$;

(s) $\frac{3cx^2+bx-a}{2x^{3/2}}$;

(t) $4\cdot 3^{4x}\log 3-\frac{1}{\sqrt[3]{x^4}}$;

(u) $5^{32}.2x$;

(v) $-\frac{15}{2}(3-5x)^{1/2}$;

(w) $5x^{1/4}+\frac{1}{2}x^{-3/4}-\frac{1}{8}x^{-1/2}-\frac{3}{8}x^{-5/2}+7$.

3. (a) $2acx + 7a + bc$; $10^x x^9(10+x\log_e 10)$; $2^x\left(\frac{1}{x}+\log_e x\log_e 2\right)$;

(b) $\frac{2t^2(aB-bA)+2t(aC-Ac)+(bC-Bc)}{(At^2+2Bt+C)^2}$;

(c) $\frac{6(x^2-2)}{(x+1)^2(x+2)^2}$;

(d) $-\frac{2a}{(a+x)^2}$;

(e) $\frac{-35}{(5+7x)^2}$;

(f) $\frac{4a^2u}{(a^2-u^2)^2}$;

(g) $6t^{7/3}-16t+4t^{7/3}$;

(h) $\frac{2(2x^2-4x-1)}{(1+2x^2)^2}$;

(i) $\frac{8t}{(2-t^2)^2}$;

(j) $\frac{4x}{(x^2+1)^2}$;

(k) $\frac{-40}{(5+4x)^2}$;

(l) $\frac{x^2-2x+5}{(x-1)^2}$

(m) $2xe^x+e^x(x^2+1)$;

(n) $\frac{(1-x)e^x-1}{(e^x-1)^2}$;

(o) $\frac{x^2-2x-1}{(x-1)^2}$;

(p) $2x^3e^{3x^2}(2+3x^2)$;

(q) $7^{x^2+2x}\cdot 2(x+1)\log_e 7$.

4. (a) $\frac{1}{2\sqrt{x-1}}$;

(b) $\frac{x}{\sqrt{a^2+x^2}}$;

(c) $\frac{x}{(a^2-x^2)^{3/2}}$;

(d) $-\frac{36}{x^3}\left(5+\frac{6}{x^3}\right)^2$;

(e) $\frac{a^2+2x^2}{\sqrt{a^2+x^2}}$;

(f) $\frac{4ax+5bx^2}{2\sqrt{a+bx}}$;

(g) $\frac{a^2}{(a^2-x^2)^{3/2}}$;

(h) $\frac{4}{(2+3x)^{2/3}(2-3x)^{4/3}}$;

(i) $-\sqrt[3]{\frac{y}{x}}$;

(j) $\frac{6v^5}{\sqrt{x}}$;

(k) $\frac{1}{2\sqrt{u}}(3x^2-2x)$;

(l) $\frac{4ab}{(b+u)^2(a+x)^2}$;

(m) $\frac{1}{2\sqrt{ax^2+bx+c}}(2ax+b)$;

(n) $\frac{7ax^4+4bx}{2\sqrt{ax^3+b}}$;

(o) $(2ax+b)e^{ax^2+bx+c}$;

(p) $-\frac{3(2x-3)}{2(3x^2-9x+7)^{3/2}}$;

(q) $\frac{2e^x}{(e^x+1)^2}$.

5. (a) $\frac{4}{y}$;

(b) $-\frac{x}{y}$;

(c) $-\frac{b^2x}{a^2y}$;

(d) $-\frac{x+g}{y+f}$;

(e) $-\frac{2x+y}{4y+x}$;

(f) $-\frac{x^2+2xy}{x^2+y^2}$;

(g) $-\sqrt{\frac{y}{x}}\left\{\frac{4x^{3/2}+7y^{1/2}}{4y^{3/2}+7x^{1/2}}\right\}$;

(h) $-\frac{2\sqrt{x}-a^2\sqrt{y}}{2\sqrt{y}-a^2\sqrt{x}}$;

(i) $\frac{ay-x^2}{y^2-ax}$;

(j) $\sqrt{\frac{y}{x}}$;

(k) $-\left(\frac{b}{a}\right)^{2/3}\left(\frac{y}{x}\right)^{1/3}$;

(l) $-\left(\frac{y}{x}\right)^{1/3}$;

(m) $-\frac{y}{x}\left(\frac{\sqrt{y}+2\sqrt{x}}{\sqrt{x}+2\sqrt{y}}\right)$;

(n) $-\frac{x(2x^2+y^2)}{y(x^2+2y^2)}$;

(o) $\frac{y}{x}$;

(p) $-\frac{3}{2}$;

(q) $-\frac{x^2(4x+3y^3)}{y^2(3x^3+4y)}$.

6. (a) $\frac{1}{t}$;

(b) 1;

(c) $\frac{-1}{t^2}$;

(d) $-\frac{1}{2}$;

(e) $-\frac{1}{2}$;

(f) $\frac{1}{t}$;

(g) $\frac{1}{3}$;

(h) -5;

(i) $\frac{1}{3-3t^2}$;

(j) $\frac{2t-t^4}{1-2t^3}$;

(k) $-\sqrt{3}$;

(l) $-\frac{1}{t^6}$;

(m) $\left(\frac{1+t}{1-t}\right)^2$;

(n) 1.

7. (a) $\frac{2x}{x^2+a}$;

(b) $\frac{1-x^2}{x(1+x^2)}$;

(c) $\frac{x}{x^2-1}$;

(d) $\frac{2x}{1-x^4}$;

(e) $\frac{5}{5x+7}$;

(f) $\frac{3a}{ax+b}$;

(g) $\frac{3}{x}(\log x)^2$;

(h) $\frac{3}{x}$;

(i) $\frac{2ax+b}{ax^2+bx+c}$;

(j) $\frac{1}{\sqrt{x^2-a^2}}$;

(k) $1+\log_e x$;

(l) $\frac{ab}{a^2-b^2x^2}$;

(m) $10^x \log_e 10$,
$10^{5x}(\log_e 10)\times 5$;

(n) ae^{ax};

(o) $2xe^{x^2}$;

(p) $-a/e^x$;

(q) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$;

(r) $2b^{2y}\cdot\log_e b$;

(s) $(1-\log_e x)/x^2$;

(t) $\frac{2e^x}{(e^x+1)^2}$;

(u) $e^{-x}(2x-x^2)$;

(v) $-\frac{2}{\sqrt{x^2+a^2}}$;

(w) $\frac{4}{(e^x+e^{-x})^2}$;

(x) $\frac{1}{x\log_e 10}$;

(y) $\frac{2-4\log x}{x^3}$;

(z) $x^{3/2}\left(\frac{5}{2}\log x+1\right)$;

(aa) $\frac{x-y}{x\log(xye)^2}$;

(ab) $\frac{x-y}{x\log(x/ey)}$;

(ac) $\frac{y(x-2y)}{x(x+3y)}$;

(ad) $\frac{y(x-y)}{x(x+y)}$.

8. (a) $x^x(1+\log x)$;

(b) $\frac{x^{\sqrt{x}}(2+\log x)}{2\sqrt{x}}$;

(c) $\left(\frac{a}{x}\right)^x\left(\log\frac{a}{x}-1\right)$;

(d) $\frac{2x^2-10x+11}{(x-1)^{1/2}(x-2)^{1/2}(x-3)^{3/2}(x-4)^{3/2}}$;

(e) $y\left[\frac{1}{x}+\frac{5}{3(5x+7)}-\frac{1}{2x+8}\right]$;

(f) $y\left[\frac{x}{4+x^2}-\frac{1}{x}+\frac{x}{4-x^2}\right]$;

(g) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}\left[\frac{1}{2}\log x+\frac{1}{\sqrt{x}}\right]$;

(h) $\frac{y}{x}$;

(i) $-\frac{y}{x}\frac{x\log y+y}{y\log x+x}$;

(j) $\frac{y(x-y)}{x(x+y)}$;

(k) $\frac{y}{x}\cdot\frac{x-y}{x+y}$;

(l) $(x)^{x^x}\cdot x^x\left\{\frac{1}{x}+\log x(1+\log x)\right\}$;
$(x^x)^x\{x+2x\log x\}$; $x^x(1+\log x)$;

(m) $\frac{-2a^2x^3}{(x^2+a^2)^{1/2}(x^2-a^2)^{3/2}}+2x\sqrt{\frac{x^2+a^2}{x^2-a^2}}$;

(n) $\frac{y}{x}\cdot\frac{(y-1)}{(1-xy-x)}$;

(o) $-\frac{y}{x}\cdot\frac{(xy^{-1}+1)}{(x^{y-1}\log x+1)}$;

(p) $(1+x)^x\left[\log(1+x)+\frac{x}{1+x}\right]$;

(q) $\frac{a^x\log_e a+y\cdot x^{y-1}}{x^y\log x}$;

(r) $-\frac{(yx^{y-1}+y^x\log y)}{(xy^{x-1}+x^y\log x)}$;

(s) $\left(\frac{y}{x}-\log y\right)/\left(\frac{x}{y}-\log x\right)$
or $\frac{y}{x}\cdot\frac{(y-x\log y)}{(x-y\log x)}$;

(t) $x^x(1+\log x)+e^{ax^2+bx+c}\cdot(2ax+b)$;

(u) $(1+x)^{1+x}\{1+\log(1+x)\}$.

9. (a) $x=2$ or 1;

(c) $2x^3$;

(d) $x=3$ or 2;

10. (a) $\frac{1}{2}\cdot\frac{1-x^2}{1+x^4}$;

(g) $\left(\frac{x}{a^2}+\frac{y}{k}\right)/\left(\frac{x}{k}-\frac{y}{b^2}\right)$;

(j) -1.

11. -16; yes.

15. $5\left(10-\frac{\sqrt{2}}{8}\right)$.

Chapter 4

Second and Higher Order Derivatives

4.1 Introduction

We have observed that the derivative of a function of x is itself a function of x. Thus, if $y = f(x)$, the derivative $\frac{dy}{dx}$ or $f'(x)$ has values at different values of x in the domain. Thus, function $f'(x)$ may be again derivable and the derivative of $f'(x)$ is denoted by

$$f''(x) \text{ or, } \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of $f(x)$. Similarly, the derivative of the second derivative, denoted by

$$f'''(x) \text{ or, } \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

is called the *third derivative* of $f(x)$ and so on.

[If $y = 3x^5$, then $\dfrac{dy}{dx} = 15x^4, \dfrac{d^2y}{dx^2} = 60x^3, \dfrac{d^3y}{dx^3} = 180x^2$, etc.]

In general, nth derivative is denoted by $f^n(x)$ or $\dfrac{d^ny}{dx^n}$.

Another notation:

$$y_1 = \frac{dy}{dx},\ y_2 = \frac{d^2y}{dx^2},\ y_3 = \frac{d^3y}{dx^3},\ \cdots,\ y_n = \frac{d^ny}{dx^n}.$$

4.2 Illustrative Examples

Example 1. *Second, third, fourth and fifth order derivatives of* $y = 3x^4 - 2x^3 + 6x$ *are given below.*

$$y = 3x^4 - 2x^3 + 6x.$$

$$\frac{dy}{dx} = \mathbf{12x^3 - 6x^2 + 6};\ \frac{d^2y}{dx^2} = \mathbf{36x^2 - 12x}.$$

$$\frac{d^3y}{dx^3} = \mathbf{72x - 12};\ \frac{d^4y}{dx^4} = \mathbf{72};\ \frac{d^5y}{dx^5} = \mathbf{0}.$$

Example 2. *Find the second and the third order derivatives of* $y = \sqrt{a+bx}$.

Solution: We have $y=\sqrt{a+bx}$.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\ \left(\text{where } u=a+bx \text{ and } y=u^{1/2}\right)$$
$$=\frac{1}{2\sqrt{u}}\cdot b=\frac{b}{2\sqrt{a+bx}}.$$
$$\therefore\ \frac{d^2y}{dx^2}=\frac{-b}{2}\cdot\frac{1}{2}(a+bx)^{-(1/2)-1}\times\frac{d}{dx}(a+bx)=\frac{-b^2}{4(a+bx)^{3/2}}.$$

Check. $\dfrac{d^3y}{dx^3}=\dfrac{3b^3}{8(a+bx)^{5/2}}$.

Example 3. *Find the second order derivative of the Implicit Function:* $x^2+y^2=a^2$.

Solution: Differentiating once w.r.t. x, we obtain $2x+2y\frac{dy}{dx}=0$; i.e., $\frac{dy}{dx}=-\frac{x}{y}$.

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2}=-\frac{y-x(-x/y)}{y^2},\ \left(\text{putting } \frac{dy}{dx}=-x/y\right)$$
$$=-\frac{x^2+y^2}{y^3}=-\frac{\boldsymbol{a^2}}{\boldsymbol{y^3}}.$$

Example 4. *Find* $\dfrac{d^2y}{dx^2}$ *for the Parametric Functions:* $x=at^2$, $y=2at$.

Solution: We have $x=at^2$ at $y=2at$.

$$\therefore\ \frac{dx}{dt}=2at,\ \frac{dy}{dt}=2a;\ \text{hence, } \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{2a}{2at}=\frac{1}{t}.$$
$$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{t}\right)=\frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{dt}{dx}=-\frac{1}{t^2}\cdot\frac{1}{2at}=-\frac{\mathbf{1}}{\boldsymbol{2at^3}}.$$

Example 5. *Second order derivative at a given* x. *To find* $\dfrac{d^2y}{dx^2}$ *at* $x=0$, *if* $y=(2x+3)^{10}$.

Solution:
$$\frac{dy}{dx}=10(2x+3)^{10-1}\cdot\frac{d}{dx}(2x+3)=20(2x+3)^9$$
$$\frac{d^2y}{dx^2}=20\cdot 9(2x+3)^8\frac{d}{dx}(2x+3)=360(2x+3)^8.$$

$\therefore$ at the value $x=0$, $\frac{d^2y}{dx^2}=360\times 3^8$.

Example 6. (a) *If* $y=e^{ax}$, *then* $\dfrac{dy}{dx}=ae^{ax}$; $\dfrac{d^2y}{dx^2}=a^2\cdot e^{ax}$; $\dfrac{d^3y}{dx^3}=a^3\cdot e^{ax}$.

(b) *If* $y=e^{-2x}$, *then* $\dfrac{dy}{dx}=-2e^{-2x}$ *and* $\dfrac{d^2y}{dx^2}=(-2)(-2)e^{-2x}=4e^{-2x}$.

Let as exercise to the students.

Example 7. *If* $y = \log x$, *then* $\frac{dy}{dx} = \frac{1}{x}$; $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$; $\frac{d^3y}{dx^3} = \frac{(-1)(-2)}{x^3} = \frac{2}{x^3}$.

Example 8. (i) *If* $y = ke^{2x} + l\,e^{-2x}$, *prove that* $\frac{d^2y}{dx^2} = 4y$.

(ii) *If* $y = e^{2mx} + e^{-2mx}$, *show that* $\frac{d^2y}{dx^2} - 4m^2y = 0$. [C.U. B.Com.(H) 2006]

Solution: (i) $\frac{dy}{dx} = 2ke^{2x} - 2l\,e^{-2x}$.

$$\frac{d^2y}{dx^2} = 4ke^{2x} + 4l\,e^{-2x} = 4\left(ke^{2x} + l\,e^{-2x}\right) = \mathbf{4y}.$$

(ii) $\frac{dy}{dx} = 2me^{2mx} - 2me^{-2mx}$.

$$\therefore\ \frac{d^2y}{dx^2} = 4m^2e^{2mx} + 4m^2e^{-2mx} = 4m^2\left(e^{2mx} + e^{-2mx}\right) = 4m^2y.$$

Hence,

$$\boldsymbol{\frac{d^2y}{dx^2} - 4m^2y = 0.}$$

Example 9. (i) *If* $y = x^k$ $(x > 0)$, *k having any rational value, then*

$$y_n = k(k-1)(k-2)(k-3)\cdots[k-(n-1)]x^{k-n}. \tag{I}$$

In case k is a positive integer, then

$$y_n = \frac{k!}{(k-n)!}x^{k-n}. \tag{II}$$

(ii) $y = \frac{1}{1+x}$; *to prove* $y_n = \frac{d^ny}{dx^n} = \frac{(-1)^n n!}{(1+x)^{n+1}}$.

Solution: (i) $y = x^k$. Differentiating, we get

$$y_1 = kx^{k-1}.$$

Differentiating again, we obtain

$$y_2 = k(k-1)\,x^{k-2}.$$

Similarly,

$$y_3 = k(k-1)(k-2)\,x^{k-3}.$$

We infer

$$y_n = k(k-1)(k-2)\cdots[k-(n-1)]x^{k-n}$$

for all positive integers n.

Thus, the result holds by actual differentiation for $n = 1, 2, 3$. We, therefore, assume that the result (I) is true for a particular positive integer, i.e., we assume:

$$y_r = k(k-1)\cdots(k-r+1)\,x^{k-r}.$$

Now, differentiating once more, we get

$$y_{r+1} = [k(k-1)\cdots(k-r+1)](k-r)\,x^{k-r-1},$$
$$\text{i.e.,}\quad y_{r+1} = k(k-1)\cdots(k-r)\,x^{k-r-1},$$

i.e., the results (I) for $n = r+1$, if assume it for $n = r$.

But the result was proved to be true for $n = 1$ or, $n = 2$.

$\therefore$ the result will be true for $n = 3$ and hence for $n = 4$ and so on (for all positive integral values of n).

Particular Case

- If k is a positive integer $< n$, then

$$k(k-1)\cdots(k-n+1) = \frac{k(k-1)\cdots(k-n+1)(k-n)(k-n-1)\cdots 3\cdot 2\cdot 1}{(k-n)(k-n-1)\cdots 3\cdot 2\cdot 1}$$
$$= \frac{k!}{(k-n)!} \text{ and so } y_n = \frac{k!}{(k-n)!}x^{k-n}.$$

- If k is a positive integer $= n$, then $y_n = n!$.
- If k is a positive integer and $n > k$, then $y_n = 0$, e.g., if $y = x^3$, then $y_4 = 0$.

(ii) $y_1 = \frac{-1}{(1+x)^2}$; $y_2 = \frac{(-1)(-2)}{(1+x)^3}$; $y_3 = \frac{(-1)(-2)(-3)}{(1+x)^4} = \frac{(-1)^3 3!}{(1+x)^4}$; and so on.

Thus,

$$y_n = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

This last result can be established by the method of induction.

[This is true for $n = 1$ or, $n = 2$. Assume it to be true for $n =$ a certain positive integer k, i.e., assume

$$y_k = \frac{(-1)^k k!}{(1+x)^{k+1}};$$

differentiating once more,

$$y_{k+1} = (-1)^k k!\left(-\overline{k+1}\right)(1+x)^{-k-1-1} = \frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}.$$

Thus, if the result is assumed to be true for $n = k$, then it is found to be true for $n = k+1$. But it is true for $n = 1$ or 2; hence true for $n = 2+1 = 3$. If true for $n = 3$, it is true for $n = 4$ and so on.]

Example 10. *Find* $\frac{d^2y}{dx^2}$ *at* $x = 1, y = 1$, *if* $x^3 - 2x^2y^2 + 5x + y - 5 = 0$.

Solution: Differentiating w.r.t. x, we get

$$3x^2 - 4xy^2 - 4x^2y\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0. \tag{1}$$

Differentiating again w.r.t. x, we get

$$6x - 4y^2 - 8xy\frac{dy}{dx} - 8xy\frac{dy}{dx} - 4x^2\left(\frac{dy}{dx}\right)^2 - 4x^2y\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = 0. \tag{2}$$

From eq. (1),

$$\frac{dy}{dx}=\frac{3x^2-4xy^2+5}{4x^2y-1}=\frac{3\cdot 1-4\cdot 1+5}{4\cdot 1-1}=\frac{4}{3} \text{ at } x=1, y=1.$$

Putting $x=1$, $y=1$, $\frac{dy}{dx}=\frac{4}{3}$ in (2), we get

$$6-4-8\times\frac{4}{3}-8\times\frac{4}{3}-4\left(\frac{4}{3}\right)^2-4\frac{d^2y}{dx^2}+\frac{d^2y}{dx^2}=0;$$

or, $$-\frac{238}{27}=\frac{d^2y}{dx^2}; \quad \therefore \frac{d^2y}{dx^2}=-8\frac{22}{27}.$$

Example 11. *If* $y=x^4-3x^3+3x^2+5x+1$, *prove that* $\frac{d^2y}{dx^2}$ *is negative, when* x *lies between* $\frac{1}{2}$ *and* 1.

Solution: We have

$$\frac{dy}{dx}=4x^3-9x^2+6x+5;$$

$$\frac{d^2y}{dx^2}=12x^2-18x+6=6\left(2x^2-3x+1\right)$$

$$=6(2x-1)(x-1)=\mathbf{12}\left(\boldsymbol{x}-\frac{\mathbf{1}}{\mathbf{2}}\right)(\boldsymbol{x}-\mathbf{1}).$$

$\frac{d^2y}{dx^2}$ is negative, if $\left(x-\frac{1}{2}\right)(x-1)$ is negative, i.e., if x lies between $\frac{1}{2}$ and 1, because then $x-\frac{1}{2}$ is positive and $x-1$ is negative and their product is negative.

Example 12. *If* $y=\sqrt{3x+2}$, *prove that* $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$.

Solution: We have

$$y=\sqrt{3x+2} \text{ or, } y^2=3x+2. \tag{1}$$

Differentiating both sides w.r.t. x, $2y\cdot\frac{dy}{dx}=3$ or, $y\frac{dy}{dx}=\frac{3}{2}$.

Differentiating both sides again w.r.t. x,

$$\frac{dy}{dx}\cdot\frac{dy}{dx}+y\frac{d^2y}{dx^2}=0 \text{ or, } \boldsymbol{y}\frac{\boldsymbol{d^2y}}{\boldsymbol{dx^2}}+\left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}}=\mathbf{0}.$$

Example 13. *If* $y=2x+\frac{4}{x}$, *prove that* $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$.

Solution: We have $y=2x+\frac{4}{x}$.

$$\therefore \frac{dy}{dx}=2\cdot 1+4\cdot\left(-\frac{1}{x^2}\right)=2-\frac{4}{x^2} \text{ and } \frac{d^2y}{dx^2}=0-4\cdot\frac{-2}{x^3}=\frac{8}{x^3}. \tag{1}$$

LHS $=x^2\times\frac{8}{x^3}+x\left(2-\frac{4}{x^2}\right)-\left(2x+\frac{4}{x}\right)=\frac{8}{x}+2x-\frac{4}{x}-2x-\frac{4}{x}=0=$ RHS **(Proved).**

Example 14. (a) *If* $y=\left(x+\sqrt{1+x^2}\right)^m$, *prove that* $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-m^2y=0.$ [C.U. B.Com.(H) 2009]

(b) *If* $y=\log\left(x+\sqrt{1+x^2}\right)$, *prove that* $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0.$ [C.U. B.Com. 2010]

Solution: (a) $y=\left(x+\sqrt{1+x^2}\right)^m$.

$$\therefore \quad y_1=m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)\quad\left(y_1=\frac{dy}{dx}\right)$$

$$\text{or,}\quad y_1\sqrt{1+x^2}=m\left(x+\sqrt{1+x^2}\right)^m.$$

Differentiating again (writing $y_2=\frac{d^2y}{dx^2}$),

$$\sqrt{1+x^2}\,y_2+\frac{x}{\sqrt{1+x^2}}y_1=m^2\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\text{or,}\quad \left(1+x^2\right)y_2+xy_1=m^2\left(x+\sqrt{1+x^2}\right)^m=m^2y,$$

$$\text{i.e.,}\quad \boldsymbol{\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0.}$$

(b) We have $y=\log\left(x+\sqrt{1+x^2}\right)$.

$$\therefore\ \frac{dy}{dx}=\frac{1}{x+\sqrt{1+x^2}}\times\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)=\frac{1}{x+\sqrt{1+x^2}}\times\left\{1+\frac{1}{2}\left(1+x^2\right)^{\frac{1}{2}-1}\times(0+2x)\right\}$$

$$=\frac{1}{x+\sqrt{1+x^2}}\times\left(1+\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{\left(x+\sqrt{1+x^2}\right)}\times\frac{\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

or, $\sqrt{1+x^2}\times\dfrac{dy}{dx}=1$ or, $\left(1+x^2\right)\left(\dfrac{dy}{dx}\right)^2=1.$

Differentiating both sides w.r.t. x, we get

$$\left(1+x^2\right)\cdot 2\left(\frac{dy}{dx}\right)^{2-1}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+(0+2x)\left(\frac{dy}{dx}\right)^2=0,$$

$$\text{or,}\quad \left(1+x^2\right)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=0.$$

Dividing both sides by $2\frac{dy}{dx}$, we get

$$\boldsymbol{\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.}$$

Example 15. *If* $2x=y^{1/m}+y^{-1/m}$, *prove that* $\left(x^2-1\right)y_2+xy_1-m^2y=0.$ [C.U. B.Com. 2008]

Solution: Given:

$$2x = y^{1/m} + \frac{1}{y^{1/m}} \quad \text{or,} \quad \left(y^{1/m}\right)^2 - 2xy^{1/m} + 1 = 0$$

$$\text{or,} \quad y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$\text{or,} \quad y = \left(x + \sqrt{x^2-1}\right)^m \rightarrow \text{Case I}$$

(See Ex. 14 given above)

$$\text{or,} \quad y = \left(x - \sqrt{x^2-1}\right)^m \rightarrow \text{Case II.}$$

In case I, $y = \left(x + \sqrt{x^2-1}\right)^m$. Now proceed as in Ex. 14(a).

We discuss the Case II:

$$y = \left(x - \sqrt{x^2-1}\right)^m \Rightarrow y_1 = m\left(x - \sqrt{x^2-1}\right)^{m-1}\left(1 - \frac{x}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \sqrt{x^2-1}\, y_1 = -my$$

$$\Rightarrow \sqrt{x^2-1}\, y_2 + \frac{x}{\sqrt{x^2-1}} y_1 = (-m)\left[-\frac{my}{\sqrt{x^2-1}}\right].$$

Hence,

$$\left(x^2-1\right) y_2 + xy_1 = m^2 y \text{ (Proved).}$$

EXERCISES ON CHAPTER 4

(Successive Derivatives)

Prove each of the following differentiations (1–16):

1. $y = x^{10} + 3x^8 + 4x^2 - 7x + 8$; $\dfrac{d^2y}{dx^2} = 90x^8 + 168x^6 + 8$.

2. $y = \sqrt{x}$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4\sqrt{x^3}}$ and $y = \dfrac{1}{\sqrt{x}}$; $\dfrac{d^2y}{dx^2} = \dfrac{3}{4\sqrt{x^5}}$.

3. (a) (i) If $y = e^{7x}$, then $\dfrac{d^2y}{dx^2} = 49e^{7x}$; (ii) If $y = Ae^x + Be^{-x}$, show that $\dfrac{d^2y}{dx^2} - y = 0$.

   (b) If $y = \log(2x+3)$; then $\dfrac{d^2y}{dx^2} = -\dfrac{2}{(2x+3)^2}$. **[B.U. B.Com.(H) 2008]**

   (c) If $y = Ae^{2x} + Bxe^{2x}$; where a and b are constants, then show that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

[**Hints:** $y/e^{2x} = A + Bx$ or, $ye^{-2x} = A + Bx$. Now differentiate.]

4. $y=x^2(1-x)^2$; $\dfrac{d^2y}{dx^2}=2-12x+12x^2$.

5. $y=\dfrac{1+x}{1-x}$; $\dfrac{d^2y}{dx^2}=\dfrac{4}{(1-x)^3}$.

6. $y=x\sqrt{a^2-x^2}$, $\dfrac{d^2y}{dx^2}=\dfrac{2x^3-3a^2x}{(a^2-x^2)^{3/2}}$.

7. $y=\dfrac{a+bx}{a-bx}$; $y_2=\dfrac{4ab^2}{(a-bx)^3}$.

8. $f(x)=\sqrt{a^2+x^2}$; $f''(x)=\dfrac{a^2}{(a^2+x^2)^{3/2}}$.

9. $y=\dfrac{x^2}{a+x}$; $\dfrac{d^2y}{dx^2}=\dfrac{2a^2}{(a+x)^3}$.

10. $s=\dfrac{t}{\sqrt{2t+1}}$; $\dfrac{d^2s}{dt^2}=-\dfrac{(t+2)}{(2t+1)^{5/2}}$.

11. $y=\dfrac{x^8-2x^2}{1-x}$; $y_4=\dfrac{d^4y}{dx^4}=-\dfrac{4!}{(1-x)^5}$.

12. $y=\dfrac{a}{x+1}$; $y_n=\dfrac{(-1)^n\cdot a\cdot n!}{(x+1)^{n+1}}$.

13. $y^2=4ax$; $y_2=-\dfrac{4a^2}{y^3}$.

14. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; $y_2=-\dfrac{b^4}{a^2y^3}$ and $y_3=-\dfrac{3b^6x}{a^4y^5}$. [C.U. B.Com. 2008]

15. $ax^2+2hxy+by^2=1$; $\dfrac{d^2y}{dx^2}=\dfrac{h^2-ab}{(hx+by)^3}$.

16. $x^4+2x^2y^2=4$; $\dfrac{d^2y}{dx^2}=\dfrac{2y^4-x^2y^2-x^4}{x^2y^3}$.

17. (a) If $y=\sqrt{25-3x}$, prove that $y_2=-\dfrac{9}{256}$ at $x=3$.

(b) If $y=x\sqrt{x^2+9}$, show that $y_1=\dfrac{41}{5}$ and $y_2=\dfrac{236}{125}$ at $x=4$.

18. Prove that $\dfrac{d^2}{dx^2}\left(x^4e^{5x}\right)=e^{5x}\left(25x^4+40x^3+12x^2\right)$.

19. (a) If $y=\log\left(x+\sqrt{x^2+a^2}\right)$, prove that $\left(a^2+x^2\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0$.

(b) If $e^y=x+\sqrt{a^2+x^2}$, show that $\left(a^2+x^2\right)\dfrac{d^24}{dx^2}+x\dfrac{dy}{dx}=0$. [B.U. B.Com.(H) 2006]

20. (a) If $y=\left(x+\sqrt{1+x^2}\right)^m$, prove that $\left(1+x^2\right)y_2+xy_1=m^2y$.

(b) If $y=\log\left(x+\sqrt{1+x^2}\right)$, prove that $\left(1+x^2\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0$.

[B.U. B.Com.(H) 2002; C.U. B.Com. 2010]

21. $x^2+y^2=25$, show that $\dfrac{\left(1+y_1^2\right)^{3/2}}{y_2}=-5$.

22. Eliminate λ from $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ and obtain $\left(a^2-b^2\right)y_1=(x+yy_1)(xy_1-y)$.

23. (a) If $2x=y^{1/m}+y^{-1/m}$, prove that $\left(x^2-1\right)y_2+xy_1=m^2y$.

(b) If $y^{1/3}+y^{-1/3}=2x$, prove that $\left(x^2-1\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=9y$.

[See worked-out Exs 14 and 15.]

24. If $y=\left(x^2-1\right)^n$, prove that $\left(x^2-1\right)y_2+2(1-n)xy_1-2ny=0$. [B.U. B.Com.(H) 2008]

25. Given, $y = x\sqrt{x^2+9}$, prove that $\dfrac{d^2y}{dx^2} = 1\dfrac{111}{125}$, when $x = 4$.

26. (a) If $y = x^3 \log\dfrac{1}{x}$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\cdot\dfrac{dy}{dx} + 3x = 0$. [CA 1991]

(b) If $y = x^2 \log_e\left(x^2\right)$, find $\dfrac{d^2y}{dx^2}$, when $x = 1$.

[Hints: $\dfrac{dy}{dx} = 2x\log x^2 + x^2 \cdot \dfrac{1}{x^2}\cdot\dfrac{d}{dx}\left(x^2\right)$.

$\therefore\ \dfrac{d^2y}{dx^2} = 2\cdot\left[1\cdot\log\left(x^2\right) + x\cdot\left(0 + \dfrac{1}{x^2}\cdot\dfrac{d}{dx}\left(x^2\right)\right)\right] = 2[1+\log x^2 + 2] = 2\left(3+\log x^2\right)$.

At $x = 1$, $\dfrac{d^2y}{dx^2} = 2(3+\log 1) = 2\times 3 = 6$.]

27. Given, $x = t + \dfrac{1}{t}$ and $y = t - \dfrac{1}{t}$; find the value of $\dfrac{d^2y}{dx^2}$ at the point $t = 20$. [*Ans.* $-4\left(\dfrac{20}{399}\right)^3$]

[C.U. B.Com.(H) 2003; B.U. B.Com.(H) 2002]

28. If $(a+bx)e^{y/x} = x$, prove that $x^3\dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$.

29. If $y' = \dfrac{dy}{dx}$, prove that $\dfrac{dy'}{dx} = y'\dfrac{dy'}{dy}$.

30. Show that for any circle $(x-a)^2 + (y-b)^2 = r^2$, $\dfrac{\left(1+y_1^2\right)^{1/2}}{y_2}$ is always a constant.

31. If $y = f(u)$, $u = g(x)$, show that $\dfrac{d^2y}{dx^2} = \dfrac{d^2u}{dx^2}\cdot\dfrac{dy}{du} + \left(\dfrac{du}{dx}\right)^2\cdot\dfrac{d^2y}{du^2}$.

32. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$, then find $\dfrac{d^2y}{dx^2}$ at $x = 1, y = 1$. [C.U. B.Com. 2004]

[*Ans.* $-238/27$]

33. If $y = x^{n-1}\log x$, prove that $x^2y_2 + (3-2n)\,xy_1 + (n-1)^2 y = 0$.

34. If $y = x\log\left(\dfrac{x}{a+bx}\right)$, prove that $x^3y_2 = (y - xy_1)^2$, where $y_1 = \dfrac{dy}{dx}, y_2 = \dfrac{d^2y}{dx^2}$.

[Hints: $\dfrac{y}{x} = \log x - \log(a+bx)$. Differentiate both sides w.r.t. x to obtain

$$\frac{x\frac{dy}{dx} - y\cdot 1}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\times b \text{ or, } x\frac{dy}{dx} - y = x - \frac{bx^2}{a+bx} = \frac{ax}{a+bx} \quad (1)$$

Differentiating again,

$$x\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)\cdot a - ax(b\cdot 1)}{(a+bx)^2}$$

$$\text{or, } x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2} \text{ or, } x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(x\frac{dy}{dx} - y\right)^2 \text{ [by (1)].]}$$

Chapter 5

Partial Derivatives: Euler's Theorem

5.1 Introduction

We have so far discussed functions of one independent variable. We now turn to functions of more than one independent variable. We call such functions as **functions of several variables**. In the present chapter we shall be mainly concerned with algebraic functions and their partial derivatives. As an application of such partial derivatives we shall introduce the concept of envelope of a family of curves (family of straight lines, in particular).

5.2 Functions of Several Variables

We begin with some simple examples of functions that depend on more than one independent variable.

Illustration 1. *The volume V of a right circular cylinder is given by $V=\pi x^2 y$, where x is the radius and y is the altitude of the cylinder. See that if the values of x and y are known, then only we can find the value of V. Moreover, the values of x and y can be chosen* **independently**. *In such case, we say that V is a function of two independent variables x and y, and we write $V=f(x,y)=\pi x^2 y$.*

Illustration 2. *Consider a relation $z=x^3+3x^2y+y^3$.*

Here also x,y being known, we can find z and the choices of the values of x and y are independent, i.e., the choice of one does not depend on the other.

Thus, z is a function of two independent variables x and y. We write $z=f(x,y)$ or, $z=\phi(x,y)$, etc.

Illustration 3. *Let $w=xy+yz+zx$. Here x,y,z being known independently, w can be determined uniquely. We say that w is a function of three independent variables x,y,z, a fact which we denote by $w=f(x,y,z)=xy+yz+zx$.*

We shall mainly restrict our discussions of functions of two variables only.

Example 1. *Let $f(x,y)=x^3+3x^2y+y^3$. What is the value of $f(1,2)$?*

Solution: Here we require the value of the function when $x=1, y=2$.

Thus,

$$f(1,2)=1^3+3\cdot 1^2\cdot 2+2^3=15.$$

Here 15 is called the **functional value** when $x=1$ and $y=2$.

Limit and Continuity

(i) $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = A$. By this statement we mean that when the points (x,y) on a plane are taken in such a manner that their distances from a fixed point (a,b) are sufficiently small, then the corresponding functional values $f(x,y)$ will have arbitrarily small differences from the fixed number A.

$$\text{e.g.,}\quad \lim_{\substack{x\to 2\\ y\to 4}}(x+3y) = \lim_{\substack{h\to 0\\ k\to 0}}[2+h+3(4+k)] = 14.$$

(ii) A function $f(x,y)$ is said to be continuous at $(x=a, y=b)$, if $\lim_{\substack{x\to a\\ y\to b}} f(x,y) =$ functional value $f(a,b)$, no matter in what way x and y approach independently a and b, respectively.

Roughly speaking, a very small change in one or both the variables produces a very small change in the value of the function.

We shall be considering continuous functions only.

5.2.1 Different Types of Functions

1. **Polynomials:**

   (a) *A polynomial function in two variables x and y* ***of degree one*** *is of the form*: $z = ax + by + c$ $(a, b, c$ are constants). Such a function is called a *linear function.*

   (b) *Polynomial function in two variables x and y* ***of degree two*** *is of the form*:

   $$z = ax^2 + by^2 + cxy + dx + ey + f.$$

   (c) *General polynomial* is a sum of a finite number of terms like $x^m y^n$, where m and n are positive integers or zero. We say that the degree of the term $x^m y^n$ is $m+n$.

   (d) **Homogeneous Polynomials**: All the terms have the same degree,

   $$\text{e.g.,}\quad x^2 + 3xy + y^2$$

   is a *homogeneous polynomial in x and y of degree two.*

   Polynomial functions are defined for all values of x and y, and they are always continuous.

2. **Rational Functions:** Quotient of two polynomials,

   $$\text{e.g.,}\quad z = \frac{3x+2y+8}{x-y}.$$

   Such a function is defined for all points (x,y) except where the denominator (in this case $x-y$) is zero. z is defined for every pair (x,y) except those where $x-y=0$.

3. **Algebraic Functions:** Extracting roots of rational functions,

   $$\text{e.g.,}\quad z_1 = \sqrt{\frac{x-y}{x+y}},\quad z_2 = \sqrt{\frac{(x+y)^2}{x^2+xy^2}},\quad z_3 = \sqrt{(x-2)(y+3)}.$$

   They are defined where the quantities under the root sign are not negative.

We have functions of other types — Trigonometric functions, logarithmic functions or exponential functions, etc. We shall, however, be concerned *with algebraic functions, rational functions* and *polynomials, logarithmic and exponential functions.*

Example 2. *In which region of the xy-plane, the function $z=\sqrt{1-x^2-y^2}$ is defined?*

Solution: z is defined for those points (x,y) for which $1-x^2-y^2 \geq 0$ or, $x^2+y^2 \leq 1$, i.e., z is defined for all points on and inside a circle whose centre is $(0,0)$ and radius is 1.

5.3 Partial Derivatives

Let $z=f(x,y)$. Take $x=a, y=b$ so that the value of $z=f(a,b)$. We keep the value of y fixed at b; we only change the value of x from $x=a$ to $x=a+h$. The new value of $z=f(a+h,b)$.

If now

$$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} \text{ exists,}$$

then we call it the **partial derivative of $f(x,y)$ w.r.t. x at the point (a,b)** and denote it by

$$\left(\frac{\partial f}{\partial x}\right)_{x=a,y=b} \quad \text{or, } f_x(a,b).$$

Similarly, keeping x fixed at a, we give an increment k to y so that y changes from $y=b$ to $y=b+k$. Then

$$\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}, \quad \text{if it exists,}$$

is called the **partial derivative of $f(x,y)$ w.r.t. y at the point $(x=a,\ y=b)$**, denoted by

$$\left(\frac{\partial f}{\partial y}\right)_{x=a,y=b} \quad \text{or, } f_y(a,b).$$

Instead of a given (a,b) we may take **any point** (x,y). We thus have

$$\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}, \quad \text{if it exists} = f_x(x,y) \text{ or, } \frac{\partial f}{\partial x}$$

$$= \text{partial derivative of } f(x,y) \text{ w.r.t. } x$$

and $$\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}, \quad \text{if it exists} = f_y(x,y) \text{ or, } \frac{\partial f}{\partial y}$$

$$= \text{partial derivative of } f(x,y) \text{ w.r.t. } y.$$

Rule 1. To find $\frac{\partial f}{\partial x}$, take the ordinary derivative of $f(x,y)$ w.r.t. x, as if y is a constant,

$$\text{e.g., if } f(x,y)=x^2+xy+y^2, \quad \text{then } \frac{\partial f}{\partial x}=2x+y.$$

Rule 2. To find $\frac{\partial f}{\partial y}$, take the ordinary derivative of $f(x,y)$ w.r.t. y, as if x is a constant,

$$\text{e.g., if } f(x,y)=x^2+xy+y^2, \quad \text{then } \frac{\partial f}{\partial y}=x+2y.$$

5.3.1 Partial Derivatives of Higher Order

(i) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}(x,y)$ or simply f_{xx}.

e.g., if $f(x,y) = x^2 + xy + y^2$, then $f_x = 2x + y$ and $f_{xx} = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}(2x+y) = 2$.

(ii) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x} = f_{yx}(x,y)$ or simply f_{yx}.

e.g., if $f(x,y) = x^2 + xy + y^2$, then $f_x = 2x + y; f_{yx} = \dfrac{\partial}{\partial y}(f_x) = \dfrac{\partial}{\partial y}(2x+y) = 1$.

(iii) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = f_{xy}(x,y)$ or simply f_{xy}.

e.g., if $f(x,y) = x^2 + xy + y^2$, then $f_y = x + 2y; \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(f_y\right) = \dfrac{\partial}{\partial x}(x+2y) = 1$.

Wee see here that

$$\frac{\partial^2 f}{\partial x \partial y} = 1 = \frac{\partial^2 f}{\partial y \partial x}, \text{ i.e., } f_{xy} = f_{yx}.$$

Usually this equality holds. The students are allowed to assume this equality in almost all their problems of this text.

(iv) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = f_{yy}(x,y)$ or simply f_{yy}.

e.g., if $f(x,y) = x^2 + xy + y^2$, then $f_y = x + 2y; f_{yy} = \dfrac{\partial}{\partial y}\left(f_y\right) = \dfrac{\partial}{\partial y}(x+2y) = 2$.

Here, $\dfrac{\partial^2 f}{\partial x \partial y}$ or $\dfrac{\partial^2 f}{\partial y \partial x}$ are called **Mixed Partial Derivatives.**

5.4 Illustrative Examples

Example 3.

(a) *Let* $z = ax^2 + by^2 + cxy + dx + ey + f$, *($a,b,c,d,e,f$ are constants). Find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$.

(b) *If* $u = \dfrac{y}{x^2}$, *find* u_x *and* u_y. **[B.U. B.Com.(H) 2008]**

Solution:

(a) $\dfrac{\partial z}{\partial x} = \boldsymbol{2ax + cy + d}$ and $\dfrac{\partial z}{\partial y} = \boldsymbol{2by + cx + e}$.

(b) $u = \dfrac{y}{x^2}$; $\therefore u_x = \dfrac{\partial u}{\partial x} = y \times \dfrac{-2}{x^3} = -\dfrac{2y}{x^3}$ and $u_y = \dfrac{\partial u}{\partial y} = \dfrac{1}{x^2} \cdot \dfrac{\partial u}{\partial y} = \dfrac{1}{x^2} \cdot 1 = \dfrac{1}{x^2}$.

Example 4. *Let* $f(x,y) = \dfrac{5x+4y}{7x+8y}$. *Find* $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial f}{\partial y}$.

Solution:
$$\frac{\partial f}{\partial x}=\frac{(7x+8y)\frac{\partial}{\partial x}(5x+4y)-(5x+4y)\frac{\partial}{\partial x}(7x+8y)}{(7x+8y)^2}$$
$$=\frac{5(7x+8y)-7(5x+4y)}{(7x+8y)^2}=\frac{\mathbf{12y}}{\mathbf{(7x+8y)^2}}.$$

Similarly,
$$\frac{\partial f}{\partial y}=-\frac{\mathbf{12x}}{\mathbf{(7x+8y)^2}}.$$

Example 5. *Let $w=xy+yz+zx$. Find $\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$.*

Solution: Then $\frac{\partial w}{\partial x}=\boldsymbol{y+z}$ (y,z remain constant here).

$\frac{\partial w}{\partial y}=\boldsymbol{x+z}$ (x,z are constant here) and $\frac{\partial w}{\partial z}=\boldsymbol{y+x}$ (y,x remain constant here)

(y,x remain constant here).

Example 6. *Let $z=2x^2-3xy+4y^2$. To find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x=2,y=3$.*

Solution: We first evaluate $\frac{\partial z}{\partial x}=4x-3y$ (y remains constant);

$$\frac{\partial z}{\partial y}=-3x+8y \text{ (} x \text{ remains constant).}$$

$\therefore$ at $x=2,y=3$,
$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=4\times2-3\times3=\mathbf{-1}\\ \text{and}\quad \frac{\partial z}{\partial y}&=-3\times2+8\times3=\mathbf{18}\end{aligned}\right\}$$

Example 7.

(a) *Let $z=\frac{x^2y^2}{x+y}$. To prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z$.*

(b) *If $z=\log(x^2+y^2)$, prove that $z_{xx}+z_{yy}=0$.* [B.U. B.Com.(H) 2007]

Solution: (a)
$$\frac{\partial z}{\partial x}=\frac{(x+y)\frac{\partial}{\partial x}(x^2y^2)-x^2y^2\frac{\partial}{\partial x}(x+y)}{(x+y)^2}=\frac{2xy^2(x+y)-x^2y^2}{(x+y)^2}=\frac{x^2y^2+2xy^3}{(x+y)^2}.$$
$$\frac{\partial z}{\partial y}=\frac{(x+y)\frac{\partial}{\partial y}(x^2y^2)-x^2y^2\frac{\partial}{\partial y}(x+y)}{(x+y)^2}=\frac{2x^2y(x+y)-x^2y^2}{(x+y)^2}=\frac{x^2y^2+2x^3y}{(x+y)^2}.$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\cdot\frac{x^2y^2+2xy^3}{(x+y)^2}+y\cdot\frac{x^2y^2+2x^3y}{(x+y)^2}=\frac{x^3y^2+2x^2y^3+x^2y^3+2x^3y^2}{(x+y)^2}$$
$$=\frac{3(x^3y^2+x^2y^3)}{(x+y)^2}=\frac{3x^2y^2(x+y)}{(x+y)^2}=\frac{3x^2y^2}{(x+y)}=3z \textbf{ (Proved).}$$

(b) $z=\log(x^2+y^2)$; $\therefore z_x=\dfrac{1}{x^2+y^2}\cdot\dfrac{\partial}{\partial x}(x^2+y^2)=\dfrac{2x}{x^2+y^2}$ and

$$z_{xx}=\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)=2\left[\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}\right]=\frac{2(-x^2+y^2)}{(x^2+y^2)^2}.$$

Similarly,
$$z_{yy}=\frac{2(x^2-y^2)}{(x^2+y^2)}.$$

Hence,
$$z_{xx}+z_{yy}=\frac{2(-x^2+y^2)}{(x^2+y^2)^2}+\frac{2(x^2-y^2)}{(x^2+y^2)^2}=\frac{2\times 0}{(x^2+y^2)^2}=0 \text{ (Proved).}$$

Example 8. *Let* $f(x,y)=\log\dfrac{x^2+y^2}{xy}$. *Show that* $f_{xy}=f_{yx}$.

Solution: $f(x,y)=\log\dfrac{x^2+y^2}{xy}=\log(x^2+y^2)-\log x-\log y.$

$$\therefore f_x=\frac{2x}{x^2+y^2}-\frac{1}{x}=\frac{x^2-y^2}{x(x^2+y^2)};\quad f_y=\frac{2y}{x^2+y^2}-\frac{1}{y}=\frac{y^2-x^2}{y(x^2+y^2)}$$

$$\therefore f_{xy}=\frac{\partial}{\partial x}(f_y)=\frac{1}{y}\left[\frac{-2x(x^2+y^2)-(y^2-x^2)2x}{(x^2+y^2)^2}\right]=\frac{1}{y}\times\frac{-4xy^2}{(x^2+y^2)^2}=\frac{-4xy}{(x^2+y^2)^2}$$

and
$$f_{yx}=\frac{\partial}{\partial y}(f_x)=\frac{1}{x}\left[\frac{-2y(x^2+y^2)-(x^2-y^2)2y}{(x^2+y^2)^2}\right]=\frac{1}{x}\times\frac{-4x^2y}{(x^2+y^2)^2}=\frac{-4xy}{(x^2+y^2)^2}$$

Hence,
$$\boldsymbol{f_{xy}=f_{yx}=-\frac{4xy}{(x^2+y^2)^2}}.$$

Example 9. *If* $u=\log(x^3+y^3+z^3-3xyz)$, *show that* $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\dfrac{3}{x+y+z}$.

[V.U. B.Com.(H) 2008]

Solution: Since $u=\log(x^3+y^3+z^3-3xyz)$,

$$\therefore \frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)$$
$$=\frac{1}{x^3+y^3+z^3-3xyz}\times 3(x^2-yz).$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{1}{x^3+y^3+z^3-3xyz}\times 3(y^2-zx)$$

and
$$\frac{\partial u}{\partial z}=\frac{1}{x^3+y^3+z^3-3xyz}\times 3(z^2-xy).$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3[x^2-yz+y^2-zx+z^2-xy]}{x^3+y^3+z^3-3xyz}$$
$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}=\frac{3}{x+y+z} \text{ (Proved).}$$

5.5 Homogeneous Functions: Euler's Theorem

Definition 1. *If a function $f(x,y)$ of two independent variables x and y can be written in the form $x^n\phi\left(\frac{y}{x}\right)$ or $y^n\phi\left(\frac{x}{y}\right)$, we say that $f(x,y)$ is a homogeneous function in x and y of degree n.*

Illustration 1.

(i) $z=\dfrac{x^2y^2}{x+y}=\dfrac{x^4\cdot\left(\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}=x^3\cdot\dfrac{\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}=x^3\times$ *a function of* $\frac{y}{x}$.

$\therefore$ *z is a homogeneous function in x and y of degree 3.*

(ii) $f(x,y)=ax^2+2hxy+by^2=x^2\left[a+2h\left(\dfrac{y}{x}\right)+b\left(\dfrac{y}{x}\right)^2\right]=x^2\times$ *a function of* $\dfrac{y}{x}$.

$\therefore$ *$f(x,y)$ is a homogeneous function in x and y of degree 2.*

We may define homogenous functions in the following way:

Definition 2. *A function $f(x,y)$ of two variables x and y is said to be homogeneous of degree n when $f(tx,ty)=t^nf(x,y)$ for every positive quantity t.*

Illustration 2. *Let* $f(x,y)=\dfrac{x^3+y^3}{x+y}$.

Then $f(tx,ty)=\dfrac{t^3x^3+t^3y^3}{tx+ty}=t^2\cdot\dfrac{x^3+y^3}{x+y}=t^2f(x,y)$.

Hence $f(x,y)$ is a homogeneous function in x and y of degree 2.

The two definitions of homogeneity of the function $f(x,y)$ are equivalent. See that when $t=\dfrac{1}{x}$, $f(tx,ty)=t^nf(x,y)$ gives

$$f\left(1,\frac{y}{x}\right)=\frac{1}{x^n}f(x,y),\quad \text{i.e.,}\quad f(x,y)=x^nf\left(1,\frac{y}{x}\right)=x^n\phi\left(\frac{y}{x}\right).$$

Euler's Theorem on Homogeneous Function of Two Variables [B.U. B.Com.(H) 2008]

If $z=f(x,y)$ be a homogeneous function of x and y of degree n having partial derivatives f_x and f_y, then

$$\boldsymbol{xf_x+yf_y=nf}\quad \text{or,}\quad \boldsymbol{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial x}=nf(x,y)}.$$

Verification: Since $z=f(x,y)$ is a homogeneous function of degree n, we may write,

$$z=x^n\phi\left(\frac{y}{x}\right),\quad \text{i.e.,}\quad f(x,y)=x^n\phi\left(\frac{y}{x}\right).$$

$$\therefore\ \frac{\partial z}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right)\times\frac{\partial}{\partial x}\left(\frac{y}{x}\right).$$

[Here dash indicates the ordinary derivative of $\phi(u)$ w.r.t. u, u being $\frac{y}{x}$.]

$$\text{i.e.,}\quad \frac{\partial z}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right)$$

$$\text{or,}\ x\frac{\partial z}{\partial x}=nx^n\phi\left(\frac{y}{x}\right)-yx^{n-1}\phi'\left(\frac{y}{x}\right). \tag{1}$$

Also
$$\frac{\partial z}{\partial y}=x^n\phi'\left(\frac{y}{x}\right)\times\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=x^n\phi'\left(\frac{y}{x}\right)\times\frac{1}{x}=x^{n-1}\phi'\left(\frac{y}{x}\right).$$
$$\therefore\quad y\frac{\partial z}{\partial y}=x^{n-1}y\phi'\left(\frac{y}{x}\right). \qquad (2)$$

Hence, adding (1) and (2), we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)-yx^{n-1}\phi'\left(\frac{y}{x}\right)+x^{n-1}y\phi'\left(\frac{y}{x}\right)=nx^n\phi\left(\frac{y}{x}\right)=\boldsymbol{nf(x,y)}.$$

This proves Euler's Theorem on homogeneous functions of two variables.

Alternative method. Since $f(x,y)$ is a homogeneous function of degree n, we can write:

$$f(tx,ty)=t^n f(x,y). \qquad (3)$$

Differentiating both sides w.r.t. t, we get

$$\frac{\partial f}{\partial(tx)}\cdot x+\frac{\partial f}{\partial(ty)}\cdot y=nt^{n-1}f(x,y)\ [\because\ x,y \text{ are independent of } t].$$

Putting $t=1$, we get
$$\boldsymbol{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).}$$

Note: If $f(x,y,z)$ be a homogeneous function of x,y,z of degree n, then we can similarly prove that

$$\boldsymbol{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z).}$$

Example 10. *If $u=x^2-y^2+3xy$, verify Euler's Theorem.* [V.U. B.Com.(H) 2007]

Solution: Since
$$u=x^2-y^2+3xy=x^2\left[1-\frac{y^2}{x^2}+3\frac{y}{x}\right]=x^2\phi\left(\frac{y}{x}\right).$$

Thus u is a *homogeneous function in x and y of degree* 2. In order to verify Euler's Theorem, we are to prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u.$$

Here
$$\frac{\partial u}{\partial x}=2x+3y,\quad x\frac{\partial u}{\partial x}=2x^2+3xy;\quad \frac{\partial u}{\partial y}=-2y+3x,\quad y\frac{\partial u}{\partial y}=-2y^2+3xy.$$
$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2x^2+6xy-2y^2=2\left(x^2+3xy-y^2\right)=\boldsymbol{2u}.$$

Thus **Euler's Theorem is verified.**

Example 11. *If $f(x,y)=2x^3-11x^2y+3y^3$, show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f(x,y)$.* [C.U. B.Com.(H) 1999]

Solution: $f(x,y)=2x^3-11x^2y+3y^3$.

$$\therefore \quad \frac{\partial f}{\partial x}=6x^2-11\cdot(2xy)+0=6x^2-22xy$$

$$\text{and} \quad \frac{\partial f}{\partial y}=0-11x^2\cdot 1+9y^2=9y^2-11x^2.$$

$$\therefore x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x(6x^2-22xy)+y\left(9y^2-11x^2\right)=6x^3-22x^2y+9y^3-11x^2y$$

$$=6x^3-33x^2y+9y^3=3\left(2x^3-11x^2y+3y^3\right)=3f(x,y)\ \textbf{(Proved).}$$

Example 12.

(a) *Show that* $z=\dfrac{2x-3y}{5y}$ *is homogeneous in x and y, and prove that* $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$.

(b) *If* $u=\dfrac{x-y}{x+y}$, *find the value of* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. [B.U. B.Com.(H) 2008]

Solution: (a) Let

$$z=f(x,y)=\frac{2x-3y}{5y}=\frac{2x}{5y}-\frac{3y}{5y}=\frac{2}{5}\frac{1}{\frac{y}{x}}-\frac{3}{5}=x^0\phi\left(\frac{y}{x}\right).$$

$\therefore$ *z is a homogeneous of degree zero.*

Otherwise. Replace x by tx, y by ty, where $t>0$. Then

$$f(tx,ty)=\frac{2tx-3ty}{5ty}=t^0f(x,y).$$

$\therefore$ *z is a homogeneous function of degree zero.*

Now, we find

$$\frac{\partial z}{\partial x}=\frac{2}{5y}\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial x}\left(\frac{3}{5}\right)=\frac{2}{5y}. \qquad \therefore x\frac{\partial z}{\partial x}=\frac{2x}{5y}.$$

Again,

$$\frac{\partial z}{\partial y}=\frac{2x}{5}\frac{\partial}{\partial y}\left(\frac{1}{y}\right)-\frac{\partial}{\partial y}\left(\frac{3}{5}\right)=-\frac{2x}{5y^2}. \qquad \therefore y\frac{\partial z}{\partial y}=-\frac{2xy}{5y^2}=-\frac{2x}{5y}.$$

Hence,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{2x}{5y}-\frac{2x}{5y}=0\ \textbf{(Proved).}$$

(b) $u=\dfrac{x\left(1-\frac{y}{x}\right)}{x\left(1+\frac{y}{x}\right)}=x^0\phi\left(\dfrac{y}{x}\right)$. Now proceed as in Ex. 12. The value of $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ will be **0**.

Example 13. *Verify Euler's Theorem for the functions:*

(i) $f(x)=\dfrac{x^3+y^3}{x-y}$; [C.U. B.Com.(H) 2001]

(ii) $f(x)=\dfrac{x^3+y^3}{x^2+y^2}$. [C.U. B.Com.(H) 2002; V.U. B.Com.(H) 2009]

Solution:

(i) $f(x,y)=\dfrac{x^3+y^3}{x-y}$.

$$\therefore\ f(tx,ty)=\frac{t^3x^3+t^3y^3}{tx-ty}=\frac{t^3(x^3+y^3)}{t(x-y)}=\frac{t^2(x^3+y^3)}{x-y}.$$

Thus
$$f(tx,ty)=t^2f(x,y).$$

This shows that $f(x,y)$ *is a homogeneous function in* x,y *of degree* 2. To verify Euler's Theorem, we have to show that

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f(x,y).$$

Now,
$$\frac{\partial f}{\partial x}=\frac{(x-y)\cdot 3x^2-(x^3+y^3)\cdot 1}{(x-y)^2}=\frac{2x^3-3x^2y-y^3}{(x-y)^2}$$

and
$$\frac{\partial f}{\partial y}=\frac{(x-y)\cdot 3y^2-(x^3+y^3)\cdot(-1)}{(x-y)^2}=\frac{x^3+3xy^2-2y^3}{(x-y)^2}.$$

$$\therefore\ x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{2x^4-3x^3y-xy^3+x^3y+3xy^3-2y^4}{(x-y)^2}=\frac{2x^4-2x^3y+2xy^3-2y^4}{(x-y)}$$
$$=\frac{2x^3(x-y)+2y^3(x-y)}{(x-y)^2}=\frac{2(x-y)(x^3+y^3)}{(x-y)^2}=2\left(\frac{x^3+y^3}{x-y}\right)=\boldsymbol{2f(x,y)}.$$

Thus Euler's Theorem is verified.

(ii) See that $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}$ is a homogeneous function in x and y of degree one and proceed as in (i).

Example 14. *Prove that* $f(x,y)=\dfrac{\sqrt{x}+\sqrt{y}}{x+y}$ *is a homogeneous function of degree* $-\frac{1}{2}$ *and then verify Euler's Theorem.*

Solution: $f(x,y)=\dfrac{\sqrt{x}+\sqrt{y}}{x+y}=\dfrac{\sqrt{x}\left\{1+\sqrt{\dfrac{y}{x}}\right\}}{x\left\{1+\dfrac{y}{x}\right\}}=x^{-(\frac{1}{2})}\phi\left(\dfrac{y}{x}\right).$

$\therefore\ f(x,y)$ *is a homogeneous function in* x *and* y *of degree* $-\left(\frac{1}{2}\right)$. Now, in order to verify Euler's Theorem, we are to establish

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-\frac{1}{2}f(x,y).$$

Now,
$$\frac{\partial f}{\partial x}=\frac{(x+y)\frac{\partial}{\partial x}x^{\frac{1}{2}}+y^{\frac{1}{2}}-\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}=\frac{(x+y)\frac{1}{2}x^{-\frac{1}{2}}-\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\cdot 1}{(x+y)^2}$$
$$=\frac{-\frac{1}{2}x^{\frac{1}{2}}+\frac{1}{2}yx^{-\frac{1}{2}}-y^{\frac{1}{2}}}{(x+y)^2}..$$

$$\therefore\ x\frac{\partial f}{\partial x}=\frac{-\frac{1}{2}x^{\frac{3}{2}}+\frac{1}{2}yx^{\frac{1}{2}}-xy^{\frac{1}{2}}}{(x+y)^2}.$$

Also,
$$\frac{\partial f}{\partial y}=\frac{(x+y)\frac{\partial}{\partial y}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)-\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\frac{\partial}{\partial y}(x+y)}{(x+y)^2}=\frac{(x+y)\frac{1}{2}y^{-\frac{1}{2}}-\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\cdot 1}{(x+y)^2}$$
$$=\frac{\frac{1}{2}xy^{-\frac{1}{2}}-\frac{1}{2}y^{\frac{1}{2}}-x^{\frac{1}{2}}}{(x+y)^2}.$$

$$\therefore\ y\frac{\partial f}{\partial y}=\frac{\frac{1}{2}xy^{\frac{1}{2}}-\frac{1}{2}y^{\frac{3}{2}}-yx^{\frac{1}{2}}}{(x+y)^2}.$$

Hence,
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{-\frac{1}{2}x^{\frac{3}{2}}+\frac{1}{2}yx^{\frac{1}{2}}-xy^{\frac{1}{2}}+\frac{1}{2}xy^{\frac{1}{2}}-\frac{1}{2}y^{\frac{3}{2}}-yx^{\frac{1}{2}}}{(x+y)^2}=\frac{-\frac{1}{2}[x^{\frac{3}{2}}+yx^{\frac{1}{2}}+xy^{\frac{1}{2}}+y^{\frac{3}{2}}]}{(x+y)^2}$$
$$=\frac{-\frac{1}{2}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)(x+y)}{(x+y)^2}=-\frac{1}{2}\frac{\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)}{x+y}=-\frac{1}{2}f(x,y).$$

Thus **Euler's Theorem is verified.**

Example 15. *Let u be a homogeneous function in x and y of degree n. Prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Solution: Since u is a homogeneous function in x and y of degree n, we obtain from Euler's Theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu. \tag{1}$$

Differentiating (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}\text{ or, }x\frac{\partial^2 y}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}.$$

Multiplying both sides by x,

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=(n-1)\,x\frac{\partial u}{\partial x}. \tag{2}$$

Differentiating (1) w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}.$$

Then multiplying both sides by y,

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)y\frac{\partial u}{\partial y}. \tag{3}$$

Adding (2) and (3), and assuming $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$
$$= (n-1)nu \quad \text{[using (1)]}$$
$$= n(n-1)u \quad \textbf{(Proved).}$$

Example 16. *If* $z(x+y) = x^2 + y^2$, *show that* $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

[C.U. B.Com.(H) 1998; V.U. B.Com.(H) 2010]

Solution: We have $\qquad z(x+y) = x^2 + y^2$ or, $z = \dfrac{x^2+y^2}{x+y}$. (1)

Now,
$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2} = \frac{(x+y)\cdot 2x - (x^2+y^2)\cdot 1}{(x+y)^2}$$
$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

and
$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2} = \frac{(x+y)\cdot 2y - (x^2+y^2)\cdot 1}{(x+y)^2}$$
$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}.$$

$$\therefore \text{ LHS} = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left\{\frac{x^2 + 2xy - y^2 - (y^2 + 2xy - x^2)}{(x+y)^2}\right\}^2 = \left\{\frac{2(x^2-y^2)}{(x+y)^2}\right\}^2$$
$$= 4\left\{\frac{(x+y)(x-y)}{(x+y)^2}\right\}^2 = 4\left(\frac{x-y}{x+y}\right)^2.$$

Again

$$\text{RHS} = 4\left\{1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right\} = 4\left\{\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right\}$$
$$= 4\left\{\frac{(x+y)^2 - 4xy}{(x+y)^2}\right\} = 4\frac{(x-y)^2}{(x+y)^2} = 4\left(\frac{x-y}{x+y}\right)^2.$$

Hence, LHS = RHS (Proved).

Derivative of Implicit Functions

If $f(x,y) = 0$ defines y as a function of x, say $y = \phi(x)$, then we say, $y = \phi(x)$ is defined implicitly by $f(x,y) = 0$, e.g., $3x + 2y - 6 = 0$ is an implicit relation. It defines y as a function of x, given by $y = \frac{1}{2}(6-3x)$.

For such implicit functions we can find the derivative of y w.r.t. x by the following formula:

$$\frac{dy}{dx} = -\left[\frac{\partial f}{\partial x} \bigg/ \frac{\partial f}{\partial y}\right].$$

Example 17. *Using Euler's theorem on homogeneous function and* $u(x,y)=\log_e \dfrac{x^3+y^3}{x^2+y^2}$, *prove that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$ [C.U. B.Com.(H) 2003; (P) 2009]

Solution: Given, $u(x,y)=\log_e \dfrac{x^3+y^3}{x^2+y^2}$; $\quad\therefore\quad e^u=\dfrac{x^3+y^3}{x^2+y^2}$, where $u=u(x,y)$.

Let $e^u=z$; thus

$$z=\frac{x^3+y^3}{x^2+y^2}=\frac{x^3\left\{1+\left(\frac{y}{x}\right)^3\right\}}{x^2\left\{1+\left(\frac{y}{x}\right)^2\right\}}=x^1\left\{\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^2}\right\}.$$

$\therefore\quad z$ is a homogeneous function in x,y of degree 1.

By Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1\cdot z=e^u,\text{ where } z=e^u. \tag{1}$$

Now $z=e^u$;

$$\therefore\quad \frac{\partial z}{\partial x}=e^u\frac{\partial u}{\partial x}\quad\text{and}\quad\frac{\partial z}{\partial y}=e^u\frac{\partial u}{\partial y}.$$

$\therefore$ from (1), we get

$$x\cdot e^u\frac{\partial u}{\partial x}+y\cdot e^u\frac{\partial u}{\partial y}=e^u.$$

Dividing both sides, by e^u, we get

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.}$$

Example 18.

(a) *Verify Euler's theorem for the function* $u=f(x,y)=x^3+y^3+6x^2y+6xy^2$. [C.U. B.Com.(H) 2002]

(b) *If* $u=\sqrt{x^2+y^2}$, *prove that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{1}{u}$. [B.U. B.Com.(H) 2008]

Solution:

(a) $u=x^3\left[1+\left(\dfrac{y}{x}\right)^3+6\dfrac{y}{x}+6\left(\dfrac{y}{x}\right)^2\right]=x^3\phi\left(\dfrac{y}{x}\right)$

$\therefore$ u is a homogeneous function in x and y of degree 3.

To verify Euler's theorem, i.e., to verify

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u.$$

Observe that

$$\frac{\partial u}{\partial x}=3x^2+12xy+6y^2;\quad\therefore\quad x\frac{\partial u}{\partial x}=3x^3+12x^2y+6xy^2$$

$$\text{and}\quad\frac{\partial u}{\partial y}=3y^2+6x^2+12xy;\quad\therefore\quad y\frac{\partial u}{\partial y}=3y^3+6x^2y+12xy^2.$$

Hence $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3[x^3+y^3+6x^2y+6xy^2]=3u.$$

Thus *Euler's theorem is verified.*

(b) $\dfrac{\partial u}{\partial x}=\dfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot\dfrac{\partial}{\partial x}(x^2+y^2)=\dfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\times 2x=x(x^2+y^2)^{-\frac{1}{2}}.$

$$\begin{aligned}\therefore\quad \frac{\partial^2 u}{\partial x^2}&=1\cdot\left(x^2+y^2\right)^{-\frac{1}{2}}+x\times-\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}-1}\times 2x\\ &=\frac{1}{\sqrt{x^2+y^2}}-\frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}=\frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}}\\ &=\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}.\end{aligned}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2}=\frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}.$$

Hence $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2+x^2}{(x^2+y^2)\sqrt{x^2+y^2}}=\frac{1}{\sqrt{x^2+y^2}}=\frac{1}{u}\ \textbf{(Proved)}.$$

EXERCISES ON CHAPTER 5

(Partial Derivatives)

1. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $x=3, y=2$, if (a) $f(x,y)=x^2+y^2+3xy$; (b) $f(x,y)=\dfrac{x^3+y^3}{x+y}$.

2. Verify the following results:

(a) $\dfrac{\partial}{\partial x}\left(3x^2-4x^2y^2+6y^2\right)=2x\left(3-4y^2\right);\dfrac{\partial}{\partial y}\left(3x^2-4x^2y^2+6y^2\right)=4y\left(3-2x^2\right).$

(b) If $z=3x^4-4x^3y+6x^2y^2$, then $\dfrac{\partial z}{\partial x}=12x\left(x^2-xy+y^2\right),\dfrac{\partial z}{\partial y}=4x^2(3y-x).$

(c) If $f(x,y)=\dfrac{x+2y}{y+2x}$, then $\dfrac{\partial f}{\partial x}=\dfrac{-3y}{(y+2x)^2},\dfrac{\partial f}{\partial y}=\dfrac{3x}{(y+2x)^2}.$

(d) Let $z=\dfrac{x}{x^2+y^2}$. Then $\dfrac{\partial z}{\partial x}=\dfrac{y^2-x^2}{(y^2+x^2)^2},\dfrac{\partial z}{\partial y}=-\dfrac{2xy}{(x^2+y^2)^2}.$

(e) If $f(x,y)=e^{x+2y}$, then $\dfrac{\partial f}{\partial x}=f(x,y)$ and $\dfrac{\partial f}{\partial y}=2f(x,y).$

(f) If $w=\log\left(x^2+y^2\right)$, then $\dfrac{\partial w}{\partial x}=\dfrac{2x}{x^2+y^2}$ and $\dfrac{\partial w}{\partial y}=\dfrac{2y}{x^2+y^2}.$

(g) If $f(x,y)=2x^3-11x^2y+3y^3$, then $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=3f(x,y).$ [C.U. B.Com.(H) 1999]

(h) If $f(x,y)=(2x+3)(y-2)$, then $\dfrac{\partial f}{\partial x}=2(y-2)$ and $\dfrac{\partial f}{\partial y}=(2x+3)$.

(i) Find f_{yx} for the function $f(x,y)=x^3+y^3+2xy^2$. [B.U. B.Com.(H) 2005]

3. If $f(x,y)=\dfrac{2x}{x-y}$, prove that $\dfrac{\partial f}{\partial x}=-\dfrac{1}{2}$ and $\dfrac{\partial f}{\partial y}=\dfrac{3}{2}$ at $x=3, y=1$.

4. (a) If $z=ax^4+2bx^2y^2+cy^4$, prove that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=4z$.

(b) If $u=\dfrac{ax^n+by^n}{cx^2+dy^2}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=(n-2)u$.

(c) If $z=f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$.

5. Verify Euler's Theorem for each of the following functions:

(a) $z=2x^2+4xy+y^2$;

(b) $f(x,y)=x^4+x^2y^2-y^4$;

(c) $z=ax^2+2hxy+by^2$; [N.B.U. B.Com.(H) 2007]

(d) $z=x^4-x^3y+2x^2y^2-xy^3+y^4$; [C.U. B.Com.(H) 1991]

(e) $f(x,y)=x^3y^2+x^2y^3$; [C.U. B.Com.(H) 1995]

(f) $z=\dfrac{x^3+y^3}{x^2+y^2}$; [C.U. B.Com.(H) 2000; V.U. B.Com.(H) 2009]

(g) $z=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$;

(h) $f(x,y)=-\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$; [B.U. B.Com.(H) 2008]

(i) $f(x,y)=\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$;

(j) $z=\dfrac{x-3y}{2\sqrt{x}+\sqrt{y}}$; [V.U. B.Com.(H) 2011]

(k) $f(x,y)=x^3+y^3+3ax^2y$;

(l) $f=\dfrac{x+y}{x-y}$; [B.U. B.Com.(H) 2008]

(m) $f(x,y)=\dfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$; [B.U. B.Com.(H) 2008]

(n) $f(x,y)=\dfrac{x^3+y^3}{x^2+xy+y^2}$. [C.U. B.Com.(H) 2005]

6. Using the definitions of partial derivatives, find $f_x(1,2)$ and $f_y(1,2)$, where

$$f(x,y)=x^2+xy+y^2.$$

7. (a) If $V=\log\left(x^3+y^3+z^3\right)$, find $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$.

(b) If $u=x^2+y^2+z^2$, then prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=2u$. [C.U. B.Com.(H) 2008]

8. Find $\dfrac{dy}{dx}$ from the following relation by using the implicit function.
$x^4-x^2y^2+y^4=80$.

9. Use the implicit function rule to find $\dfrac{dy}{dx}$, if y is defined implicitly by the following relations: (a) $ax^2 + 2hxy + by^2 = 1$; (b) $y^x + x^y = c$; (c) $x^3 + y^3 - 3axy = 0$.

10. If $u = \log\sqrt{x^2+y^2}$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2}$ at $x = 1, y = 1$.

[C.U. B.Com.(H) 1996; B.U. B.Com.(H) 2007]

[Hints: $u = \frac{1}{2}\log\left(x^2+y^2\right)$; $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$; $\therefore\ x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{x^2+y^2}$, etc.]

11. If $z(x+y) = x^2+y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

[C.U. B.Com.(H) 1998; V.U. B.Com.(H) 2010]

12. (a) If $u = \sqrt{xy}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{1}{4}\dfrac{x^2+y^2}{(xy)^{\frac{3}{2}}}$.

(b) If $u = \log\left(x^2+y^2\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. [B.U. B.Com.(H) 2007]

13. If $u = (ax+by)^2 - \left(x^2+y^2\right)$, where $a^2+b^2 = 2$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$ and $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = Ku$, show that $K = \dfrac{3}{2}$.

[See that u is a homogeneous function of x and y of degree $\frac{3}{2}$. Hence, by Euler's theorem, we would obtain $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$ and hence K should be $\frac{3}{2}$.]

15. (a) $u = \left(x^2+y^2\right)\dfrac{1}{3}$, show that $\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^2 u = -\dfrac{4}{9}u$.

[Hints: $u = \left(x^2+y\right)^{1/3} = x^{2/3}\left(1+\frac{y^2}{x^2}\right)^{1/3} = x^{2/3}\phi\left(\frac{y}{x}\right) \Rightarrow u$ is homogeneous function in x, y of degree 2/3.

$\therefore$ by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}u \tag{1}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u \equiv \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(\frac{2}{3}u\right)$$

$$= \frac{2}{3}\cdot\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{2}{3}\times\frac{2}{3}u = \frac{4}{9}u \quad \text{[using (1)]]}$$

(b) If $f(x,y) = \sqrt{x^2+y^2+z^2}$, then find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial z^2}$ and hence show that $f_{xx} + f_{yy} + f_{zz} = \dfrac{2}{f}$.

[**Hints:** $f(x,y)=\sqrt{x^2+y^2+z^2}=\sqrt{u}$, where $u=x^2+y^2+z^2$.

$\therefore\ f_x=\dfrac{1}{2\sqrt{u}}\times\dfrac{\partial u}{\partial x}=\dfrac{1}{2\sqrt{u}}\times 2x=\dfrac{x}{\sqrt{u}}$ and $f_{xz}=\dfrac{1\cdot\sqrt{u}-x\times\frac{1}{2\sqrt{u}}\times 2x}{(\sqrt{u})^2}=\dfrac{u-x^2}{u\cdot\sqrt{u}}=\dfrac{y^2+z^2}{u\sqrt{u}}$.

Similarly, $f_{yy}=\dfrac{z^2+x^2}{u\sqrt{u}}$ and $f_{zz}=\dfrac{x^2+y^2}{u\sqrt{u}}\Rightarrow f_{xx}+f_{yy}+f_{zz}=\dfrac{2u}{u\sqrt{u}}=\dfrac{2}{\sqrt{u}}$.]

16. If $u=\dfrac{Ax^n+By^n}{Cx^2+Dy^2}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=(x-2)u$.

17. If $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}, x^2+y^2+z^2\neq 0$, show that $u_{xx}+u_{yy}+u_{zz}=0$. [N.B.U. B.Com.(H) 2007]

[Hints: $u_x=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}-1}\times\dfrac{\partial}{\partial x}\left(x^2+y^2+z^2\right)=-x\cdot\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$

and $u_{xx}=-\left[1\cdot\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+x\times-\frac{3}{2}\cdot\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}\times 2x\right]$

$$=-\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right]=-\left[\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right]$$

$$=-\frac{(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

Similarly, $u_{yy}=-\dfrac{(z^2+x^2-2y^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$ and $u_{zz}=-\dfrac{(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$.

Now find $u_{xx}+u_{yy}+u_{zz}$.]

18. If V be a homogeneous function in x and y of degree n, then prove that $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ are each homogeneous of degree $(n-1)$.

[Hints: $\because$ V is homogeneous of degree n;

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=nV.$$

Differentiating w.r.t. x, we get

$$x\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}+y\frac{\partial^2 V}{\partial x\partial y}=n\frac{\partial V}{\partial x}\text{ or, }x\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right)=(n-1)\frac{\partial V}{\partial x};$$

hence $\dfrac{\partial V}{\partial x}$ is homogeneous of degree $(n-1)$.

Similarly, $\dfrac{\partial V}{\partial y}$ is homogeneous of degree $(n-1)$.]

ANSWERS

1. (a) 12, 13; (b) 4, 1.

2. (i) $4y$.

6. 4 and 5

7. (a) $\dfrac{3x^2}{x^3+y^3+z^3}, \dfrac{3y^2}{x^2+y^2+z^2}, \dfrac{3z^2}{x^2+y^2+z^2}$.

8. $\dfrac{x(2x^2-y^2)}{y(x^2-2y^2)}$.

9. (a) $-\dfrac{(ax+hy)}{(hx+by)}$;

(b) $-\left(\dfrac{yx^{y-1}+y^x\log|y|}{xy^{x-1}+x^y\log|x|}\right)$;

(c) $-\left(\dfrac{x^2-ay}{y^2-ax}\right)$.

10. 0.

15. (b) $\dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}, \dfrac{z^2+x^2}{(x^2+y^2+z^2)^{3/2}}, \dfrac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$.

5.6 Envelope of a Family of Straight Lines

We begin with an example:

$$y = mx + m^2. \tag{1}$$

This equation represents different straight lines for different values of m. We say that the eq. (1) represents a **family of straight lines.** The quantity m has a fixed value for a particular line but it is different for different lines of the family. We call m, a *variable parameter* or simply, *a parameter.* We shall usually write the eq. (1) in the implicit form:

$$f(x, y, m) = mx + m^2 - y = 0. \tag{2}$$

Now we look for a curve at any point of which the equation of the tangent line is a member of (1) for some value of m. At the same time the curve is such that any particular line of the family (1) touches the curve at some point. Such a curve will be called **envelope of the family of lines.**

General Rule for Finding the Equation of the Envelope of a Family of Lines

First Step. Write the equation of the given family of lines in the implicit form:

$$f(x, y, m) = 0, \text{ where } m \text{ is a variable parameter.}$$

Second Step. Obtain $\frac{\partial f}{\partial m}$ and equate it to zero.

Third Step. Eliminate m from $f(x, y, m) = 0$ and $\frac{\partial f}{\partial m} = 0$.

This eliminant is the required envelope.

Particular Case. If $f(x, y, m) = 0$ is a quadratic equation in m, this elimination of m from $f(x, y, m) = 0$ and $\frac{\partial f}{\partial m} = 0$ amonts to writing *the discriminant of the quadratic equation in m equated to zero.*

Example 1. *Given a family of lines* $y = mx + \frac{a}{m}$ *(a is a fixed constant and m is the only variable parameter). To find the envelope of this family of lines.* [C.U. B.Com.(H) 2001]

Solution: Using the general rule: We write the given equation as

$$f(x, y, m) = 0 \text{ or, } mx + \frac{a}{m} - y = 0. \tag{1}$$

We equate to zero $\frac{\partial f}{\partial m}$,

$$\text{i.e., } \frac{\partial f}{\partial m} = x - \frac{a}{m^2} = 0 \text{ or, } x = \frac{a}{m^2} \text{ or, } m = \sqrt{\frac{a}{x}}. \tag{2}$$

Eliminating m between (1) and (2), we get

$$\sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{a/x}} - y = 0 \quad \text{or,} \quad 2\sqrt{a}\sqrt{x} = y, \quad \text{or,} \quad y^2 = 4ax. \quad \text{(a parabola)}$$

Thus *the envelope of the family of lines* $y = mx + \frac{a}{m}$ *is the parabola* $\mathbf{y^2 = 4ax}$.

[*Otherwise.*

$$y = mx + \frac{a}{m} \text{ or, } m^2x - my + a = 0. \tag{1}$$

Differentiating both sides w.r.t. parameter m, we get $2mx - y + 0 = 0$ or, $m = \frac{y}{2x}$; putting $m = \frac{y}{2x}$ in (1), we get $\frac{y^2}{4x} - \frac{y^2}{2x} + a = 0$ or, $-\frac{y^2}{4x} + a = 0$ or, $\boldsymbol{y^2 = 4ax}$.]

Geometrical Interpretation: At any point (x,y) of the parabola $y^2=4ax$, the equation of the tangent line is of the form $y=mx+\frac{a}{m}$ for some value of m, i.e., the tangent line is a member of the family of lines (1). At the same time for every value of m, the line $y=mx+\frac{a}{m}$ touches the parabola at some point (x,y) of the curve. Hence, *$y^2=4ax$ is the envelope of the family of lines $y=mx+\frac{a}{m}$*.

Using the Rule for the Particular Case: The given family of lines can be written as a quadratic in the variable parameter m; thus $m^2x-my+a=0$.

$\therefore$ required envelope is obtained by equating to zero the discriminant of this equation:

$$\text{Discriminant}=(-y)^2-4\cdot x\cdot a=0 \text{ or, } \mathbf{y^2=4ax}.$$

We draw a diagram: $y^2=4ax$ is a parabola with focus at $(a,0)$ and axis as +ve x-axis if $a>0$.

At any point P of the curve the equation of the tangent is $y=m_1x+\dfrac{a}{m_1}$ for some $m=m_1$. At another point Q the equation of the tangent will be $y=m_2x+\dfrac{a}{m_2}$ for another value of $m=m_2$ and so on. Also any line $y=mx+\dfrac{a}{m}$ will touch the curve for every value of m. Hence, the parabola is the envelope of the given family of lines.

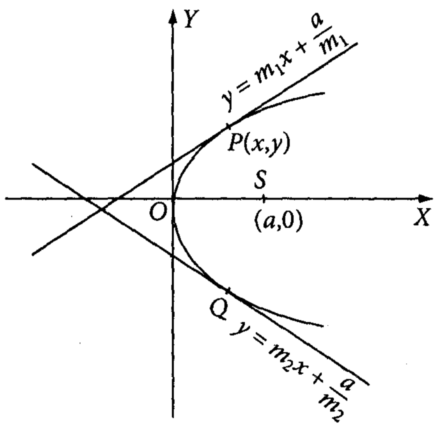


Fig. 8.1

Example 2. *Find the envelope of the straight lines $\dfrac{x}{m}+\dfrac{my}{3}=1$, where m is a variable parameter.*

[C.U. B.Com.(H) 1999]

Solution: We have

$$\frac{x}{m}+\frac{my}{3}=1 \text{ or, } 3x+m^2y=3m. \tag{1}$$

Differentiating both sides of (1) partially w.r.t. the parameter m, we get

$$0+2my=3 \text{ or, } m=\frac{3}{2y}.$$

Substituting $m=\frac{3}{2y}$ in (1), we get

$$3x+\left(\frac{3}{2y}\right)^2y=3\cdot\frac{3}{2y} \text{ or, } 3x+\frac{9}{4y}=\frac{9}{2y} \text{ or, } 12xy+9=18 \text{ or, } 12xy=9 \text{ or, } \mathbf{4xy=3},$$

which is the required envelope.

Example 3. *Find the envelope of the family of straight lines $\dfrac{x}{a}+\dfrac{y}{b}=1$, where the parameters a and b are connected by the relation $ab=4$.*

[C.U. B.Com.(H) 2000]

Solution: We consider one of the parameters, say b, as a function of the other parameter a.

Differentiating $\frac{x}{a}+\frac{y}{b}=1$ w.r.t. a (x, y being a fixed point on the envelope)

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \text{ or, } \frac{db}{da}=-\frac{b^2}{a^2}\frac{x}{y}. \tag{1}$$

Also from the given relation, $ab=4$, we differentiate w.r.t. a and we get

$$b+a\frac{db}{da}=0 \text{ or, } \frac{db}{da}=-\frac{b}{a}. \tag{2}$$

Eliminating $\frac{db}{da}$ from (1) and (2), we get

$$-\frac{b^2}{a^2}\frac{x}{y}=-\frac{b}{a} \text{ or, } \frac{x}{a}=\frac{y}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{2}=\frac{1}{2} \text{ or, } a=2x,\ b=2y.$$

$\therefore$ $\boldsymbol{ab}=4$ or, $4xy=4 \Rightarrow \boldsymbol{xy=1}$ *is the required envelope.*

Example 4. *Find the envelope of the family of lines $x/a+y/b=1$ where the parameters are connected by $a^2+b^2=c^2$ (c being a given constant).*

Solution: As before, we obtain $\frac{db}{da}=-\frac{b^2}{a^2}\frac{x}{y}$ and from $a^2+b^2=c^2$, we get $\frac{db}{da}=-\frac{a}{b}$.

$$\therefore \frac{x}{y}\frac{b^2}{a^2}=\frac{a}{b} \text{ or, } \frac{x}{y}=\frac{a^3}{b^3} \text{ or, } \frac{x/a}{a^2}=\frac{y/b}{b^2}=\frac{x/a+y/b}{a^2+b^2}=\frac{1}{a^2+b^2}=\frac{1}{c^2}.$$

$$\therefore \frac{x}{a^3}=\frac{1}{c^2} \text{ or, } a^3=c^2x \text{ or, } a=\left(c^2x\right)^{1/3}$$

$$\text{and } \frac{y}{b^3}=\frac{1}{c^2} \text{ or, } b^3=c^2y \text{ or, } b=\left(c^2y\right)^{1/3}.$$

Since $a^2+b^2=c^2$ or, $\left(c^2x\right)^{\frac{2}{3}}+\left(c^2y\right)^{\frac{2}{3}}=c^2$ or, $\boldsymbol{x^{2/3}+y^{2/3}=c^{2/3}}$ is the required envelope.

Try in a similar manner the following problems

Example 5. *Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters a and b are connected by*

	Required envelope
(i) $a+b=c$	(i) $\sqrt{x}+\sqrt{y}=\sqrt{c}$
(ii) $ab=c^2$	(ii) $4xy=c^2$
(iii) $a^n+b^n=c^n$	(iii) $x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=c^{\frac{n}{n+1}}$

Example 6. *Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the parameters a and b are connected by*

	Required envelope
(i) $a+b=k$	(i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$
(ii) $ab=c^2$	(ii) $2xy=c^2$
(iii) $\sqrt{a}+\sqrt{b}=\sqrt{c}$	(iii) $x^{\frac{2}{5}}+y^{\frac{2}{5}}=c^{\frac{2}{5}}$

Example 7. *Find the envelope of the family of lines $y = mx + \sqrt{1+m^2}$, m being the parameter.*

[V.U. B.Com.(H) 2008]

Solution: We have $y = mx + \sqrt{1+m^2}$, where m is the parameter. (1)

Differentiating partially w.r.t. parameter m, we get

$$0 = x + \frac{1}{2\sqrt{1+m^2}} \times \frac{d}{dm}\left(1+m^2\right) = x + \frac{1}{2\sqrt{1+m^2}} \times 2m = x + \frac{m}{\sqrt{1+m^2}}$$

$$\text{or, } -x = \frac{m}{\sqrt{1+m^2}} \text{ or, } \sqrt{1+m^2} = -\frac{m}{x}. \tag{2}$$

Putting the value of $\sqrt{1+m^2}$ in (1), we get

$$y = mx - \frac{m}{x} \text{ or, } m\left(x - \frac{1}{x}\right) = y \text{ or, } m = \frac{xy}{x^2-1}.$$

$\therefore$ from (2), we get

$$1 + m^2 = \frac{m^2}{x^2} \text{ or, } 1 + \frac{x^2y^2}{\left(x^2-1\right)^2} = \frac{x^2y^2}{\left(x^2-1\right)^2 x^2}$$

$$\text{or, } \left(x^2-1\right)^2 + x^2y^2 = y^2 \text{ or, } \left(x^2-1\right)^2 + y^2\left(x^2-1\right) = 0$$

$$\text{or, } x^2 - 1 + y^2 = 0 \text{ or, } \mathbf{x^2 + y^2 = 1},$$

which is the required envelope.

EXERCISES ON CHAPTER 5

(Envelope of a family of a Straight Lines)

Find the envelopes of the following family of straight lines (m, t, θ, wherever occurs are variable parameters):

1. $y = \frac{x}{m} + m^2$. 2. $y = m^2x - 2m^3$. 3. $y = 2mx + m^4$. 4. $y = tx - t^2$.
5. $y = t^2x + t$. 6. $y = tx - 2t^2; y = mx + \frac{1}{m}$. 7. $y = t^2x + t^3$. 8. $y = mx + m^2$.

9. $x\cos\theta + y\sin\theta = p\sin\theta\cos\theta$, where p is a fixed constant and θ is the variable parameter.

10. $x = mx + \sqrt{a^2m^2 + b^2}$ (a, b are fixed constants).

11. Find the envelope of the family of lines $\frac{x}{a} + \frac{ay}{4} = 1$, where a is a variable parameter.

[C.U. B.Com.(H) 1996]

ANSWERS

1. $27x^2 = 4y^3$.
2. $27y = x^3$.
3. $16y^3 + 27x^4 = 0$.
4. $x^2 = 4y$.
5. $4xy = -1$.
6. $x^2 - 8y = 0$, $y^2 = 4x$.
7. $4x^3 = 27y$.
8. $x^2 + 4y = 0$.
9. $x^{2/3} + y^{2/3} = p^{2/3}$ (Astroid).
10. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Ellipse).
11. $xy = 1$.

Chapter 6

Applications of the Derivative: Maxima and Minima, Demand, Cost, Revenue and Profit Functions, etc.

6.1 Some Specific Functions Useful in Business and Economics

In this chapter, we first define *some specific functions* useful in Business and Economics with illustrations. Then we shall discuss various applications of Derivatives.

Cost Function

The *Cost function* written as $C(x)$, is the *total cost* of producing x units of an item by a firm (or a company). It consists of two parts: (i) *Fixed costs* and (ii) *Variable costs.*

Fixed costs include costs which do not change when the level of production/sales changes, i.e., they are incurred even if nothing is produced or sold. Some examples of Fixed costs are rent, office expenses, overhead expenses, insurance, etc. Fixed cost is denoted by F or $F(0)$ or $C(0)$ or FC.

Variable costs of producing x units, written as $V(x)$, is defined by $V(x) = C(x) - F$, which depends on the number of units (x) produced. Some examples of variable costs are cost of materials, cost of labour, cost of fuel, etc.

Thus the *cost function* (or *total cost* of x units) is given by $C(x) = F + V(x)$, where F is independent of x.

Average cost (AC) $= \dfrac{C(x)}{x} = \dfrac{\text{Total cost}}{\text{Output}}$.

Average variable cost (AVC) $= \dfrac{V(x)}{x} = \dfrac{\text{Variable cost}}{\text{Output}}$.

If cost function is known, we can draw the corresponding cost curve. For a *linear cost function*, the cost curve is a *straight line* and for a *quadratic cost function*, it is a *parabola*. If the cost of production increases with the output x, then the cost curve rises towards the right side.

Revenue Function

The *revenue function*, written as $R(x)$, is the total amount of money generated from the sales of x units of a good (or an item) by a firm (or *a company*). It depends on the level of sales, i.e., on the price per unit and the number of units sold. If x units be sold at ₹p per unit, then the total revenue (or *total turnover*) $R(x)$ is given by $R(x) = px$, where $p > 0$ and $x > 0$.

Note that if $p =$ ₹2, then $R(x) = 2x$, i.e., the revenue curve is a *straight line.*

If $p =$ ₹$(3 - 2x)$, then $R(x) = (3 - 2x)\,x$ or $R(x) = 3x - 2x^2$, i.e., the revenue curve is a *parabola.*

Average Revenue (AR) $= \dfrac{R(x)}{x} = \dfrac{px}{x} = p$, i.e., *Average Revenue and Price (or price per unit) are the same.*

Profit Function

Profit function, written as $P(x)$, is the amount of money available to a firm (or company) from the sale of its product (say, x units) after all costs have been deducted, and is defined by

$$P(x) = R(x) - C(x),$$

where $C(x)$ = total cost = $F + V(x)$.

Thus *profit function is the difference between the total revenue and total cost.*

The average profit = profit per unit of output = $\dfrac{P(x)}{x}$.

Example 1. *A textbook publisher finds that the production costs to each book are ₹25 and that the fixed costs are ₹15,000. If each book can be sold for ₹45, determine:* (i) *the cost function,* (ii) *the revenue function,* (iii) *the break-even point.*

Solution: If $C(x)$ be the *cost function* and $R(x)$ the *revenue function* when x number of books are produced and sold, then

(i) $C(x) = F + V(x)$ = fixed cost + variable cost, i.e., $\mathbf{C(x) = 15000 + 25x}$, which is the required *cost function;*

(ii) $R(x)$ = total turnover = px = $\mathbf{45x}$ [$\because$ p = ₹45], i.e., $\mathbf{R(x) = 45x}$, which is the required *revenue function*;

(iii) For the *break-even point,* total revenue = total cost, i.e., $R(x) = C(x)$
or, $45x = 15000 + 25x$ or, $45x - 25x = 15000$ or, $20x = 15000$; $\therefore$ $\mathbf{x = 750.}$

Hence, the *break-even point is* 750 **copies.**

Example 2. *For the first year the fixed cost for setting up a new electronic pocket calculators manufacturing company is ₹300000. The variable cost for producing a calculator is ₹70. The company expects the revenue from the sales of the calculators to be ₹270 per calculator.*

(i) *Construct the revenue function.*

(ii) *Find the break-even point.*

(iii) *Also find the number of calculators produced for which the company will suffer a loss.* [CA May 1990]

Solution: (i) If x be the number of calculators produced and sold in a year, then the cost function $C(x)$ is given by

$$C(x) = \text{fixed cost} + \text{variable cost} = 300000 + 70x$$

and the *revenue function* $R(x)$ is given by

$$R(x) = px = \mathbf{270x} \quad [\because\ p = ₹270]$$

(ii) For the *break-even point,* total revenue = total cost or, $R(x) = C(x)$,

$$\text{i.e., } 270x = 3,00,000 + 70x \text{ or, } 200x = 3,00,000$$

$$\therefore\ x = \frac{3,00,000}{200} = \mathbf{1500.}$$

Hence, the *break-even point is at the production level of* **1500 calculators** per annum.

(iii) At the *break-even point,* the company will neither make a profit nor incur a loss. *Hence, the company would suffer loss if the number of calculators produced is less than* 1500 per annum.

Example 3. *A shoe manufacturer is planning production of new varieties of shoes. For the first year the fixed costs for setting up the new production line are ₹1.25 lac. Variable costs for producing each pair of shoes are ₹35. The sales department projects that* 1500 *pairs can be sold in the first year at the rate of* ₹160 *a pair.*

(i) *Determine the profit function $P(x)$ for the profit from the sales of x pairs of shoes.*

(ii) *If* 1500 *pairs are actually sold, what profit or loss the company would incur?*

(iii) *Determine the break-even point.* [CA Nov. 1990]

Solution: (i) If x pairs of shoes are produced and sold, then the cost function $C(x)$ and the revenue function $R(x)$ are given by $C(x) = 1,25,000 + 35x$ and $R(x) = 160x$ [$\because$ $p =$ ₹160]

If $P(x)$ be the *profit function*, then

$$P(x) = R(x) - C(x) = 160x - (125000 + 35x) = 125x - 125000.$$

Hence, the *required profit function is* $\mathbf{P(x) = 125x - 125000}$.

(ii) If $x = 1500$, then profit $= P(1500) = 125 \times 1500 - 125000 = 125(1500 - 1000)$
$= 125 \times 500 =$ ₹62,500.

(iii) For the *break-even point*, total revenue = total cost, i.e., $R(x) = C(x)$
or, $160x = 125000 + 35x$ or, $125x = 125000$; $\therefore$ $x = 1000$.

Hence, the *break-even point* is the production and sale of 1000 pairs of shoes.

Example 4. *If the cost function $C(x)$ of producing x quantities of a product is given by $C(x) = 500x^2 + 2500x + 5000$ and if each unit of the product is sold at* ₹6000, *then what are the break-even points?*

[CA May 1991]

Solution: We have $C(x) = 500x^2 + 2500x + 5000$.

If $R(x)$ be a revenue function, then $R(x) = 6000x$ [$\because$ $p =$ ₹6000]

For the *break-even points*,

$$
\begin{aligned}
R(x) = C(x) \quad \text{or,} \quad & 6000x = 500x^2 + 2500x + 5000 \\
\text{or,} \quad & 500x^2 - 3500x + 5000 = 0 \\
\text{or,} \quad & x^2 - 7x + 10 = 0 \\
\text{or,} \quad & x^2 - 5x - 2x + 10 = 0 \\
\text{or,} \quad & x(x - 5) - 2(x - 5) = 0 \\
\text{or,} \quad & (x - 5)(x - 2) = 0. \quad \therefore \ \mathbf{x = 5 \text{ or } 2}.
\end{aligned}
$$

Hence, the *break-even point is the level at which* 5 *or* 2 *units of the product are produced.*

Example 5. *The cost of manufacturing a particular type of shirt by a company is found to be a quadratic function of total number of shirts produced. For producing* 10, 20 *and* 30 *shirts, cost works out to be* ₹3690, ₹2590 *and* ₹1690 *respectively. Determine the functional relation between cost per shirt and the number of shirts produced.* [CA PE-I Nov. 2002]

Solution: If x be the number of shirts produced, then the functional relation is

$$C(x) = ax^2 + bx + c, \tag{1}$$

where a, b, c are constants. Given, $C(10) = 3690$, $C(20) = 2590$ and $C(30) = 1690$.

$$\therefore \quad 100a+10b+c=3690 \tag{2}$$
$$400a+20b+c=2590 \tag{3}$$
$$\text{and} \quad 900a+30b+c=1690. \tag{4}$$
$$(2)-(3) \Rightarrow -300a-10b=1100 \text{ or, } 30a+b=-110 \tag{5}$$
$$(4)-(3) \Rightarrow 500a+10b=-900 \text{ or, } 50a+b=-90 \tag{6}$$
$$(6)-(5) \Rightarrow 20a=20 \text{ or, } a=1.$$

Substituting $a=1$ in (5), $30\times 1+b=-110$ or, $b=-140$.

Substituting $a=1$, $b=-140$ in (2),

$$100\times 1+10\times(-140)+c=3690 \text{ or, } c=3690+1300=4990.$$

Hence, from (1), *the required functional relation* between cost $C(x)$ per shirt and the number of shirt (x) is $\boldsymbol{C(x)=x^2-140x+4990}$.

Example 6. *A computer software company wishes to start the production of floppy disks. It was observed that the company had to spend ₹2 lac for the technical information. The cost of setting up the machine is ₹88000 and the cost of producing each unit is ₹30, while each floppy could be sold at ₹45. Find:* (i) *the total cost function for producing x number of floppies; and* (ii) *the break-even point.*

Solution: Clearly, fixed cost (F) = ₹200000 + ₹88000 = ₹288000. Let $C(x)$ be the total cost in rupees for producing x floppies. Then $C(x)=V(x)+F$, where
$V(x)$ = variable cost for x units = $30x$.

$$\therefore \ C(x)=30x+288000, \tag{1}$$

which is the total cost function.

If $R(x)$ be the total revenue function for sales of x units, then $R(x)=45x$.

At the break-even point, $R(x)=C(x)$ [$\because$ there is neither any profit nor any loss.]

$$\text{or,} \quad 45x=30x+288000 \Rightarrow 45x-30x=288000$$
$$\text{or,} \quad 15x=288000 \Rightarrow x=\frac{288000}{15}=19200.$$

Hence, to get the **break-even point, 19200 floppies are to be produced.**

EXERCISES ON CHAPTER 6(I)

(Specific Functions: Break-Even Point)

1. A textbook publisher finds that the production cost of each book is ₹30 and that the fixed cost is ₹18000. If each book can be sold for ₹50, determine:

   (a) the cost function, (b) the revenue function, (c) the break-even point.

2. For the first year the fixed cost for setting up a new electronic pocket transistors company is ₹2,50,000. The variable cost for producing a transistor is ₹75. The company expects the revenue from the sales of the transistors to be ₹275 per transistor.

(a) Construct the revenue function.

(b) Find the break-even point.

(c) Also find the number of transistors to be produced for which the company will suffer a loss.

3. A shoe company is planning to produce new varieties of shoes. For the first year the fixed cost for setting up the new production line is ₹1.75 lac. Variable cost for producing each pair of shoes is ₹55. The sales department projects that 1400 pairs can be sold in the first year @ ₹230 a pair.

   (a) Determine the profit function $P(x)$ for the profit from the sales of x pairs of shoes.

   (b) If 1400 pairs are actually sold, what profit or loss the company would incur?

   (c) Determine the break-even point.

4. (a) If the cost function $C(x)$ of producing x units of a product is given by $C(x) = 400x^2 + 2200x + 4800$ and if each unit of the product is sold at ₹5000, then what are the break-even points?

   (b) The fixed cost and the variable cost of x units of a manufactured product of a company are ₹4,00,000 and ₹$80x$ respectively. If each unit is sold for ₹280, what is the break-even point? [CA Nov. 1991]

   [Hints: $C(x) = 4,00,000 + 80x$ and $R(x) = 280x$; at the break-even point, $C(x) = R(x)$.]

5. The daily cost of production C for x units of a manufactured product is given by

$$C(x) = ₹3.5x + ₹12000.$$

   (a) If each unit is sold for ₹6, determine the minimum number of units that should be produced and sold to ensure no loss.

   (b) If the selling price is increased by half of a rupee per unit, what would be the break-even point?

   (c) If 6000 units are sold daily, what price per unit should be charged to guarantee no loss?

6. For a manufacturer of dry cells, the daily cost of production C for x cells is given by $C(x) = ₹2.05x + ₹550$. If the price of a cell is ₹3, determine the minimum number of cells that must be produced and sold daily to ensure no loss.

7. A book publisher finds that the production cost of a book is ₹30 and the fixed cost per year amounts to ₹25,000. If each book is sold at the rate of ₹50, find:

   (a) the cost function,

   (b) the revenue function,

   (c) the minimum number of books to be sold per year in order that there is no loss.

8. The pricing policy of a company follows the demand equation $p = D(x)$, $D(x)$ being the price per unit when x units are demanded. After studying the market trends the company determines the price function to be given by $D(x) = 800 - 2x$.

   If the product is to be marketed, the company will incur a fixed cost of ₹20,000 and will have to bear ₹220 for each unit that is produced. What sales level the company can expect to recover its cost?

   [CA PE–I Nov. 2002]

[**Hints**: The cost function $C(x)$ is $C(x) = 220x + 2000$ and the revenue function is

$$R(x) = px = (800 - 2x)\,x = 800x - 2x^2.$$

Just to recover cost, we must have $R(x) = C(x)$, i.e., $220x + 20000 = 800x - 2x^2$ or, $2x^2 - 580x + 20000 = 0$ or, $x^2 - 290x + 10000 = 0$ or, $(x-40)(x-250) = 0 \Rightarrow x = 40, 250$.]

9. The daily cost of production C for x units of an assembly is given by

$$C(x) = ₹12.5x + ₹6400.$$

(a) If each unit is sold for ₹25, determine the minimum number of units that should be produced and sold to ensure no loss.

(b) If the selling price is reduced by ₹2.50 per unit, what would be the break-even point?

(c) If it is known that 500 units can be sold daily, what price per unit should be charged to guarantee no loss?

10. A company sells x tins of talcum powder each day at ₹10 per tin. The cost of manufacturing is ₹6 per tin and the distributor charges Re. 1 per tin. Besides these the daily overhead cost comes to ₹600. Determine the profit function. What is the profit if 500 tins are manufactured and sold a day? How do you interpret the situation, if the company manufactures and sells 100 tins a day? What is the break-even point?

[**Hints**: $C(x) = 600 + 6x + x = 600 + 7x$ and $R(x) = 10x$; $\therefore$ $P(x)$ = profit function = $R(x) - C(x) = 3x - 600$ and $P(500) = 3 \times 500 - 600 = ₹900$. $P(100) = 300 - 600 = -300$. Negative value of $P(100)$ indicates that the company will incur a loss of ₹300 daily if it manufactures and sells 100 tins a day. For the *break-even point*, $P(x) = 0$ or, $3x - 600 = 0$, etc.]

11. A cottage toy industry has 29 workers. The cost of producing a unit of toy is ₹2.07. Other fixed cost including bonus is ₹30 per worker.

(a) If each toy is sold for ₹6, determine the number of toys that must be produced and sold daily to ensure no loss.

(b) To promote sale, if price is reduced by 50 paise per toy, what would be the break-even point and if at this rate 500 toys are sold daily, what would be the profit? [CA Final May 1999]

12. A leather company starts production of a new variety of ladies bag. For the first year, the fixed cost for setting up the infrastructure comes to ₹1,40,000. Variable cost for the production of each bag is ₹75. But the company gives production bonus to its employees. So, the variable cost further increases by 50 paise per bag. Each bag is sold at ₹250.50. What is the profit function $P(x)$ for x bags, produced and sold in the first year?

If 700 bags are produced and sold in the first year, what profit or loss would the company incur? What is the break-even point? [CA Final May 2000]

ANSWERS

1. (a) $C(x) = 18000 + 30x$;
   (b) $R(x) = 50x$;
   (c) 900 copies of the book.
2. (a) $R(x) = 275x$;
   (b) 1250 transistors;
   (c) less than 1250.
3. (a) $P(x) = 175x - 175000$;
   (b) Profit of ₹70000;
   (c) 1000 pairs of shoes.
4. (a) 3 or 4 units;

(b) 2000 units.

5. (a) 4800;

(b) 4000 units;

(c) ₹5.50 or more.

6. 579 cells.

7. (a) $C(x) = 25000 + 30x$;

(b) $R(x) = 20x - 2500$;

(c) 1250.

8. 40 or 250.

9. (a) 512;

(b) 640 units;

(c) ₹25.30 or more.

10. $P(x) = 3x - 600$; ₹900; 200 tins.

11. (a) 222;

(b) 254 units, ₹845.

12. $175x - 140000$; ₹17500 (loss); 800 bags.

6.2 Geometrical Interpretation of Derivative

We have the following fundamental result:

Theorem 1. *The value of the derivative at any point of a curve is equal to the slope of the tangent line to the curve at that point.* [B.U. B.Com.(H) 2008]

Proof. Let $P(x, y)$ be a given point on a continuous curve whose equation is $y = f(x)$, the system of axes being rectangular [Fig. 6.1].

Let $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point of the curve. Then the slope of the chord PQ

$$= \frac{\text{Rise}}{\text{Run}} = \frac{NQ}{PN} = \frac{\Delta y}{\Delta x} = \frac{\text{increment in } y}{\text{increment in } x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

As Q tends to P along the curve, the chord PQ may tend to a definite position PT (called the *Tangent* to the curve at P).

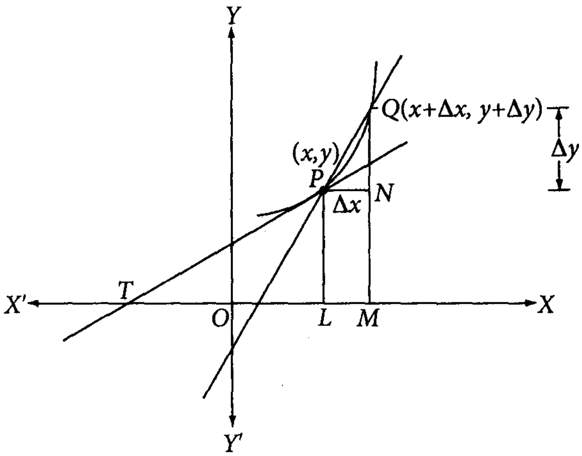


Fig. 6.1

In the limit as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

and hence the slope of the tangent $PT = f'(x)$ = derivative of $f(x)$ at $P(x, y)$.

Remember. *The slope of the curve at any point P is defined by the slope of the tangent at P and hence the slope of a curve at a point is equal to the derivative of $f(x)$ at the point $P(x,y)=f'(x)$.*

Observation

- $f'(x)=0$ implies that the tangent line at $P(x,y)$ is parallel to the X-axis.
- $1/f'(x)=0$ implies that the tangent line at $P(x,y)$ is perpendicular to X-axis, i.e., parallel to Y-axis.

Recalling our knowledge of coordinate geometry, the equation of the tangent PT at (x,y) of the curve $y=f(x)$ is $Y-y=f'(x)(X-x)$, where (X,Y) are the current coordinates, i.e., the coordinates of any point on PT. The equation of the tangent at a point (x_1,y_1) may also be written as

$$y-y_1=-\left[\frac{dy}{dx}\right]_{(x_1-y_1)}(x-x_1).$$

The equation of the normal at $P(x,y)$ [i.e., a line through P perpendicular to the tangent PT] is given by $Y-y=-\dfrac{1}{f'(x)}(X-x)$, if $f'(x)\neq 0$, where (X,Y) is any point on the normal at P. The *equation of the normal at* $P(x_1,y_1)$ *may be written as* $x-x_1+\left[\dfrac{dy}{dx}\right]_{(x_1,y_1)}(y-y_1)=0$.

Example 1. *Find the slopes of the tangents to the parabola $y=x^2$ at the vertex and at the point, where $x=1/2$.*

Solution: $\dfrac{dy}{dx}=2x=$ slope of the tangent at any point (x,y) of the curve.

- The vertex of the given parabola is at the origin $(0,0)$. Putting $x=0$, *the slope of the tangent at the vertex is* **0**.
- The slope of the tangent at the point, where $x=1/2$, is the value of $\dfrac{dy}{dx}$ at $x=\dfrac{1}{2}$, i.e., *the slope* $=2\times\dfrac{1}{2}=1$.

Example 2. *Find the slope of the parabola $y=4x^2$ at the vertex and also at $(1/2,1)$. Obtain the equations of the tangent and normal at $(1/2,1)$.*

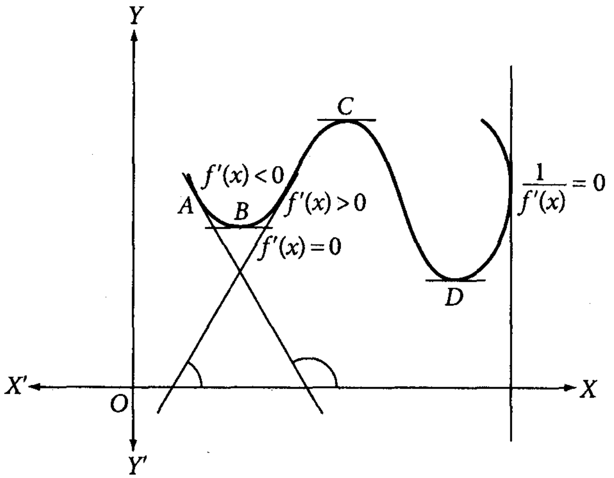


Fig. 6.2

Solution: (i) The slope of $y = 4x^2$ at the vertex $(0,0)$ is the value of $\frac{dy}{dx}$ at $x = 0$. We see that $\frac{dy}{dx} = 8x$; at $x = 0$, $\frac{dy}{dx} = 0$.

$\therefore$ the slope at the vertex is zero, i.e., **the tangent at the vertex is the X-axis itself.**

(ii) The slope of $y = 4x^2$ at $(1/2, 1)$ is the value of $\frac{dy}{dx}$ at $x = \frac{1}{2}$.

Since $\frac{dy}{dx} = 8x$ and at $x = \frac{1}{2}$, this value = 4.

$\therefore$ the slope of the curve at $(1/2, 1)$ is **4.**

(iii) The equations of the tangent and normal at the point $\left(\frac{1}{2}, 1\right)$ are respectively

$Y - 1 = 4\left(X - \frac{1}{2}\right)$ and $Y - 1 = -\frac{1}{4}\left(X - \frac{1}{2}\right)$ or, $\boldsymbol{Y = 4X - 1}$ **and** $\boldsymbol{8Y = -2X + 9}$**.**

Example 3. *Find the gradient of the curve $y = \log x$ at $(1,0)$ and at $(e^3, 3)$.*

Solution: The equation of the curve is $y = \log x$, where $x > 0$.

$\therefore \frac{dy}{dx} = \frac{1}{x}$. Also $(1,0)$ lies on the curve $y = \log x$.

Geometrically, *the gradient* of the curve at any point (x, y) is $\frac{dy}{dx} = \frac{1}{x}$.

At the point $(1,0)$, $\frac{dy}{dx} = \frac{1}{1} = 1$, i.e., **gradient of the curve at $(1,0)$ is 1.**

The point $(e^3, 3)$ lies on the curve $y = \log x$ and at $(e^3, 3)$, $\frac{dy}{dx} = \frac{1}{e^3}$.

Hence, the **gradient of the curve at** $(e^3, 3)$ is $\left[\frac{dy}{dx}\right]_{(e^3,3)} = \boldsymbol{\frac{1}{e^3}}$.

Example 4. *Find the gradient of the tangent to the parabola $y^2 = 4x$ at the point $(1,2)$ and hence find the equation of the tangent and normal at $(1,2)$.*

Solution: The equation of the parabola is

$$y^2 = 4x. \tag{1}$$

Differentiating both sides w.r.t. x, we get $2y\frac{dy}{dx} = 4$ or, $\frac{dy}{dx} = \frac{2}{y}$.

At $(1,2)$, gradient of the tangent to the parabola $= \left[\frac{dy}{dx}\right]_{(1,2)} = \frac{2}{2} = \mathbf{1}$.

The equation of the tangent at $(1,2)$ is $y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1,y_1)} (x - x_1)$ or, $y - 2 = 1 \cdot (x - 1)$ or, $y = x - 1 + 2$, or, $\boldsymbol{y = x + 1}$.

The equation of the normal at $(1,2)$ is $(x - x_1) + \left[\frac{dy}{dx}\right]_{(x_1,y_1)} (y - y_1) = 0$
or, $(x - 1) + 1 \cdot (y - 2) = 0$ or, $x - 1 + y - 2 = 0$ or, $\boldsymbol{x + y = 3}$.

Example 5. *Find the slope of the curve $x = y^2 - 4y$ at the points where it crosses the Y-axis.*

Solution: For the point of intersection of the curve and the Y-axis, we have $x=0$ and $x=y^2-4y$, i.e., $y^2-4y=0$ or, $y(y-4)=0$ or, $y=0,4$.

The points of intersection are $(0,0)$ and $(0,4)$.

Now, the equation of the curve is $x=y^2-4y$; $\therefore \dfrac{dx}{dy}=2y-4$.

$$\therefore \frac{dy}{dx}=1\Big/\frac{dx}{dy}=\frac{1}{2y-4}.$$

At the point $(0,0)$, *slope of the curve is* $\left[\dfrac{dy}{dx}\right]_{(0,0)}=\dfrac{1}{2\times 0-4}=-\dfrac{1}{4}$ *and at the point* $(0,4)$, *slope of the curve is* $\dfrac{1}{2\times 4-4}=\dfrac{1}{4}$.

Example 6. *The gradient of the curve* $xy+ax+by=0$ *at* $(1,1)$ *is* 2. *Find the values of* a *and* b.

Solution: The equation of the curve is

$$xy+ax+by=0. \tag{1}$$

If the curve (1) passes through the point $(1,1)$, then

$$1+a+b=0. \tag{2}$$

Differentiating both sides of (1) w.r.t. x we get

$$1\cdot y+x\frac{dy}{dx}+a+b\frac{dy}{dx}=0 \text{ or, } (x+b)\frac{dy}{dx}=-(a+y) \text{ or, } \frac{dy}{dx}=-\frac{a+y}{x+b}.$$

At the point $(1,1)$, gradient of the curve $=\left[\dfrac{dy}{dx}\right]_{(1,1)}=-\left(\dfrac{a+1}{b+1}\right)$.

$$\therefore -\left(\frac{a+1}{b+1}\right)=2 \text{ or, } a+1=-2b-2 \text{ or, } a+2b+3=0. \tag{3}$$

Subtracting (2) from (3), we get $b+2=0$ or, $b=-2$.
Substituting $b=-2$ in (2), we get $1+a-2=0$ or, $a=1$.
Hence $\boldsymbol{a=1}$ and $\boldsymbol{b=-2}$.

EXERCISES ON CHAPTER 6(II)

(Slope of a Curve)

1. Give the geometrical meaning of the derivative of a function at a point.
2. (a) Find the gradient of the curve $y=3x^2-5x+4$ at the point $(1,2)$.
   (b) Find the slope of the curve $f(x)=\dfrac{2x}{x^2+1}$ at $(0,0)$.
3. Find the gradient of the curve $y=\log_e x$ at $(e,1)$ and find the point at which the gradient is $\dfrac{1}{2}$.

4. Find the gradient of the curve $y^2 = 2x$ at the point $(2,2)$ and hence find the equation of the tangent at $(2,2)$.

5. Determine the points on the curve $y = x + \frac{1}{x}$, where the tangent is parallel to the X-axis.

6. Find the gradient of the tangent to the curve $y = x(x+3)$ at the point where it crosses the X-axis.

7. The curve $y = ax + bx^2$ passes through the point $(2,0)$ and its gradient at that point is -1. Find the values of a and b.

8. Gradient of the curve $xy + 2ax + 3by = 0$ at the point $(3,2)$ is $-\frac{2}{3}$. Find the values of a and b.

9. (a) Find the slope of the tangent line at the point $(0,2)$ of the curve

$$y = \tfrac{1}{8}\left(x^3 - 12x + 16\right).$$

Write down the equation of the tangent line. What is the equation of the normal at $(0,2)$? At what points on the curve the slope of the tangent is equal to $\frac{9}{2}$ and at what points are tangents parallel to X-axis?

(b) Find the slope of the curve at the point $t = 2$ when $x = t^2 - 3$, $y = 2t + 1$. [C.U. B.Com. 2004]

10. At what points do the curve $y = x^2 - 1$ and $y = -2x^2 + 2$ intersect? Which curve is steeper at these points?

11. Find the area of the triangle formed by the X-axis, the tangent at $(5,5)$ and normal at $(5,5)$ to the curve $y = 6x - x^2$.

12. Find the equations of the tangent and normal at the given point:

(a) $6y = x^2$ at $(6,6)$;

(b) $9x^2 + 4y^2 = 72$ at $(2,3)$;

(c) $x^2 - 4y^2 = 9$ at $(5,2)$.

13. Find the equations of the tangent and normal at the given point:

(a) $y = x^3 - 3x$ at $(2,2)$;

(b) $y = \dfrac{2x+1}{3-x}$ at $(2,5)$.

14. Prove that the equations of the tangent and normal to

$$y(x-2)(x-3) - x + 7 = 0$$

at the point, where it cuts the X-axis are respectively $x - 20y = 7$ and $20x + y = 140$.

15. Prove that $y = mx + c$ touches the parabola $y^2 = 4a(x+a)$, if $c = am + \frac{a}{m}$.

16. Prove that $y = mx + c$ touches the parabola $x^2 = 4by$, if $bm^2 + c = 0$.

ANSWERS

2. (a) 1; (b) 3.
3. $1/e; (2, \log_e 2)$.
4. $1/2; x-2y+2=0$.
5. $(1,2), (-1,0)$.
6. $3, -3$.
7. $a=1, b=-1/2$.
8. $a=b=-1/2$.
9. (a) $-3/2$; $2y+3x=4$; $2x-3y+6=0$; $(-4,0)$ and $(4,4)$; $(2,0)$ and $(-2,4)$;
   (b) $1/2$.
10. $(1,0)$ and $(-1,0)$; the second curve is steeper.
11. $1/8 \times 425$.
12. (a) $2x-y=6, x+2y=18$;
    (b) $2y+3x=12$; $3y-2x=5$;
    (c) $5x-8y=9, 8x+5y=50$.
13. (a) $9x-y=16, x+9y=20$;
    (b) $7x-y=9, x+7y=37$.

6.2.1 An important note: Signs of derivative

- If at any point (x,y) of the curve $y=f(x)$ we see that $f'(x)$ is positive (i.e., $f'(x)>0$), then we can conclude that the tangent line at (x,y) makes an *acute angle* with the positive direction of the X-axis. But if $f'(x)$ is negative (i.e., if $f'(x)<0$), then the tangent line at (x,y) makes an *obtuse angle* with the positive direction of the X-axis.
- If $f'(x)=0$, the tangent line is parallel to the X-axis; the tangents at (B,C,D) are parallel to X-axis; if $\dfrac{1}{f'(x)}=0$, the tangent line is parallel to the Y-axis.

6.3 Increasing and Decreasing Functions

1. A function $y=f(x)$ is said to be increasing in a certain interval, say in $[a,b]$, if as x increases, y also increases, i.e., if $a \le x_1 < x_2 \le b$, then $f(x_2) > f(x_1)$.

   Graphically, it means that as we move along the curve from left to right in $[a,b]$, the curve is rising. See that in case the curve is rising, then
   - the increment in y (i.e., Δy) and the increment in x (i.e., Δx) have the same sign (both positive or both negative). The ratio $\Delta y/\Delta x$ is always positive. In the limit $\Delta y/\Delta x$ when $\Delta x \to 0$, i.e., the derivative $\dfrac{dy}{dx}$ at any point of such a rising curve is positive, i.e., $\dfrac{dy}{dx}>0$ at the point.
   - the tangent at any point makes an acute angle with the positive direction of the X-axis.

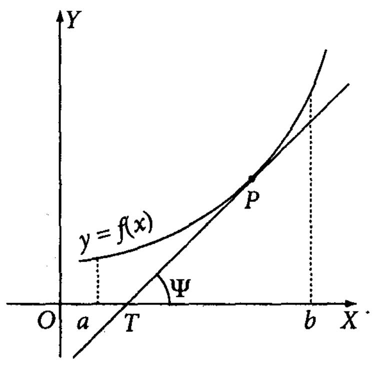


Fig. 6.3

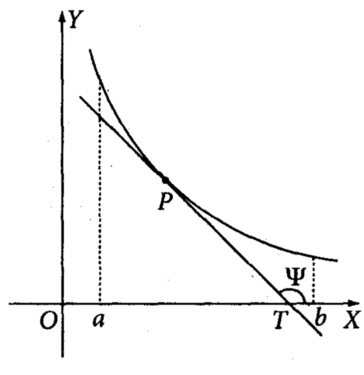


Fig. 6.4

2. A function $y=f(x)$ is said to be decreasing in a certain interval, say in $[a,b]$, if as x increases, y decreases, i.e., if $a \le x_2 < x_1 \le b$, then $f(x_2) < f(x_1)$.

Graphically, when $y = f(x)$ is decreasing in $[a, b]$, the curve is falling there. We also see that

- the increment in y (i.e., Δy) and the increment in x (i.e., Δx) have opposite signs; ultimately

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

is negative, i.e., *the derivative* $\frac{dy}{dx}$ *at any point of a falling curve is negative.*

- the tangent at any point makes an obtuse angle with the positive direction of the X-axis.

From the above discussion, it follows that

(a) *if* $f'(c) > 0$, *then* $f(x)$ *is increasing at* c, *i.e.,* $f(x)$ *has values greater than* $f(c)$ *on the right of* $x = c$ *and* (b) *if* $f'(c) < 0$, *then* $f(x)$ *is decreasing at* c, *i.e.,* $f(x)$ *has values greater than* $f(c)$ *on the left of* $x = c$.

Illustration 1. $f(x) = \left(x - \frac{1}{x}\right)$ *is* increasing for all values of $x (x \neq 0)$ *since* $f'(x) = 1 + \frac{1}{x^2} > 0$, *for all real values of* x *except* 0.

Illustration 2. $f(x) = \left(1 + \frac{1}{x}\right)$ *is always a* decreasing function *of* x *for all* $x(\pm 0)$, *since* $f'(x) = -\frac{1}{x^2} < 0$ *for all real values of* x $(\neq 0)$.

3. A function $y = f(x)$ may at some point be neither increasing nor decreasing. We then say that the *curve is stationary at that point.*

At a stationary point, the derivative is zero, i.e., $f'(x) = 0$ and *the tangent is parallel to the X-axis.* A stationary point is also called a *critical point.* We explain these ideas through the following examples:

Example 1. *Find the range of values of* x *for which the function* $y = f(x) = 2x^3 - 9x^2 + 12x - 3$ (i) *increases with* x, (ii) *decreases with* x. *Point out the stationary points, if there be any.*

Solution: First observe: $y = -3$ when $x = 0; y = 2$ when $x = 1; y = 1$ when $x = 2$.

Here, $f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$.

(i) When $x < 1$, both $(x - 1)$ and $(x - 2)$ are negative and consequently $f'(x)$ itself is positive; i.e., from $x = -\infty$ to $x = 1$, the function is increasing and the curve is rising; at every point of this interval the tangent makes acute angle with the positive direction of the X-axis [See the curve of Fig. 6.5 on the left of the point A].

Thus *in the interval* $(\infty, 1)$, $f(x)$ *increases with* x.

(ii) When $1 < x < 2$, we find that $x - 1$ is $+$ve but $x - 2$ is $-$ve and as such $f'(x)$ is negative and the function $f(x)$ is decreasing and the curve is falling there; at every point of this interval the tangent makes obtuse angle with the positive direction of the X-axis [See the curve of Fig. 6.5 from A to B].

Thus in $(1, 2)$, $f(x)$ *decrease if* x *increases.*

(iii) When $x > 2$, clearly $f'(x)$ is positive, the function $f(x)$ is increasing and the curve is again rising from $x = 2$ to $x = +\infty$ [See the curve of Fig. 6.5 on the right of the point B].

Thus in $(2, \infty)$, $f(x)$ *is again increasing with* x.

(iv) At $x = 1$ and at $x = 2$, $f'(x)$ is zero, i.e., *the curve is stationary at these two points and the tangents there are parallel to the X-axis* [See the points A and B of Fig. 6.5].

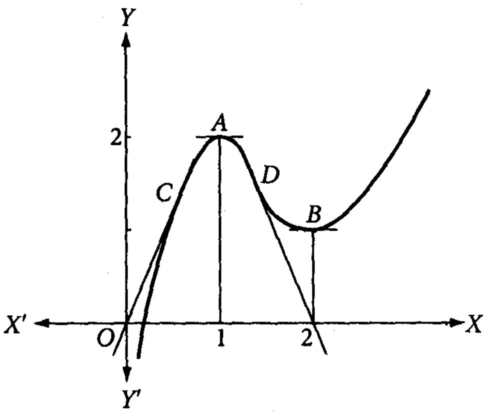


Fig. 6.5

Example 2. *If* $f(x)=2x^3+3x^2-12x+7$, *find the stationary points of* $f(x)$.

Solution: We have $f(x)=2x^3+3x^2-12x+7$.

$$\therefore\ f'(x)=6x^2+6x-12=6\left(x^2+x-2\right).$$

At the stationary points of $f(x)$, we have $f'(x)=0$,

$$\begin{aligned}
\text{i.e.,}\quad & 6\left(x^2+x-2\right)=0 \text{ or, } x^2+x-2=0\\
\text{or,}\quad & x^2+2x-x-2=0 \text{ or, } x(x+2)-1(x+2)=0\\
\text{or,}\quad & (x+2)(x-1)=0; \quad \therefore\ x=-2,1.
\end{aligned}$$

Hence *the required stationary points are* $\boldsymbol{x=-2,1}$.

Example 3. (i) *If* $f(x)=\log(1+x)+\dfrac{1}{1+x}$, $x>0$, *show that* $f(x)$ *increases with* x. (ii) *If* $x>0$, *show that* $\log(1+x)>x-\dfrac{x^2}{2}$. [C.U. B.Com.(H) 2003]

Solution: (i) Let $f(x)=\log_e(1+x)+\dfrac{1}{1+x}$, $x>0$.

Then

$$\begin{aligned}
f'(x)&=\frac{1}{1+x}\times(0+1)-\frac{1}{(1+x)^2}\times(0+1)=\frac{1}{1+x}-\frac{1}{(1+x)^2}\\
&=\frac{1+x-1}{(1+x)^2}=\frac{x}{(1+x)^2}>0, \text{ for all } x>0.
\end{aligned}$$

i.e., $f'(x)>0$, for all $x>0$. Hence, $\boldsymbol{f(x)}$ **increases with** $\boldsymbol{x}$.

(ii) Let

$$f(x)=\log(1+x)-x+\frac{x^2}{2}, \text{ where } x>0.$$

Then

$$f'(x)=\frac{1}{1+x}-1+x=\frac{1-1-x+x+x^2}{1+x}=\frac{x^2}{1+x}>0,\ \text{if } x>0.$$

$\therefore$ $f(x)$ is increasing, if $x>0$; $\therefore$ $f(x)>f(0)$, if $x>0$.

But, $f(0)=\log 1-0+0=0$.

$\therefore$ $f(x)>0$, if $x>0$, i.e., $\log(1+x)-x+\frac{x^2}{2}>0$, if $x>0$.

Hence, $\boldsymbol{\log(1+x)>x-\frac{x^2}{2}}$, if $x>0$.

EXERCISES ON CHAPTER 6(III)

(Increasing and Decreasing Functions)

1. (a) Show that $f(x)=x^3$ is always an increasing function of x.

   (b) Show that $y=\frac{1}{x}$ is always a decreasing function of x for all x ($\neq 0$).

2. Show that if $x>\frac{1}{2}$, then $x(4x^2-3)$ is steadily increasing.

3. Show that $\phi(x)=x^4$ steadily decreases from $x=-\infty$ to $x=0$ and then from $x=0$ to $x=+\infty$, $\phi(x)$ increases.

4. Verify that

$$f(x)=\begin{cases} x, & \text{when } -1\leq x<1 \\ x-2, & \text{when } 1\leq x<2 \end{cases}$$

   is increasing at $x=0, x=1.5$.

   Is $f(x)$ increasing or decreasing at $x=1$?

   [Hints: At $x=0, x=1.5$, $f'(x)=1$ (positive) and hence $f(x)$ increases there.
   At $x=1$, R.H. derivative = 1, L.H. derivative does not exist; hence no definite answer is possible.]

5. If $f(x)=(x-1)e^x+1$; show that $f(x)$ is positive for all $x>0$.

6. Find the range of x for which the function $x^3-6x^2-36x+7$ increases with x.

7. Separate the intervals in which $f(x)=2x^3-15x^2+36x+1$ is increasing or decreasing. Draw the graph.

8. Show that:

   (a) $\frac{x}{1+x}<\log(1+x)<x$, when $x>0$;

   (b) $x-\frac{1}{2}x^2<\log(1+x)<x$, when $x>0$.

9. Show that the function $f(x)=\frac{2x-5}{3x+2}$ is increasing for all real values of x.

10. Prove that $4x^3+6x^2-24x+1$ decreases in the interval $(-2,1)$.

11. Find the intervals in which the function $2x^3 - 9x^2 + 12x + 30$ is increasing or decreasing.

12. Find the intervals in which the function $f(x) = 6 + 12x + 3x^2 - 2x^3$ is increasing or decreasing.

[**Hints**: $f(x) = 6 + 12x + 3x^2 - 2x^3$; $\therefore f'(x) = 12 + 6x - 6x^2 = 6(1+x)(2-x)$.
If x lies between -1 and 2, then both $(1+x)$ and $(2-x)$ are positive so that $f'(x) > 0$ and $f(x)$ *is increasing in* $(-1, 2)$. But if $-\infty < x < -1$, then $(1+x)$ is negative and $(2-x)$ is positive so that $f'(x) < 0$ and if $2 < x < \infty$, then $(1+x)$ is positive and $(2-x)$ is negative so that $f'(x) < 0$ and $f(x)$ *is decreasing in* $(-\infty, -1)$ *and in* $(2, \infty)$.]

ANSWERS

6. Increases when $x > 3$ or when $x < 2$; decreases when $2 < x < 3$, stationary at $x = 2, 3$.
7. For $2 < x < 3$, $f(x)$ decreases; otherwise it increases.
8. Stationary points $x = 0$, $x = +\sqrt{3/2}$, $x = -\sqrt{3/2}$; maximum at $x = 0$; minimum at $x = \pm\sqrt{3/2}$.
11. Increasing in $(-\infty, 1)$ and $(2, \infty)$, but decreasing in $(1, 2)$.
12. Increasing in $(-1, 2)$, but decreasing in $(-\infty, -1)$ and $(2, \infty)$.

6.4 Maximum and Minimum Values of a Function

Definition:

1. A function $f(x)$ is said to have a relative maximum or a local maximum at $x = c$, if $f(c) >$ every other $f(x)$ in some suitably chosen small neighbourhood of c, i.e., if there exists a positive δ such that $f(c) >$ every $f(x)$, where $c - \delta < x < c + \delta (x \neq c)$.

2. A function $f(x)$ is said to have a relative minimum or a local minimum at $x = c$, if $f(c) <$ every other $f(x)$ in some suitably chosen small neighbourhood of c, i.e., if there exists a positive δ such that $f(c) <$ every $f(x)$, where $c - \delta < x < c + \delta (x \neq c)$.

   In Fig. 6.5 see that $f(x)$ has a maximum at $x = 1$ (at A) and a minimum at $x = 2$ (at B).

   The definitions stated above lead us to conclude the following:

3. (a) If $f(x)$ is increasing when $x < c$ (but sufficiently near c) and is decreasing when $x > c$ (but sufficiently near c), then $f(x)$ has a maximum at $x = c$.
   (b) Graphically, on the left neighbourhood of c, if the curve $y = f(x)$ is rising and on the right neighbourhood of c the curve is falling, then $f(x)$ has a maximum at $x = c$.
   (c) In terms of first derivative $f'(x)$, the result can be stated as:
   If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ (in both cases x is sufficiently near c), then $f(x)$ has a maximum at $x = c$.

4. Similar statements can be framed for a minimum point.
   (a) If $f(x)$ is decreasing when $x < c$ (but sufficiently near c) and is increasing when $x > c$ (but sufficiently near c), then $f(x)$ has a minimum at $x = c$.
   (b) Graphically, on the left of $x = c$, if the curve $y = f(x)$ is falling and on the right of $x = c$, if the curve $y = f(x)$ is rising, then $f(x)$ has a minimum at $x = c$.
   (c) *In the language of first derivative*: If $f'(x) < 0$ for $x < c$ (sufficiently near c) and $f'(x) > 0$ for $x > c$ (sufficiently near c), then $f(x)$ has a minimum at $x = c$.

Note: The expression, extreme value or turning value, covers both cases — maximum value as well as minimum value. The corresponding points of the independent variable are called *extreme points* or *turning points*.

6.4.1 A necessary condition for Maxima and Minima

Theorem 2. *At an extreme point, $f'(x) = 0$, provided $f'(x)$ exists there.*

Proof. Suppose, $f(x)$ has a maximum at $x = c$ and suppose $f'(c)$ exists. To prove $f'(c) = 0$.

Since $f'(c)$ exists, $f'(c)$ is either > 0 or $f'(c) < 0$, if $f'(c) \neq 0$.

But if $f'(c)$ were > 0, then $f(x)$ would have been increasing at c and there would have been values of $f(x) > f(c)$ on the right of $x = c$. Thus, $f(x)$ would not be a maximum at $x = c$ which contradicts our assumption.

Similarly, we would get a contradiction if $f'(c)$ were < 0.

$\therefore$ the only possibility, when $f'(c)$ exists, is that $f'(c) = 0$.

If $f(x)$ has a minimum at $x = c$ and $f'(c)$ exists, then also we can similarly show that $f'(c) = 0$.

Note: An extreme value can occur only at that point of the independent variable where either the derivative vanishes or where the derivative does not exist.

5. **First-Derivative Test for an Extremum**

If at $x = c$, the first derivative of $f(x)$, i.e., $f'(c) = 0$, then $f(c)$ will be

(a) a maximum value, if the derivative $f'(x)$ changes its sign from positive to negative from the left of the point c to the right;

(b) a minimum value, if the derivative $f'(x)$ changes its sign from negative to positive from the left to right of the point $x = c$;

(c) neither a maximum nor a minimum, if $f'(x)$ has the same sign (+ ve or − ve) on both sides of the point $x = c$. Such a point will then give rise to what we call a point of inflexion (See later).

Working Rules: First-Derivative Test

- If at $x = c$, $f'(c) = 0$, then c is a critical value of x and $f(c)$ is a stationary value of the function $f(x)$.
- A critical value of x may give a maximum or a minimum value of $f(x)$ or it may give a point of inflexion.
- Change of derivative-sign in the neighbourhood of a critical value gives an extremum; no change of derivative-sign gives a point of inflexion.

If $f'(x)$ changes sign from positive to negative as it passes through c, then $f(c)$ is a maximum value. In case $f'(x)$ changes sign from negative to positive, then $f(c)$ is a minimum value of $f(x)$.

Example 1. *Show that the function $f(x) = x^3 - 6x^2 + 9x - 8$ has a maximum value at $x = 1$ and a minimum value at $x = 3$.*

Solution. We have $f(x) = x^3 - 6x^2 + 9x - 8$.

$$\therefore \; f'(x) = 3x^2 - 12x + 9 = 3\left(x^2 - 4x + 3\right) = 3(x-1)(x-3). \tag{1}$$

At $x = 1, f'(1) = 3(1-1)(1-3) = 3 \cdot 0 \cdot (-2) = 0.$

We see that when $x < 1$ but sufficiently close to 1, $(x-1)$ and $(x-3)$ are both negative so that their product is positive and from (1), $f'(x) > 0$, i.e., the curve is rising on the left of $x = 1$.

When $x > 1$ but sufficiently close to 1, $(x-1)$ is positive and $(x-3)$ is negative so that, from (1), $f'(x) < 0$, i.e., the curve is *falling on the right of* $x = 1$.

Here $\boldsymbol{f(x)}$ **is maximum at** $\boldsymbol{x = 1}$.

At $x = 3$, $f'(3) = 3 \times 2 \times 0 = 0$.

When $x < 3$ but sufficiently close to 3, $(x-1)$ is positive and $(x-3)$ is negative so that, from (1), $f'(x) < 0$, i.e., the curve is *falling on the left of* $x = 3$.

Again, when $x > 3$ (but very close to 3), $(x-1)$ and $(x-3)$ are both positive so that, from (1), $f'(x) > 0$, i.e., the *curve is rising on the right of* $x = 3$.

Hence, $\boldsymbol{f(x)}$ **is minimum at** $\boldsymbol{x = 3}$.

Example 2. *Examine the maxima or minima of the function* $f(x) = x^3 - 12x^2 + 36x + 8$.

Solution: Here

$$\begin{aligned} f'(x) &= 3x^2 - 24x + 36 \\ &= 3\left(x^2 - 8x + 12\right) \\ &= 3(x-2)(x-6). \end{aligned}$$

The critical values of x are those where $f'(x) = 0$.

Here the critical values are $x = 2$, $x = 6$.

At $x = 2$. We observe that:

- when $x < 2$ (but sufficiently near 2), $(x-2)$ and $(x-6)$ are both negative and hence $f'(x)$ becomes positive, i.e., the curve is rising there.
- when $x > 2$ (but sufficiently near 2), $x - 2$ is positive but $x - 6$ is negative and hence $f'(x)$ becomes negative, i.e., the curve is falling there.

$\therefore$ at the critical value $x = 2$, $f(x)$ has a maximum value $f(2) = 40$.

At $x = 6$. We observe that:

- when $x < 6$ (but sufficiently near 6), $x-2$ is positive, $x-6$ is negative and hence $f'(x)$ is negative, i.e., the curve is falling there.
- when $x > 6$ (but sufficiently near 6) the derivative $f'(x)$ is positive and the curve rises there.

Hence *at the critical value* $x = 6$, $f(x)$ *has a minimum value* $\boldsymbol{f(6) = 8}$.

Example 3. *Examine the function* $(x-1)^2(x+1)^3$ *for maximum and minimum values.*

Solution: Let $f(x) = (x-1)^2(x+1)^3$.

Then

$$\begin{aligned} f'(x) &= 2(x-1)(x+1)^3 + 3(x+1)^2(x-1)^2 \\ &= (x-1)(x+1)^2\{2x + 2 + 3x - 3\} \\ &= (x-1)(x+1)^2(5x-1) \\ &= 5(x - 1/5)(x-1)(x+1)^2. \end{aligned}$$

$\therefore x = 1, -1, 1/5$ are three critical values of x.

At $x = 1$.

- When $x < 1$ (sufficiently near 1), $f'(x) = 5(+\text{ ve})(-\text{ ve})(+\text{ ve})^2 = -\text{ve}$.
- When $x > 1$ (sufficiently near 1), $f'(x) = 5(+\text{ ve})(+\text{ ve})(+\text{ ve})^2 = +\text{ve}$.

$\therefore$ as we pass through $x = 1$ from left to right, $f'(x)$ changes sign from negative to positive.

Hence, **$f(x)$ has a minimum value $f(1) = 0$.**

At $x = 1/5$.

- Verify that $f(x)$ has a **maximum value $f(1/5) = 1.11$.**

At $x = -1$.

- When $x < -1$ (sufficiently near 1), $f'(x) = 5(-\text{ ve})(-\text{ ve})(-\text{ ve})^2 = +\text{ ve}$, i.e., the curve $y = f(x)$ rises on the left of $x = -1$.
- When $x > -1$ (sufficiently near -1), $f'(x) = (-\text{ ve})(-\text{ ve})(+\text{ ve})^2 = +\text{ ve}$, i.e., the curve $y = f(x)$ rises also on the right of $x = -1$.

Therefore, *when $x = -1$, the function has neither a maximum nor a minimum value.*

6. **Second-Derivative Test for an Extremum: $f(x)$ has a maximum at $x = c$, if**

   (a) $f'(c) = 0$ and

   (b) $f'(x)$ changes from $+$ ve to $-$ ve as x passes through c from left to right;

   (c) implies that $f'(x)$ is a decreasing function and hence the derivative of $f'(x)$, i.e., $f''(x) < 0$ [if $f''(x)$ exists].

Conclusion. If at $x = c$, $f'(c) = 0$, then the function $f(x)$ has a maximum at $x = c$, if $f''(c) < 0$ [provided $f''(c)$ exists].

Similarly, we may conclude:

If at $x = c$, $f'(c) = 0$, then the function $f(x)$ has a minimum at $x = c$, if $f''(c) > 0$ [provided $f''(c)$ exists].

General Case: nth Derivative Test for Maxima and Minima

Let $f(x)$ be a function such that $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$. Then $f(x)$ has (i) a maximum at $x = c$, if $f^n(c) < 0$ and n is even, (ii) a minimum at $x = c$, if $f^n(c) > 0$ and n is even and (iii) neither a maximum nor a minimum at $x = c$, if n is odd.

This criterion involving the sign of the derivatives of second and higher orders will be used to find the extreme values of a function $f(x)$.

Example 4. *Let $M = x^2 + \dfrac{432}{x}$. To examine maximum and minimum.*

Solution: Given that

$$M = x^2 + \frac{432}{x}. \tag{1}$$

$$\therefore \frac{dM}{dx} = 2x - \frac{432}{x^2} \text{ and } \frac{d^2M}{dx^2} = 2 + \frac{864}{x^3}.$$

For maximum and minimum values of M, we must have

$$\frac{dM}{dx} = 0 \text{ or, } 2x - \frac{432}{x^2} = 0 \text{ or, } 2x^3 = 432 \text{ or, } x^3 = 216 \text{ or, } x = 6.$$

Now, at $x = 6$, $\dfrac{d^2M}{dx^2} = 2 + \dfrac{864}{6^3}$ = a positive quantity > 0.

Hence, M has a minimum value at $x = 6$ and from (1), the minimum value of M is $6^2 + \dfrac{432}{6} = 36 + 72 = \mathbf{108}$.

Example 5. $f(x) = x^3 - 3x^2 - 9x + 5$. *To examine for maximum and minimum of* $f(x)$.

Solution:

First-Derivative Method:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

$$\therefore\ f'(x) = 3x^2 - 6x - 9 = 3\left(x^2 - 2x - 3\right) = 3(x-3)(x+1).$$

$\therefore$ critical values are where $f'(x) = 0$, i.e., $x = 3$, $x = -1$.

At $x = 3$.

- When $x < 3$ (sufficiently near 3), $f'(x) = 3(-\text{ve})(+\text{ve}) = -\text{ve}$.
- When $x > 3$ (sufficiently near 3), $f'(x) = 3(+\text{ve})(+\text{ve}) = +\text{ve}$.

Thus, $f'(x)$ changes sign from negative to positive as we pass through $x = 3$ from left to right. The curve $y = f(x)$ falls and then rises.

Hence, $f(x)$ *is minimum at* $x = 3$ *and the minimum value is*
$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5 = 27 - 27 - 27 + 5 = \mathbf{-22}$.

At $x = -1$.

- When $x < -1$ (sufficiently near -1), $f'(x) = 3(-\text{ve})(-\text{ve}) = +\text{ve}$.
- When $x > -1$ (sufficiently near -1), $f'(x) = 3(-\text{ve})(+\text{ve}) = -\text{ve}$, i.e., $f'(x)$ changes sign from positive to negative as we pass through $x = -1$ from left to right. The curve $y = f(x)$ rises and then falls.

Hence, $f(x)$ *is maximum at* $x = -1$ *and the maximum value is*
$f(-1) = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 5 = -1 - 3 + 9 + 5 = \mathbf{10}$.

Using Second Derivative: $f'(x) = 3x^2 - 6x - 9$, $f''(x) = 6x - 6$.

For maximum and minimum values of $f(x)$, $f'(x) = 0$, i.e., $3x^2 - 6x - 9 = 0$, or, $x^2 - 2x - 3 = 0$ or, $(x-3)(x+1) = 0$;

$\therefore\ x = 3, -1$.

At $x = 3$

$f''(3) = 6 \times 3 - 6 = 18 - 6 = 12 > 0$.

Hence, $f(x)$ *is minimum at* $x = 3$ *and the minimum value is*
$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5 = 27 - 27 - 27 + 5 = \mathbf{-22}$.

At $x=-1$,

$f''(-1)=6\times(-1)-6=-12<0.$

Hence, *$f(x)$ is maximum at $x=-1$ and the maximum value is*
$f(-1)=(-1)^3-3\times(-1)^2-9\times(-1)+5=-1-3+9+5=\mathbf{10}.$

Observations. It may appear that second-derivative test is much more easy to apply but the concept and understanding will be better if first derivative test is applied. Moreover, it may happen that second derivative may not exist, though the first derivative happens to be zero.

Example 6. *Find for what values of x, the following expression is maximum and minimum respectively:* $2x^3-21x^2+36x-20.$

Find also the maximum and the minimum values.

Solution: Let

$$f(x)=2x^3-21x^2+36x-20. \tag{1}$$

Then $f'(x)=6x^2-42x+36$ and $f''(x)=12x-42.$

For maximum and minimum values of $f(x)$, we must have

$f'(x)=0$, i.e., $6x^2-42x+36=0$ or, $x^2-7x+6=0$ or, $x^2-6x-x+6=0$,
or, $x(x-6)-1(x-6)=0$ or, $(x-6)(x-1)=0$;

$\therefore\ x=6,1.$

At $x=6$, $f''(6)=12\times6-42=72-42=30>0.$

Hence, *$f(x)$ is minimum at $x=6$ and the minimum value is*
$f(6)=2\times6^3-21\times6^2+36\times6-20=432-756+216-20=\mathbf{-128}.$

At $x=1$, $f''(1)=12\times1-42=-30<0.$

Hence, *$f(x)$ is maximum at $x=1$ and the maximum value is*
$f(1)=2\times1^3-21\times1^2+36\times1-20=2-21+36-20=\mathbf{-3}.$

Example 7. *Show that the function $x^3-6x^2+12x+50$ is neither a maximum nor a minimum at $x=2$.*

Solution: Let $f(x)=x^3-6x^2+12x+50.$

Then $f'(x)=3x^2-12x+12$, $f''(x)=6x-12$ and $f'''(x)=6.$

At $x=2$, $f'(2)=3\cdot2^2-12\cdot2+12=12-24+12=0$, $f''(2)=6\cdot2-12=0$ and $f'''(2)=6\neq0.$

Thus $f'(2)=0$ and $f''(2)=0$, but $f'''(2)\neq0.$

Hence, **$f(x)$ has neither a maximum nor a minimum at $x=2$.**

Example 8. *Determine whether the function $f(x)=\frac{2}{3}x^3-6x^2+20x-5$ has a maximum or a minimum.* [C.U. B.Com.(H) 1998]

Solution: We have

$$f(x)=\tfrac{2}{3}x^3-6x^2+20x-5. \tag{1}$$

$$\begin{aligned}\therefore\ f'(x) &= \tfrac{2}{3}\times 3x^2 - 6\times 2x + 20\times 1 - 0\\ &= 2x^2 - 12x + 20\\ &= 2\left(x^2 - 6x + 10\right).\end{aligned}$$

For $f(x)$ to have a maximum or minimum, we must have $f'(x)=0$.

Now, $f'(x)=0$ gives $x^2-6x+10=0$.

$$\therefore\ x = \frac{6\pm\sqrt{36-4\cdot 1\cdot 10}}{2\cdot 1} = \frac{6\pm\sqrt{-4}}{2}.$$

which are not real, i.e., $f'(x)=0$ gives *non-real values of* x.

Hence, **$f(x)$ has neither a maximum nor a minimum value.**

6.5 Concepts of Concavity and Point of Inflexion

When the point $P(x,y)$ traces a curve $y=f(x)$, the slope of the tangent line at P varies.

When the tangent line is below the curve, the arc is concave upward (Fig. 6.6(a)); when the curve is below the tangent line, the arc is concave downward (Fig. 6.6(b)).

In Fig. 6.6(a) the slope increases as x increases, i.e., $f'(x)$ is an increasing function of x and hence its derivative $f''(x)$ is positive. On the other hand, in Fig. 6.6(b) the slope decreases as x increases and hence $f''(x)$ is negative.

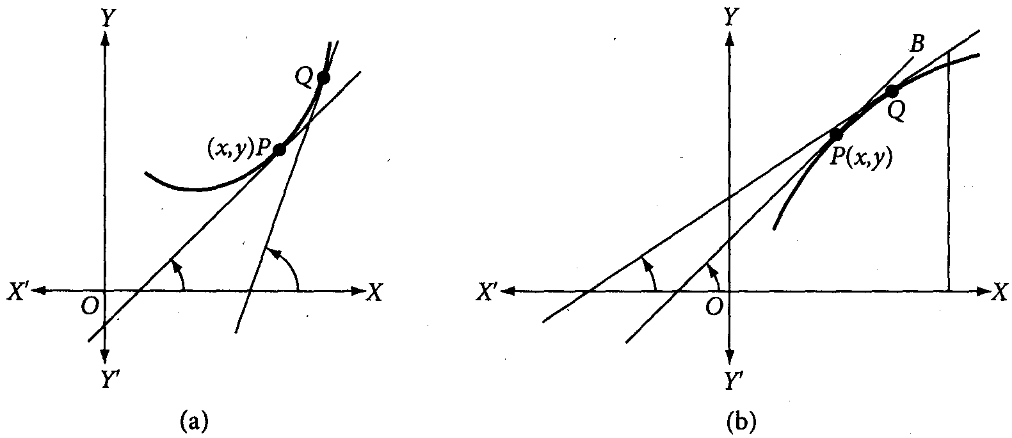


Fig. 6.6

Hence, we have the following criterion of determining the direction attaining of a curve at a point. *The curve $y=f(x)$ is concave upward, if the second derivative $f''(x)$ is positive; concave downward, if the second derivative $f''(x)$ is negative.*

We call a point P of a curve a *point of inflexion*, if it separates arcs having opposite directions of bending (i.e., from concave upward to concave downward or vice versa). *On the two sides of a point of inflexion the second derivative will have opposite sign and at that point* **second derivative vanishes** (assuming that the second-derived function is continuous).

At a point of inflexion the tangent crosses the curve.

Example 9. $f(x) = 3x^4 - 4x^3 + 1$. *To examine for points of inflexion.*

Solution: We have

$$f'(x) = 12x^3 - 12x^2,$$
$$f''(x) = 36x^2 - 24x = 36x(x - 2/3),$$
$$\text{i.e., } f''(x) = 0, \text{ when } x = 2/3 \text{ or, } x = 0.$$

When $x < 2/3$ (but sufficiently near 2/3)

$$f''(x) = 36(+\text{ ve})(-\text{ ve}) = -\text{ ve},$$

i.e., the curve is concave downward for $x < 2/3$.

When $x > 2/3$ (but sufficiently near 2/3)

$$f''(x) = 36(+\text{ ve})(+\text{ ve}) = +\text{ ve},$$

i.e., the curve is concave upward for $x > 2/3$.

$\therefore$ *at* $x = 2/3$, *there is a point of inflexion of the curve.* Similarly, we can give arguments to establish that at $x = 0$, *there is also a point of inflexion of the curve* (Fig. 6.7).

The curve is concave upwards everywhere to the left of $A(0, 1)$, concave downward from $A(0, 1)$ to $B(2/3, 11/27)$ and concave upward everywhere to the right of B.

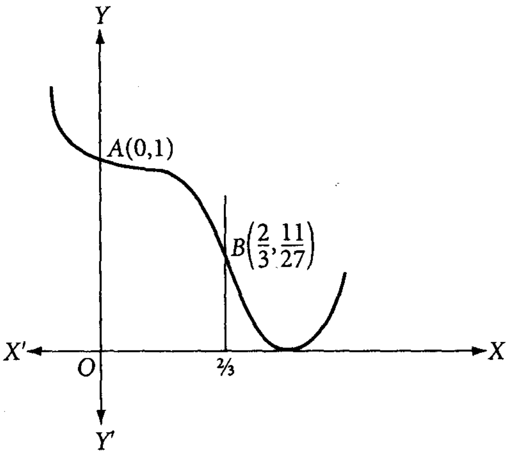


Fig. 6.7

Rules for Finding Points of Inflexion

- Find $f''(x)$. Make $f''(x) = 0$ and solve for real roots of this equation.
- Test $f''(x)$ for values of x sufficiently near each such root. *If $f''(x)$ changes sign, we have a point of inflexion.*
- **The conditions for a point $x = a$ to be a point of inflexion of $y = f(x)$ are $f''(a) = 0$ and $f'''(a) \neq 0$.**

Remember.

- When $f''(x) = +\text{ ve}$, the curve is concave upward.
- When $f''(x) = -\text{ ve}$, the curve is concave downward.

It is assumed that $f'(x)$ and $f''(x)$ are continuous.

Even if $f'(x)$ and $f''(x)$ do not exist finitely at a certain point, still we may sometimes obtain the point of inflexion at that point.

Example 10. *Show that the point* $(4,2)$ *is a point of inflexion of the curve given by* $(y-2)^3 = x-4$ *though both* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *do not exist at that point.*

Solution: We have $(y-2)^3 = x-4$ or, $y-2 = \sqrt[3]{x-4}$ or, $y = 2+\sqrt[3]{x-4}$;

$$\therefore \frac{dy}{dx} = \frac{1}{3}(x-4)^{-2/3} \text{ and } \frac{d^2y}{dx^2} = -\frac{2}{9}(x-4)^{-5/3}.$$

When $x=4$, both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ become infinite (i.e., they do not exist finitely).

Now, when $x<4$, $\frac{d^2y}{dx^2} = +\text{ve}$; when $x>4$, $\frac{d^2y}{dx^2} = -\text{ve}$.

We, therefore, conclude that the tangent to the curve at the point $(4,2)$ is perpendicular to the X-axis and, moreover, to the left of $(4,2)$, the curve is concave upward and to the right of $(4,2)$, the curve is concave downward.

Therefore, **$(4,2)$ is a point of inflexion of the curve.**

6.5.1 Further Illustrative Examples on Maxima and Minima

Example 11. *Prove that the function* $f(x) = 12-24x-15x^2-2x^3$ *has a maximum at* $x=-1$, *minimum at* $x=-4$ *and point of inflexion at* $x=-\frac{5}{2}$. [C.U. B.Com.(H) 1994]

Solution: We have

$$f(x) = 12-24x-15x^2-2x^3. \tag{1}$$

$$\therefore f'(x) = 0-24\cdot1-15\cdot2x-2\cdot3x^2 = -24-30x-6x^2,$$
$$f''(x) = 0-30\cdot1-6\cdot2x = -30-12x \text{ and } f'''(x) = -12.$$

At $x=-1$,

$$f'(-1) = -24+30-6 = 0$$
$$\text{and } f''(-1) = -30-12\times(-1) = -18 < 0.$$

Hence, by second-derivative test, **$f(x)$ has a maximum at $x=-1$.**

At $x=-4$,

$$f'(-4) = -24-30\times(-4)-6\times(-4)^2 = -24+120-96 = 0$$
$$\text{and } f''(-4) = -30-12\times(-4) = -30+48 = 18 > 0.$$

Hence, **$f(x)$ has a minimum at $x=-4$.**

At $x=-\frac{5}{2}$,

$$f''\left(-\frac{5}{2}\right) = -30-12\times\left(-\frac{5}{2}\right) = -30+30 = 0$$
$$\text{and } f'''\left(-\frac{5}{2}\right) = -12 \neq 0.$$

Hence, **$f(x)$ has a point of inflexion at $x=-\frac{5}{2}$.**

Example 12. (a) *Show that the maximum value of the function* $2x + \frac{1}{2x}$ *is less than its minimum value.*

[C.U. B.Com.(H) 2003; V.U. B.Com.(H) 2009]

(b) *Show that the maximum value of* $x^3 + \frac{1}{x^3}$ *is less than its minimum value.* [C.U. B.Com. 2010]

Solution: (a) Let

$$f(x) = 2x + \frac{1}{2x}, \text{ where } x \neq 0.$$

Then

$$f'(x) = 2\cdot 1 + \frac{1}{2} \times \left(-\frac{1}{x^2}\right) = 2 - \frac{1}{2x^2} \text{ and } f''(x) = 0 - \frac{1}{2} \times \frac{-2}{x^3} = \frac{1}{x^3}.$$

For maximum value of $f(x)$, we must have

$$f'(x) = 0 \text{ or, } 2 - \frac{1}{2x^2} = 0, \text{ or, } 2 = \frac{1}{2x^2}, \text{ or, } 4x^2 = 1 \text{ or, } x^2 = \frac{1}{4}, \text{ or, } x = \pm\frac{1}{2}.$$

At $x = \frac{1}{2}$,

$$f''\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^3} = 8 > 0.$$

Hence, $f(x)$ *is minimum at* $x = \frac{1}{2}$ *and the minimum value is*

$$f\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + \frac{1}{2 \times \frac{1}{2}} = 1 + \frac{1}{1} = \mathbf{2}.$$

At $x = -\frac{1}{2}$,

$$f''\left(-\frac{1}{2}\right) = \frac{1}{\left(-\frac{1}{2}\right)^3} = -8 < 0.$$

Hence, $f(x)$ *is maximum at* $x = \left(-\frac{1}{2}\right)$ *and the maximum value is*

$$f\left(-\frac{1}{2}\right) = 2 \times \left(-\frac{1}{2}\right) + \frac{1}{2 \times \left(-\frac{1}{2}\right)} = -1 + \frac{1}{-1} = -1 - 1 = \mathbf{-2}.$$

But $-2 < 2$, i.e., *maximum value < minimum value.*

Hence, the result follows.

(b) Let $f(x) = x^3 + \frac{1}{x^3}$; then $f'(x) = 3x^2 - \frac{3}{x^4}$ and $f''(x) = 6x + \frac{12}{x^5}$.

For maximum or minimum value of $f(x)$, $f'(x) = 0$

i.e., $3x^2 - \frac{3}{x^4} = 0$ or, $x^6 - 1 = 0$ or, $(x^2)^3 - 1 = 0$ or, $(x^2 - 1)(x^4 + x^2 + 1) = 0$.

$\therefore x^2 - 1 = 0$ or, $x^4 + x^2 + 1 = 0$, $x^2 - 1 = 0$ gives $x = 1, -1$.

$x^4 + x^2 + 1 = 0$ gives $(x^2+1)^2 - x^2 = 0$ or, $(x^2 + x + 1)(x^2 - x + 1) = 0$ which gives $x^2 + x + 1 = 0$ and $x^2 - x + 1 = 0$; none of these two equations gives real roots.

At $x = 1$,

$$f''(1) = 6\cdot 1 + \frac{12}{1} = \mathbf{18 > 0},$$

i.e., $f(x)$ is minimum at $x = 1$ and the minimum value is $f(1) = 1^3 + \frac{1}{1^3} = \mathbf{2}$.

At $x = -1$,

$$f''(-1) = 6(-1) + \frac{12}{(-1)^5} = -6 - 12 = \mathbf{-18 < 0.}$$

$\therefore$ $f(x)$ is maximum at $x = -1$ and the maximum value is $f(-1) = -1 - 1 = -2$.
We see that $-2 < 2$, i.e., *Maximum value < Minimum value.*
Hence, the result follows.

Example 13. *Find the maximum and minimum values of* $y = \dfrac{x}{(x-1)(x-4)}$. [C.U. B.Com.(H) 1996]

Solution: We have

$$y = \frac{x}{(x-1)(x-4)} = \frac{x}{x^2 - 5x + 4}. \tag{1}$$

$$\therefore \frac{dy}{dx} = \frac{1.(x^2 - 5x + 4) - x.(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{4 - x^2}{(x^2 - 5x + 4)^2}. \tag{2}$$

For maximum or minimum value of y, we must have

$$\frac{dy}{dx} = 0 \text{ or, } 4 - x^2 = 0 \text{ or, } x^2 = 4 \text{ or, } x = \pm 2.$$

At $x = 2$, $\quad \dfrac{dy}{dx} = \mathbf{0.}$

- If $x < 2$ but close to 2, then $\dfrac{dy}{dx} > 0$ and y is increasing.
- If $x > 2$ and close to 2, then $\dfrac{dy}{dx} < 0$ and y is decreasing.

Hence, by *first derivative test, y is maximum at $x = 2$ and the maximum value of y is*

$$\frac{2}{(2-1)(2-4)} = \mathbf{-1.}$$

- If $x < -2$ but close to -2, then $\dfrac{dy}{dx} < 0$ and y is decreasing.
- If $x > -2$ and close to -2, then $\dfrac{dy}{dx} > 0$ and y is increasing.

Hence, *y is minimum at $x = -2$ and the minimum value is*

$$\frac{-2}{(-2-1)(-2-4)} = -\frac{\mathbf{1}}{\mathbf{9}}.$$

Otherwise. Find $\dfrac{d^2y}{dx^2}$ from (2) and apply *second-derivative test.*

Example 14. (i) *If $x + y = 16$, find the maximum value of xy.*
(ii) *If the sum of two numbers is 18, find the maximum value of their product.*

Solution: (i) Given, $x+y=16$ or, $y=16-x$. Let $P=xy$.

Then

$$P=x(16-x)=16x-x^2.$$

$$\therefore \frac{dP}{dx}=16\cdot 1-2x=16-2x \text{ and } \frac{d^2P}{dx^2}=-2.$$

For maximum or minimum value of P, we must have

$$\frac{dP}{dx}=0 \text{ or, } 16-2x=0, \text{ or, } 2x=16, \text{ or, } x=8.$$

At $x=8$, $y=16-x=16-8=8$ and $\frac{d^2P}{dx^2}=-2<0$.

Hence, P, i.e., xy is maximum at $x=8$, $y=8$ and the maximum value of xy is $8\cdot 8=\mathbf{64}$.

(ii) Let the two numbers be x and y. Then $x+y=18$ or, $y=18-x$.

Let P be the product of the two numbers. Then $P=x\cdot y=x(18-x)=18x-x^2$. Now proceed as in (i) and obtain the maximum value as $9\cdot 9=81$.

Applications of Maxima and Minima in Business Economics and other fields

Example 15. *The total cost function C for producing x units of an article is given by*

$$C=₹\left(400-16x+2x^2\right).$$

Find the average cost function and the level of output at which this function is minimum.

[C.U. B.Com.(H) 1990]

Solution: We have

$$C=₹\left(400-16x+2x^2\right),$$

which gives total cost of x units. If $f(x)$ be the average cost function, then

$$f(x)=\frac{400-16x+2x^2}{x}=\frac{\mathbf{400}}{\boldsymbol{x}}-\mathbf{16}+\mathbf{2}\boldsymbol{x}.$$

$$\therefore f'(x)=-\frac{400}{x^2}-0+2=2-\frac{400}{x^2}$$

$$\text{and } f''(x)=0-400\times\left(-2x^{-3}\right)=\frac{800}{x^3}.$$

For average cost function $f(x)$ to be minimum, we must have

$$f'(x)=0, \text{ i.e., } 2-\frac{400}{x^2}=0 \text{ or, } 2=\frac{400}{x^2} \text{ or, } x^2=200 \text{ or, } x=\sqrt{200}=10\sqrt{2}.$$

At $x=10\sqrt{2}$,

$$f''\left(10\sqrt{2}\right)=\frac{800}{\left(10\sqrt{2}\right)^3}>0.$$

Hence, the required level of output at which $f(x)$ is minimum is $\mathbf{10\sqrt{2}}$ **units.**

Example 16. *The demand function of a firm is given by the relation $2p+3x=60$, where p is the price per unit and x is the number of units demanded. Find the level of output which maximizes the total revenue and also find the maximum total revenue in rupees.* [C.U. B.Com. 2004]

Solution: Price p per unit is given by $2p+3x=60$ or, $2p=60-3x$ or, $p=\dfrac{60-3x}{2}$.

If $R(x)$ be the total revenue, then $R(x)=px=\left(\dfrac{60-3x}{2}\right)\cdot x=30x-\dfrac{3}{2}x^2$.

$\therefore \dfrac{dR}{dx}=30-3x^2$ and $\dfrac{d^2R}{dx^2}=-6x$.

For the total revenue R to be maximum, we must have $\dfrac{dR}{dx}=0$, i.e., $30x-3x^2=0$ or, $x(x-10)=0 \Rightarrow x=10$, since $x \neq 0$.

At $x=10$, $\dfrac{d^2R}{dx^2}=-6\times 10=\mathbf{-60<0}$.

Hence $R(x)$ is maximum at $x=10$, i.e., *the level of output which maximizes the total revenue $R(x)$ is* 10.

The *maximum value of $R(x)$ is* $R(10)=30\times 10-\dfrac{3}{2}\times(10)^2=300-150=$ **₹150.**

Example 17. *A firm produces x units of output per week at a total cost of*

$$₹\left(\tfrac{1}{3}x^3-x^2+5x+3\right).$$

Find the output levels at which the marginal cost and the average variable cost attain their respective minima. (Marginal cost is the rate of change of total cost w.r.t. x).

Solution: If $C(x)$ be the total cost of x units, then

$$C(x)=₹\left(\tfrac{1}{3}x^3-x^2+5x+3\right).$$

If $f(x)$ be the marginal cost (MC), then

$$\text{Marginal cost (MC)}=f(x)=\frac{dC}{dx}=\frac{1}{3}\times 3x^2-2x+5\cdot 1+0=x^2-2x+5.$$

$\therefore\ f'(x)=2x-2$ and $f''(x)=2$.

For $f(x)$, i.e., MC to be minimum, we must have

$$f'(x)=0,\ \text{i.e., } 2x-2=0 \text{ or, } x=1.$$

Hence, the *required output level at which MC is minimum is $x=1$*, i.e., **1 unit.**

If $g(x)$ be the average variable cost (AVC), then

$$g(x)=\frac{\text{variable cost}}{x}=\frac{\frac{1}{3}x^3-x^2+5x}{x}=\frac{1}{3}x^2-x+5.$$

$$\therefore\ g'(x)=\frac{1}{3}\times 2x-1=\frac{2}{3}x-1 \text{ and } g''(x)=\frac{2}{3}.$$

For average variable cost (AVC) to be minimum, we must have

$$g'(x) = 0 \text{ or, } \frac{2}{3}x - 1 = 0, \text{ or, } \frac{2}{3}x = 1, \text{ or, } x = \frac{3}{2} = 1.5.$$

At $x = 1.5$,

$$g''(1.5) = \frac{2}{3} > 0.$$

Hence, $g(x)$, i.e., AVC *is minimum at output level of* **1.5 units.**

Example 18. *A manufacturer can sell x items per month at a price $p = (300 - 2x)$* rupees. *Manufacturer's cost of production, y* rupees, *of x items is given by $y = 2x + 1000$. Find the number of items to be produced p.m. to yield the maximum profit.* [Given, Profit = Total Sale – Total Cost.]

Solution: If $R(x)$ be the total sale of x items and $C(x)$ be the total cost, then

$$R(x) = ₹px = (300 - 2x)\,x = ₹\left(300x - 2x^2\right)$$

and $y = ₹(2x + 1000)$.

If $P(x)$ be the total profit on x units, then

$$P(x) = R(x) - y = 300x - 2x^2 - (2x + 1000)$$
$$= 298x - 2x^2 - 1000.$$

$\therefore$ $P'(x) = 298 - 4x$ and $P''(x) = -4$.

For $P(x)$, i.e., profit to be maximum, we must have

$$P'(x) = 0, \text{ i.e., } 298 - 4x = 0, \text{ or, } 4x = 298, \text{ or, } x = 74.5.$$

At $x = 74.5$, $P''(x) = -4$, i.e., $P''(74.5) = -4 < 0$.

Hence, *the required number of items to be produced p.m. to yield maximum profit is* 74.5.

Example 19. *Show that among rectangles of given perimeter, the square has the greatest area.*

Solution: Let x and y be the length and breadth of the rectangle. Then

$2(x + y)$ = perimeter of the rectangle = constant = c (say) or, $y = \frac{c}{2} - x$.

If A be the area of the rectangle, then

$$A = xy = x\left(\frac{c}{2} - x\right) = \frac{cx}{2} - x^2.$$

$$\therefore \quad \frac{dA}{dx} = \frac{c}{2} - 2x \text{ and } \frac{d^2A}{dx^2} = -2.$$

For the area A to be greatest (i.e., maximum), we must have

$$\frac{dA}{dx} = 0 \text{ or, } \frac{c}{2} - 2x = 0, \text{ or, } 2x = \frac{c}{2}, \text{ or, } x = \frac{c}{4}.$$

At $x = \frac{c}{4}$,

$$y = \frac{c}{2} - \frac{c}{4} = \frac{c}{4} \text{ and } \frac{d^2A}{dx^2} = -2 < 0.$$

Hence, *the area A is greatest when $x = y = \frac{c}{4}$, i.e., when the rectangle is a square.*

Example 20. *A farmer can afford to buy* 800 metres *of wire of fencing. He wishes to enclose a rectangular field of largest possible area. What should be the dimensions of the field?* [C.U. B.Com. 2000]

Solution: Let x and y be the length and breadth of the rectangular field in metres. Then $2(x+y)=800$ or, $y=400-x$.

If A be the area of the field, then $A=xy=x(400-x)=400x-x^2$.

$\therefore \dfrac{dA}{dx}=400-2x$ and $\dfrac{d^2A}{dx^2}=-2$.

For *largest possible area,* we have $\dfrac{dA}{dx}=0$, i.e., $400-2x=0$ or, $x=200$.

At $x=200$, $\dfrac{d^2A}{dx^2}=-2<0$, and $y=400-200=200$.

Hence, the rectangular field is largest when $x=200$ and $y=200$, i.e., **the dimensions of the field are length = breadth = 200 metres.**

Point of Inflexion

Example 21. *Examine whether the curve* $y-3=6\cdot 5(x-2)^5$ *has a point of inflexion at* $(2,3)$.

[C.U. B.Com.(H) 2002; (P) 2010]

Solution: The curve is

$$y-3=6(x-2)^5. \tag{1}$$

$$\therefore \frac{dy}{dx}-0=6\cdot 5(x-2)^4\frac{d}{dx}(x-2)=30(x-2)^4.1=30(x-2)^4$$

$$\text{and } \frac{d^2y}{dx^2}=30\cdot 4(x-2)^3\cdot\frac{d}{dx}(x-2)=120(x-2)^3.$$

If $x=2$, from eq. (1), $y-3=0$ or, $y=3$, i.e., $(2,3)$ is a point on the curve.

At $x=2$,

$$\frac{d^2y}{dx^2}=0.$$

Now, if $x<2$ but sufficiently close to 2, $\dfrac{d^2y}{dx^2}<0$ and

if $x>2$ and sufficiently close to 2, then $\dfrac{d^2y}{dx^2}>0$,

i.e., $\dfrac{d^2y}{dx^2}$ changes sign from $-$ to $+$ when passing through the point $(2,3)$ from left to right of $x=2$.

Hence, **the curve eq. (1) has a point of inflexion at (2, 3).**

EXERCISES ON CHAPTER 6(IV)

(Maximum and Minimum Values of Functions: Point of Inflexion)

1. State (without proof) the criterion for determining the maximum and minimum value of a single-valued function.

2. Find for what values of x, the function $x^2(x-1)^3$ has a maximum or minimum. Write down the minimum and maximum values. [C.U. B.Com. 2009]

3. (a) Find the minimum and maximum values of the function, $y=x^3-3x+1$. [C.U. B.Com.(H) 2007]

(b) Show that the function $f(x) = x^3 - 6x^2 + 9x - 8$ has a maximum value at $x = 1$ and a minimum value at $x = 3$.

(c) Find the extreme values of the function $y = f(x) = x(2x^2 - 12) + 3x^2$. [N.B.U. B.Com.(H) 2007]

(d) Find the maximum and minimum values of $x(12 - 2x)^2$. [V.U. B.Com.(H) 2008]

[**Hints:** Let $f(x) = x(12 - 2x)^2 = 4x(6 - x)^2 = 144x - 48x^2 + 4x^3$.
Then $f'(x) = 144 - 96x + 12x^2$ and $f''(x) = -96 + 24x$.
$f'(x) = 0$ gives $12(12 - 8x + x^2) = 0$ or, $(x - 2)(x - 6) = 0$ or, $\boldsymbol{x = 2, 6}$.
At $x = 2$, $f''(x) = -96 + 48 = -48 < 0$, i.e., $f(x)$ is maximum at $x = 2$ and maximum value is $8 \times 4^2 = 128$.
At $x = 6$, $f''(6) = -96 + 24 \times 6 = 48 > 0$, i.e., $f(x)$ is minimum at $x = 6$ and the minimum value $= 24 \times 0 = 0$.]

4. Show that the maximum value of the function $2x + \dfrac{1}{2x}$ is less than its minimum value. [C.U. B.Com.(H) 2003; V.U. B.Com. 2008]

5. (a) Show that the maximum value of $x + \dfrac{1}{x}$ is less than its minimum value. [C.U. B.Com.(H) 2002 (Old); B.U. B.Com.(H) 2008]

(b) Show that the maximum value of $x^3 + \dfrac{1}{x^3}$ is less than its minimum value. [C.U. B.Com.(H) 2001; (P) 2010]

(c) The sum of two numbers is 12. Find the maximum value of their product. [C.U. B.Com.(H) 1992]

6. Find at what points each of the following functions has a maximum or a minimum. Write down the maximum/minimum values:

(a) $x^2 - 8x + 15$;
(b) $x^3 - 27x + 15$;
(c) $x^3 - 3x^2 + 5$;
(d) $x + \dfrac{1}{x}$;
(e) $\dfrac{x}{x^2 - 16}$;
(f) $\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$;
(g) $2x^3 - 21x^2 + 36x - 20$;
(h) $\dfrac{x^2 - 7x + 6}{x - 10}$;
(i) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$;
(j) $2 - 9x + 6x^2 - x^3$;
(k) $x^2(x - 1)^3$.

7. Examine each of the following functions for extreme values, obtain them whenever they exist:

(a) $x^2 e^{-x}$;
(b) $x^{2/3} - x$;
(c) $x^{1/x}$;
(d) $\dfrac{x}{\log x}$;
(e) $\left(\dfrac{1}{x}\right)^x$;
(f) $\dfrac{x}{(x-1)(x-4)}$.

[C.U. B.Com.(H) 1996]

8. (a) Verify that $x^3 - 6x^2 + 12x + 50$ has neither a maximum nor a minimum at $x = 2$.

(b) $f(x) = |x|$; show that the function has a minimum at $x = 0$ although $f'(0)$ does not exist.

(c) Show that $y = x^3 - 3x^2 + 5$ has a maximum value at $x = 0$ and a minimum value at $x = 2$.

(d) Show that the curve $y = x^2(3 - x)$ has a point of inflexion at the point $(1, 2)$. [N.B.U. B.Com.(H) 2006]

(e) Examine whether the curve has a point of inflexion at $(3, 2)$, when $y - 2 = 4(y - 3)^5$ is the given curve. [C.U. B.Com. 2010]

(f) Find the point of inflexion of the curve $y + 5 = x^3 - 3x^2 + 9x$. [C.U. B.Com. 2008]

9. Examine the following curves for points of inflexion and explain where the curve is concave upward and where it is concave downward: In (c) below, find extreme value, if any.

(a) $(y-2)^3 = x-4$;
(b) $y = x^2$;
(c) $y = 5-2x-x^2$; [C.U. B.Com. 2006]
(d) $y = x^3$;
(e) $y = x^4$;
(f) $y = 2x^3-3x^2-36x+25$;
(g) $y+5 = x^3-3x^2+9x$. [C.U. B.Com. 2008]

10. Verify the following statements:

(a) The curve $y = x^3-9x^2+24x-7$ has a maximum at $x=2$, minimum at $x=4$, a point of inflexion at $x=3$.

(b) The curve $y = \frac{1}{3}\left(x^3-3x^2-9x+11\right)$ has a maximum at $x=-1$; minimum at $x=3$; point of inflexion at $x=1$. Show that the equations of the tangent and normal at the point of inflexion are $4x+y=4$ and $x-4y=1$ respectively.

(c) The curve $y = 12-24x-15x^2-2x^3$ has a maximum at $x=-1$, minimum at $x=-4$; point of inflexion at $x=-5/2$.

(d) The curve $y = x^4-8x^2$ has a maximum at $x=0$, minimum at $x=\pm 2$, points of inflexion at $x = \pm\frac{2}{3}\sqrt{3}$.

(e) Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{1/e}$. [C.U. B.Com.(H) 1990; B.U. B.Com.(H) 2008]

11. (a) If $x+y=20$, find the maximum value of xy. [C.U. B.Com. 2006]

(b) If $2x+3y=4$, find the maximum or minimum value of xy.

(c) If $x+y=2$, show that the maximum value of $\dfrac{4}{x}+\dfrac{36}{y}$ is less than its minimum value. [C.U. B.Com. 2007]

ANSWERS

2. At $x=0$, maximum (maximum value $=0$); at $x=2/5$ minimum (minimum value $=-108/3125$); at $x=1$, neither maximum nor minimum, but it is a point of inflexion.

3. (a) At $x=1$, minimum value $=-1$; at $x=-1$, maximum value $=3$.
   (c) Maximum value $=20$ at $x=-2$ and minimum value $=-7$ at $x=1$.
   (d) Maximum value $=128$ and minimum value $=0$.

6. (a) $x=4$ minimum;
   (b) $x=3$ minimum; $x=-3$ maximum;
   (c) $x=0$ maximum value; $x=2$ minimum value;
   (d) $x=-1$ maximum; $x=1$ minimum;
   (e) Neither maximum nor minimum;
   (f) $x=-2$ maximum; $x=2$ minimum;
   (g) $x=1$ maximum; $x=6$ minimum;
   (h) $x=4$ maximum; $x=16$ minimum;
   (i) $x=1$ maximum; $x=-1$ minimum;
   (j) $x=3$ maximum; $x=+1$ minimum;
   (k) $x=0$ maximum; $x=2/5$ minimum; $x=1$ neither maximum nor minimum

7. (a) 0 minimum; $4/e^2$ maximum;
   (b) 0 minimum; 4/27 maximum;
   (c) $e^{1/e}$ maximum;
   (d) e minimum;
   (e) Maximum $e^{1/e}$;
   (f) $-1; -1/9$.

8. (e) The curve has a point of inflexion at $(3,2)$.
   (f) $(1,2)$.

9. (a) On the left of $(4,2)$ concave upward; on the right of $(4,2)$ concave downward; and the point $(4,2)$ is a point of inflexion;
   (b) Concave upward everywhere;
   (c) Concave downward everywhere;
   (d) On the left of $(0,0)$ concave downward and on the right concave upward;
   (e) Concave upwards everywhere;
   (f) Concave downward to the left and concave upward to the right of $x=1/2$.
   (g) $(1,2)$.

11. (a) 100;
   (b) Maximum value $=2/3$ at $x=1$.

6.6 Elementary Ideas of Curvature

The shape of a curve at a point (i.e., its flatness or sharpness) depends on the rate of change of its direction, i.e., the rate at which the tangent turns w.r.t. the length of the arc described. This rate is called the *curvature of the curve* at the point and is denoted by κ (Greek Kappa). We refer to Fig. 6.8.

Let P be a point of a curve. Q be another point on the curve near P. We measure the arc $AP = s$, (A being taken a fixed point on the curve from which the length of the arc is measured). Let arc $AQ = s + \Delta s$ so that the arc $PQ = \Delta s$.

Let the tangent at P make an angle ψ with the positive direction of x-axis and let the tangent at Q make an angle $\psi + \Delta\psi$ with the positive direction of X-axis. Thus, when the point of contact of the tangent line describes the arc Δs, the tangent line turns through an angle $\Delta\psi$. $\Delta\psi/\Delta s$ is called the average curvature of the arc PQ.

The curvature at $P(=\kappa)$ is defined by

$$\kappa = \lim_{\Delta x \to 0} \frac{\Delta\psi}{\Delta s} \text{ (i.e., when } Q \to P \text{ along the curve)}$$

$$\text{i.e., } \kappa = \frac{d\psi}{ds} = \text{curvature of the curve at } P.$$

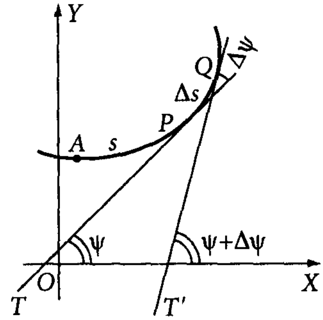


Fig. 6.8

Theorem 1. *The curvature of a circle at any point is always same and is equal to the reciprocal of radius* ($\kappa = \frac{1}{R}$, R = *radius of the circle*).

Proof. We refer to Fig. 6.9. In the figure (as in the Fig. 6.8) the angle $\Delta\psi$ between the tangent lines at P and Q equals $\angle PCQ$ at the centre (angle between the tangents at P and Q equals the angle between their normals, i.e., angle between the radii CP and CQ).

[We measure the arc lengths from a fixed point A on the circle. Here arc $AP = s$, arc $AQ = s + \Delta s$, so that arc $PQ = \Delta s$.

Note that we have measured $\angle PCQ$ in radians so that $\angle PCQ = \frac{\text{arc } PQ}{\text{radius } R}$]

Hence,

$$\frac{\Delta\psi}{\Delta s} = \frac{\text{angle } PCQ}{\Delta s} = \frac{\Delta s/R}{\Delta s} = \frac{1}{R},$$

where R is the radius of the circle.

$$\therefore \kappa = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{1}{R} \text{ (constant for all positions } P \text{ on the circle).}$$

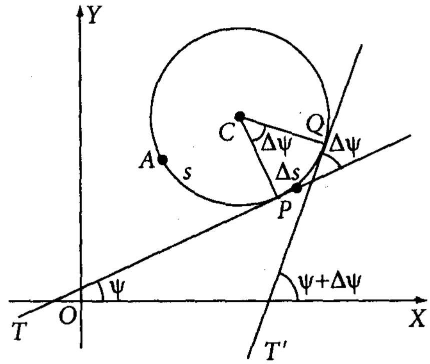


Fig. 6.9

Points to Remember

1. Curvature at any point of a circle = $\frac{1}{\text{radius}}$ = constant. In other words, a circle bends at uniform rate (= 1/radius).

Obviously the curvature of a straight line is everywhere zero.

3. We shall assume, *without proof*, that when the equation of a curve is given in rectangular coordinates, then the **curvature** at any point of a curve can be shown to be

$$\kappa = \frac{d^2y}{dx^2} \Bigg/ \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2},$$

if x is the independent variable, i.e.,

$$\kappa = \frac{y_2}{\left(1+y_1^2\right)^{3/2}}, \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}.$$

But use

$$\kappa = \frac{d^2x}{dy^2} \Bigg/ \left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2},$$

if y is the independent variable and if $\frac{dy}{dx}$ becomes infinite but $\frac{dx}{dy}$ is finite.

Example 1. *For a straight line $y = mx + c$, $\dfrac{d^2y}{dx^2} = 0$ and hence $\kappa = 0$ at any point (x, y) of the line.*

Example 2. *For a circle $x^2 + y^2 = a^2$, $\dfrac{dy}{dx} = \dfrac{-x}{y}$ and $\dfrac{d^2y}{dx^2} = \dfrac{x\frac{dy}{dx} - y}{y^2} = \dfrac{-(x^2+y^2)}{y^3} = \dfrac{-a^2}{y^3}$ and*
$\kappa = \dfrac{-a^2/y^3}{\left\{1+(-x/y)^2\right\}^{3/2}} = \dfrac{-a^2/y^3}{\frac{(x^2+y^2)^{3/2}}{y^3}} = -\dfrac{a^2}{a^3} = -\dfrac{1}{a}$*; numerical value of κ is $\dfrac{1}{a}$.*

Example 3. *Find the curvature of the parabola $y^2 = 4x$ (a) at the point $(1, 2)$, (b) at the vertex $(0, 0)$.*

[V.U. B.Com.(H) 2007]

Solution:

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}} \quad \left[\text{Here } \frac{dy}{dx} = \frac{2}{y}; \frac{d^2y}{dx^2} = \frac{-2}{y^2}\cdot\frac{dy}{dx} = \frac{-4}{y^3}\right]$$

$$\therefore\ \kappa = \frac{-4/y^3}{\left(1+\frac{4}{y^2}\right)^{3/2}} = \frac{-4}{(y^2+4)^{3/2}} = \frac{-4}{(4x+4)^{3/2}} = -\frac{1}{2}\cdot\frac{1}{(x+1)^{3/2}}.$$

(a) At $(1, 2)$,

$$\kappa = \frac{-1}{2\cdot 2^{3/2}} = -\frac{1}{2^{5/2}} = -\frac{1}{4\sqrt{2}} = \frac{-\sqrt{2}}{8};$$

we get the numerical value of $\kappa = \dfrac{\sqrt{2}}{8}$.

(b) At $(0, 0)$, $\dfrac{dy}{dx}$ becomes infinite. So we use the formula

$$\kappa = \frac{d^2x}{dy^2} \Bigg/ \left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}.$$

Here $x = \dfrac{y^2}{4}$, $\dfrac{dx}{dy} = \dfrac{y}{2}$ and it is $= 0$ at $(0, 0)$, but $\dfrac{d^2x}{dy^2} = \dfrac{1}{2}$ for all points.

$\therefore\ \kappa$ at $(0, 0) = \dfrac{1}{2}$.

Radius of Curvature (ρ)

The radius of curvature (ρ) at a point on a curve equals the reciprocal of the curvature at that point (provided the curvature at the point is not zero). Hence, we have the formula

$$\rho = \frac{1}{k} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \bigg/ \frac{d^2y}{dx^2} = \frac{\left(1+y_1^2\right)^{3/2}}{y_2},$$

where $y_1 = \dfrac{dy}{dx}, y_2 = \dfrac{d^2y}{dx^2}$.

Example 4. *Find the radius of curvature at any point (x, y) of the curve $y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right)$.*

Solution:
$$y_1 = \frac{1}{2}\left(e^{x/c} - e^{-x/c}\right); y_2 = \frac{1}{2c}\left(e^{x/c} + e^{-x/c}\right) = \frac{y}{c^2}.$$

$$1 + y_1^2 = 1 + \frac{1}{4}\left(e^{x/c} - e^{-x/c}\right)^2 = \frac{1}{4}\left(e^{x/c} + e^{-x/c}\right)^2 = \frac{y^2}{c^2}.$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{(y^2/c^2)^{3/2}}{y/c^2} = \frac{y^2}{c},$$

i.e., $\rho \propto y^2$.

Circle of Curvature

At any point P of a curve Fig. 6.10 we draw the normal to the curve on the *concave side* of the curve. We cut off a length PC = radius of curvature at $P = \rho$ (say). With centre C, we draw a circle passing through P. The curvature of this circle is then $\kappa = \dfrac{1}{\rho}$ which equals the curvature of the curve itself at P.

The circle thus constructed is called the *circle of curvature* for the point P on the curve. The centre and radius of this circle are respectively centre of curvature and radius of curvature of the curve at P.

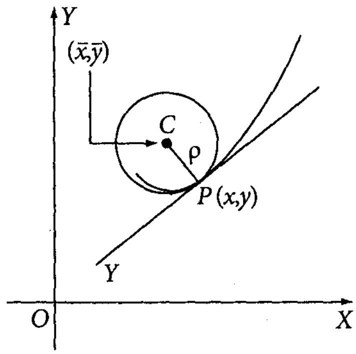


Fig. 6.10

The coordinates $(\bar{x}, \bar{y})$ of the centre of curvature for the point $P(x, y)$ are given by

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}, \bar{y} = y + \frac{1+y_1^2}{y_2},$$

where $y_1 = \dfrac{dy}{dx}, y_2 = \dfrac{d^2y}{dx^2}$.

Example 5. *Find the coordinates of the centre of curvature of the parabola $y^2 = 8x$ at any point (x, y) of the curve.*

Solution: We have $y^2 = 8x$.

$$\therefore y_1 = \frac{4}{y} \text{ and } y_2 = -\frac{4}{y^2} \cdot y_1 = -\frac{16}{y^3}$$

so that

$$\frac{y_1}{y_2} = -\frac{y^2}{4};\ 1 + y_1^2 = 1 + \frac{16}{y^2} = \frac{y^2+16}{y^2}.$$

$$\therefore\ \bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right) = x + \frac{y^2}{4} \times \frac{y^2+16}{y^2} = \frac{y^2}{8} + \frac{y^2+16}{4} = \frac{3y^2+32}{8} = 3x+4.$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \left(\frac{y^2+16}{y^2}\Big/ -\frac{16}{y^3}\right) = y - \frac{1}{16}\left(y^2+16\right)y = -\frac{y^3}{16}.$$

$\therefore$ **the coordinates of the centre of curvature at $P(x,y)$ of the parabola $y^2 = 4x$ are $\left(3x+4, -\frac{y^3}{16}\right)$.**

6.6.1 Further Illustrative Examples on Curvature and Radius of Curvature

Example 6. *Find the radius of curvature of the curve $y = e^{-x^2}$ at $(0,1)$.* [C.U. B.Com.(H) 1995]

Solution: The curve is $y = e^{-x^2}$. (1)

$$\therefore\ y_1 = \frac{dy}{dx} = e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$$

and

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2\frac{d}{dx}\left(x \cdot e^{-x^2}\right) = -2\left[1 \cdot e^{-x^2} + xe^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right)\right]$$
$$= -2e^{-x^2}(1 + x \times -2x) = 2e^{-x^2}(2x^2-1).$$

At $(0,1)$,

$$y_1 = -2 \cdot 0 \cdot e^{-0} = 0 \text{ and } y_2 = 2e^{-0}(2.0-1) = -2.$$

$$\therefore\ \rho = \text{radius of curvature} = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}.$$

$$\therefore \text{ At } (0,1), \text{ the radius of curvature} = \frac{(1+0)^{3/2}}{-2} = -\frac{1}{2} = \mathbf{\frac{1}{2}} \text{ (numerically).}$$

Example 7. *Find the radius of curvature of the curve $x^2 + y^2 = 4$ at any point (x,y).* [C.U. B.Com.(H) 1996]

Solution: The curve is $x^2 + y^2 = 4$. (1)

Differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \text{ or, } 2y\frac{dy}{dx} = -2x \text{ or, } \frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore\ \frac{d^2y}{dx^2} = -\left\{\frac{1 \cdot y - x\frac{dy}{dx}}{y^2}\right\} = -\frac{\left\{y - x \times -\frac{x}{y}\right\}}{y^2} = -\frac{(x^2+y^2)}{y^3} = -\frac{4}{y^3} \text{ [by (1)]}$$

$\therefore$ the radius of curvature at any point (x,y)

$$= \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{x^2}{y^2}\right)^{3/2}}{-\frac{4}{y^3}} = \frac{(x^2+y^2)^{3/2}}{y^3} \times -\frac{y^3}{4} = -\frac{(4)^{3/2}}{4} = -\frac{8}{4} = -2 = 2 \text{ (numerically).}$$

Example 8. *Find the curvature of the curve* $9x^2+4y^2=36x$ *at* $(2,3)$. [C.U. B.Com.(H) 1997]

Solution: We have

$$9x^2+4y^2=36x. \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$18x+8y\frac{dy}{dx}=36 \text{ or, } 8y\frac{dy}{dx}=36-18x \text{ or, } \frac{dy}{dx}=\frac{18(2-x)}{8y}=\frac{9}{4}\left(\frac{2-x}{y}\right).$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left\{\frac{9}{4}\left(\frac{2-x}{y}\right)\right\}=\frac{9}{4}\cdot\left\{\frac{-1\cdot y+(2-x)\frac{dy}{dx}}{y^2}\right\}=\frac{9}{4y^2}\left\{-y+(2-x)\times\frac{9}{4}\left(\frac{2-x}{y}\right)\right\}.$$

At (2,3),

$$\frac{dy}{dx}=\frac{9}{4}\left(\frac{2-2}{3}\right)=0 \text{ and } \frac{d^2y}{dx^2}=\frac{9}{4\cdot3^2}\{-3+0\}=-\frac{3}{4}.$$

Hence, the required curvature of the curve (κ) at $(2,3)$

$$=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}=\frac{-\frac{3}{4}}{(1+0)^{3/2}}=-\frac{3}{4}=\mathbf{\frac{3}{4}} \text{ (numerically).}$$

Example 9. *Determine the curvature of the curve* $y^2=4ax$ *at the point* $(at^2,2at)$. [C.U. B.Com.(H) 1994]

Solution: The curve is

$$y^2=4ax. \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=4a\cdot1 \text{ or, } \frac{dy}{dx}=\frac{2a}{y} \text{ and } \frac{d^2y}{dx^2}=-\frac{2a}{y^2}\cdot\frac{dy}{dx}=-\frac{2a}{y^2}\times\frac{2a}{y}=-\frac{4a^2}{y^3}.$$

At $(at^2,2at)$,

$$\frac{dy}{dx}=\frac{2a}{2at}=\frac{1}{t} \text{ and } \frac{d^2y}{dx^2}=-\frac{4a^2}{(2at)^3}=-\frac{1}{2at^3}.$$

Hence, the required curvature of the curve at the point $\left(at^2,2at\right)$

$$=\frac{\frac{d^2y}{dx^2}\,at\,(at^2,2at)}{\left[1+\left\{\frac{dy}{dx}\,at\,(at^2,2at)\right\}^2\right]^{3/2}}=\frac{-\frac{1}{2at^3}}{\left(1+\frac{1}{t^2}\right)^{3/2}}$$

$$=-\frac{1}{2a\left(t^2+1\right)^{3/2}}=\frac{\mathbf{1}}{\mathbf{2a\left(t^2+1\right)^{3/2}}} \text{ (numerically).}$$

Example 10. *Find the curvature of* $\sqrt{x}+\sqrt{y}=1$ *at the point* $\left(\frac{1}{4},\frac{1}{4}\right)$. [C.U. B.Com.(H) 2000]

Solution: The curve is

$$\sqrt{x}+\sqrt{y}=1. \qquad (1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0 \text{ or, } \frac{dy}{dx}=-\frac{y^{1/2}}{x^{1/2}}.$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{\left\{x^{1/2}\cdot\frac{1}{2}y^{-1/2}\frac{dy}{dx}-y^{1/2}\cdot\frac{1}{2}x^{-1/2}\right\}}{\left(x^{\frac{1}{2}}\right)^2}=\frac{1}{2x}\left(\frac{x^{1/2}}{y^{1/2}}\times\frac{y^{1/2}}{x^{1/2}}+\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$=\frac{1}{2x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)=\frac{1}{2x}\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right)=\frac{1}{2x\sqrt{x}} \text{ [by (1)]}.$$

At $\left(\frac{1}{4},\frac{1}{4}\right)$,

$$\left[\frac{dy}{dx}\right]_{\left(\frac{1}{4},\frac{1}{4}\right)}=-\frac{\frac{1}{2}}{\frac{1}{2}}=-1 \text{ and } \left[\frac{d^2y}{dx^2}\right]_{\left(\frac{1}{4},\frac{1}{4}\right)}=\frac{1}{2\cdot\frac{1}{4}\cdot\frac{1}{2}}=4.$$

Hence, the required curvature of the curve (1) at the point $\left(\frac{1}{4},\frac{1}{4}\right)$

$$=\frac{\left[\frac{d^2y}{dx^2}\right]_{\left(\frac{1}{4},\frac{1}{4}\right)}}{\left\{1+\left(\frac{dy}{dx}\right)^2_{\left(\frac{1}{4},\frac{1}{4}\right)}\right\}^{3/2}}=\frac{4}{\{1+(-1)^2\}^{3/2}}=\frac{4}{(2)^{3/2}}=\frac{4}{2\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}.$$

EXERCISES ON CHAPTER 6(V)

(Concept of Curvature)

1. Find the radius of curvature at any point (x,y) of the curve $x^2y=x^3-a^3$.
2. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$. What is the curvature at $\left(\frac{1}{4},\frac{1}{4}\right)$?

   [C.U. B.Com.(H) 2000]
3. Prove that the radius of curvature at any point (x,y) of the catenary $y\frac{1}{2}a\left(e^{x/a}+e^{-x/a}\right)$ varies as the square of the ordinate (i.e., varies as y^2).
4. Find the coordinates of the centre of curvature at (x,y) on the parabola.
5. Find the centre of curvature of the following curves at the points indicated:
   (a) $y=3x^3+2x^2-3$ at $(0,-3)$; (b) $y=x^3-6x^2+3x+1$ at $(1,-1)$.
6. Verify the following results:

Equation of the curve	Curvature at (x, y)
(a) Circle : $x^2+y^2=a^2$	$\frac{1}{a}$ [C.U. B.Com.(H) 1996]
(b) Parabola : $y^2=4ax$	$\sqrt{a}/2(a+x)^{3/2}$ [C.U. B.Com.(H) 1994]
(c) Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{ab}{(a^2-e^2x^2)^{3/2}}[b^2=a^2(1-e^2)]$
(d) Hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{ab}{(e^2x^2-a^2)^{3/2}}[b^2=a^2\left(e^2-1\right)]$
(e) Rectangular hyperbola : $xy=c^2$	$\frac{2c^2}{(x^2+y^2)^{3/2}}$
(f) Straight line : $y=mx+c$	0.

7. Find the equation of the circle of curvature of the curve $2xy+x+y=4$ at the point $(1,1)$.

[**Hints**: Find $\rho=\frac{3\sqrt{2}}{2}, \bar{x}=\frac{5}{2}, \bar{y}=\frac{5}{2}$. Then the required eqn. is $(x-\bar{x})^2+(y-\bar{y})^2=(\rho)^2$]]

8. Verify that the radius of curvature of the curve

(a) $2y=x^2$ at $(0,0)$ is 1; (b) $6y=x^3$ at $(2,4/3)$ is $\frac{5}{2}\sqrt{5}$;
(c) $y^2=x^3$ at $(1,1)$ is $\frac{13}{6}\sqrt{13}$; (d) $y=e^x$ at $(0,1)$ is $2\sqrt{2}$.

9. Find the curvature of the curve $9x^2+4y^2=36x$ at $(2,3)$.

10. Find the curvature of the curve $x^2-y^2=1$ at any point (x,y). [C.U. B.Com.(H) 1999; V.U. B.Com.(H) 2008]

[**Hints**: $x^2-y^2=1$; $\therefore$ $2x-2y\cdot y_1=0$ or, $y_1=\frac{x}{y}$ and $y_2=\frac{1\cdot y-x\cdot y_1}{y^2}=\frac{y-x\times\frac{x}{y}}{y^2}=\frac{-(x^2-y^2)}{y^2}=-\frac{1}{y^3}$.

$\therefore$ $\kappa=\frac{y_2}{\left(1+y_1^2\right)^{3/2}}=-\frac{1}{y^3}\Big/\left\{1+\frac{x^2}{y^2}\right\}^{3/2}=-\frac{y^3(x^2+y^2)^{3/2}}{y^3}=-\left(x^2+y^2\right)^{3/2}=\left(x^2+y^2\right)^{3/2}$ (numerically)]

Note: At $(3,0)$, $\kappa=27$.

11. Find the curvature of the curve $x^2=4y$ at the point $(0,0)$. [V.U. B.Com.(H) 2008]

ANSWERS

1. $\frac{(a^4+9x^4)^{3/2}}{6a^4x}$.

2. $1/\sqrt{2}$; curvature $=\sqrt{2}$.

4. $\bar{x}=2a+3x, \bar{y}=-2a^{-(1/2)}x^{3/2}$.

5. (a) $\left(0,-\frac{11}{4}\right)$; (b) $\left(-36,-\frac{43}{6}\right)$.

7. $4x^2+4y^2-20x-20y+32=0$.

9. 0.75.

10. $\left(x^2+y^2\right)^{-3/2}$.

6.7 Derivative as Rate of Change of Dependent Variable w.r.t. the Independent Variable x

We begin with an example: Consider a functional relation, say, $y = x^2$.

Let us take a fixed value for $x = 4$, then the corresponding $y = 16$. Give an increment to the value $x = 0.5$ (say). We shall write $\Delta x = 0.5$.

Then the corresponding increment of $y = \Delta y = (4.5)^2 - 4^2 = 20.25 - 16 = 4.25$.

$\therefore$ increment ratio $= \dfrac{\Delta y}{\Delta x} = \dfrac{4.25}{0.5} = 8.5$.

We say that the average rate of change of y w.r.t. $x = 8.5$, when x increases from $x = 4$ to $x = 4.5$ (increment = 0.5).

More generally, let $y = f(x)$ be a function of x.

Choose a fixed value $x = x_1$; the corresponding value of y is $y = f(x_1)$.

Give an increment Δx to x, i.e., take a new value of $x = x_1 + \Delta x$, then the corresponding value of y is $y = f(x_1 + \Delta x)$.

$\therefore$ increment of $y = \Delta y = f(x_1 + \Delta x) - f(x_1)$.

We now say, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ = average rate of change of y w.r.t. x, when x changes from x_1 to $x_1 + \Delta x$.

Instantaneous Rate of Change

Suppose, now we go on choosing our increments of x smaller and smaller. Ultimately when $\Delta x \to 0$, the limit to which $\Delta y/\Delta x$ approaches as $\Delta x \to 0$ (provided that such a limit exists) is called the *instantaneous rate of change* of y w.r.t. x at the value $x = x_1$. Since according to our definition of derivative this limit is nothing but the derivative of y w.r.t. x at $x = x_1$, we now state:

> $\dfrac{dy}{dx}$ or $f'(x)$ at $x = x_1$ is the instantaneous rate of change of $y = f(x)$ w.r.t. x at the value $x = x_1$.

e.g., let $y = x^2$. Then $\dfrac{dy}{dx} = 2x$.

At $x = 4$, the instantaneous rate of change of $y = x^2$ is $2 \times 4 = 8$ units per unit change in x.

Example 1. *Elementary economics tells us that, given a total cost function $C = f(Q)$, where C denotes the total cost and Q the output, the marginal cost is defined as the change in total cost per unit change in output.*

Solution: If ΔQ be a small change in Q from Q_0 to $Q_0 + \Delta Q$ and if ΔC be the corresponding increment of C, then $\Delta C/\Delta Q$ is the average marginal cost. If ΔQ be made smaller and smaller and ultimately $\Delta Q \to 0$, then

$$\lim_{\Delta Q \to 0} \frac{\Delta C}{\Delta Q} = \frac{dC}{dQ}$$

is the marginal cost at $Q = Q_0$, e.g., if $C = Q^3 - 4Q^2 + 10Q + 75$, the marginal cost at $Q = 10$ units is obtained by putting $Q = 10$ in $\dfrac{dC}{dQ}$.

We see that
$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10.$$

Putting $Q = 10$, this value becomes $= 230$.

$\therefore$ the marginal cost is 230 units per unit change in Q, when Q stands at 10 units.

Example 2. *Let the functional relationship between V and P be given by* $V = \frac{600}{P}$. *Find the average rate of change of V w.r.t. P when P increases from* 30 *to* 30.5 *units. What is the instantaneous rate of change of V at* $P = 30$?

Solution: We have

$$\frac{\Delta V}{\Delta P} = \text{average rate of change of } V \text{ w.r.t. } P$$

$$= \left\{\frac{600}{30.5} - \frac{600}{30}\right\} \Big/ (30.5 - 30) = \left\{\frac{1200}{61} - 20\right\} \Big/ 0.5$$

$$= -\frac{40}{61} = -0.66 \text{ (approx.) units/unit change in } P$$

$$\frac{dV}{dP} = -\frac{600}{P^2}. \text{ At } P = 30, \frac{dV}{dP} = -\frac{600}{30^2} = -0.667,$$

i.e., instantaneous rate of change = -0.667 units/unit change in P. The negative sign indicates the decrease rate (i.e., decrease of V per unit increase in P).

6.8 Time-Rate of Change

If the independent variable is time t, then we call the rates *time-rates* — average or instantaneous as the case may be. Suppose, the dependent variable is the displacement s of a moving particle at any instant of time t. Then $s = f(t)$.

We call

$$\frac{\Delta s}{\Delta t} = \frac{\text{Increment of } s}{\text{Increment of } t} = \begin{array}{l}\text{average velocity when } t \text{ changes from a fixed} \\ \text{value, say from } t_0 \text{ to } t_0 + \Delta t\end{array}$$

and $\left(\frac{ds}{dt} \text{ at } t = t_0\right)$ = instantaneous rate of change of s w.r.t. t at $t = t_0$ = instantaneous velocity at the instant $t = t_0$ or simply, velocity at $t = t_0$.

We write,

$$v = \left(\frac{ds}{dt}\right)_{t=t_0},$$

i.e., the velocity at any instant is the derivative of the displacement w.r.t. time (or time-rate of change of displacement).

Example 3. *A point moves in a straight line so that its distance s* (*in* cm) *measured from a fixed point O and a line at time t* seconds *reckoned from some fixed epoch is given by* $s = t^3 - 6t^2 - 15t$. *Find* (a) *the velocity v at any instant t, at the end of first* second; (b) *the average velocity while t changes from* $t = 1$ *to* $t = 6$; (c) *when and where the body stops.*

Solution: (a) $\frac{ds}{dt} = (3t^2 - 12t - 15)$ cm/sec is the velocity at any instant t.

At $t = 1$, $\frac{ds}{dt} = 3 \cdot 1^2 - 12 \cdot 1 - 15 = -24$ cm/sec (velocity at the end of first second)

(b)
$$\frac{\Delta s}{\Delta t} = \frac{(6^3 - 6 \cdot 6^2 - 15 \cdot 6) - (1^3 - 6 \cdot 1^2 - 15 \cdot 1)}{6 - 1}$$

$$= \frac{-90 - (-20)}{5} = -14 \text{ cm/sec}$$

= *average velocity while t changes from* 1 *to* 6.

(c) The body stops when $\frac{ds}{dt} = 0$, i.e., when $3\left(t^2 - 4t - 5\right) = 3(t+1)(t-5) = 0$, i.e., when $t = -1$ or, $t = 5$.

Again, if $t = -1$, $s = 8$ and if $t = 5$, $s = -100$.

Interpretations. $t = -1$ means one second before the fixed epoch from which the time is reckoned. We have observed that at such an instant the point was at rest being at a distance $s = 8$ cm from O (from where the distance is measured).

Again, at $t = 5$, i.e., five seconds after the time-epoch the point is again at rest, being at a distance of 100 cm ($s = -100$) from O but this time the point is on the other side of O (s is negative here).

Example 4. *If the radius of a circle increases uniformly at the rate of* 4 cm *per* second, *find the rate of increase of its area when the radius is* 5 cm. [C.U. B.Com.(H) 2000]

Solution: Let r cm be the radius of the circle at any time t sec. Then

$$\frac{dr}{dt} = 4. \tag{1}$$

If A be the area of a circle of radius r cm, then $A = \pi r^2$.

$$\therefore \quad \frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt} = 2\pi r \times 4 = 8\pi r \text{ [by eq. (1)]}.$$

When $r = 5$ cm,

$$\frac{dA}{dt} = 8\pi \times 5 = 40\pi \text{ sq cm/sec.}$$

Hence, *the required rate of increase of the area* = **40π sq cm/sec.**

Example 5. *If the area of a circle increases at a constant rate, show that the rate of increase of its perimeter is inversely proportional to its radius.* [C.U. B.Com.(H) 1996]

Solution. Let the area of the circle be A sq units and its perimeter be P units when its radius is r units. Then

$A = \pi r^2$ and $P = 2\pi r$.

$$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \tag{1}$$

$$\text{and} \quad \frac{dP}{dt} = 2\pi \cdot 1 \cdot \frac{dr}{dt} = 2\pi\frac{dr}{dt}. \tag{2}$$

By the given condition, $\frac{dA}{dt} = \text{constant} = c$ (say).

From (1), we get $c = 2\pi r\frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{c}{2\pi r}$.

From (2), we get $\frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt} = 2\pi \times \frac{c}{2\pi r} = \frac{c}{r}$, where c is a constant.

Hence $\frac{dP}{dt} \propto \frac{1}{r}$, i.e., *the rate of increase of the perimeter is inversely proportional to its radius.*

Example 6. *The radius of a spherical balloon is increasing uniformly at the rate of* 2 cm *per second. Find at what rate the volume of the balloon is increasing when its radius is* 6 cm. [C.U. B.Com.(H) 1997]

Solution: Let r cm be the radius of a spherical balloon at any time t sec. Let v cu cm be the volume at time t. Then $\frac{dr}{dt}=2$.

Now,
$$v=\frac{4}{3}\pi r^3; \therefore \frac{dv}{dt}=\frac{4}{3}\pi\times 3r^2\frac{dr}{dt}=4\pi r^2\times 2=8\pi r^2.$$

When $r = 6$ cm, $\frac{dv}{dt}=8\pi\cdot 6^2=288\pi$ cu cm/sec.

Hence, *the required rate at which the volume is increasing when radius is* 6 cm *is* **288π cu cm/sec.**

6.9 Related Rates

In many problems several variables (each is a function of the time) are involved. Relations between the variables are established by the given conditions of the problem. The relations between their time-rates of change are then found by differentiation w.r.t. time.

Example 7. *A metal cube expands on heating. At the instant when each edge measures* 3 cm *the volume of the cube increases at the rate of* $0.015\,\text{cm}^3$ *per* second. *At what rate is the length of the edge increasing at the instant?*

Solution: Let the length of the edge of the cube be x cm and its volume be V cu. cm.

Then $V=x^3$;
$$\therefore \frac{dV}{dt}=3x^2\cdot\frac{dx}{dt}. \tag{1}$$

At $x=3$ cm., $\frac{dV}{dt}=0.015$, and from (1), we get
$$0.015=3\cdot 3^2\cdot\frac{dx}{dt} \text{ or, } \frac{dx}{dt}=\frac{0.015}{27}=\frac{1}{1800}.$$

Hence, the required rate at which the length of the edge is increasing $=\frac{1}{1800}$ cm per sec.

Example 8. *A man is walking at the rate of* 5 km *per* hour *towards the foot of a tower* 60 metres *high. At what rate is he approaching the top of the tower when he is* 80 metres *from the foot of the tower?*

[C.U. B.Com.(H) 2003; (P) 2009]

Solution: See the Fig. 6.11. F is the foot and T is the top of the tower so that $TF=60$ m (given). Let x = distance of the position of the man M from the foot, and y = distance of the position of the man M from the top at any instant t.

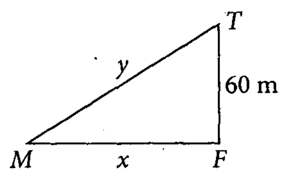


Fig. 6.11

From the right-angled triangle MFT,
$$y^2=x^2+60^2 \text{ or, } 2y\frac{dy}{dt}=2x\frac{dx}{dt}, \text{ i.e., } \frac{dy}{dt}=\frac{x}{y}\frac{dx}{dt},$$

i.e., rate of change of y w.r.t. $t=\frac{x}{y}\times$ rate of change of x w.r.t. t.

See that when $x = 80$,
$$y = \sqrt{80^2 + 60^2} = 100.$$

Also it is given that $\frac{dx}{dt} = -5$ km/hr (negative sign, x decreases as t increases).

$$\therefore \frac{dy}{dt} = -\frac{80}{100} \times 5 \text{ km/hr} \left(\text{Note that } \frac{x}{y} = \frac{80\,\text{m}}{100\,\text{m}} = \frac{80}{100}\right)$$
$$= -4 \text{ km/hr},$$

i.e., *the man is approaching the top at the* **rate of 4 km/hr.**

6.10 Elementary Business Problems

Recapitulation: **Some business terms and problems.**

1. In a business situation profit is given by
$$\text{Profit} = \text{Revenue} - \text{Cost}.$$

2. The manufacturing cost of a particular commodity consists of two components, viz., fixed cost F and the variable cost V. Fixed costs include rent, insurance and other overhead expenses that exist even with no production. Variable costs are the expenses with a direct bearing on each unit being produced, viz., man-hour, raw material, fuel, etc. It depends on the number (x) of units produced. Thus we have

   cost function $C(x)$ = fixed cost $F(x)$ + variable cost $V(x)$.

   If p be the selling price per unit of the commodity, then $R(x)$ = total revenue $= px$

   and the profit function $P(x)$ = revenue function $R(x)$ − cost function $C(x)$
   $$= R(x) - [F(x) + V(x)]$$
   $$= R(x) - F(x) - V(x).$$

3. The price p of a quantity bought and the amount x of a quantity demanded are related by the equation
$$p = D(x), \text{ known as the demand equation.}$$
   The function D is called the price function and $D(x)$ is the price per unit, when x units are demanded.

Example 9. *The overhead expenditure in a workshop is ₹200 and the cost for each piece manufactured is ₹10. Construct the cost function $C(x)$. What is the domain and range of $C(x)$?*

Solution: $C(x)$ = cost function, where x is the number of pieces manufactured in the workshop.

Then
$$C(x) = 10x + 200.$$

x cannot be negative or fractions. So the **domain** of x is whole numbers $\{0, 1, 2, 3, \ldots\}$.

The *range of $C(x)$ consists of* $\{C(0), C(1), C(2), C(3), \ldots\} = \mathbf{\{200, 210, 220, 230, \ldots\}}$.

Example 10. *A biscuit company sells x boxes of biscuits each day at ₹20 per box. The cost of manufacturing and selling these boxes is ₹10 per box plus a fixed daily overhead cost of ₹600. What is the profit function? What is the total profit, if 500 boxes are manufactured and sold per day? What happens if the company manufactures and sells only 50 boxes per day?*

Solution: The revenue earned by the company per day is given by

$$R = 20x \text{ (}x\text{ boxes are manufactured and sold).}$$

Total cost of x boxes manufactured each day is given by $C = 10x + 600$.

$\therefore$ Profit per day = Revenue − Cost.

$$P(x) = 20x - (10x + 600) = 10x - 600,$$

i.e., *the profit function is* $\mathbf{P(x) = 10x - 600}$.

If 500 boxes are manufactured and sold per day, profit is given by

$$P(500) = 10 \times 500 - 600 = ₹4400.$$

But

$$P(50) = 10 \times 50 - 600 = -₹100.$$

It indicates that the company would incur a loss of **₹100 per day** if the company manufactures and sells only 50 boxes per day.

Break-Even Analysis

Profit function $P(x)$ = Revenue function $R(x)$ − Cost function $C(x)$.

If $R(x) = C(x)$, then $P(x) = 0$, i.e., when the revenue equals the cost, there is no profit, no loss; in this case, *the value of x is called* **Break-Even Point.**

Example 11. *The daily cost of production C for x units of production is given by*

$$C(x) = ₹3.50x + ₹12000$$

(i) *If each unit is sold for* ₹6, *determine the minimum number of units that should be produced to ensure no loss.*

(ii) *If the selling price is increased by* 50 paise *per unit, what would be the break-even point?*

(iii) *If* 6000 *units are sold daily, what price per unit should be charged to guarantee no loss?*

Solution: (i) The revenue $R(x) = ₹6x$.

In order that there is no loss,

$$C(x) = R(x), \text{ i.e., } 3.50x + ₹12000 = 6x \text{ or, } \frac{12000}{2.50} = x \text{ or, } x = 4800.$$

Thus, *at least* 4800 *units must be produced daily to ensure no loss.*

(ii) In the second case, the revenue R from sales is given by

$$R(x) = ₹6.50x.$$

For break-even point:

$$6.50x = 3.50x + 12000, \quad \text{or,} \quad 6.50x - 3.50x = 12000,$$
$$\text{or,} \quad 3x = 12000, \text{ i.e., } x = 4000.$$

With the increased selling price, *the break-even point is* 4000.

(iii) If 6000 units are sold daily, the price per unit of production needed to ensure no loss is given by

$$6000p = 3.50 \times 6000 + 12000 = 21000 + 12000 = 33000$$

$$\therefore\ p = \frac{33}{6} = 5.50,$$

i.e., *price per unit should be* ₹5.50.

6.11 Applications of Differential Calculus in Economics and Business

In differential calculus we have observed that $\dfrac{dy}{dx}$ is the rate of change of the dependent variable y w.r.t. the independent variable x.

Marginal cost

For any cost function $C = f(q)$, that gives the total cost of producing and marketing q units of a product, if the production level is increased by Δq units to $q + \Delta q$, C changes to $f(q + \Delta q)$.

$$\frac{\text{Change in total cost } C}{\text{Change in output } q} = \frac{\Delta C}{\Delta q} = \frac{f(q+\Delta q) - f(q)}{\Delta q}.$$

This is called the *average rate of change* of total cost C w.r.t. the output q over the interval $(q, q + \Delta q)$.

- The Marginal Cost is then defined as the instantaneous rate of change of C w.r.t. q.

  Thus, **marginal cost** (MC) is given by

$$\text{MC} = \lim_{\Delta q \to 0} \frac{\Delta C}{\Delta q} = \frac{dC}{dq}.$$

$$\textbf{Marginal Average cost (MAC)} = \frac{d(\text{AC})}{dq}.$$

Interpretation of Marginal Cost: It is the approximate change in cost resulting from one additional unit of output.

Marginal Revenue

- We may similarly define **Marginal Revenue** (MR) as $\dfrac{dR}{dq}$, where $R = f(q)$ is the total revenue function.

  Under monopoly (or equilibrium) MR = MC.

- The consumption function $C = f(I)$ expresses a relationship between the total income I and the total consumption C.

  The Marginal Propensity to consume = rate of change of consumption w.r.t. income = dC/dI.

- Savings S = Income I − Consumption C.

  Then $\dfrac{dS}{dI}$ = **Marginal Propensity to save**, i.e., *an indication of how fast savings change w.r.t. income.*

Example 12. *If the average cost function* $\bar{c} = 0.01q + 10 + \dfrac{400}{q}$, *where* q *is the number of units produced, find the marginal cost function. What is the marginal cost when* 200 *units are produced?*

Solution: The total cost

$$C = q\bar{c} = q\left(0.01q + 10 + \frac{400}{q}\right) = 0.01q^2 + 10q + 400.$$

$\therefore$ *Marginal cost function*

$$\frac{dC}{dq} = \mathbf{0.02q + 10.}$$

When $q = 200$,

$$\left[\frac{dC}{dq}\right]_{q=200} = 0.02 \times 200 + 10 = 14.$$

This means that if C is in rupees *and the production is increased by one unit from $q = 200$ to $q = 201$, then the cost of additional unit is approximately ₹14.*

Example 13. *Revenue R out of sale of q units of a product has the relation $R = 30q - 0.5q^2$.*

(i) *How fast R changes w.r.t. q? What is the marginal revenue* (MR)?

(ii) *When $q = 20$, find marginal revenue and the relative rate of change of R and also find the percentage rate of change of R.*

Solution: (i) $\frac{dR}{dq}$ = rate of change of R w.r.t. q is $30 - q$, what is also the marginal revenue (MR).

(ii) Marginal revenue $= 30 - q = 30 - 20 = \mathbf{10}$. Relative rate of change of R when $q = 20$, is

$$\frac{\left[\frac{dR}{dq}\right]_{q=20}}{[R]_{q=20}} = \frac{30 - 20}{30 \times 20 - 0.5 \times 20^2} = \frac{10}{400} = \frac{1}{40}.$$

Percentage rate of change of R is $\frac{1}{40} \times 100 = 2.50\%$.

Example 14. *The total cost C, of making x units of a product, is*

$$C = ax^n + b, \quad (a, b, n \text{ are constants}).$$

Find the marginal cost and marginal average cost. [C.U. B.Com.(H) 2002]

Solution: Total cost $C = ax^n + b$.

$\therefore$ *Marginal Cost* (MC) $= \frac{dC}{dx} = nax^{n-1} + 0 = \boldsymbol{nax^{n-1}}$.

Again, Average Cost (AC) $= \frac{C}{x} = \frac{ax^n + b}{x} = ax^{n-1} + bx^{-1}$.

$\therefore$ *Marginal Average Cost* $= \frac{d}{dx}(\text{AC}) = \frac{d}{dx}\left(ax^{n-1} + bx^{-1}\right) = \boldsymbol{a(n-1)\,x^{n-2} - \frac{b}{x^2}}$.

Example 15. *If* AR *and* MR *denote the average and marginal revenues at any output level, show that the elasticity of demand is equal to* AR/(AR−MR). *Verify this for the linear demand law $p = a + bx$.*

Solution: We have $R=px, \therefore$ AR$=\frac{R}{x}=p$ and MR $=\frac{dR}{dx}=p+x\frac{dp}{dx}$.

Now

$$\frac{\text{AR}}{\text{AR}-\text{MR}}=\frac{p}{p-\left(p+\frac{x dp}{dx}\right)}=-\frac{p}{x}\cdot\frac{dx}{dp}.$$

ϵ_d = elasticity of demand $=-\frac{p}{x}\cdot\frac{dx}{dp}=\frac{\text{AR}}{\text{AR}-\text{MR}}$.

2nd Part: $p=a+bx; \therefore \frac{dp}{dx}=b$ or, $\frac{dx}{dp}=\frac{1}{b}$.

Now $R=px=ax+bx^2; \therefore$ AR$=\frac{R}{x}=a+bx$ and MR $=\frac{dR}{dx}=a+2bx$.

$$\therefore \quad \frac{\text{AR}}{\text{AR}-\text{MR}}=\frac{a+bx}{a+bx-(a+2bx)}=-\frac{a+bx}{bx}$$

and $\quad \epsilon_d=$ elasticity of demand $=-\frac{p}{x}\cdot\frac{dx}{dp}=-\frac{a+bx}{x}\times\frac{1}{b}=-\frac{a+bx}{bx}$.

Hence, $\epsilon_d=\frac{\text{AR}}{\text{AR}-\text{MR}}$.

6.12 Applications of Derivatives in Economics

We shall discuss some applications of Calculus in Elementary Economics. For this purpose we first define some economic terms with notations.

Demand Function

If q be the quantity (or *demand*) and p the price of a commodity, then the demand function is given by $q=f(p)$, where p,q are both positive. Here income of consumer, price of other commodity, taste, habits of the consumer remain constant.

Normally, p and q are inversely related, i.e., if p increases, q decreases and if p decreases, q increases. The demand curve is a falling curve in which $\frac{dq}{dp}<0$.

[Usually, if price increases, supply increases but demand decreases.]

Supply Function

If x be the amount of supply when p is the price (i.e., if x denotes amounts of a particular commodity that sellers offer in the market at various prices p), then the supply function is given by $x=F(p)$, where $x>0, p>0$, and other things remain constant.

The supply curve is a rising curve in which $\frac{dx}{dp}>0$.

Elasticity of Demand and Supply

The elasticity of demand for a commodity is defined as the ratio of the proportionate change in quantity demanded to the proportionate change in price.

Thus, if the quantity demanded changes from x to $x+\Delta x$ when price changes from p to $p+\Delta p$, then *elasticity of demand* ϵ_d is given by

$$\epsilon_d=\lim_{\Delta p\to 0}\frac{\Delta x/x}{\Delta p/p},$$

[where Δx = change in quantity demanded, when Δp = change in price.]

$$= \frac{p}{x} \cdot \lim_{\Delta p \to 0} \frac{\Delta x}{\Delta p} = \frac{p}{x} \cdot \frac{dx}{dp}.$$

Usually, the demand curve has a negative slope at all prices and, therefore, $\frac{dx}{dp}$ is negative at all prices. To keep the sign of ϵ_d always positive (since we consider only positive values of ϵ_d), we write,

$$\epsilon_d = -\frac{p}{x} \cdot \frac{dx}{dp} = \frac{-\frac{dx}{dp}}{x/p} = \frac{-\text{Marginal value}}{\text{Average value}}. \tag{1}$$

ϵ_d is independent of the units in which x and p are expressed, but it varies at different points of the demand curve.

Thus, if $x = f(p)$, then (i) marginal function demanded $= \frac{dx}{dp}$; (ii) average function demanded $= \frac{x}{p}$; and (iii) $|\epsilon_d| = \frac{dx}{dp} \Big/ \frac{x}{p}$.

Note: (i) The formula (1) gives the *elasticity of demand ϵ_d at any point* on the demand curve and hence ϵ_d is known as *point-elasticity of demand.*

(ii) Since ϵ_d is a ratio of marginal value and average value, it is a *pure number.*

(iii) By definition, ϵ_d is the negative of the ratio of proportionate change in quantity demand to proportionate change in price.

Income Elasticity of Demand

If x be the quantity demanded and y the income per head in a particular group of people, then income elasticity of demand ϵ_y is given by

$$\epsilon_y = \frac{y}{x} \cdot \frac{dx}{dy}.$$

This gives the elasticity of quantity demanded in response to a change in income.

If $\epsilon_y > 1$, then the goods are *luxury goods.*

If $0 < \epsilon_y < 1$, then the goods are *necessities of life.*

If $\epsilon_y < 0$, then the goods are *inferior goods.*

Elasticity of Supply

It is the relative change in supply corresponding to a relative change in price. If x be the amount of supply when p is the price, then the supply function is $x = f(p)$ and, as before, the elasticity of supply ϵ_s, is given by

$$\epsilon_s = \text{Elasticity of supply} = \frac{p}{x} \cdot \frac{dx}{dp}.$$

As the supply curve is a rising curve, the slope of the supply curve is always positive and $\epsilon_s > 0$.

Note: In both demand and supply equations, price p is an *independent variable* and quantity as demanded or supplied is a *dependent* *variable*, depending on p.

Equilibrium Price and Quantity (Market Equilibrium)

Equilibrium price is the price at which quantity demanded is equal to quantity supplied.

Equilibrium quantity is the quantity obtained substituting the value of equilibrium price in any one of the given demand and supply functions.

Market equilibrium between demand and supply for a commodity is attained when the quantity supplied is exactly equal to the quantity demanded.

Monopoly price is the price at which profit is maximum. It can be shown that the price at the point of equilibrium is the same as the monopoly price.

Example 16. *Find the equilibrium price and quantity for the following demand and supply functions:* $Q_d = 3 - 0.04p$ *and* $Q_s = 0.4 + 0.09p$*, where* Q_d = *quantity demanded and* Q_s = *quantity supplied.*

Solution: For the equilibrium price,

$$Q_d = Q_s,\ \text{i.e., } 3 - 0.04p = 0.4 + 0.09p,$$

$$\text{or,}\quad -0.04p - 0.09p = 0.4 - 3 \text{ or, } -0.13p = -2.6,$$

$$\text{or,}\quad p = \frac{2.6}{0.13} = 20.$$

Now, $Q_d = 3 - 0.04p$.

If $p = 20$, then $Q_d = 3 - 0.04 \times 20 = 3 - 0.8 = 2.2$.

∴ *equilibrium price* = **20** *and equilibrium quantity* = **2.2.**

Example 17. *Show that the elasticity of demand at all points on the curve* $xy = c^2$ *(y being the price) will be numerically equal to one.*

Solution: We have $xy = c^2$ or, $x = \frac{c^2}{y}$; $\therefore\ \frac{dx}{dy} = -\frac{c^2}{y^2}$.

$$\epsilon_d = \text{elasticity of demand} = -\frac{y}{x}\cdot\frac{dx}{dy} = -\frac{y}{c^2/y} \times \left(-\frac{c^2}{y^2}\right) = \mathbf{1}\ \textbf{(numerically)}.$$

Hence *the result follows.*

Example 18. *A demand function is given by* $xp^n = k$*, where n and k are constants. Calculate price elasticity of demand.*

Solution: We have $xp^n = k$ or, $x = kp^{-n}$.

$$\therefore\ \frac{dx}{dp} = k \times (-n)\cdot p^{-n-1} = -nkp^{-n-1}.$$

$$\therefore\ \epsilon_d = \textit{price elasticity of demand} = -\frac{p}{x}\cdot\frac{dx}{dp} = -\frac{p}{kp^{-n}} \times -nkp^{-n-1} = \boldsymbol{n}.$$

Example 19. *Find the elasticity of demand w.r.t. price for the following demand functions:*

(i) $p = \sqrt{a - bx}$, *a and b being constants;* (ii) $x = \frac{8}{p^{3/2}}$.

Solution: (i) We have $p=\sqrt{a-bx}=(a-bx)^{1/2}$.

$$\therefore\ \frac{dp}{dx}=\frac{1}{2}(a-bx)^{\frac{1}{2}-1}\times(-b)=-\frac{b}{2\sqrt{a-bx}};\quad \therefore\ \frac{dx}{dp}=-\frac{2\sqrt{a-bx}}{b}.$$

$$\therefore\ \epsilon_d=\text{price elasticity of demand}=-\frac{p}{x}\cdot\frac{dx}{dp}=-\frac{\sqrt{a-bx}}{x}\times\frac{-2\sqrt{a-bx}}{b}=\frac{2(a-bx)}{bx}.$$

(ii) We have $x=\dfrac{8}{p^{3/2}}=8\cdot p^{-(3/2)}$.

$$\therefore\ \frac{dx}{dp}=8\times\left(-\frac{3}{2}\right)\cdot p^{-(5/2)}=-12p^{-(5/2)}.$$

$$\therefore\ \epsilon_d=\textit{elasticity of demand}=\frac{p}{x}\cdot\frac{dx}{dp}=\frac{p}{8p^{-(3/2)}}\times(-12)p^{-(5/2)}=-\frac{\mathbf{3}}{\mathbf{2}}.$$

6.13 Miscellaneous Illustrative Examples

Example 20. *The side of a square is* 4 cm *and is increasing at the rate of* 2 cm/sec. *At what rate is the area increasing?* [C.U. B.Com.(H) 2005]

Solution: Let the length of a side of the square at time t seconds be x cm and let y be the area of the square in sq cm. Then

$$y=x^2 \text{ and } \frac{dx}{dt}=2. \text{ Also } \frac{dy}{dt}=2x\frac{dx}{dt}=2x\times 2=4x.$$

When $x=4$ cm, $\dfrac{dy}{dt}=4\times 4=16$ cm^2/sec.

Hence *the area is increasing at the rate of* **16 cm^2/sec.**

Example 21. *Find the slope of the curve*

$$x^2-2xy-y^2-2x+4y+4=0$$

at the point $(2,-2)$ *and hence write down the equations of the tangent and the normal at the point.*

Solution: Differentiating the given equation, w.r.t. x, we get

$$2x-2x\frac{dy}{dx}-2y-2y\frac{dy}{dx}-2+4\frac{dy}{dx}=0 \text{ or, } 2(x-y-1)=\frac{dy}{dx}\cdot 2(x+y-2),$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{x-y-1}{x+y-2}=\text{slope of the curve at } (x,y).$$

$$\therefore \text{ slope of the curve at } (2,-2)=\left[\frac{dy}{dx}\right]_{(2,-2)}=\frac{2-(-2)-1}{2+(-2)-2}=-\frac{\mathbf{3}}{\mathbf{2}}.$$

$\therefore$ equation of the tangent at $(2,-2)$ is

$$y+2=\left(\frac{dy}{dx}\right)_{(2,-2)}\cdot(x-2) \text{ or, } y+2=-\frac{3}{2}(x-2) \text{ or, } \mathbf{3x+2y=2}$$

and equation of the normal at $(2,-2)$ is

$$y+2=-\frac{1}{\left[\dfrac{dy}{dx}\right]_{(2,-2)}}(x-2) \text{ or, } y+2=\frac{2}{3}(x-2) \text{ or, } \mathbf{2x-3y=10.}$$

Example 22. *Find where the tangent is parallel to the X-axis for the curve*

$$y=x^2+2x+4.$$

Solution: $\dfrac{dy}{dx}=2x+2=$ slope of the tangent to the curve at any point (x,y).

$\therefore$ the tangent is parallel to the X-axis when $\dfrac{dy}{dx}=0$, i.e., when $2x+2=0$ or, $x=-1$.

In that case

$$y=(-1)^2+2(-1)+4=3.$$

$\therefore$ the tangent is parallel to the X-axis at the point $\mathbf{(-1,3)}$.

Example 23. *Prove that the line $y=mx+c$ will touch the parabola $y^2=4a(x+a)$, if $c=am+a/m$.*

Solution: We first find the equation of tangent at any point (x,y) of the given parabola.

Here slope of the curve at (x,y) is

$$\frac{dy}{dx}=\frac{2a}{y}.$$

$\therefore$ the equation of the tangent at (x,y) is

$$Y-y=\frac{2a}{y}(X-x),$$

where (X,Y) is any point (current coordinates) of the tangent line. This gives

$$Yy-2aX=y^2-2ax=4ax+4a^2-2ax=2ax+4a^2.$$

This equation is identical with the given line $Y-mX=c$ (taking (X,Y) as current coordinates)

$\therefore$ comparing, we get

$$\frac{y}{1}=\frac{2a}{m}=\frac{2a(x+2a)}{c} \text{ or, } y=\frac{2a}{m}$$

and

$$\frac{c}{m}=x+2a \text{ or, } x=\frac{c}{m}-2a.$$

But (x,y) is any point on $y^2=4a(x+a)$; putting $y=\dfrac{2a}{m}$ and $x=\dfrac{c}{m}-2a$ we get

$$\therefore \left(\frac{2a}{m}\right)^2=4a\left\{\frac{c}{m}-2a+a\right\} \text{ or, } \frac{a}{m^2}=\frac{c-am}{m} \text{ or, } c-am=\frac{a}{m}$$

$$\text{or, } c=am+\frac{a}{m} \textbf{ (Proved).}$$

Example 24. *Give an example, where a maximum value may become less than its minimum value.*

[V.U. B.Com.(H) 2010]

Solution: There are many such examples, we give one below:

Let

$$f(x) = x + \frac{1}{x}.$$

Then

$$f'(x) = 1 - \frac{1}{x^2}.$$

which vanishes when $x = 1$ or $x = -1$. We use the *second-derivative test.*

$$f''(x) = \frac{2}{x^3}.$$

which is equal to 2, when $x = 1$ (i.e., $f''(x)$ is positive at $x = 1$). But $f''(x) = -2$ (negative), when $x = -1$.

$\therefore f(x)$ has a maximum at $x = -1$, the maximum value being -2 and $f(x)$ has a minimum at $x = 1$, the minimum value being $+2$.

Thus, **maximum value is less than the minimum value.**

Example 25. *The cost of fuel in running a locomotive is proportional to the square of the speed in* km *per* hour *and is* ₹48 *per* hour *for a speed of* 16 kilometres *per* hour. *Other cost amounts to* ₹300 *per* hour. *What is the most economical speed?*

Solution: Let V be the speed in km/hr and C be the cost of fuel per hour. Then $C = KV^2$ (K is the constant of variation).

Since, $V = 16$ km/hr, when $C =$ ₹48 per hour; we have

$$48 = K(16)^2 \Rightarrow K = \frac{3}{16}. \therefore\ C = \frac{3}{16}V^2.$$

Since other costs amount to ₹300 per hour,

total running cost per hour $= 300 + \frac{3}{16}v^2$, where v is the speed in km/hr.

If the total journey be d km with speed v, then the time required for the journey $= \frac{d}{v}$

$$\begin{aligned} \therefore\ T &= \text{total cost of the journey} \\ &= \frac{d}{v}\left(300 + \frac{3}{16}v^2\right) = d\left(\frac{300}{v} + \frac{3}{16}v\right) \\ \frac{dT}{dv} &= d\left(-\frac{300}{v^2} + \frac{3}{16}\right) = 0, \text{ if } v = 40 \text{ km/hr} \\ \text{and } \frac{d^2T}{dv^2} &= 2d \times \frac{300}{v^3} = \text{positive, for } v = 40; \end{aligned}$$

$\therefore T$ is minimum, if $v = 40$ km/hr.

So *the most economical speed is* **40 km/hr.**

Example 26. *A radio manufacturer produces x sets per week at a total cost of* ₹$\left(x^2+78x+2500\right)$. *He is a monopolist and the demand function for his product is* $x=\dfrac{600-p}{8}$, *where* ₹p *is the price per set. Show that the maximum net revenue (i.e., profit) is obtained when 29 sets are produced per week. What is the monopoly price, i.e., the price at which the profit is maximum?* [V.U. B.Com.(H) 2008]

Solution: Demand is x units. Price per unit = ₹p.

$\therefore$ total sale price $= px = x(600-8x)$ $\left[\because x=\dfrac{600-p}{8} \text{ gives } 8x=600-p\right]$

and total cost price for x sets = ₹$\left(x^2+78x+2500\right)$.

$$\therefore \text{ Profit} = x(600-8x)-\left(x^2+78x+2500\right).$$

$$P=-9x^2+522x-2500.$$

For maximum profit, $\dfrac{dP}{dx}=0$, which gives

$$-18x+522=0 \text{ or, } x=29$$

and since

$$\frac{d^2P}{dx^2}=-18 \text{ (negative).}$$

we get the maximum profit when $x=29$.

The monopoly price $=600-8x=600-8\times 29=600-232=368$, i.e., **₹368.**

Example 27. *A steel plant produces x tons of steel per week at a total cost of* ₹$\left(\dfrac{1}{3}x^3-7x^2+111x+50\right)$. *Find the output level at which* (i) *the marginal cost* (MC) *attains its minimum,* (ii) *average variable cost attains its minimum.*

Solution: We have $C(x)$ = total cost of x tons = $\dfrac{1}{3}x^3-7x^2+111x++50$.

(i) If Marginal Cost (MC) be $f(x)$, then

$$f(x)=\frac{dC}{dx}=\frac{1}{3}\times 3x^2-14x+111=x^2-14x+111.$$

$$\therefore\ f'(x)=\frac{d}{dx}\left(\frac{dC}{dx}\right)=2x-14 \text{ and } f''(x)=2.$$

For the marginal cost $f(x)$ to be minimum, we must have $f'(x)=0$ or, $2x-14=0$, or, $x=7$.

At $x=7$, $f''(x)=2$, i.e., $f''(7)=2>0$.

$\therefore$ $f(x)$ is minimum at $x=7$.

Hence, the marginal cost (MC), i.e., $f(x)$ is minimum at $x=7$, i.e., at output level of 7 tons.

(ii) Average variable cost (AVC) $=\dfrac{\text{variable cost}}{x}=\dfrac{\frac{1}{3}x^3-7x^2+111x}{x}=\dfrac{1}{3}x^2-7x+111.$

If $g(x)=\text{AVC}=\dfrac{1}{3}x^2-7x+111$, then $g'(x)=\dfrac{2}{3}x-7$ and $g''(x)=\dfrac{2}{3}$.

For AVC $g(x)$ to be minimum, we must have $g'(x)=0$, i.e., $\frac{2}{3}x-7=0$ or, $2x=21$, or, $x=\frac{21}{2}=10.5$.

At $x=10.5, g''(x)=\frac{2}{3}>0$, i.e., $g(x)$ is minimum at $x=10.5$.

Hence **the average variable cost (AVC) is minimum at output level of 10.5 tons.**

Example 28. *A wholesaler sells pencils at ₹30 per dozen on an order of 50 dozens or less. For orders in excess of 50 dozens, the price is reduced by 50 paise per dozen in excess of 50 dozens. Find the size of the order which will maximize his total revenue (TR).*

Solution: Let the size of the order be x dozens. Then if $x \leq 50$, $TR=30x$. If $x>50$, then price per dozen is $p=30-0.50(x-50)=55-0.5x$.

$\therefore$ $R(x)=TR$ = total revenue = $px=(55-0.5x)\,x=55x-0.5x^2$.

[We see that $R(x)$ will be maximum where $x>50$, because if $x \leq 50$,

$R(x)=TR=30x$ is a straight line and hence there is no maximum value.]

Now, $R'(x)=55-0.5\times 2x=55-x$ and $R''(x)=-1$.

For $R(x)$ to be maximum, we must have $R'(x)=0$ or, $55=x=0$ or, $x=55$.

At $x=55$, $R''(x)=-1<0$, i.e., $R(x)$ is maximum at $x=55$.

Hence **the total revenue *R* will be maximum when the size of the order is 55 dozen.**

Example 29. *For a certain establishment, the total cost function C and the total revenue function R are given by $C(x)=x^2-12x^2+48x+11$ and $R(x)=83x-4x^2-21$, where x = output. Obtain the output level for which the profit is maximum and find the maximum profit.*

Solution: We have

$$C(x)=x^3-12x^2+48x+11 \text{ and } R(x)=83x-4x^2-21.$$

$$\therefore\ P(x)=\text{total profit on } x \text{ units}=R(x)-C(x)$$

$$=83x-4x^2-21-\left(x^3-12x^2+48x+11\right)$$

$$=35x+8x^2-x^3-32.$$

$$\therefore P'(x)=35+16x-3x^2 \text{ and } P''(x)=16-6x.$$

For profit $P(x)$ to be maximum, we must have

$P'(x)=0$, i.e., $35+16x-3x^2=0$ or, $35+21x-5x-3x^2=0$ or, $7(5+3x)-x(5+3x)=0$

or, $(5+3x)(7-x)=0$ or, $x=7$. [$\because$ x cannot be negative]

At $x=7$, $P''(x)=16-6\times 7=-26<0$.

Hence *profit is maximum at x* = **7**, *i.e., at output level of* **7 units** and *the maximum profit is* $P(7)$ = $35\times 7+8\times 49-343-32=$ **262.**

EXERCISES ON CHAPTER 6(VI)

(Applications of Maxima and Minima)

A

[*Suggestions:* 1. From the given conditions construct a function whose maximum or minimum value is desired. 2. If the problem contains more than one independent variable, the problem itself will suggest how the dependent variable can be expressed in terms of only one independent variable (i.e., the relation, between independent variables will be given). 3. A diagram, whenever possible, will be helpful.]

1. (a) A firm produces x tonnes of a valuable metal per month at a total cost C given by: $C = ₹\left(\frac{1}{3}x^3 - 5x^2 + 75x + 10\right)$. Find at what level of output the marginal cost attains its minimum.

   (b) The cost C of manufacturing a certain article is given by the expression $C = 5 + \frac{48}{x} + 3x^2$, where x is the number of articles manufactured. Find the minimum value of C. [V.U. B.Com.(H) 2011]

   [**Hints**: $C = 5 + \frac{48}{x} + 3x^2$; $\therefore$ $\frac{dC}{dx} = -\frac{48}{x^2} + 6x$ and $\frac{d^2C}{dx^2} = \frac{96}{x^3} + 6$.
   For minimum value of C, $\frac{dC}{dx} = 0$ or, $6x^3 = 48$ or, $x^3 = 8$ or, $x = 2$.
   At $x = 2$, $\frac{d^2C}{dx^2} = \frac{96}{8} + 6 = 18 > 0$; $\therefore$ C is minimum at $x = 2$ and the minimum values of C is $5 + \frac{48}{2} + 3 \times 2^2 = 5 + 24 + 12 = 41$.]

   (c) The sum of two numbers is 12. Find the maximum value of their product.

   (d) Show that of all rectangles of given areas, the square has the least perimeter.

   [C.U. B.Com.(H) 1992]

   [**Hints**: Let x and y be the length and breadth of the rectangle whose given area is A. Then $xy = A$, where A is constant, or, $y = \frac{A}{x}$. If P be the perimeter, then $P = 2(x+y) = 2\left(x + \frac{A}{x}\right) \Rightarrow \frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$ and $\frac{d^2P}{dx^2} = \frac{4A}{x^3}$.
   For P to be minimum, $\frac{dP}{dx} = 0 \Rightarrow x^2 = A \Rightarrow x = \sqrt{A}$, since $x > 0$ and $y = \frac{A}{\sqrt{A}} = \sqrt{A}$.
   At $x = \sqrt{A}$, $\frac{d^2P}{dx^2} = \frac{4A}{(\sqrt{A})^3} > 0$ and $x = y = \sqrt{A}$. Hence P is minimum when **the rectangle is a square**.]

2. The telephone directorate finds that there is a net profit of ₹15 per instrument if an exchange has 1000 subscribers or less. But if there are over 1000 subscribers, the profits per instrument decrease by one paise for each subscriber above 1000. How many subscribers would give the maximum net profit?

   [**Hints**: Let the number of subscribers above 1000 be x. Then net profit per ins ument $= ₹\left(15 - \frac{x}{100}\right)$. Total profit $P = (1000 + x)\left(15 - \frac{x}{100}\right)$. To maximize this function the required number 1250 is obtained by writing $P' = 0$ and showing $P'' < 0$.]

3. A steel plant is capable of producing x tons per day of a low-grade steel and y tons per day of a high-grade steel, y and x are related by $y = (40 - 5x)/(10 - x)$. If the fixed market price of the low-grade steel is half that of the high-grade steel, find how many tons of low-grade steel are produced per day for maximum revenue.

   [**Hints**: Total receipts per day $= E = ax + by$, when a = price per ton of low-grade steel and b = price per ton of high-grade steel. Given: $2a = b$ and $y = (40 - 5x)/(10 - x)$. For maximum revenue $dE/dx = 0$. Now proceed.]

4. Assuming that the petrol burnt in driving a motor-boat varies as the cube of its speed, show that the most economical speed when going against a current of c km/hr is $3c/2$ km/hr.

   [**Hints**: Total amount y of petrol burnt in covering a given distance $d = kv^3/(v - c)$, where v is the velocity of the boat.]

5. The cost of fuel in running a locomotive is proportional to the square of the speed and is ₹48 per hour for a speed of 16 km/hr. Other costs amount to ₹300 per hour. What is the most economical speed?

6. It is desired to make an open box with square base out of a square piece of card-board of side 1 metre by cutting equal square of the corners and then folding up the card-board to form the sides. What must be the length of the side of the square cut out in order that the volume be a maximum?

7. In a certain office, examination and analysis of past record shows that there is a relationship between the number of clerks employed and the average cost of processing an order for new business. If x is the number of clerks employed, average cost is given by $C = \dfrac{3}{2(x-4)} + 24x$. What value of x will minimise this expression and how do you interpret this result?

8. The cost of erecting an office building is ₹50000 for the first story, ₹52500 for the second, ₹55000 for the third and so on. Other expenses are ₹35000. The net annual income is ₹5000 for each story. How many stories will give the greatest rate of interest on the investment?

9. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius r.

10. Find the altitude of a right circular cylinder of maximum volume that can be inscribed in a given right circular cone.

11. A radio manufacturer finds that he can sell x instruments/week at ₹p each, where $x = 75 - (3/5)p$. The cost of production is ₹$\left(500 + 15x + \dfrac{1}{5}x^2\right)$. Show that the maximum profit is obtained when the production is about 30 instruments per week.

   Suppose, there is Government tax of ₹t per instrument. The manufacturer adds the tax to his cost and determines the output and price under the new conditions.

   (a) Show that the price increases by a little less than half the tax;

   (b) Express the receipts from the tax in terms of t and determine the tax for maximum return;

   (c) When the tax determined in (b) is imposed, show that the price is increased by about 33%.

12. (a) A firm produces x units of output per week at a total cost of ₹$\left(\dfrac{1}{3}x^3 - x^2 + 5x + 3\right)$.

   Find the output levels at which the marginal cost and the average variable cost attain their respective minima. **[B.U. B.Com.(H) 2005; V.U. B.Com.(H) 2009]**

   (b) A radio manufacturer finds that he can sell x radios per week at p each, where $p = 2\left(100 - \dfrac{x}{4}\right)$. His cost of production of x radios per week is ₹$\left(120x + \dfrac{x^2}{2}\right)$. Show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week.

   [V.U. B.Com.(H) 2010]

13. A radio manufacturer produces x sets per week at a total cost of ₹$\left(x^2 + 70x + 1600\right)$. He is a monopolist and the demand function for his product is $x = \dfrac{500 - p}{4}$, when the price is ₹p per set. Show that the maximum net revenue (i.e., profit) is obtained when 43 sets are produced per week. What is the monopoly price? **[V.U. B.Com.(H) 2008 Type]**

[**Hints:** The monopoly price here implies the price at which profit is maximum.]

14. A manufacturer can sell x items per month at a price $p = 300 - 2x$ rupees. Manufacturer's cost of production y rupees of x items is given by $y = 2x + 1000$. Find the number of items to be produced per month to yield the maximum profit. (Profit = Total Sale Value−Total Cost).

15. The cost C of manufacturing a certain article is given by the formula $C = 5 + \frac{48}{x} + 3x^2$, where x is the number of articles manufactured. Find minimum value of C.

16. (a) The sum of two numbers is 18. Find the maximum value of their product.

 (b) Divide 24 into two parts such that their product is maximum.

17. (a) The total cost function $C(x)$ for producing x units of an item is given by $C(x) = ₹\left(200 - 18x + 2x^2\right)$. Find the average cost function and the level of output at which this function is minimum.

 [**Hints:** If $f(x)$ is the average cost function, then $f(x) = \frac{200 - 18x + 2x^2}{x}$ or, $f(x) = \frac{200}{x} - 18 + 2x$; $\therefore f'(x) = -\frac{200}{x^2} + 2$ and $f''(x) = \frac{400}{x^3}$.

 For the average cost (AC) to be minimum, we must have $f'(x) = 0$, i.e., $-\frac{200}{x^2} + 2 = 0$, or, $x^2 = 100$ or, $x = \pm 10$. But x cannot be negative.

 $\therefore x = 10$ and at $x = 10$, $f''(x) = \frac{400}{(10)^3} = \frac{2}{5} > 0$.

 Hence AC is minimum at $x = \mathbf{10}$, i.e., at output level of **10 units.**]

 (b) The total cost function C for producing x units of an article per day is given by $C = ₹\left(x^2 - 16x + 400\right)$. Find the average cost function and the level of output at which this function is minimum. [C.U. B.Com. 2001]

18. A farmer can afford to buy 800 metres of wire fencing. He wishes to enclose a rectangular field of largest possible area. What should the dimensions of the field be? [C.U. B.Com.(H) 2000]

19. The total profit ₹P from the manufacture and sale of x units of production is given by $P(x) = -\frac{x^2}{400} + 2x - 80$. How many units must be produced and sold to achieve maximum profit? What is the profit per unit when this maximum is achieved?

20. If the total cost function is $C = \frac{q^2}{3} + 2q + 300$, where q is the number of units produced, then at what level of output will the average cost per unit be a minimum?

 [**Hints:** Average cost $\bar{C} = \frac{C}{q} = \frac{q}{3} + 2 + \frac{300}{q}$. Then prove C is minimum if $q = 30$.]

21. A tour operator charges ₹150 per passenger for 100 passengers with a discount of ₹5 for each 10 passengers in excess of 100. Find the number of passengers that will maximize the amount of money the tour operator receives.

 [**Hints:** Revenue $R(x)$ for x passengers (assuming $x > 100$) is given by

$$R(x) = x\left[150 - \frac{5}{10}(x - 100)\right] = 150x - \frac{1}{2}x^2 + 50x = 200x - \frac{1}{2}x^2.$$

 See that $R(x)$ is maximum for $x = 200$.]

22. A firm produces x tonnes of output at a total cost $C = ₹\left(\frac{1}{10}x^3 - 5x^2 + 10x + 5\right)$. At what level of output will the marginal cost and the average variable cost attain their respective minima?

[V.U. B.Com.(H) 2007]

23. The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$, where C is the total cost and x is output. A tax @ ₹2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where ₹p is the price per unit of output, find the profit maximising output and price.

B

(Derivative as a Rate Measurer)

1. If the rate of change of y w.r.t. x is 5 and x is changing at 3 units/sec, how fast is y changing?

2. (a) A balloon (always takes spherical) has a variable radius. Find the rate at which its volume is increasing with the radius at the instant when radius = 10 cm.

   (b) The side of a square metal is expanding at the rate of 5 cm per minute when heated. At what rate is the area of the square sheet expanding when the side is 70 cm long? [C.U. B.Com. 2004]

   [**Hints:** If x cm be the length of the side of the square, then $\frac{dx}{dt} = 5$ and A = area of the sheet $= x^2 \Rightarrow \frac{dA}{dt} = 2x\frac{dx}{dt} =$ $2 \times 70 \times 5$ at $x = 70$ cm. Hence, the required rate = **700 sq cm per minute.**]

3. A kite 80 feet high with 100 feet of cord starts moving away horizontally @ 4 miles/hr. How fast is the cord being moved out?

4. A man is walking @ 5 km/hr towards the foot of a tower 60 metres high. At what rate is he approaching the top when he is 80 metres from the foot of the tower?

5. A point moves on a parabola $y = \frac{1}{6}x^2$ in such a way that when $x = 6$ the abscissa is increasing @ 2 cm/sec. At what rate is the ordinate increasing at that instant?

6. A circular plate of metal expands by heat in such a way that its radius increases @ 0.01 cm/sec. At what rate is the surface increasing when the radius is 2 cm?

7. Find x for which the rate of change of $x^3 - 12x^2 + 45x - 13$ is zero.

8. If $y = 4x - x^3$ and $\frac{dx}{dt} = 1/3$ cm/sec, find how fast the slope of the graph is changing at the instant when $x = 2$ cm.

9. If r = radius of a sphere, S = its surface and V = its volume, prove the relation $\frac{dV}{dt} = \frac{r}{2} \cdot \frac{dS}{dt}$.

10. (a) If the area of a circle increases at a constant rate, show that the rate of increase of its perimeter is inversely proportional to its radius. [C.U. B.Com.(H) 1996]

(b) If the area of a circle increases at a constant rate, show that the rate of increase of its perimeter is inversely proportional to its radius. [C.U. B.Com.(H) 1996]

[**Hints:** If r be the radius of a circle of area A and perimeter P, then $A=\pi r^2$ and $P=2\pi r$.

$$\text{Given } \frac{dA}{dt}=\text{constant}=C \text{ or, } 2\pi r\frac{dr}{dt}=C \text{ or, } \frac{dr}{dt}=\frac{C}{2\pi r}. \quad (1)$$

Again, $\dfrac{dP}{dt}=2\pi\dfrac{dr}{dt}=2\pi\times\dfrac{C}{2\pi r}=\dfrac{C}{r}\Rightarrow\dfrac{dP}{dt}\propto\dfrac{1}{r}$.]

11. The radius of a spherical balloon is increasing uniformly at the rate of 2 cm per second. Find at what rate the volume of the balloon is increasing when its radius is 6 cm. [C.U. B.Com.(H) 1997]

12. (a) A particle moves along a straight line according to the law: $s^2=at^2+bt+c$. Prove that the acceleration varies as $1/s^3$.

(b) If $s=at^2+bt+c$, where a,b,c are constants and s is the distance traversed in time t, prove that $4a(s-c)=v^2-b^2$.

[**Hints:** $v=\dfrac{ds}{dt}=2at+b$ or, $v^2-b^2=4a^2t^2+4abt=4a(at^2+bt)=$ etc.]

C

(AC, AVC, MC, MR, Demand and Supply)

1. A firm produces x tonnes of output at a total cost $C(x)=₹\left(\dfrac{1}{10}x^3-9x^2+85x+17\right)$. Find the Average Cost, Average Variable Cost and Average Fixed Cost in terms of x. Find the value of each of these at the level of output of 10 tonnes.

2. Find the equilibrium price and quantity for the following demand and supply functions:

$$Q_d=4-0.05p \text{ and } Q_s=0.8+0.11p.$$

3. (a) Show that the elasticity of demand at all points on the curve $xy^2=c$, where y represents price, will be numerically equal to 2.

(b) A demand curve is $xp^m=b$, where m and b are constants. Calculate the price elasticity of demand.

4. If AR and MR denote the average and marginal revenues at any output level, show that elasticity of demand is equal to AR/(AR−MR). Verify this for the linear demand laws $p=a+bx$.

5. The demand y for a commodity when its price is x, is given by $y=\dfrac{x+2}{x-1}$; find the elasticity of demand when the price is 3 units.

6. The supply of certain goods is given by $x_s=a\sqrt{p-b}$, when p is price and a and b are positive constants ($p>b$), find an expression for elasticity of supply e_s. Show that e_s decreases as price and supply increases and becomes unity at the price $=2b$.

7. Suppose, the total cost function is given by $C=a+bx+cx^2$, where x is the quantity of output produced. Show that the slope of the average cost curve is $\dfrac{1}{x}$ (MC−AC), where MC = Marginal Cost and AC = Average Cost.

8. Let p be the price of a commodity, q be the quantity so that the total revenue R is given by $R = pq$. Compute the elasticity of demand $\eta\left(= -\dfrac{p}{q}\cdot\dfrac{dq}{dp}\right)$ for the demand curve $3q = 20 - 7p$. Show that $\dfrac{dR}{dp} = q(1-\eta)$.

9. Let p be the price per unit of a certain product, when there is a sale of q units. The relation between p and q is given by $p = \dfrac{100}{3q+1} - 4$.

   (a) Find the Marginal Revenue function.

   (b) When $q = 10$, find the relative change of Revenue R, i.e., (rate of change of R w.r.t. q)/R, and also the percentage rate of change of R at $q = 10$. [CA Foun. Nov. 2000]

10. The cost of making x units of a product is $C(x) = 8x + 12\sqrt{x+4}$; find the level of output at which the marginal cost is ₹10.

[Hints: $C(x) = 8x + 12\sqrt{x+4}$ and MC = marginal cost = $\dfrac{dC}{dx}$.

$$\therefore\ M = \frac{dC}{dx} = \frac{d}{dx}\left(8x + 12\sqrt{x+4}\right) = 8 + 12\times\frac{1}{2\sqrt{x+4}}\times 1 = 8 + \frac{6}{\sqrt{x+4}}.$$

By the given condition, $8 + \dfrac{6}{\sqrt{x+4}} = 10$ or, $\dfrac{6}{\sqrt{x+4}} = 2$ or, $\sqrt{x+4} = 3$ or, $x+4 = 9$ or, $x = 5$.

Hence, the required level of output is 5 units.]

11. The price p and quantity q of a commodity are given by a relation $q = 35 + 6p - p^2$. Find the marginal revenue at $p = 2$.

[Hints: The total revenue R is given by $R = pq = p\left(35 + 6p - p^2\right)$, i.e., $R = 35p + 6p^2 - p^3$ and $\dfrac{dR}{dp} = 35 + 12p - 3p^2$.

Now $q = 35 + 6p - p^2$; $\therefore\ \dfrac{dq}{dp} = 6 - 2p$.

$\therefore$ MR = the marginal revenue = $\dfrac{dR}{dq} = \dfrac{dR}{dp}\times\dfrac{dp}{dq} = \dfrac{35 + 12p - 3p^2}{6 - 2p}$.

At $p = 2$, MR = $\dfrac{35 + 12\times 2 - 3\times 2^2}{6 - 2\times 2} = \dfrac{47}{2} = 23.5$.]

ANSWERS

A

1. (a) 5 tons.; (b) 41; (c) 36.
2. 1250.
3. About $5\frac{1}{2}$ tons.
5. 40 km/hr.
6. 1/6 m.
7. $x = 4\frac{1}{4}$; 5 clerks to be employed or some part-time workers besides 4 permanent clerks to be employed.
8. 17.
9. $4r/3$.
10. 1/3 hr.
12. (a) $x = 1$ and $x = 3/2$; (b) ₹1600.
13. ₹328.
14. 75.
15. 41.
16. (a) 81; (b) 12.12.
17. (a) x = 10 units; (b) x = 20 units.

18. Length = 200 m and breadth = 200 m.
19. (a) 400 units; (b) 80 paise per unit.
20. 30 units.
21. 200.
22. $\frac{50}{3}$ units and 25 units.
23. 50 units; ₹2280.

B

1. 15 units/sec.
2. (a) 1256.14 cc/unit increase in radius;
   (b) 700 sq cm per man.
3. 2.4 m.p.h.
4. 4 km/hr.
5. 4 cm/sec.
6. 0.04π sq cm/sec.
7. 3 and 5.
8. Decrease 4 cm/sec.
11. 288π cc/sec.

C

1. ₹$\left(\frac{1}{10}x^2 - 9x + 85 + \frac{17}{x}\right)$, ₹$\left(\frac{1}{10}x^2 - 9x + 85\right)$,
   ₹$17/x$; ₹6.70, ₹5, ₹1.70.
2. 20; 3.
3. (b) m.
5. 0.90.
6. $\frac{p}{2(p-b)}$.
8. $\eta = \frac{7p}{3q}$.
9. $\frac{100}{(3q+1)^2} - 4$; $\frac{100 - 4(3q+1)^2}{(3q+1)(96q - 12q^2)}$; 50.3%.
10. 5 units.
11. 23.5 units.

Chapter 7

Rolle's Theorem: Taylor's and Maclaurin's Theorems

7.1 Introduction

We shall, in the present chapter, consider some important theorems, which involve the derivative in a range of values of x. The most fundamental of such theorems is known as *Rolle's Theorem*. From Rolle's Theorem we can deduce a very useful theorem called *Mean Value Theorem* (MVT). More generalised versions of MVT are *Taylor's Theorem* and *Maclaurin's Theorem*. We shall conclude this chapter with the *expansions of functions by Taylor's* and *Maclaurin's infinite series*. The rigorous proofs of these theorems require knowledge of advanced calculus. We, therefore, shall make general discussions of these theorems and concentrate on their applications.

7.2 Rolle's Theorem

Statement. *Suppose that $f(x)$ is continuous in $a \le x \le b$ and has a derivative in $a < x < b$. Suppose also that $f(a) = f(b)$. Then there exists at least one value c, satisfying $a < c < b$ such that $f'(c) = 0$.*

Geometrically we can interpret this theorem in the following manner:

If the graph of the function $y = f(x)$ has a tangent at each point x within a range of values a to b and if there is no break of the curve within this range and if the curve has the same level at the two points a and b, then according to Rolle's Theorem there exists at least one point (if not more) where the tangent is parallel to the x-axis (i.e., the tangent is horizontal). In Fig. 7.1 see that the levels at A and B (at $x = a$ and $x = b$ respectively) are same (i.e., $AL = BM$) and there is no break of the curve between A and B, and at every point of the curve within this range a tangent can be drawn; then we can easily see that there are points (C, D, E in the figure) where the tangents are horizontal.

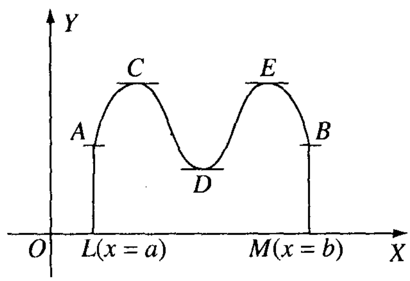


Fig. 7.1

In particular, if $f(a) = f(b) = 0$, we may state Rolle's Theorem in a more simpler language:

If $f(x)$ is continuous in $a \le x \le b$ and if $f'(x)$ exists for each x in $a < x < b$ and if $f(x)$ vanishes at the two ends a and b, then $f'(x) = 0$ for at least one value of x within (a, b), i.e., between two roots of $f(x) = 0$ there exists at least one root of $f'(x) = 0$, provided $f(x)$ is continuous and derivable within those two roots.

Example 1. *Let $f(x) = (x-a)^3(x-b)^2$. Show that there exists a point c in $a < c < b$ which divides the segment (a, b) in the ratio* $3:2$.

Solution: $f(x)=(x-a)^3(x-b)^2$ is a polynomial[1] in x and we know that a polynomial is always continuous and derivable. So we can assume that $f(x)$ is continuous and derivable in the range (a,b).

Moreover, $f(x)=0$ for $x=a$ and $x=b$, i.e., a and b are the two roots of $f(x)=0$.

$\therefore$ by Rolle's Theorem, there exists a point c, satisfying $a<c<b$ such that $f'(c)=0$.

Now

$$\begin{aligned} f'(x) &= 3(x-a)^2(x-b)^2+2(x-a)^3(x-b) \\ &= (x-a)^2(x-b)(3x-3b+2x-2a) \\ &= (x-a)^2(x-b)(5x-3b-2a) \\ \text{and } f'(c) &= (c-a)^2(c-b)(5c-3b-2a). \end{aligned}$$

Now
$$f'(c)=0 \Rightarrow (5c-3b-2a)=0 \text{ or, } c=\frac{\mathbf{3b+2a}}{\mathbf{3+2}},$$

i.e., c divides the segment (a,b) in the ratio 3 : 2.

Example 2. (i) *Verify Rolle's Theorem by finding the values of x for which $f(x)$ and $f'(x)$ vanish if $f(x)=x^3-3x$.* (ii) *Verify Rolle's Theorem for $f(x)=x^3-6x^2+11x-6$ in $[1,3]$.* [C.U. B.Com.(H) 2000]

Solution: (i) $f(x)=x^3-3x=x\left(x^2-3\right)=x\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$
$f'(x)=3x^2-3=3\left(x^2-1\right)=3(x+1)(x-1)$.

We know that every polynomial is always continuous and derivable. Hence, the polynomial $f(x)=x^3-3x$ is everywhere continuous and derivable.

$f(x)=0$ if $x=0$, $x=-\sqrt{3}$, $x=\sqrt{3}$. [Three roots of $f(x)=0$].

The conditions of Rolle's Theorem are satisfied in $\left[-\sqrt{3},0\right]$, $\left[0,\sqrt{3}\right]$, $\left[-\sqrt{3},\sqrt{3}\right]$.

Now we see that $f'(x)=0$ if $\boldsymbol{x=-1, x=1}$.

But -1 lies between $-\sqrt{3}$ and 0, and that 1 lies between 0 and $\sqrt{3}$.

Clearly, $-1, 1$ lie between $-\sqrt{3}$ and $\sqrt{3}$.

$\therefore$ between any two roots of $f(x)=0$ there exists *at least one root* of $f'(x)=0$.

Rolle's Theorem is thus verified.

(ii) $f(x)=x^3-6x^2+11x-6$ is **continuous and derivable in $\boldsymbol{1\le x\le 3}$.** In fact, $f(x)$ being a polynomial is everywhere continuous and derivable.

We find that $f(1)=0$ and $f(3)=27-54+33-6=0$, i.e., $x=1$ and $x=3$ are two roots of $f(x)=0$.

We see that all the conditions of Rolle's Theorem are satisfied in the range $[1,3]$.

$f'(x)=3x^2-12x+11=0$ if $x=\dfrac{12\pm\sqrt{144-4\cdot3\cdot11}}{6}$, i.e., if $x=2\pm\dfrac{1}{\sqrt{3}}=2\pm0.58=2.58$ or 1.42.

Both these values lie between 1 and 3, i.e., there exists two points $x=1.42$ and $x=2.58$ in between 1 and 3 where $f'(x)$ vanishes. Hence, **Rolle's Theorem is thus verified.**

Example 3. *Verify Rolle's Theorem for the function $f(x)=x^4+x^2-2$ in $-1\le x\le 1$.* [C.U. B.Com.(H) 1996]

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[1]A function $f(x)$ of the form $f(x)=a+bx+cx^2+\cdots+kx^m$ $(a,b,c,\ldots,k$ are numerical constants), where m is a **positive integer**, is called a *polynomial* in x of degree m. If some value, say $x=\alpha$, satisfies $f(\alpha)=0$, then α is called a root of $f(x)=0$ or α is a **zero** of the polynomial $f(x)$.

Solution: We have $f(x)=x^4+x^2-2$ in $-1\leq x\leq 1$.

We see that (i) $f(x)$ is continuous in $-1\leq x\leq 1$, since polynomial function is always continuous,

(ii) $f'(x)=4x^3+2x$ exists in $-1<x<1$, and

(iii) $f(-1)=(-1)^4+(-1)^2-2=1+1-2=0$ and $f(1)=1+1-2=0$ so that $f(-1)=f(1)$.

Thus, *all the conditions of Rolle's Theorem are satisfied.* To verify Rolle's Theorem, we have to find a c in $-1<x<1$ such that $f'(c)=0$.

Now $f'(c)=0$ gives $4c^3+2c=0$ or, $2c(2c^2+1)=0$; $\therefore$ $c=0$. Also $-1<0<1$, i.e., $c=0$ lies in $-1<x<1$.

Hence *Rolle's theorem is verified.*

Points to remember about Rolle's Theorem

1. The theorem asserts the existence of *at least one value* of $x=c$ which is *strictly* between two points a and b where $f'(x)=0$, i.e., where the tangent is parallel to the x-axis. We may get more than one such point [See Fig. 7.1, or Ex. 2(ii) above].

2. In order to get some value $x=c$ which satisfies the theorem the hypothesis cannot be relaxed:

(i) If $f(x)$ is continuous in $a\leq x\leq b$ but $f'(x)$ does not exist for some x in $a<x<b$, then the theorem may not be true [See Fig. 7.2]. Here $f(x)$ is continuous throughout the interval $[a,b]$ and $f(a)=f(b)=0$ but $f'(x)$ does not exist for $x=c$ [Here derivative is not finite]. We see that at no point of the graph the tangent is parallel to the X-axis, i.e., the theorem is not true.

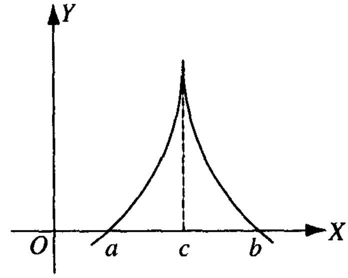

Fig. 7.2

(ii) If $f(x)$ is not continuous at some point between a and b or even if $f(x)$ is not continuous just at a or at b, other conditions (derivability and equality at a and b) are satisfied, still the theorem may not be true [See Figs 7.3 (a) and (b)]. See that in both cases there exists no point where the tangent is parallel to the X-axis, i.e., the theorem is not true.

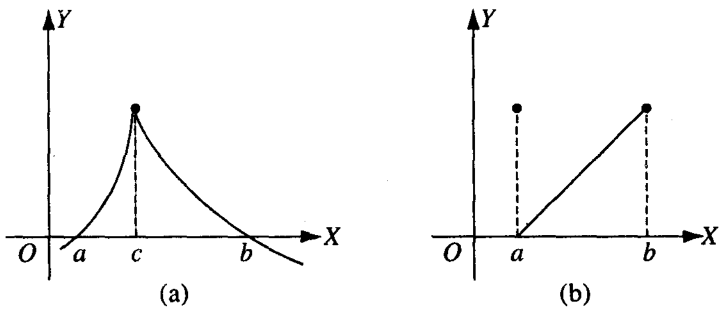

Fig. 7.3

3. In order just to get $f'(c)=0$, i.e., for a tangent at a point to become parallel to the X-axis, we need not require *any of the conditions of Rolle's Theorem* [See Fig. 7.4].

In this case $f(a)\neq f(b)$, $f(x)$ is not continuous or derivable at c, still there are many points in the range (a,b) where the tangents are parallel to the X-axis.

What we mean to say is that — *If $f(x)$ satisfies all the conditions of Rolle's Theorem in $[a,b]$, then there must exist at least one point c, satisfying $a<c<b$ such that $f'(c)=0$, if any of the conditions are violated, then Rolle's Theorem will not necessarily be true; but it may still be true but the truth is not guaranteed.*

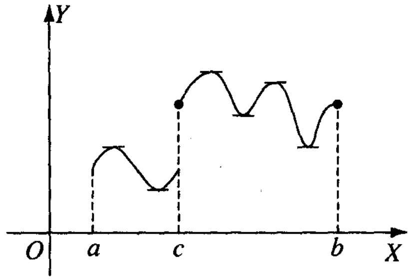

Fig. 7.4

In other words, the conditions of Rolle's Theorem *are sufficient but not necessary* for getting a point where derivative vanishes.

The following two examples illustrate these facts:

Example 4. *Is Rolle's Theorem applicable to the function $f(x)=|x|$ in $[-1,1]$, i.e., $f(x)=x$, when $x \geq 0$ and $f(x)=-x$, when $x<0$?* [C.U. B.Com.(H) 1998]

Solution: Here (i) $f(x)$ is continuous at $x=0$

$$\left[\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x=0;\ \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-x)=0 \text{ and } f(0)=0\right];$$

in fact, $f(x)$ is continuous in $-1 \leq x \leq 1$.

(ii) $f'(x)=\begin{cases} 1, & \text{if } 0<x\leq 1 \\ -1, & \text{if } -1\leq x<0 \end{cases}$ and $f'(0)$ does not exist.

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h-0}{h}=1$$
$$\text{and } \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h-0}{h}=-1,$$

i.e., right-hand derivative $\neq$ left-hand derivative and as such $f'(0)$ does not exist.

(iii) $f(1)=f(-1)=1$.

Here $f'(x)$ nowhere vanishes and as such *Rolle's Theorem is not true here.* The reason is that $|x|$ is not derivable in $-1<x<1$, other conditions are satisfied.

Example 5. $f(x)=\dfrac{1}{x}+\dfrac{1}{1-x}$ *in* $[0,1]$. *Examine whether all the conditions of Rolle's Theorem hold.*

Solution: Here (i) $f(x)$ is continuous in $(0,1)$ (but not in $0\leq x\leq 1$; at $x=0, x=1$ the function is not even defined — no question of continuity there).

(ii) $f'(x)=-\dfrac{1}{x^2}+\dfrac{1}{(1-x)^2}$ exists in $0<x<1$.

(iii) $f(0)\neq f(1)$ (they are not defined).

Thus, *all the conditions of Rolle's Theorem do not hold.* Still we find $f'(x)=0$, if $x=\dfrac{1}{2}$, where $0<\dfrac{1}{2}<1$.

7.3 Mean Value Theorem

MVT states: *If a function $f(x)$ is continuous in $a \le x \le b$ and if $f(x)$ possesses a determinate derivative at every point in $a < x < b$, then there exists a point c, satisfying $a < c < b$ such that* $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

Geometrical Interpretation: In Fig. 7.5 we draw the graph of $y = f(x)$ from $x = OL = a$ to $x = OM = b$.

$AL = f(a)$, $BM = f(b)$.

$\therefore$ $BN = BM - NM = BM - AL = f(b) - f(a)$ and $LM = OM - OL = b - a$.

$\therefore$ $\tan\angle BAN = \dfrac{BN}{AN} = \dfrac{BN}{LM} = \dfrac{f(b)-f(a)}{b-a}$ = slope of the chord AB; slope of the tangent at C (when $OK = c$) $= f'(c)$.

The theorem asserts that (under the restrictions — continuity or no break of the curve from A to B and existence of tangent to the curve for every point *between* A and B) *there is some point C between A and B where the tangent to the curve is parallel to the chord AB.*

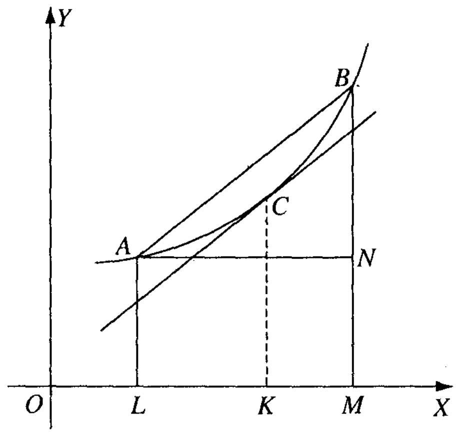

Fig. 7.5

7.4 Taylor's Theorem and Maclaurin's Theorem

Our main objective is to expand a function $f(x)$ in a series of the form (i.e., in powers of x): $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are constants. Such a series is called a *Power Series* in x.

Such expansions may be possible: (i) For all values of x, *e.g., Exponential Series:* $f(x) = e^x$.

(1) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$.

This expansion is valid for all values of x.

(2) $a^x = e^{x\log a} = 1 + \dfrac{x}{1!}\log a + \dfrac{x^2}{2!}(\log a)^2 + \cdots + \dfrac{x^n}{n!}(\log a)^n + \cdots (a > 0)$.

This series is valid for all values of x and $a > 0$.

(ii) For certain values of x, *e.g., Logarithmic Series and Binomial Series*

(1) $\log_e(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^{n-1}\dfrac{x^n}{n} + \cdots$

This expansion is valid for all values of x lying between -1 and $+1$, and also valid for $x = 1$, i.e., the range of validity is $-1 < x \le 1$.

Binomial Series

(2) $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + \cdots + \dfrac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + \cdots$ where m is any real number.

This expansion is valid for values of x given by $-1 < x < 1$.

Maclaurin's Series: Let $f(x)$ be a function of x which admits of expansion in a convergent power series for all values of x within a certain range, say $(-\alpha, \alpha)$. Thus, we may write

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots.$$

Maclaurin's Series gives the values of the coefficients $a_0, a_1, a_2, \ldots$ in terms of derivatives at $x=0$. In fact, $a_n = \frac{1}{n!} f^n(0)$, where $f^n(0)$ is the nth successive derivative of $f(x)$ at $x=0$.

Maclaurin's Series or Maclaurin's formula can thus be written as

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots \tag{1}$$

Example 6. *Expand e^x in ascending powers of x by Maclaurin's infinite series.* [C.U. B.Com.(H) 1994]

Solution: $f(x) = e^x, \quad f(0) = 1$
$f'(x) = e^x, \quad f'(0) = 1$
$f''(x) = e^x, \quad f''(0) = 1$
etc. etc.
$f^n(x) = e^x, \quad f^n(0) = 1$

By Maclaurin's infinite series (1),

$$\boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.}$$

Without going into the rigorous investigation, we assume that *this expansion is valid for all values of x.*

Example 7. *Expand e^{-x} in ascending powers of x by using the Maclaurin's infinite series.*
[C.U. B.Com.(H) 1999; V.U. B.Com.(H) 2007]

Solution: Let $f(x) = e^{-x}$; then

$$f'(x) = e^{-x} \cdot \frac{d(-x)}{dx} = (-1)e^{-x}$$
$$f''(x) = (-1) \cdot e^{-x} \cdot (-1) = (-1)^2 e^{-x}$$
$$f'''(x) = (-1)^2 \cdot e^{-x} \cdot (-1) = (-1)^3 e^{-x}, \text{ and so } f^n(x) = (-1)^n e^{-x}.$$

Putting $x=0$, we get $f(0) = e^{-0} = 1$, $f'(0) = -1 \cdot e^{-0} = -1$, $f''(0) = 1$, $f'''(0) = -1$ and so on, $f^n(0) = (-1)^n$.

By Maclaurin's infinite series, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

$$\text{or, } e^{-x} = 1 + \frac{x}{1!} \times -1 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times -1 + \cdots + \frac{x^n}{n!} \times (-1)^n + \cdots$$

$$\boldsymbol{= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \cdot \frac{x^n}{n!} + \cdots \infty}$$

This series is valid for all values of x.

Example 8. *Expand* $\log_e(1+x)$ *in ascending powers of x by Maclaurin's infinite series. Write down the condition of validity.* [C.U. B.Com.(H) 1997; V.U. B.Com.(H) 2009]

Solution: $f(x)=\log(1+x)$, $\quad f(0)=\log 1=0$

$$f'(x)=\frac{1}{1+x}, \qquad f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2}, \qquad f''(0)=-1$$

$$f'''(x)=\frac{2!}{(1+x)^3}, \qquad f'''(0)=2!$$

....

$$f^n(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}, \quad f^n(0)=(-1)^{n-1}(n-1)!$$

Now from Maclaurin's infinite series, we get

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$$

$$\text{or, } \log_e(1+x)=0+\frac{x}{1!}+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}\cdot 2!-\cdots+\cdots+\frac{x^n}{n!}(-1)^{n-1}(n-1)!+\cdots$$

$$=\boldsymbol{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots.}$$

Without going into rigorous investigations we assume that *this expansion is valid for all values of x in* $-1<x\le 1$.

This is the range of validity of logarithmic series.

Example 9. *Expand $(1+x)^n$ (where n is not necessarily a positive integer) in ascending powers of x by Maclaurin's infinite series. State the range of validity.* [C.U. B.Com.(H) 1995]

Solution:

Case I. *Suppose n is a positive integer.*

We shall prove that

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\frac{n(n-1)(n-2)\cdots(n-n+1)}{n!}x^n.$$

This expansion is valid for all values of x.

Let $f(x)=(1+x)^n$, $\quad f(0)=1$

$$f'(x)=n(1+x)^{n-1}, \qquad f'(0)=n$$

$$f''(x)=n(n-1(1+x)^{n-2}, \qquad f''(0)=n(n-1)$$

$$f'''(x)=n(n-1)(n-2)(1+x)^{n-3}, \qquad f'''(0)=n(n-1)(n-2)$$

....

General Case: $f^r(x)=n(n-1)(n-2)\cdots(n-r+1)(1+x)^{n-r}$;

$$f^r(0)=n(n-1)(n-2)\cdots(n-r+1)$$

....

If n *is a positive integer*, then, we get

$$f^n(x) = n(n-1)\cdots(n-n+1)(1+x)^{n-n} = n!\ (\text{constant});\ f^n(0) = n!.$$

$\therefore$ all higher order derivatives $f^{n+1}(x), f^{n+2}(x)$, etc., vanish.
So, when n is a positive integer, we get the expansion up to $(n+1)$th terms; thus

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}x^n \\ &= \mathbf{1} + \boldsymbol{nx} + \frac{\boldsymbol{n(n-1)}}{\mathbf{2!}}\boldsymbol{x^2} + \cdots + \boldsymbol{x^n} \text{ (valid for all values of } x). \end{aligned}$$

Case II. *n is any real number other than a positive integer.*

In this case, for no value of r, $f^r(0) = 0$. So we shall obtain an infinite series:

$$\begin{aligned} (1+x)^n &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^r}{r!}f^r(0) + \cdots \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots \end{aligned}$$

This expansion is valid for $\mathbf{-1 < x < 1}$.

Example 10. *Assuming that within a certain interval of x, $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. What should be the form of a_n in terms of successive derivatives of $f(x)$ at $x = 0$?*

[In this problem we justify Maclaurin's formula given above.]

Solution: We assume that a power series can be differentiated term by term, any number of times, within its interval convergence.

Now

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots + a_{n+1}x^{n+1} + \cdots \\ f'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + (n+1)a_{n+1}x^n + \cdots \\ f''(x) &= 2a_2 + 3\cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots \\ f'''(x) &= 3.2.a_3 + 4.3.2a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3} + \cdots \\ &\quad \text{etc.} \end{aligned}$$

Putting $x = 0$, we get $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2$ or, $a_2 = \frac{1}{2!}f''(0)$.

$$f'''(0) = 3a_3 \text{ or, } a_3 = \frac{1}{3!}f'''(0), \ldots,\ f^n(0) = n!a_n \text{ or, } \boldsymbol{a_n = \frac{1}{n!}f^n(0)}.$$

Hence, we have the Maclaurin's formula in the form

$$\boldsymbol{f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots}.$$

Taylor's theorem for a finite number of terms [C.U. B.Com.(H) 2001]

If $f(x)$ possesses derivatives of the first $(n-1)$ order for all x in $[a, a+h]$ and nth derivative exists in $(a, a+h)$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(c),$$

where $a<c<a+h$. [*We can write, $c=a+\theta h$, where $0<\theta<1$.*]

In practice, if $f(x)$ possesses derivatives of the first n orders for all x, we can apply Taylor's Theorem.

Taylor's Series: We now write down an expansion of a function $f(x)$ in powers of $x-a$, where a is a fixed number.

We assume that conditions exist for which such an expansion is possible. Then

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)+\cdots. \tag{1}$$

This is called *Taylor's Series* (or *Taylor's Formula*).

Example 11. *Expand* $\log x$ *in powers of* $(x-1)$ *by Taylor's formula.*

Solution: Let

$$f(x)=\log x,\ f'(x)=\frac{1}{x},\ f''(x)=-\frac{1}{x^2},\ f'''(x)=\frac{(-1)(-2)}{x^3}=\frac{2!}{x^3},\ \ldots,\ f^n(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},$$

Hence

$$f(1)=\log 1=0,\ f'(1)=1,\ f''(1)=-1,\ f'''(1)=2!,\ \ldots,\ f^n(1)=(-1)^{n-1}(n-1)!,$$

From Taylor's formula (1) we then get

$$\log x=f(1)+\frac{(x-1)}{1!}f'(1)+\frac{(x-1)^2}{2!}f''(1)+\cdots+\frac{(x-1)^n}{n!}f^n(1)+\cdots$$

$$\text{i.e., } \mathbf{\log x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\cdots+(-1)^{n-1}\frac{(x-1)^n}{n}+\cdots.}$$

Another Form of Taylor's Series: In (1) write $a+h$ in place of x and hence for $x-a$ write h. We have the Taylor's series in the form:

$$f(a+h)=f(a)+\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)+\cdots \tag{2}$$

or putting x for a, this form reduces to

$$f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\cdots+\frac{h^n}{n!}f^n(x)+\cdots. \tag{3}$$

Interchanging x and h, we get

$$f(x+h)=f(h)+\frac{x}{1!}f'(h)+\frac{x^2}{2!}f''(h)+\cdots+\frac{x^n}{n!}f^n(h)+\cdots.$$

Example 12. *Expand e^{x+h} in ascending powers of x by Taylor's infinite series.*

[V.U. B.Com.(H) 2010]

Solution: Let $f(x)=e^x$; then $f(x+h)=e^{x+h}$, $f'(x)=e^x$, $f''(x)=e^x$, $f'''(x)=e^x, \ldots, f^n(x)=e^x$.

$$\therefore\ f(h)=e^h,\ f'(h)=e^h,\ f''(h)=e^h,\ f'''(h)=e^h,\ \ldots,\ f^n(h)=e^h.$$

By Taylor's infinite series, we get

$$f(x+h)=f(h)+\frac{x}{1!}f'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+\cdots+\frac{x^n}{n!}f^n(h)+\cdots$$

$$\text{or, } e^{x+h}=e^h+\frac{x}{1!}\cdot e^h+\frac{x^2}{2!}\cdot e^h+\frac{x^3}{3!}\cdot e^h+\cdots+\frac{x^n}{n!}\cdot e^h+\cdots$$

$$=\boldsymbol{e^h\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\right]}.$$

Example 13. *Expand $f(x)=x^4-3x^3$ in powers of $(x-2)$ by Taylor's series of the form* (1).

Solution: $f(x)=f[2+(x-2)]=f(a+h)$, where $a=2, h=x-2$.

$$\therefore\ x^4-3x^3=f(2)+\frac{(x-2)}{1!}f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+\frac{(x-2)^4}{4!}f^{IV}(2),$$

other terms will not come as $f^V(2)$, $f^{VI}(2)$, etc., all have values equal to zero.

$$\begin{aligned} f(x)&=x^4-3x^3, & f(2)&=-8\\ f'(x)&=4x^3-9x^2, & f'(2)&=-4\\ f''(x)&=12x^2-18x, & f''(2)&=12\\ f'''(x)&=24x-18, & f'''(2)&=30\\ f^{IV}(x)&=24, & f^{IV}(2)&=24 \end{aligned}$$

higher derivatives like $f^V(x)=f^{VI}(x)=\cdots=0$.

Hence, finally,

$$x^4-3x^3=-8+\frac{x-2}{1!}(-4)+\frac{(x-2)^2}{2!}\times 12+\frac{(x-2)^3}{3!}\times 30+\frac{(x-2)^4}{4!}\times 24$$

$$\text{or, } \boldsymbol{x^4-3x^3=-8-4(x-2)+6(x-2)^2+5(x-2)^3+(x-2)^4}.$$

EXERCISES ON CHAPTER 7

1. (a) Verify Rolle's Theorem by finding values of x for which $f(x)$ and $f'(x)$ vanish in each of the following cases:

   (i) $f(x)=x^3-3x$; (ii) $f(x)=6x^2-x^3$;

   (iii) $f(x)=(x-a)^m(x-b)^n$; (iv) $f(x)=\dfrac{(x-a)(x-b)}{x^2}$.

(b) Verify Rolle's Theorem in the following cases:

(i) $f(x)=x^2$ in $-1 \le x \le 1$; (ii) $f(x)=x^3-6x^2+11x-6$ in $1 \le x \le 3$;

(iii) $f(x)=x(x+3)e^{\frac{1}{2}x}$ in $[-3,0]$; (iv) $f(x)=x^4+x^2-2$ in $-1 \le x \le 1$.

[C.U. B.Com.(H) 1996]

2. In each of the following cases, find, if any, the value of c such that $f(b)=f(a)+(b-a)\,f'(c)$ (MVT):

(a) $f(x)=x^2$, $a=1$, $b=2$; (b) $f(x)=\sqrt{x}$, $a=1$, $b=4$;

(c) $f(x)=\dfrac{2}{x}$, $a=1$, $b=2$; (d) $f(x)=\dfrac{1}{x}$, $a=-1$, $b=1$;

(e) $f(x)=x^2+3x+2$, $a=1$, $b=2$.

3. Use MVT to prove the following: If $f'(x)=0$ for all values of x, within a certain range, then $f(x)=$ constant in that range.

Hence deduce that if $f(x)$ and $\phi(x)$ have equal derivatives for all values of x, within a certain range, then they can differ only by a constant.

4. Use MVT to prove $\dfrac{x}{1+x} < \log(1+x) < x$, for all $x>0$.

[Hints: Let $f(x)=\log(1+x)$. Use MVT in $[0,x]$. Thus, $f(x)=f(0)+xf'(\theta x), 0<\theta<1$ or, $\log(1+x)=\dfrac{x}{1+\theta x}$.

Since $1+\theta x>1$, $\dfrac{x}{1+\theta x}<x$. Since $1+\theta x<1+x$, $\dfrac{x}{1+\theta x}>\dfrac{x}{1+x}$.

Hence, $\log(1+x)<x$ and $\log(1+x)>\dfrac{x}{1+x}$. Hence, the result.]

5. Verify that on the curve $y=x^2+2ax+b$, the chord joining the points at $x=\alpha$ and $x=\beta$ is parallel to the tangent line drawn at the point $x=\dfrac{1}{2}(\alpha+\beta)$.

6. Use MVT to prove $0<\dfrac{1}{\log(1+x)}-\dfrac{1}{x}<1$.

[Hints: Let $f(x)=\log(1+x)$. Use MVT in the form $f(x)=f(0)+xf'(\theta x)$ or, $\log(1+x)=\dfrac{x}{1+\theta x}$ or, $\dfrac{1}{\log(1+x)}=\dfrac{1}{x}+\theta$,

i.e., $\theta=\dfrac{1}{\log(1+x)}-\dfrac{1}{x}$. Hence, $0<\theta<1$ gives $0<\dfrac{1}{\log(1+x)}-\dfrac{1}{x}<1$.]]

7. Use Maclaurin's Series to establish

(a) $f(x)=\log_e(a+x)=\log_e a+\dfrac{x}{a}-\dfrac{1}{2}\dfrac{x^2}{a^2}+\dfrac{1}{3}\dfrac{x^3}{a^3}+\cdots+\dfrac{(-1)^n x^{n-1}}{(n-1)a^{n-1}}+\cdots$

(Valid for $-a<x\le a$)

(b) $\log_e(x+\sqrt{1+x^2})=x-\dfrac{x^2}{3!}+\dfrac{9x^5}{5!}-\cdots$

8. Use Taylor's Series to prove

(a) (i) $e^x=e^a\left[1+(x-a)+\dfrac{(x-a)^2}{2!}+\dfrac{(x-a)^3}{3!}+\cdots\right]$,

(ii) $2^x = 2\left[1+\frac{(x-1)}{1!}\log 2+\frac{(x-1)^2}{2!}(\log 2)^3+\cdots\right]$;

(b) $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^3+\cdots$;

(c) $2x^3+7x^2+x-1 = 45+53(x-2)+19(x-2)^2+2(x-2)^3$.

9. Expand $\log x$ in powers of $(x-2)$ up to four terms.

10. (a) Expand e^x in powers of $(x-2)$ up to five terms.

(b) Expand e^{x+h} by Taylor's Infinite Series.

[**Hints:** See worked-out Ex. 12 in **Section 7.4.**]

11. (a) Show that (a) $\log x = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots (0 < x \le 2)$;

(b) $\frac{1}{x} = \frac{1}{2}-\frac{1}{2^2}(x-2)+\frac{1}{2^3}(x-2)^2-\cdots (0 < x < 4)$.

12. Using Maclaurin's Series expand the following functions: (a) e^x; (b) e^{-x}; (c) a^x; (d) $\log_e(1+x)$.

13. Evaluate: $\lim_{x\to 0}\frac{e^{3x}-e^{2x}+2x}{x}$

[**Hints:** Use $e^{3x} = 1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}+\cdots$; $e^{2x} = 1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+\cdots$

$$\text{Given limit} = \lim_{x\to 0}\frac{\left(1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3+\cdots\right)-\left(1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+\cdots\right)+2x}{x}$$

$$= \lim_{x\to 0}\frac{3x+\frac{5}{2}x^2+\frac{19}{6}x^3+\cdots}{x}$$ (We can cancel x since $x\to 0$ does not imply that x should be zero.)

$$= \lim_{x\to 0}\left(3+\frac{5}{2}x+\frac{19}{6}x^2+\text{ terms containing higher power of } x\right) = 3.]$$

See also worked-out Exs 6 and 7 in **Section 7.4.**]

ANSWERS

1. (a) (i) $c = \pm 1$;
   (ii) $c = 4$;
   (iii) $c = \frac{mb+na}{m+n}$;
   (iv) $c = \frac{2ab}{a+b}$;
   (b) (i) $c = 0$;
   (ii) $c = 2\pm\frac{1}{\sqrt{3}}$;
   (iii) $c = -2$.
2. (a) $c = 1.5$;
   (b) $c = 2.25$;
   (c) $c = \sqrt{2}$;
   (d) No c exists;
   (e) 1.5.

9. $\log x = \log 2 + (x-2) - \frac{1}{2^2}(x-2)^2 + \frac{1}{4}(x-2)^3 - \frac{3}{8}(x-2)^4+\cdots$.

10. $e^x = e^2\{1+(x-2)+\frac{1}{2!}(x-2)^2+\frac{1}{3!}(x-2)^3+\cdots\}$.

12. (a) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$;
   (b) $1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+(-1)^{n-1}\frac{x^{n-1}}{(n-1)!}+\cdots$;
   (c) $1+\frac{x}{1!}(\log_e a)+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)^3+\cdots+\frac{x^n}{n!}(\log_e a)^n+\cdots$;
   (d) $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots+\frac{(-1)^{n-1}}{n}\cdot x^n+\cdots$ $(-1 < x \le 1)$.

Chapter 8

Integration: Indefinite Integrals, Standard Forms and Integration by Substitution

8.1 Introduction

The word integration actually means summation.

We sum up infinitely small quantities and consider the limit of this sum. Such a limit gives the integral.

In the present chapter, however, we shall consider integration as the inverse of differentiation. With this definition in mind, we shall give different methods of evaluation of integrals (Methods 1–4 of Section 8.3).

In the next chapter, the two concepts — integration as the limit of a sum and integration as anti-derivative — will be connected by a theorem called *Fundamental Theorem of Integral Calculus.* We have not included, at any stage, problems involving trigonometric functions.

8.2 Integration as Anti-derivative Process

Let the two functions $f(x)$ and $\phi(x)$ be so related that

$$\frac{d}{dx}\{f(x)\} = \phi(x), \text{ or, differential of } f(x) = \frac{d}{dx}\{f(x)\}dx = \phi(x)\,dx,$$

i.e.,

$$d\{f(x)\} = \phi(x)\,dx. \tag{1}$$

We then say that the integral of $\phi(x)$ w.r.t. x is $f(x)$ and write,

$$\int \phi(x)\,dx = f(x).$$

In view of the fact when eq. (1) holds, it is also true that

$$d\{f(x)+c\} = \phi(x)\,dx$$

and hence we shall write,

$$\int \phi(x)\,dx = f(x)+c,$$

where c is an arbitrary constant (known as *constant of integration*); $\phi(x)$ is called the **integrand** and $f(x)$ is the **integral** [in fact, $\boldsymbol{f(x)+c}$ **is the integral of** $\boldsymbol{\phi(x)}$].

Illustration 1. *We know,* $\frac{d}{dx}\left(x^2\right) = 2x$ *or,* $d\left(x^2\right) = 2x\,dx$ *or,* $d\left(x^2+c\right) = 2x\,dx$.

$\therefore \int 2x\,dx = x^2 + c$, *where c is the constant of integration.*

Standard Formula I

(a) $\displaystyle\int \boldsymbol{x^n\,dx = \frac{x^{n+1}}{n+1} + c}$ (except when $n = -1$).

(b) $\displaystyle\int \boldsymbol{x^{-1}\,dx = \log|x| + c}$ $(x > 0)$. [B.U. B.Com.(H) 2005]

(c) $\displaystyle\int \boldsymbol{dx = x + c.}$

Justification for (a):

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)\,x^n,\ \text{or, } d\left(x^{n+1}\right) = (n+1)\,x^n\,dx,\ \text{or, } d\left(\frac{x^{n+1}}{n+1} + c\right) = x^n\,dx.$$

$$\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \text{where } c \text{ is the constant of integration.}$$

Justification for (b):

$$\frac{d}{dx}(\log x) = \frac{1}{x}(x > 0) \text{ or, } d(\log x) = \frac{1}{x}dx \text{ or, } d(\log x + c) = \frac{1}{x}dx.$$

$$\therefore \int \frac{1}{x}dx = \int x^{-1}dx = \log x + c, \text{ where } c \text{ is the constant of integration.}$$

What happens when $x < 0$?

$$\frac{d}{dx}\{\log(-x)\} = \left(\frac{1}{-x}\right)\frac{d}{dx}(-x) = \frac{1}{x}(x < 0);\ \therefore \int \frac{1}{x}dx = \log(-x) + c.$$

Hence, we remember:
$$\int \frac{1}{x}dx = \begin{cases} \log x + c, & \text{if } x > 0 \\ \log(-x) + c, & \text{if } x < 0, \end{cases}$$

i.e., $\displaystyle\int \boldsymbol{\frac{1}{x} = \log|x| + c.}$

Justification for (c):
$$\frac{d}{dx}(x) = 1 \text{ or, } d(x) = 1 \cdot dx \text{ or, } d(x + c) = 1 \cdot dx.$$

$$\therefore \int dx = x + c.$$

Standard Formula II

(a) $\frac{d}{dx}(e^x) = e^x$ or, $d(e^x + c) = e^x\,dx;\ \therefore \int e^x\,dx = e^x + c.$

Similarly, $\int e^{mx}dx = \frac{e^{mx}}{m} + c$

and $\int e^{3x+2}dx = \int e^2 \cdot e^{3x}dx = e^2 \cdot \frac{e^{3x}}{3} + c = \frac{1}{3}e^{3x+2} + c.$

(b) $\frac{d}{dx}(a^x) = a^x \log_e a (a > 0$ and $a \neq 1)$ or, $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x;$

$\therefore \int a^x\,dx = \frac{a^x}{\log_e a} + c.$

Similarly, $\int a^{mx}dx = \frac{a^{mx}}{m \log a} + c.$

In evaluation of integrals we require two theorems:

Theorem 1. *If a is a non-zero constant, then*

$$\int af(x)dx = a\int f(x)dx, \quad \text{e.g.,} \int 2x^2dx = 2\int x^2dx = 2\frac{x^3}{3} + c.$$

Theorem 2.

$$\int \{f(x) \pm \phi(x)\}dx = \int f(x)dx \pm \int \phi(x)dx.$$

Justification:

Let $\int f(x)dx = u.$ Then $du = f(x)dx.$

Let $\int \phi(x)dx = v.$ Then $dv = \phi(x)dx.$

$$d(u \pm v) = du \pm dv = f(x)dx \pm \phi(x)dx = \{f(x) \pm \phi(x)\}dx.$$

$$\therefore \int \{f(x) \pm \phi(x)\}dx = u \pm v = \int f(x)dx \pm \int \phi(x)dx.$$

We can extend this result for more than two functions.

$$\text{e.g.,} \int \left(x^3 + 3x^{12}\right)dx = \int x^3dx + \int 3x^{12}dx = \int x^3dx + 3\int x^{12}dx = \frac{x^4}{4} + \frac{3x^{13}}{13} + c.$$

Two Important Results:

1. $\int \frac{dx}{x \pm a} = \log|x \pm a| + c;$ 2. $\int \frac{dx}{ax \pm b} = \frac{1}{a}\log|ax \pm b| + c.$

Justification:

- $\dfrac{d}{dx}\{\log|x\pm a|+c\}=\dfrac{1}{x\pm a}\times\dfrac{d}{dx}(x\pm a)=\dfrac{1}{x\pm a}$, $\left[\because \dfrac{d}{dx}(c)=0\right]$

$$\text{or, } d\{\log|x\pm a|+c\}=\frac{1}{x\pm a}dx;\ \therefore \int\frac{dx}{x\pm a}=\mathbf{\log|x\pm a|+c}.$$

- $\dfrac{d}{dx}\left\{\dfrac{1}{a}\log|ax\pm b|+c\right\}=\dfrac{1}{a}\times\dfrac{1}{ax\pm b}\cdot\dfrac{d}{dx}(ax\pm b)=\dfrac{1}{ax\pm b}$,

$$\text{or, } d\left\{\frac{1}{a}\log|ax\pm b|+c\right\}=\frac{1}{ax\pm b}dx;\ \therefore \int\frac{dx}{ax\pm b}=\frac{1}{a}\mathbf{\log|ax\pm b|+c}.$$

8.3 Method 1: Evaluation of Indefinite Integrals

Using Standard Formulae I and II, and the two theorems and results of the last article:

Example 1. *Evaluate*: (i) $\displaystyle\int\left(x^3+2\sqrt{x}+\frac{1}{x^2}\right)dx$; (ii) $\displaystyle\int\left(1+\frac{2}{\sqrt{x}}+\frac{3}{x}\right)dx$.

Solution:

(i) $$\int\left(x^3+2\sqrt{x}+\frac{1}{x^2}\right)dx=\int x^3dx+2\int x^{1/2}dx+\int x^{-2}dx=\frac{x^4}{4}+2\cdot\frac{2}{3}x^{3/2}+\frac{x^{-2+1}}{-2+1}+c$$
$$=\mathbf{\frac{x^4}{4}+\frac{4}{3}x^{3/2}-\frac{1}{x}+c}\quad \text{(c is the constant of integration).}$$

(ii) $$\int\left(1+\frac{2}{\sqrt{x}}+\frac{3}{x}\right)dx=\int 1dx+2\int x^{-1/2}dx+3\int\frac{1}{x}dx=x+2\cdot\frac{x^{1/2}}{1/2}+3\log|x|+c$$
$$=\mathbf{x+4\sqrt{x}+3\log|x|+c},\quad \text{where } c \text{ is the constant of integration.}$$

Example 2. *Evaluate*: (i) $\displaystyle\int\left(\frac{x^2}{2}-\frac{2}{x^2}\right)^2dx$; (ii) $\displaystyle\int\frac{(1-x)^3}{x}dx$; (iii) $\displaystyle\int e^{2\log x}dx$. [B.U. B.Com.(H) 2008]

Solution:

(i) $$\int\left(\frac{x^2}{2}-\frac{2}{x^2}\right)^2dx=\int\left(\frac{1}{4}x^4+\frac{4}{x^4}-2\cdot\frac{x^2}{2}\cdot\frac{2}{x^2}\right)dx=\frac{1}{4}\int x^4dx+4\int x^{-4}dx-2\int dx$$
$$=\frac{x^5}{20}+\frac{4x^{-3}}{-3}-2x+c=\mathbf{\frac{x^5}{20}-\frac{4}{3}x^{-3}-2x+c},\quad \text{where } c \text{ is the constant of integration.}$$

(ii) $$\int\frac{(1-x)^3}{x}dx=\int\frac{(1-3x+3x^2-x^3)}{x}dx=\int\frac{1}{x}dx-3\int dx+3\int x\,dx-\int x^2dx$$
$$=\mathbf{\log|x|-3x+3\cdot\frac{x^2}{2}-\frac{x^3}{3}+c},\quad \text{where } c \text{ is the constant of integration.}$$

(iii) $$\int e^{2\log x}dx=\int e^{\log x^2}dx=\int x^2dx=\mathbf{\frac{1}{3}x^3+c}.$$

Example 3. *Evaluate:*

(i) $\int \frac{(x^2+1)^2}{x^3} dx$; [C.U. B.Com.(H) 1990] (ii) $\int \frac{(3x-2)^3}{\sqrt{x}} dx$.

Solution:

(i) $\int \frac{(x^2+1)^2}{x^3} dx = \int \frac{(x^4+2x^2+1)}{x^3} dx = \int x\,dx + 2\int \frac{1}{x} dx + \int \frac{1}{x^3} dx = \frac{x^2}{2} + 2\log|x| + \frac{x^{-3+1}}{-3+1} + c$

$= \frac{1}{2}x^2 + 2\log|x| - \frac{1}{2x^2} + c$, where c is the constant of integration.

(ii) $\int \frac{(3x-2)^3}{\sqrt{x}} dx = \int \frac{(27x^3-54x^2+36x-8)}{x^{1/2}} dx$

$= 27\int x^{3-1/2} dx - 54\int x^{2-1/2} dx + 36\int x^{1-1/2} dx - 8\int x^{-1/2} dx$

$= 27\int x^{5/2} dx - 54\int x^{3/2} dx + 36\int x^{1/2} dx - 8\int x^{-1/2} dx$

$= 27 \times \frac{x^{7/2}}{7/2} - 54\int \frac{x^{5/2}}{5/2} + 36 \times \frac{x^{3/2}}{3/2} - 8 \times \frac{x^{1/2}}{1/2} + c$

$= \frac{54}{7}x^{7/2} - \frac{108}{5}x^{5/2} + 24x^{3/2} - 16x^{1/2} + c$, where c is the constant of integration.

Example 4. *Evaluate:*

(i) $\int \sqrt{x}\left(x^2+3x+4\right) dx$;

(ii) $\int \frac{dx}{2x+5}$;

(iii) $\int \frac{x+3}{x+1} dx$.

Solution:

(i) $\int \sqrt{x}\left(x^2+3x+4\right) dx = \int \left(x^{1/2+2}+3x^{1/2+1}+4x^{1/2}\right) dx = \int x^{5/2} dx + 3\int x^{3/2} dx + 4\int x^{1/2} dx$

$= \frac{x^{5/2+1}}{5/2+1} + 3\cdot\frac{x^{3/2+1}}{3/2+1} + 4\cdot\frac{x^{1/2+1}}{1/2+1} + c$

$= \frac{2}{7}x^{7/2} + \frac{6}{5}x^{5/2} + \frac{8}{3}x^{3/2} + c$, where c is the constant of integration.

(ii) $\int \frac{dx}{2x+5} = \frac{1}{2}\log|2x+5| + c$, where c is the constant of integration.

(iii) $\int \frac{x+3}{x+1} dx = \int \frac{(x+1)+2}{x+1} dx = \int \left(1+\frac{2}{x+1}\right) dx = \int 1\,dx + 2\int \frac{1}{x+1} dx$

$= x + 2\log|x+1| + c$, where c is the constant of integration.

Example 5. *Evaluate:* $\int \frac{x}{x+1} dx.$ [C.U. B.Com. 2010]

Solution:

$$\int \frac{x}{x+1} dx = \int \frac{x+1-1}{x+1} dx = \int \left(1 - \frac{1}{x+1}\right) dx = \int dx - \int \frac{dx}{x+1}$$
$$= \mathbf{x - \log|x+1| + c}, \quad \text{where } c \text{ is the constant of integration.}$$

Note: In all problems of indefinite integral, we shall take c as the constant of integration.

Example 6. *Evaluate:* $\int \left(2ax^{-1/2} - bx^{-2} + 3cx^{2/3}\right) dx.$

Solution:

$$\int \left(2ax^{-1/2} - bx^{-2} + 3cx^{2/3}\right) dx = 2a \int x^{-1/2} dx - b \int x^{-2} dx + 3c \int x^{2/3} dx$$
$$= 2a \frac{x^{1/2}}{1/2} - b \frac{x^{-1}}{-1} + 3c \frac{x^{5/3}}{5/3} + c$$
$$= \mathbf{4a\sqrt{x} + \frac{b}{x} + \frac{9c}{5} \sqrt[3]{x^5} + c}.$$

Example 7. *Evaluate:* $\int \frac{x^3}{x-1} dx.$

Solution:

$$\int \frac{x^3}{x-1} dx = \int \left(\frac{x^3 - 1 + 1}{x-1}\right) dx = \int \frac{x^3 - 1}{x-1} dx + \int \frac{dx}{x-1} = \int \left(x^2 + x + 1\right) dx + \int \frac{dx}{x-1}$$
$$= \int x^2 dx + \int x\, dx + \int dx + \int \frac{dx}{x-1} = \mathbf{\frac{x^3}{3} + \frac{x^2}{2} + x + \log|x-1| + c}.$$

Example 8. *Evaluate:* $\int \frac{2x+3}{3x+8} dx.$ [C.U. B.Com.(H) 1998 Type]

Solution:

$$\int \frac{2x+3}{3x+8} dx = \int \left(\frac{2}{3} - \frac{\frac{7}{3}}{3x+8}\right) dx = \frac{2}{3} \int dx - \frac{7}{3} \int \frac{dx}{3x+8}$$
$$= \frac{2}{3}x - \frac{7}{3} \cdot \frac{1}{3} \log|3x+8| + c = \mathbf{\frac{2}{3}x - \frac{7}{9} \log|3x+8| + c}.$$

$$3x+8)\ 2x+3\ (\tfrac{2}{3}$$
$$\underline{2x + \tfrac{16}{3}}$$
$$-\tfrac{7}{3}$$

Example 9. *Prove from definition that*

(i) $\int 4^x\, dx = \frac{4^x}{\log_e 4} + c$ *and* (ii) $\int \left(5^{3x} + e^{5x}\right) dx = \frac{5^{3x}}{3\log_e 5} + \frac{e^{5x}}{5} + c.$

Solution:

(i) $\dfrac{d}{dx}\left(\dfrac{4^x}{\log_e 4}+c\right)=\dfrac{4^x \log_e 4}{\log_e 4}=4^x$, or, $d\left(\dfrac{4^x}{\log_e 4}+c\right)=4^x\,dx.$

$$\therefore \int 4^x\,dx=\frac{\mathbf{4^x}}{\mathbf{\log_e 4}}+\boldsymbol{c}\ \textbf{(Proved).}$$

(ii) $\dfrac{d}{dx}\left(\dfrac{5^{3x}}{3\log_e 5}+\dfrac{e^{5x}}{5}+c\right)=\dfrac{3\cdot 5^{3x}\log_e 5}{3\log_e 5}+\dfrac{5.e^{5x}}{5}=5^{3x}+e^{5x}.$

$$\therefore d\left(\frac{5^{3x}}{3\log_e 5}+\frac{e^{5x}}{5}+c\right)=\left(5^{3x}+e^{5x}\right)dx.$$

Hence,
$$\int\left(5^{3x}+e^{5x}\right)dx=\frac{\mathbf{5^{3x}}}{\mathbf{3\log_e 5}}+\frac{\boldsymbol{e^{5x}}}{\mathbf{5}}+\boldsymbol{c}\ \textbf{(Proved).}$$

Example 10. *Evaluate*: (i) $\displaystyle\int \frac{27^{1+x}-9^{1-x}}{3^x}dx$; (ii) $\displaystyle\int \frac{(15x^2+2x-8)}{5x+4}dx.$

Solution:

(i) $$\int \frac{27^{1+x}-9^{1-x}}{3^x}dx=\int \frac{(27\cdot 27^x-9\cdot 9^{-x})}{3^x}dx=\int \frac{27\cdot(3^3)^x-9\cdot(3^2)^{-x}}{3^x}dx$$
$$=\int\left(27\cdot 3^{3x-x}-9\cdot 3^{-2x-x}\right)dx=27\cdot\int 3^{2x}dx-9\cdot\int 3^{-3x}dx$$
$$=27\cdot\frac{3^{2x}}{2\log_e 3}-9\cdot\frac{3^{-3x}}{-3\log_e 3}+c=\frac{\mathbf{9}}{\mathbf{\log_e 3}}\left(\frac{\mathbf{3}}{\mathbf{2}}\cdot\mathbf{3^{2x}}+\frac{\mathbf{1}}{\mathbf{3}}\cdot\mathbf{3^{-3x}}\right)+\boldsymbol{c}.$$

(ii) $$\int \frac{(15x^2+2x-8)}{5x+4}dx=\int \frac{(5x+4)(3x-2)}{5x+4}dx$$
$$=\int(3x-2)\,dx=3\int x\,dx-2\int dx$$
$$=\frac{\mathbf{3x^2}}{\mathbf{2}}-\mathbf{2x}+\boldsymbol{c}.$$

$$\begin{aligned}&15x^2+2x-8\\&=15x^2+12x-10x-8\\&=3x(5x+4)-2(5x+4)\\&=(5x+4)(3x-2)\end{aligned}$$

Example 11. *The slope at any point (x,y) of a curve is $\dfrac{x+1}{y+1}$. If the curve passes through the origin, find the equation of the curve.* [C.U. B.Com.(H) 2003]

Solution: We know that $\dfrac{dy}{dx}$ geometrically represents the slope at any point (x,y) of a curve.

$$\therefore \frac{dy}{dx}=\frac{x+1}{y+1} \text{ or, } (y+1)\,dy=(x+1)\,dx.$$

Integrating both sides, we get

$$\int (y+1)\,dy = \int (x+1)\,dx + c \text{ or, } \frac{y^2}{2} + y = \frac{x^2}{2} + x + c. \quad (1)$$

If this curve passes through the origin (0, 0), then $0+0=0+0+c$, or, $c=0$.
Hence, from eq. (1), the required equation of the curve is

$$\frac{y^2}{2} + y = \frac{x^2}{2} + x \text{ or, } \boldsymbol{x^2 - y^2 + 2x - 2y = 0}.$$

EXERCISES ON CHAPTER 8(I)

(Indefinite Integrals: Their Evaluation)

Formulae to be used:

(i) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$;

(ii) $\displaystyle\int x^{-1}\,dx = \log|x| + c$;

(iii) $\displaystyle\int e^{mx}\,dx = \frac{e^{mx}}{m} + c$;

(iv) $\displaystyle\int a^x\,dx = \frac{a^x}{\log_e a} + c \; (a>0, a\neq 1)$;

(v) $\displaystyle\int \frac{dx}{x \pm a} = \log|x \pm a| + c$;

(vi) $\displaystyle\int \frac{dx}{ax \pm b} = \frac{1}{a}\log|ax \pm b| + c$.

A

Verify the following by evaluating the integrals (c denotes the constant of integration):

1. (a) $\displaystyle\int x^7\,dx = \frac{x^8}{8} + c$;

(b) $\displaystyle\int \frac{dx}{x^{12}} = \frac{-1}{11x^{11}} + c$;

(c) $\displaystyle\int \frac{(x^6-1)}{-1}\,dx = x - \frac{1}{7}x^7 + c$;

(d) $\displaystyle\int x^{-8}\,dx = -\frac{1}{7x^7} + c$;

(e) $\displaystyle\int \frac{dx}{\sqrt[3]{x^{70}}} = -\frac{3}{67}\frac{1}{x^{67/3}} + c$;

(f) $\displaystyle\int x^{-1}\,dx = \log|x| + c$;

(g) $\displaystyle\int \sqrt[5]{x^{16}}\,dx = \frac{5}{21}x^{21/5} + c$;

(h) $\displaystyle\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$;

(i) $\displaystyle\int \left(\sqrt{9\theta} + 5\theta^{2/3} + 7\right) d\theta$
$= 2\theta^{3/2} + 3\theta^{5/3} + 7\theta + c.$

2. (a) $\displaystyle\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$;

(b) $\displaystyle\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x} + c$;

(c) $\displaystyle\int 2^x\,dx = \frac{2^x}{\log_e 2} + c$;

(d) $\displaystyle\int 4^x\,dx = \frac{4^x}{\log_e 4} + c$;

(e) $\displaystyle\int \frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}\,dx$
$= 2e^{2x} - 9e^x + e^{-3x} + c$;

(f) $\displaystyle\int \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}\,dx = e^{3x} + c.$

3. (a) $\int \left(ax^4+bx^3+cx^2+dx+e\right)dx=\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}+\frac{dx^2}{2}+ex+\text{constant};$

(b) $\int (3x+2)^2 dx = 3x^3+6x^2+4x+c;$

(c) $\int \left(\frac{2}{x}-\frac{7}{x^2}-\frac{1}{3-x}\right)dx = 2\log|x|+\frac{7}{x}+\log|3-x|+c;$

(d) $\int \frac{(2x-3)^3}{x^2}dx = 4x^2-36x+54\log|x|+27x^{-1}+c;$

(e) $\int \frac{(x+1)^2}{\sqrt{x}}dx = \frac{2}{5}x^{5/2}-\frac{4}{3}x^{3/2}+2x^{1/2}+c;$ [V.U. B.Com.(H) 2010]

(f) $\int \frac{ax^2+bx+c}{x^3}dx = a\log|x|-\frac{b}{x}-\frac{c}{2x^2}+\text{constant};$

(g) $\int \frac{2x^2-14x+24}{x-3}dx = x^2-8x+c;$

(h) $\int \frac{(x+1)(x-4)}{\sqrt{x}}dx = \frac{2}{5}x^{5/2}-2x^{3/2}-8x^{1/2}+c.$

4. (a) $\int \left(x^{3/2}-2x^{2/3}+5\sqrt{x}-3\right)dx = \frac{2}{5}x^{5/2}-\frac{6}{5}x^{5/3}+\frac{10}{3}x^{3/2}-3x+c;$

(b) $\int \sqrt{x}(3x-2)\,dx = \frac{6}{5}x^{5/2}-\frac{4}{3}x^{3/2}+c;$

(c) $\int \frac{x^3-6x+5}{x}dx = \frac{x^3}{3}-6x+5\log|x|+c;$

(d) $\int t\left(2+t^2\right)^2 dt = \frac{1}{6}t^6+t^4+2t^2+c;$

(e) $\int \theta\left(\theta^2+3\right)^3 d\theta = \frac{1}{8}\theta^8+\frac{3}{2}\theta^6+\frac{27}{4}\theta^4+\frac{27}{2}\theta^2+c.$

5. The slope of a curve at (x, y) is $9x$. If it passes through the origin, show that its equation is $9x^2=2y$.

[V.U. B.Com.(H) 2011]

B

1. State the value of n for which $\int x^n dx = \frac{x^{n+1}}{n+1}+c$ does not hold. Find the value of the integral for that value of n.

2. Prove that $\int a^{mx}dx = \frac{a^{mx}}{m\log_e a}+c$ and state the restriction on a, if any.

3. Evaluate by using the definition as anti-derivative:

(a) $\int e^{3x}dx$; (b) $\int x^5 dx$; (c) $\int \frac{3}{x}dx$; (d) $\int 3^{2x}dx$; (e) $\int \frac{3}{\sqrt[3]{x^2}}dx$.

4. Evaluate:

(a) $\int \left(x^4+\frac{5}{x}\right)dx$;

(b) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$;

(c) $\int \left(\sqrt[3]{x}+e^{(1/2)x}\right)dx$;

(d) $\int \left(3\sqrt{x}+5+\frac{2}{x}\right)dx$;

(e) $\int (x^a+a^x+a^a)dx$;

(f) $\int \frac{(2-x)^3}{x}dx$;

(g) $\int \frac{x^4-1}{x-1}dx$;

(h) $\int \frac{(x+1)^2}{\sqrt{x}}dx$; [V.U. B.Com.(H) 2010]

(i) $\int \left(x^{2/3}+a^{2/3}\right)^3 dx$;

(j) $\int \frac{x^3+1}{x+1}dx$;

(k) $\int \sqrt{x}\left(x^2+3x+4\right)dx$;

(l) $\int \frac{x^2+1}{x+1}dx$; [C.U. B.Com.(H) 1990]

(m) $\int \frac{(2x^2+7x+3)}{x+3}dx$;

(n) $\int \frac{(x^2+3)^2}{\sqrt{x}}dx$;

(o) $\int \frac{x^3}{x+1}dx$;

(p) $\int \frac{x^3}{x-1}dx$;

(q) $\int \frac{x^2}{x+2}dx$;

(r) $\int \left(3^x+\frac{3}{x}-\frac{2}{3\sqrt[3]{x}}\right)dx$;

(s) $\int \frac{(e^x+1)^3}{e^{2x}}dx$.

5. Integrate:

(a) $x\sqrt{x}-\frac{2}{\sqrt[3]{x}}+\frac{5}{\sqrt[3]{x^2}}$; (b) $\frac{(2-3x^2)^3}{x\sqrt[3]{x}}$; (c) $\frac{x^2+5x+2}{x+2}$.

6. Evaluate:

(a) $\int \sqrt{x}\left(x^3+\frac{4}{x}\right)dx$;

(b) $\int \frac{e^{2x}+e^{4x}}{e^x+e^{-x}}dx$;

(c) $\int \frac{2^x+2^{2x}+2^{3x}}{2^{2x}}dx$;

(d) $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3 dx$;

(e) $\int \frac{e^{5\log x}-e^{3\log x}}{e^{3\log x}-e^{\log x}}dx$;

(f) $\int \left(e^{3\log x}-e^{x\log 3}\right)dx$;

(g) $\int \frac{8^{1+x}-4^{1-x}}{2^x}dx$;

(h) $\int \frac{(12x^2+x-20)}{4x-5}dx$.

7. (a) If slope of the tangent at any point (x, y) of a curve is $3x^2+2x+1$, find the equation of the curve.

[Hints: Slope = $\frac{dy}{dx}$; $\therefore$ $\frac{dy}{dx}=3x^2+2x+1$ or, $dy=\left(3x^2+2x+1\right)dx$.]

(b) If slope of a curve at (x, y) is $-x/y$ and the curve passes through $(2, 3)$, find its equation.

[V.U. B.Com.(H) 2007]

ANSWERS

[Constant of integration c, which is not written, is to be added to each answer]

B

1. $n \neq -1$; $\log|x|$.
2. $a > 0, a \neq 1$.
3. (a) $\frac{1}{3}e^{3x}$;
   (b) $\frac{1}{6}x^6$;
   (c) $3\log|x|$;
   (d) $\frac{3^{2x}}{2\log_e 3}$;
   (e) $9\sqrt[3]{x}$.
4. (a) $\frac{1}{5}x^5 + 5\log|x|$;
   (b) $\frac{2}{3}x^{3/2} + 2\sqrt{x}$;
   (c) $\frac{3}{4}x^{4/3} + 2e^{(1/2)x}$;
   (d) $2x^{3/2} + 5x + 2\log|x|$;
   (e) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + a^a x$;
   (f) $8\log|x| - 12x + 3x^2 - \frac{1}{3}x^3$;
   (g) $\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$;
   (h) $\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2\sqrt{x}$;
   (i) $\frac{1}{3}x^3 + \frac{9}{7}a^{2/3}x^{7/3} + \frac{9}{5}a^{4/3}x^{5/3} + a^2x$;
   (j) $\frac{1}{3}x^3 - \frac{1}{2}x + x$;
   (k) $\frac{2}{7}x^{7/2} + \frac{6}{5}x^{5/2} + \frac{8}{3}x^{3/2}$;
   (l) $\frac{1}{2}x^2 - x + 2\log|x+1|$;
   (m) $x^2 + x$;
   (n) $\frac{2}{9}x^{9/2} + \frac{12}{5}x^{5/2} + 18x^{1/2}$;
   (o) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \log|x+1|$;
   (p) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \log|x-1|$;
   (q) $\frac{1}{2}x^2 - 2x + 4\log|x+2|$;
   (r) $\frac{3^x}{\log_e 3} + 3\log|x| - x^{2/3}$;
   (s) $e^x + 3x - 3e^{-x} - \frac{1}{2}e^{-2x}$
5. (a) $\frac{2}{5}x^{5/2} - 3x^{2/3} + 15\sqrt[3]{x}$;
   (b) $-24x^{-1/3} - \frac{108}{5}x^{5/3} + \frac{162}{11}x^{11/3} - \frac{81}{17}x^{17/3}$.
6. (a) $\frac{2}{9}x^{9/2} + 8\sqrt{x}$;
   (b) $\frac{1}{3}e^{3x}$;
   (c) $-\frac{2^{-x}}{\log_e 2} + x + \frac{2^x}{\log_e 2}$;
   (d) $\frac{2}{5}x^{5/2} - 2x^{3/2} + 6\sqrt{x} + \frac{2}{\sqrt{x}}$;
   (e) $\frac{1}{3}x^3$;
   (f) $\frac{1}{4}x^4 - \frac{3^x}{\log_e 3}$;
   (g) $\frac{4}{\log_e 2}\left(2^{2x} + \frac{1}{3}\cdot 2^{-3x}\right)$;
   (h) $\frac{3}{2}x^2 + 4x$.
7. (a) $y = x^3 + x^2 + x$;
   (b) $x^2 + y^2 = 13$.

8.4 Method 2: Integration by Substitution (Change of Variable): Formulae

I. $\displaystyle\int f(ax+b)\,dx = \frac{1}{a}\int f(z)\,dz$, **where** $z = ax + b$.

Proof. Put $ax + b = z$; then $a\,dx = dz$ or, $dx = \frac{1}{a}dz$.

$$\therefore \int f(ax+b)\,dx = \int f(z)\cdot\frac{1}{a}dz = \frac{1}{a}\int f(z)\,dz.$$

Corollary 1. *For integrals of the types* $\int (ax+b)^n dx$, $\int \sqrt[n]{ax+b}$ *and* $\int \frac{dx}{(ax+b)^n}$, *suitable substitution is* $ax+b=z$ *and the results are given below.*

(i) $\int (ax+b)^n dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+c;$ (iii) $\int \frac{dx}{(ax+b)^n} = \frac{1}{a}\cdot\frac{(ax+b)^{-n+1}}{-n+1}+c.$

(ii) $\int \sqrt[n]{ax+b}\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{\frac{1}{n}+1}}{\frac{1}{n}+1}+c;$

In (i) $n \neq -1$, in (ii) $n \neq -1$ and in (iii) $n \neq 1$. [Students may prove these formulae]

II. $\int \{f(x)\}^n \cdot f'(x)\,dx = \frac{\{f(x)\}^{n+1}}{n+1}+c.$

Proof. Put $f(x)=z$; then $f'(x)dx = dz$.

$$\therefore \int \{f(x)\}^n \cdot f'(x)\,dx = \int z^n dz = \frac{z^{n+1}}{n+1}+c = \frac{\{f(x)\}^{n+1}}{n+1}+c.$$

Illustration 1.

$$\int \left(x^2+3x+5\right)^6 \cdot (2x+3)\,dx = \frac{\left(x^2+3x+5\right)^{6+1}}{6+1}+c = \frac{\left(x^2+3x+5\right)^7}{7}+c.$$

Note: $\frac{d}{dx}\left(x^2+3x+5\right)=2x+3.$

III. $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c.$

Proof. Let $f(x)=z$; then $f'(x)dx=dz$.

$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{dz}{z} = \log|z|+c = \log|f(x)|+c.$$

Illustration 2.

$$\int \frac{(3x^2+5)\,dx}{x^3+5x+7} = \log|x^3+5x+7|+c \quad \left[\because \frac{d}{dx}\left(x^3+5x+7\right)=3x^2+5\right].$$

8.4.1 Illustrative Examples

Example 1. *Evaluate*: (i) $\int (3x+7)^5 dx;$ (ii) $\int \sqrt{3x+2}\,dx.$

Solution: We have

(i) $\int (3x+7)^5 dx = \int z^5 \cdot \frac{1}{3}dz = \frac{1}{3}\int z^5 dz = \frac{1}{3}\cdot\frac{z^{5+1}}{5+1}+c$

Put $3x+7=z$; then $3dx = dz$ or, $dx=\frac{1}{3}dz$.

$$= \frac{1}{18}z^6+c = \frac{1}{18}(3x+7)^6+c.$$

(ii) $\int \sqrt{3x+2}\,dx = \int \sqrt{z}\cdot\frac{1}{3}dz = \frac{1}{3}\int z^{1/2}dz = \frac{1}{3}\cdot\frac{z^{1/2+1}}{\frac{1}{2}+1}+c$

Put $3x+2 = z$; then $3dx = dz$ or, $dx = \frac{1}{3}dz$.

$= \frac{2}{9}z^{3/2}+c = \mathbf{\frac{2}{9}(3x+2)^{3/2}+c.}$

Otherwise. Put $3x+2=z^2$; then $3dx = 2z\,dz$ or, $dx = \frac{2}{3}z\,dz.$

$$\therefore \int \sqrt{3x+2}\,dx = \int z\cdot\frac{2}{3}z\,dz = \frac{2}{3}\int z^2dz = \frac{2}{3}\cdot\frac{z^3}{3}+c = \mathbf{\frac{2}{9}(3x+2)^{3/2}+c.}$$

Example 2. *Evaluate:* (i) $\int \sqrt[5]{2x+3}\,dx;$ (ii) $\int \frac{dx}{(3x+5)^3}.$

Solution:

(i) $\int \sqrt[5]{2x+3}\,dx = \int \sqrt[5]{z^5}\cdot\frac{5}{2}z^4dz = \frac{5}{2}\int z.z^4dz = \frac{5}{2}\int z^5dz$

Put $2x+3 = z^5$; then $2dx = 5z^4dz$ or, $dx = \frac{5}{2}z^4dz.$

$= \frac{5}{2}\times\frac{z^6}{6}+c = \mathbf{\frac{5}{12}(2x+3)^{6/5}+c.}$

Otherwise. Put $2x+3=z$; then $2dx = dz$ or, $dx = \frac{1}{2}dz.$

$$\therefore \int \sqrt[5]{2x+3}\,dx = \int \sqrt[5]{z}\cdot\frac{1}{2}dz = \frac{1}{2}\int z^{1/5}dz = \frac{1}{2}\cdot\frac{z^{\frac{1}{5}+1}}{\frac{1}{5}+1}+c = \frac{1}{2}\times\frac{5}{6}\cdot z^{6/5}+c$$

$$= \mathbf{\frac{5}{12}(2x+3)^{6/5}+c.}$$

(ii) $\int \frac{dx}{(3x+5)^3} = \int \frac{\frac{1}{3}dz}{z^3} = \frac{1}{3}\int\frac{dz}{z^3} = \frac{1}{3}\int z^{-3}dz = \frac{1}{3}\cdot\frac{z^{-3+1}}{-3+1}+c$

Put $3x+5 = z$; then $3dx = dz$ or, $dx = \frac{1}{3}dz.$

$= \frac{1}{3}\times -\frac{1}{2}z^{-2}+c = -\frac{1}{6z^2}+c = \mathbf{-\frac{1}{6(3x+5)^2}+c.}$

Remember

- $\int (ax+b)^n dx$, put $ax+b=z$;
- $\int \sqrt[n]{ax+b}\,dx$, put $ax+b=z^n$ or z;
- $\int \frac{ax+b}{cx+d}dx, \int \frac{ax+b}{\sqrt{cx+d}}dx, \int (ax+b)\sqrt{cx+d}\,dx,$

  put $cx+d=z$ for the first integral and for the other two, put $cx+d=z^2$;

$\int \frac{dx}{\sqrt{ax+b}+\sqrt{cx+d}}$, rationalize the denominator.

Example 3. *Evaluate:*

(i) $\int \frac{2x+3}{3x+2}dx$; [C.U. B.Com.(H) 1998]

(ii) $\int \frac{1-x}{\sqrt{1+x}}dx$;

(iii) $\int \frac{2x-3}{\sqrt{4x-1}}dx$. [C.U. B.Com.(H) 1991]

Solution:

(i) $\int \frac{2x+3}{3x+2}dx = \int \frac{\left\{2\left(\frac{z-2}{3}\right)+3\right\}}{z} \times \frac{1}{3}dz$ | Put $3x+2=z$; then $3dx = dz$ or, $dx = \frac{1}{3}dz$. Also $x = \frac{z-2}{3}$.

$$= \frac{1}{9}\int \frac{2z+5}{z}dz = \frac{1}{9}\int \left(2+\frac{5}{z}\right)dz = \frac{2}{9}z + \frac{5}{9}\log|z| + c'$$

$$= \frac{2}{9}(3x+2) + \frac{5}{9}\log|3x+2| + c' = \frac{2}{3}x + \frac{5}{9}\log|3x+2| + \frac{4}{9} + c'$$

$$= \mathbf{\frac{2}{3}x + \frac{5}{9}\log|3x+2| + c}, \text{ where } c = \frac{4}{9} + c'.$$

Another method: See worked-out Ex. 8 in Section 8.3.

(ii) $\int \frac{1-x}{\sqrt{1+x}}dx = \int \frac{1-(z^2-1)}{z} \times 2z\,dz = 2\int \left(2-z^2\right)dz$ | Put $1+x = z^2$; then $dx = 2z\,dz$, $x = z^2-1$ and $z = (1+x)^{1/2}$.

$$= 2\left(2z - \frac{z^3}{3}\right) + c = \mathbf{2\left[2\sqrt{1+x} - \frac{1}{3}(1+x)^{3/2}\right] + c}.$$

(iii) $\int \frac{2x-3}{\sqrt{4x-1}}dx = \int \frac{\frac{z^2-5}{2}}{z} \times \frac{1}{2}z\,dz$ | Put $4x-1 = z^2$; then $4dx = 2z\,dz$ or, $dx = \frac{1}{2}z\,dz$ and $x = \frac{z^2+1}{4}$.

$= \frac{1}{4}\int \left(z^2-5\right)dz$ | $\therefore\ 2x-3 = \frac{z^2+1}{2} - 3 = \frac{z^2-5}{2}$. $\therefore\ z = (4x-1)^{1/2}$.

$$= \frac{1}{4}\left[\frac{z^3}{3} - 5z\right] + c = \mathbf{\frac{1}{4}\left[\frac{1}{3}(4x-1)^{3/2} - 5(4x-1)^{1/2}\right] + c}.$$

Example 4. *Evaluate:*

(i) $\int x^2\sqrt{3x^3-4}\,dx$; [C.U. B.Com.(H) 1995]

(ii) $\int \frac{x+2}{\sqrt{x-2}}dx$; [C.U. B.Com.(H) 1994]

(iii) $\int \frac{x}{\sqrt{x}+1}dx$. [C.U. B.Com.(H) 2007]

Solution:

(i) $\displaystyle\int x^2\sqrt{3x^3-4}\,dx=\int\left(\sqrt{3x^3-4}\right)x^2dx=\int\sqrt{z^2}\cdot\frac{2}{9}z\,dz$

Put $3x^3-4=z^2$; then $9x^2dx=2z\,dz$ or, $x^2dx=\frac{2}{9}z\,dz$. Also $z=\left(3x^3-4\right)^{1/2}$.

$$=\frac{2}{9}\int z^2dz=\frac{2}{9}\cdot\frac{z^3}{3}+c=\frac{2}{27}z^3+c=\mathbf{\frac{2}{27}\left(3x^3-4\right)^{3/2}+c.}$$

(ii) $\displaystyle\int\frac{x+2}{\sqrt{x-2}}dx=\int\frac{(z^2+2+2)}{\sqrt{z^2}}\times 2z\,dz=2\int\left(z^2+4\right)dz$

Put $x-2=z^2$; then $dx=2z\,dz$. Also $x=z^2+2$ and $z=\sqrt{(x-2)}$.

$$=2\left(\frac{z^3}{3}+4z\right)+c=\frac{2}{3}\left(z^2+12\right)z+c=\frac{2}{3}(x-2+12)\sqrt{x-2}+c$$

$$=\mathbf{\frac{2}{3}(x+10)\sqrt{x-2}+c.}$$

(iii) $\displaystyle\int\frac{x}{\sqrt{x}+1}dx=\int\frac{z^2.2z\,dz}{z+1}=2\int\frac{z^3}{z+1}dz$

$$=2\int\left(z^2-z+1-\frac{1}{z+1}\right)dz$$

$$=2\left[\frac{z^3}{3}-\frac{z^2}{2}+z-\log|z+1|\right]+c$$

$$=\mathbf{2\left[\frac{1}{3}x^{3/2}-\frac{1}{2}x+\sqrt{x}-\log|\sqrt{x}+1|\right]+c.}$$

Put $x=z^2$; then $dx=2z\,dz$. Also $z=x^{1/2}$ i.e., $z=\sqrt{x}$.

$$\begin{array}{rl} z+1)\;z^3 & (z^2-z+1\\ z^3+z^2 & \\ -\quad- & \\ \hline -z^2 & \\ -z^2-z & \\ +\quad+ & \\ \hline z & \\ z+1 & \\ -\quad- & \\ \hline -1 & \end{array}$$

Example 5. *Evaluate:* $\displaystyle\int\frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$ **[C.U. B.Com.(H) 1992]**

Solution: Let us rationalize the denominator of the integrand.

$$\int\frac{dx}{\sqrt{x+1}+\sqrt{x+2}}=\int\frac{\left(\sqrt{x+2}-\sqrt{x-1}\right)dx}{\left(\sqrt{x+2}+\sqrt{x+1}\right)\left(\sqrt{x+2}-\sqrt{x+1}\right)}=\int\frac{\left(\sqrt{x+2}-\sqrt{x-1}\right)dx}{x+2-(x+1)}$$

$$=\int\sqrt{x+2}\,dx-\int\sqrt{x-1}\,dx=\frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(x-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$$

$$\left[\because\int\sqrt[n]{x+a}\,dx=\frac{(x+a)^{\frac{1}{n}+1}}{\frac{1}{n}+1}+c\right]$$

$$=\frac{2}{3}(x+2)^{3/2}-\frac{2}{3}(x-1)^{3/2}+c=\mathbf{\frac{2}{3}\left[(x+2)^{3/2}-(x-1)^{3/2}\right]+c.}$$

Example 6. *Evaluate:* (i) $\displaystyle\int\frac{x}{x^2+a^2}dx$; (ii) $\displaystyle\int\frac{x}{\sqrt{x^2+a^2}}dx$; (iii) $\displaystyle\int x\sqrt{x^2+a^2}\,dx$.

Solution:

(i) $\displaystyle\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\int \frac{2x}{x^2+a^2}dx = \frac{1}{2}\log\left(x^2+a^2\right)+c.$ [by Formula III of Section 8.4.]

(ii) $\displaystyle\int \frac{x}{\sqrt{x^2+a^2}}dx = \int \frac{z\,dz}{z}$ | Put $x^2+a^2=z^2$ so that $x\,dx = z\,dz$. Also $z=\sqrt{x^2+a^2}$.

$$= \int dz = z + c = \sqrt{x^2+a^2}+c.$$

(iii) $\displaystyle\int x\sqrt{x^2+a^2}dx$ | Put $x^2+a^2=z^2$ so that $x\,dx = z\,dz$. Also $z=\left(x^2+a^2\right)^{1/2}$.

$$= \int z.z\,dz = \frac{z^3}{3}+c = \frac{(x^2+a^2)^{3/2}}{3}+c.$$

Example 7. *Evaluate:* $\displaystyle\int \frac{(x-2)}{\sqrt{2x^2-8x+5}}dx.$ [C.U. B.Com.(H) 1997]

Solution: $\displaystyle\int \frac{(x-2)\,dx}{\sqrt{2x^2-8x+5}} = \int \frac{\frac{1}{2}z\,dz}{z}$ | Put $2x^2-8x+5=z^2$; then $(4x-8)dx = 2z\,dz$ or, $4(x-2)\,dx = 2z\,dz$ or, $(x-2)dx = \frac{1}{2}z\,dz$.

$$= \frac{1}{2}\int dz = \frac{1}{2}z + c = \frac{1}{2}\sqrt{2x^2-8x+5}+c,$$

where c is the constant of integration.

Example 8. *Evaluate:* $\displaystyle\int \frac{1}{1+e^{x/2}}dx.$ [C.U. B.Com.(H) 1999]

Solution: $\displaystyle\int \frac{1}{1+e^{x/2}}dx = \int \frac{e^{-x/2}}{e^{-x/2}+1}dx = \int \frac{-2dz}{z}$ | Put $e^{-x/2}+1=z$. $\therefore\ -\frac{1}{2}e^{-x/2}dx = dz$ or, $e^{-x/2}dx = -2dz$.

$$= -2\int \frac{dz}{z} = -2\log|z|+c = -2\log\left|e^{-x/2}+1\right|+c,$$

where c is the constant of integration.

Example 9. *Evaluate:*

(i) $\displaystyle\int x\sqrt{x^2-1}\,dx;$ [C.U. B.Com.(H) 2007] (ii) $\displaystyle\int \frac{dx}{x(1+\log x)^3}.$

Solution:

(i) $\displaystyle\int x\sqrt{x^2-1}\,dx = \int \sqrt{z^2}.z\,dz = \int z^2dz = \frac{z^3}{3}+c$ | Put $x^2-1=z^2$; then $2x\,dx = 2z\,dz$ or, $x\,dx = z\,dz$ and $z=\left(x^2-1\right)^{1/2}$.

$$= \frac{1}{3}\left(x^2-1\right)^{3/2}+c.$$

(ii) $\displaystyle\int \frac{dx}{x(1+\log x)^3} = \int \frac{1}{(1+\log x)^3} \cdot \frac{1}{x} dx = \int \frac{1}{z^3} \cdot dz$ | Put $1+\log x = z$; then $\frac{1}{x} dx = dz$.

$$= \int z^{-3} dz = \frac{z^{-3+1}}{-3+1} + c = -\frac{1}{2z^2} + c = -\frac{1}{2(1+\log x)^2} + c.$$

An important result: $\displaystyle\int e^x \{f(x)+f'(x)\} dx = e^x f(x) + c.$

Proof. Put $e^x f(x) = z$; then $\{e^x.f(x)+e^x.f'(x)\} dx = dz$, i.e., $e^x\{f(x)+f'(x)\} dx = dz$

$$\therefore \int e^x \{f(x)+f'(x)\} dx = \int dz = z + c = e^x f(x) + c.$$

Example 10. *Evaluate:* $\displaystyle\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx.$ [C.U. B.Com.(H) 1997]

Solution:

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \int dz = z + c$$
$$= e^x \cdot \frac{1}{x} + c.$$

| Note that if $f(x) = 1/x$, then $f'(x) = -1/x^2$.
Put $e^x.1/x = z$; then $\left(e^x \cdot \frac{1}{x} - e^x \cdot \frac{1}{x^2}\right) dx = dz$
or, $e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx = dz$

Note: This type of integral can be easily evaluated by the Rule of Integration by Parts which will be discussed later.

Example 11. *Evaluate:* $\displaystyle\int \frac{xe^x}{(1+x)^2} dx.$ [C.U. B.Com.(H) 2001]

Solution:
$$I = \int \frac{xe^x}{(1+x)^2} dx = \int \frac{e^x(x+1-1)}{(1+x)^2} dx = \int e^x \left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} dx. \quad (1)$$

Put $e^x \cdot \dfrac{1}{1+x} = z$. Note that if $f(x) = \dfrac{1}{1+x}$, then $f'(x) = -\dfrac{1}{(1+x)^2}$.

Then
$$\left[e^x \cdot \frac{1}{1+x} + e^x\left\{-\frac{1}{(1+x)^2}\right\}\right] dx = dz \text{ or, } e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} dx = dz.$$

$\therefore$ from eq. (1),
$$I = \int dz = z + c = e^x \cdot \frac{1}{1+x} + c,$$

where c is the constant of integration.

Note: This problem can be easily solved by using Integration by Parts.

Example 12. (i) *Find a function whose derivative is* $\dfrac{x^2}{x+1}$. [C.U. B.Com.(H) 1991; V.U. B.Com.(H) 2011]

(ii) *Find a function whose differential is* $\left(\dfrac{2x+3}{x+1}\right) dx.$

Solution: (i) By the question, $$\frac{dy}{dx}=\frac{x^2}{x+1},$$

where $y=f(x)$ is a function of x.

Integrating w.r.t. x, we get

$$y=\int\frac{x^2}{x+1}dx+c \text{ or, } y=\int\frac{(x^2-1+1)}{x+1}dx+c=\int\left(x-1+\frac{1}{x+1}\right)dx+c$$
$$=\frac{x^2}{2}-x+\log|x+1|+c.$$

Hence, the required function is

$$\boldsymbol{y=\frac{1}{2}x^2-x+\log|x+1|+c.}$$

(ii) We have $$dy=\left(\frac{2x+3}{x+1}\right)dx,$$

where $y=f(x)$ is a function of x. Integrating,

$$y=\int\frac{2x+3}{x+1}dx+c=\int\left(2+\frac{1}{x+1}\right)dx+c \text{ or, } y=\boldsymbol{2x+\log|x+1|+c},$$

where c is the constant of integration.

EXERCISES ON CHAPTER 8(II)

(Integration by the Method of Substitution)

A

1. Evaluate:

(a) $\int(2x+3)^5dx;$

(b) $\int\left(x^2+5x\right)^3.(2x+5)dx;$

(c) $\int\frac{(2x+3)dx}{x^2+3x+1};$

(d) $\int\frac{1}{x\log x}dx;$

(e) $\int\sqrt{2x+5}\,dx;$

(f) $\int\sqrt[3]{x+2}\,dx;$

(g) $\int(3-2x)^4dx;$

(h) $\int\frac{dx}{(2x-3)^2};$

(i) $\int e^{2x+3}dx;$

(j) $\int e^{x^2-3x+2}(2x-3)dx;$

(k) $\int 3^{3x-2}dx;$

(l) $\int 5^{x^2-x+1}.(2x-1)dx;$

(m) $\int\frac{3x\,dx}{\sqrt{3x^2-1}};$

(n) $\int\frac{(2x-5)dx}{(x^2-5x)^3};$

(o) $\int\frac{x+2}{2x+3}dx;$

(p) $\int\frac{3x+2}{2x+1}dx;$

(q) $\int\frac{x+1}{\sqrt{x-1}}dx;$

(r) $\int\frac{4x+3}{\sqrt{2x-1}}dx;$

(s) $\int\frac{dx}{\sqrt{x+3}+\sqrt{x+2}};$

(t) $\int\frac{2x}{\sqrt{3x+5}+\sqrt{x+5}}dx;$

(u) $\int x\sqrt{x^2+1}\,x;$ (w) $\int \frac{(2x-3)\,dx}{\sqrt{2x^2-6x+1}};$ (x) $\int (2x+3)\sqrt{x^2+3x-1}\,dx.$

(v) $\int (2x-3)\sqrt{2x+5}\,dx;$

[Hints: (s) See worked-out Ex. 5 in Section .8.4.1.]

2. Evaluate:

(a) $\int \frac{dx}{e^x+1};$ (e) $\int \frac{x+1}{\sqrt[3]{x-1}}\,dx;$ (j) $\int e^x(x+1)\,dx;$

(b) $\int \frac{3^x}{3^x-1}\,dx;$ (f) $\int \frac{x^2dx}{\sqrt[3]{3x+5}};$ (k) $\int e^x\left(\log x+\frac{1}{x}\right)dx;$

(c) $\int \frac{\left(1+\frac{1}{x^2}\right)dx}{e^{x-\frac{1}{x}}};$ (g) $\int \frac{dx}{x(1+\log x)^2};$ (l) $\int \frac{dx}{1+e^x};$ [B.U. B.Com.(H) 2008]

(h) $\int \frac{(1+\log x)^5}{x}\,dx;$ (m) $\int \frac{x^4dx}{\sqrt[3]{2x^5+3}}.$ [C.U. B.Com.(H) 2005]

(d) $\int \frac{(x-2)\,dx}{\sqrt[3]{x^2-4x+5}};$ (i) $\int \frac{x^3}{x^2+4}\,dx;$

[Hints: (a) Put $e^{-x}+1=z$; then $I=\int \frac{e^{-x}}{e^{-x}-1}dx=-\int \frac{dz}{z}$. (j) Put $e^x\cdot x=z$; (k) Put $e^x\log x=z$.]

B

Evaluate each of the following integrals:

1. $\int \sqrt{c+dx}.dx.$

2. $\int \frac{dt}{\sqrt{a-bt}}.$

3. $\int \frac{x\,dx}{x^2+a^2}.$

4. $\int x\sqrt{x^2+a^2}\,dx.$

5. $\int 3x\sqrt{2x^2+5}\,dx.$

6. $\int \frac{x^3dx}{\sqrt{a^2+x^4}}.$

7. $\int \frac{x\,dx}{\sqrt{a^2-x^2}}.$

8. $\int (x+a)\left(x^2+2ax+b\right)^{3/2}dx.$

9. $\int xe^{x^2}dx;$

10. $\int x^2e^{x^3}dx.$

11. $\int \frac{e^{2x}}{e^x+1}\,dx.$ [C.U. B.Com. 2006]

12. $\int \frac{dx}{x(1+\log x)^4}.$

13. $\int \frac{(x+1)(x+\log x)^2}{x}\,dx.$

14. $\int (7x+5)^{7/8}\,dx.$

15. $\int \frac{dx}{(3-4x)^{2/3}}.$

16. $\int \frac{x}{\sqrt{3x-1}}\,dx.$

17. $\int \frac{x+3}{\sqrt[3]{x-3}}\,dx.$

18. $\int \frac{dx}{\sqrt{x}-\sqrt{x-1}}.$

19. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

20. $\int \frac{(x-2)\,dx}{\sqrt{2x^2-8x+5}}.$

21. $\int \frac{x\,dx}{\sqrt{5x+1}+\sqrt{x+1}}.$ [C.U. B.Com.(H) 1997]

22. $\int \frac{t^3 dt}{\sqrt[3]{t^4+3}}$. **23.** $\int \frac{xe^x}{(x+1)^2}dx$. [C.U. B.Com.(H) 2001]

24. Find a function whose derivative is $\frac{x^2}{x-1}$.

25. Find a function whose differential is $\left(\frac{3x+4}{4x+5}\right)dx$.

26. Evaluate: $\int \frac{x\,dx}{\sqrt{x+1}-\sqrt{x-1}}$. [C.U. B.Com.(H) 1996]

[Hints: Integrand $= \frac{x\left(\sqrt{x+1}+\sqrt{x-1}\right)}{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)} = \frac{x\left(\sqrt{x+1}+\sqrt{x-1}\right)}{2} = \frac{1}{2}\cdot\left(x\sqrt{x+1}+x\sqrt{x-1}\right)$

Integral $= \frac{1}{2}\left\{\int x\sqrt{x+1}dx + \int x\sqrt{x-1}dx\right\}$. Now put $x+1=u^2$ and $x-1=v^2$.]

27. $\int \frac{x^3dx}{(x^2+1)^3} = \int \frac{x^2\cdot xdx}{(x^2+1)^3} = \int \frac{(z-1)\times\frac{1}{2}dx}{z^3}$ etc. [N.B.U. B.Com.(H) 2006]

[Hints: Put $x^2+1=z$; then $2xdx = dz$ or, $xdx = \frac{1}{2}dz$. Also $x^2 = z-1$. The integral becomes $\int \frac{(z-1)\times\frac{1}{2}dz}{z^3} = \frac{1}{2}\int\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz =$ etc.]

28. If $g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\} = 0$ and $g(x)\neq 0$, find the relation between $f(x)$ and $g(x)$.

[N.B.U. B.Com.(H) 2006]

[Hints: $\frac{g(x)\cdot f'(x)-f(x)g'(x)}{\{g(x)\}^2} = 0$ or, $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = 0$. Integrating both sides w.r.t. x, we get $\frac{f(x)}{g(x)} = c$.]

C

1. (a) Determine the equations of the curve whose slope at any point (x,y) is $x:y$ with sign changed. Obtain the equation of that particular curve which passes through $(3,4)$.

   (b) At any point of a curve $\frac{d^2y}{dx^2} = x$, find the equation of the curve if it passes through $(3,0)$ and has the slope $7/2$ at that point.

   (c) Given: $dy = (2x+1)dx$ and $y=7$, when $x=1$. Find y, when $x=3$.

   (d) The slope of a curve at (x,y) is $9x$. It passes through the origin. Show that its equation is $9x^2 = 2y$.

   (e) The rate of change of sales of a product is proportional to the sale price $\left(\frac{dS}{dp} = -bp\right)$. Prove that the sales S is given by $S = a-(b/2)p^2$ (a is an arbitrary constant).

2. (a) Find the differentials of $3x^3+8x^2+7$; e^{5x}; $e^{m\log x}$.

   (b) Find y, if $\frac{dy}{dx} = 5x^{-a/b}$; $\frac{dy}{dx} = 0$; $\frac{dy}{dx} = 12$; $\frac{dy}{dt} = t^{-10}$.

(c) Given: $\dfrac{dv}{dt}=f$ (where f is a constant). Find v in terms of t, if it is known that $v=u$, when $t=0$.

Given: $\dfrac{ds}{dt}=u+ft$ (u, f are constants). Obtain a relation between s and t (given: $s=0$, when $t=0$).

Given: $v\dfrac{dv}{ds}=f$ (where f is a constant). Obtain a relation between v and s (given: $s=0$, when $v=u$).

(d) Let $\dfrac{d^2x}{dt^2}=2t-7$ and $\dfrac{dx}{dt}=v$. Given: $v=6$, when $x=0$, $t=0$. Obtain v and x in terms of t. Find t, when $v=0$.

(e) The rate of change of y w.r.t. x is $3x^2$. Given: $y=23$, when $x=3$. Show that $y=121$, when $x=5$.

(f) If $\dfrac{dy}{dx}=y^2$, prove that $xy+1=cy$, but if $\dfrac{dy}{dx}=x^2$, then $3y=x^3+c'$ (c, c' being arbitrary constants).

(g) If $\dfrac{ds}{dt}=3+7t-t^2$ and $s=1$, when $t=0$, obtain $s=\dfrac{1}{6}\left(18t+21t^2-2t^3+6\right)$. Find s, when $t=6$.

ANSWERS

A

[The constant of integration c, which is not written, is to be added to each answer]

1. (a) $\frac{1}{12}(2x+3)^6$;
(b) $\frac{1}{4}\left(x^2+5x\right)^4$;
(c) $\log|x^2+3x+1|$;
(d) $\log|\log x|$;
(e) $\frac{1}{3}(2x+5)^{3/2}$;
(f) $\frac{3}{4}(x+2)^{4/3}$;
(g) $-\frac{1}{10}(3-2x)^5$;
(h) $-\frac{1}{2(2x-3)}$;
(i) $\frac{1}{2}e^{2x+3}$;
(j) e^{x^2-3x+2};
(k) $\frac{1}{3}\cdot\frac{3^{3x-2}}{\log_e 3}$;
(l) $\frac{5^{x^2-x+1}}{\log_e 5}$
(m) $\sqrt{3x^2-1}$;
(n) $-\frac{1}{2(x^2-5x)^2}$;
(o) $\frac{1}{2}x+\frac{1}{4}\log|2x+3|$;
(p) $\frac{3}{2}x+\frac{1}{4}\log(2x+1)$;
(q) $2\left[\frac{1}{3}(x-1)^{3/2}+2\sqrt{x-1}\right]$;
(r) $\frac{1}{3}\sqrt{2x-1}(4x+13)$;
(s) $\frac{2}{3}\left[(x+3)^{3/2}-(x+2)^{3/2}\right]$;
(t) $\frac{2}{3}\left[\frac{1}{3}(3x+5)^{3/2}-(x+5)^{3/2}\right]$;
(u) $\frac{1}{3}\left(x^2+1\right)^{3/2}$;
(v) $\frac{1}{4}(2x+5)(2x-11)$;
(w) $\sqrt{2x^2-6x+1}$;
(x) $\frac{2}{3}\left(x^2+3x-1\right)^{3/2}$.

2. (a) $-\log\left|1+e^{-x}\right|$;
(b) $\frac{1}{\log_e 3}\log\left|3^x-1\right|$;
(c) $-\frac{1}{e^{x-\frac{1}{x}}}$;
(d) $\frac{3}{4}\left(x^2-4x+5\right)^{2/3}$;
(e) $\frac{3}{5}(x-1)^{5/3}+3(x-1)^{2/3}$;
(f) $\frac{1}{72}(3x+5)^{2/3}$ $\left\{(3x+5)^2-16(3x+5)+100\right\}$;
(g) $-\frac{1}{1+\log x}$;
(h) $\frac{1}{6}(1+\log x)^6$;
(i) $\frac{1}{2}\left[x^2+4-4\log\left(x^2+4\right)\right]$;
(j) xe^x;
(k) $e^x\log x$;
(l) $-\log\left(1+e^{-x}\right)$;
(m) $\frac{3}{20}\left(2x^5+3\right)^{2/3}$.

B

1. $\frac{2}{3d}(c+dx)^{3/2}$.
2. $-\frac{2}{b}\sqrt{a-bt}$.
3. $\frac{1}{2}\log\left(x^2+a^2\right)$.
4. $\frac{1}{3}\left(x^2+a^2\right)^{3/2}$.
5. $\frac{1}{2}\left(2x^2+5\right)^{3/2}$.
6. $\frac{1}{2}\sqrt{a^2+x^4}$.
7. $-\sqrt{a^2-x^2}$.
8. $\frac{1}{5}\left(x^2+2ax+b\right)^{5/2}$.
9. $\frac{1}{2}e^{x^2}$.
10. $\frac{1}{2}e^{x^3}$.
11. $e^x-\log(e^x+1)$.
12. $-\frac{1}{3(1+\log x)^3}$.
13. $\frac{1}{3}(x+\log x)^3$.
14. $\frac{8}{105}(7x+5)^{15/8}$.
15. $-\frac{3}{4}(3-4x)^{1/3}$.
16. $\frac{2}{27}(3x+2)\sqrt{3x-1}$.
17. $\frac{3}{5}(x+12)(x-3)^{2/3}$.
18. $\frac{2}{3}\left[x^{3/2}+(x-1)^{3/2}\right]$.
19. $2e^{\sqrt{x}}$.
20. $\frac{1}{2}\sqrt{2x^2-8x+15}$.
21. $\frac{1}{6}\left\{\frac{1}{5}(5x+1)^{3/2}-(x+1)^{3/2}\right\}$.
22. $\frac{3}{8}\left(t^4+3\right)^{2/3}$.
23. $e^x\cdot\frac{1}{x+1}$.
24. $y=\frac{1}{2}x^2+x+\log|x-1|$.
25. $\frac{3}{4}x+\frac{1}{16}\log|4x+5|$.
26. $\frac{1}{5}(x+1)^{5/2}-\frac{1}{3}(x+1)^{3/2}$ $+\frac{1}{5}(x-1)^{5/2}+\frac{1}{3}(x-1)^{3/2}$.
27. $-\frac{2x^2+1}{4(x^2+1)}$.
28. $f(x) = c\,g(x)$, where c is a constant.

C

1. (a) $x^2+y^2=c^2$; $x^2+y^2=25$;
   (b) $6y=x^3-6x-9$;
   (c) 17.
2. (a) $\left(9x^2+16x\right)dx$; $5e^{5x}dx$; $mx^{m-1}dx$;
   (b) $y=\frac{5b}{b-a}x^{(b-a)/b}$; $y=c$; $y=12x$; $y=-\frac{1}{9t^3}$;
   (c) $v=x+ft$, $s=ut+\frac{1}{2}ft^2$; $v^2=u^2+2fs$;
   (d) $v=t^2-7t+6$; $x=\frac{1}{3}t^3-\frac{7}{8}t^2+6t$; $v=0$, when $t=1$ or 6.

8.5 Some Standard Integrals

1. $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c.$$
2. $$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c.$$
3. $$\int\frac{dx}{\sqrt{x^2\pm a^2}}=\log\left|x+\sqrt{x^2\pm a^2}\right|+c.$$

Proof of 1.
$$\begin{aligned}\int\frac{dx}{x^2-a^2}&=\int\frac{dx}{(x+a)(x-a)}=\int\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx\\&=\frac{1}{2a}\left[\int\frac{1}{x-a}dx-\int\frac{1}{x+a}dx\right]=\frac{1}{2a}\left[\log|x-a|-\log|x+a|\right]+c\\&=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c.\end{aligned}$$

Proof of 2.
$$\begin{aligned}\int\frac{dx}{a^2-x^2}&=\int\frac{dx}{(a+x)(a-x)}=\int\frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx\\&=\frac{1}{2a}\left[\int\frac{1}{a+x}dx-\int\frac{-1}{a-x}dx\right]=\frac{1}{2a}\left[\log|a+x|-\log|a-x|\right]+c\\&=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c.\end{aligned}$$

Proof of 3. $I=\int \dfrac{dx}{\sqrt{x^2\pm a^2}}$.

Put $x+\sqrt{x^2\pm a^2}=z$; then

$$\left\{1+\frac{1}{2}\left(x^2\pm a^2\right)^{-1/2}\times 2x\right\}dx=dz \text{ or, } \left(1+\frac{x}{\sqrt{x^2\pm a^2}}\right)dx=dz$$

$$\text{or, } \frac{\sqrt{x^2\pm a^2}+x}{\sqrt{x^2\pm a^2}}\,dx=dz \text{ or, } \frac{z}{\sqrt{x^2\pm a^2}}\,dx=dz \text{ or, } \frac{dx}{\sqrt{x^2\pm a^2}}=\frac{dz}{z}.$$

$$I=\int\frac{dz}{z}=\log|z|+c=\log\left|x+\sqrt{x^2\pm a^2}\right|+c.$$

General Rules

- **For integrals of the form:** $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$. (1)

  Write: $ax^2+bx+c=a[(x\pm p)^2\pm q^2]$, where p, q are to be suitably determined. Then the integral will reduce to the standard form:

  $$\frac{1}{\sqrt{a}}\int\frac{du}{\sqrt{u^2\pm q^2}}, \text{ where } u=x\pm p.$$

- **For integrals of the form:** $\displaystyle\int\frac{(px+q)dx}{\sqrt{ax^2+bx+c}}$.

  Write: $px+q=l\left[\dfrac{d}{dx}\left(ax^2+bx+c\right)\right]+m$, where the constants l, m are to be determined by comparing the coefficients.

  Then the given integral $=l\underbrace{\displaystyle\int\frac{\frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}dx}_{I_1}+m\underbrace{\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}}_{I_2}$.

  For I_1, put $ax^2+bx+c=z^2$ and I_2 is similar to No. 1 above.

- **For integrals of the form:** $\displaystyle\int\frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$.

  Put $px+q=\dfrac{1}{z}$ and simplify. The integral will be reduced to a form which we have already discussed.

Example 1. *Evaluate:* (i) $\displaystyle\int\frac{dx}{x^2-4}$; (ii) $\displaystyle\int\frac{dx}{16-x^2}$; (iii) $\displaystyle\int\frac{dx}{\sqrt{x^2+9}}$.

Solution:

(i) $\int \frac{dx}{x^2-4} = \int \frac{dx}{x^2-2^2}$, which is of the form $\int \frac{dx}{x^2-a^2}$

$$= \frac{1}{2\cdot 2}\log\left|\frac{x-2}{x+2}\right| + c = \frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + c.$$

(ii) $\int \frac{dx}{16-x^2} = \int \frac{dx}{4^2-x^2}$, which is of the form $\int \frac{dx}{a^2-x^2}$

$$= \frac{1}{2\cdot 4}\log\left|\frac{4+x}{4-x}\right| + c = \frac{1}{8}\log\left|\frac{4+x}{4-x}\right| + c.$$

(iii) $\int \frac{dx}{\sqrt{x^2+9}} = \int \frac{dx}{\sqrt{x^2+3^2}}$, which is of the form $\int \frac{dx}{\sqrt{x^2+a^2}}$

$$= \log\left|x+\sqrt{x^2+3^2}\right| + c = \log\left|x+\sqrt{x^2+9}\right| + c.$$

Example 2. *Evaluate*: (i) $\int \frac{dx}{x^2+4x-5}$; (ii) $\int \frac{dx}{7+6x-x^2}$.

Solution:

(i) $\int \frac{dx}{x^2+4x-5} = \int \frac{dx}{(x+2)^2-3^2}$, which is of the form $\int \frac{dx}{x^2-a^2}$

$$\begin{aligned} & x^2+4x-5 \\ &= x^2+4x+4-9 \\ &= (x+2)^2-3^2 \end{aligned}$$

$$= \frac{1}{2\cdot 3}\log\left|\frac{(x+2)-3}{(x+2)+3}\right| + c = \frac{1}{6}\log\left|\frac{x-1}{x+5}\right| + c.$$

(ii) $7+6x-x^2 = 7-\left(x^2-6x\right) = 7-\left(x^2-2\cdot 3\cdot x+3^2-9\right) = 7+9-(x-3)^2 = 4^2-(x-3)^2.$

$\therefore \int \frac{dx}{7+6x-x^2} = \int \frac{dx}{4^2-(x-3)^2}$, which is of the form $\int \frac{dx}{a^2-x^2}$

$$= \frac{1}{2\cdot 4}\log\left|\frac{4+(x-3)}{4-(x-3)}\right| + c = \frac{1}{8}\log\left|\frac{1+x}{7-x}\right| + c.$$

Example 3. *Evaluate*: (i) $\int \frac{dx}{\sqrt{x^2-2x+5}}$; (ii) $\int \frac{dx}{\sqrt{4x^2+16x-20}}$.

Solution: We have

(i) $\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+2^2}}$, which is of the form $\int \frac{dx}{\sqrt{x^2+a^2}}$

$$= \log\left|(x-1)+\sqrt{(x-1)^2+2^2}\right| + c = \log\left|(x-1)+\sqrt{x^2-2x+5}\right| + c.$$

(ii) $$\int \frac{dx}{\sqrt{4x^2+16x-20}} = \int \frac{dx}{\sqrt{4(x^2+4x-5)}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+4x-5}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{(x+2)^2-3^2}}, \text{ which is of the form } \frac{1}{2}\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= \frac{1}{2}\log\left|(x+2)+\sqrt{(x+2)^2-3^2}\right|+c = \mathbf{\frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+c.}$$

Example 4. *Evaluate:* $\displaystyle\int \frac{dx}{x(\log x)^2+4x\log x-12x}$. [C.U. B.Com.(H) 1996]

Solution: We have

$$\int \frac{dx}{x(\log x)^2+4x\log x-12x} = \int \frac{dx}{x\left[(\log x)^2+4\log x-12\right]} \quad \left|\begin{array}{l}\text{Put } \log x = z;\\ \text{then } \frac{1}{x}dx = dz.\end{array}\right.$$

$$= \int \frac{dz}{z^2+4z-12} = \int \frac{dz}{(z+2)^2-4^2}, \text{ which is of the form } \int \frac{dx}{x^2-a^2}$$

$$= \frac{1}{2\cdot 4}\log\left|\frac{(z+2)-4}{(z+2)+4}\right|+c = \mathbf{\frac{1}{8}\log\left|\frac{\log x-2}{\log x+6}\right|+c.}$$

Example 5. *Evaluate:* $\displaystyle\int \frac{(x^2+5x+2)\,dx}{(x+2)(x+3)}$. [C.U. B.Com.(H) 2001]

Solution:

$$\int \frac{(x^2+5x+2)\,dx}{(x+2)(x+3)} = \int \frac{(x^2+5x+6-4)}{x^2+5x+6}dx = \int\left(1-\frac{4}{x^2+5x+6}\right)dx = \int 1dx - 4\int \frac{dx}{x^2+5x+6}$$

$$= x-4\int \frac{dx}{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}+c$$

$$\left[\text{since } x^2+5x+6 = x^2+2\cdot\frac{5}{2}\cdot x+\left(\frac{5}{2}\right)^2+6-\frac{25}{4} = \left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]$$

$$= x-4\cdot\frac{1}{2\cdot\frac{1}{2}}\log\left|\frac{\left(x+\frac{5}{2}\right)-\frac{1}{2}}{\left(x+\frac{5}{2}\right)+\frac{1}{2}}\right|+c = \mathbf{x-4\log\left|\frac{x+2}{x+3}\right|+c.}$$

Example 6. *Evaluate:* $\displaystyle\int \frac{dt}{2t^2+3t+1}$. [C.U. B.Com.(H) 1999]

Solution:

$$2t^2+3t+1 = 2\left(t^2+\frac{3}{2}t+\frac{1}{2}\right) = 2\left\{t^2+2\cdot t\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\frac{9}{16}+\frac{1}{2}\right\} = 2\left\{\left(t+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2\right\}.$$

$$\therefore \int \frac{dt}{2t^2+3t+1} = \int \frac{dt}{2\left\{\left(t+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2} = \frac{1}{2} \times \frac{1}{2\cdot\frac{1}{4}} \log\left|\frac{\left(t+\frac{3}{4}\right)-\frac{1}{4}}{\left(t+\frac{3}{4}\right)+\frac{1}{4}}\right| + c$$

$$= \log\left|\frac{t+\frac{1}{2}}{t+1}\right| + c = \mathbf{\log\left|\frac{2t+1}{2(t+1)}\right| + c}.$$

Note: This problem can also be solved by using partial fractions.

Example 7. *Evaluate:* $\displaystyle\int \frac{(x-2)}{\sqrt{x^2-6x+2}}\, dx.$

Solution: Let
$$x-2 = l\frac{d}{dx}\left(x^2-6x+2\right) + m = l(2x-6) + m = 2lx + m - 6l.$$

Equating coefficients of x and constant terms from both sides, we get $2l = 1$ and $m - 6l = -2$; $\therefore\ l = \frac{1}{2}$ and $m - 6 \times \frac{1}{2} = -2 \Rightarrow m = 1.$

$$\therefore \int \frac{(x-2)\,dx}{\sqrt{x^2-6x+2}} = \int \frac{l(2x-6)+m}{\sqrt{x^2-6x+2}}\,dx = \frac{1}{2}\int \frac{(2x-6)\,dx}{\sqrt{x^2-6x+2}} + 1\cdot\int \frac{dx}{\sqrt{x^2-6x+2}}$$

$$= \frac{1}{2}\int \frac{2z\,dz}{z} + \int \frac{dx}{\sqrt{(x-3)^2-\left(\sqrt{7}\right)^2}} \qquad \left|\ \begin{array}{l}\text{Put } x^2-6x+2 = z^2;\\ \text{then } (2x-6)\,dx = 2z\,dz.\end{array}\right.$$

$$= \int dz + \log\left|(x-3) + \sqrt{(x-3)^2-\left(\sqrt{7}\right)^2}\right| + c = z + \log\left|(x-3)+\sqrt{x^2-6x+2}\right| + c$$

$$= \mathbf{\sqrt{x^2-6x+2} + \log\left|(x-3)+\sqrt{x^2-6x+2}\right| + c}.$$

Example 8. *Evaluate:* $\displaystyle\int \frac{2x-8}{\sqrt{1+x+x^2}}\, dx.$ **[C.U. B.Com.(H) 2000]**

Solution: $\displaystyle I = \int \frac{2x-8}{\sqrt{1+x+x^2}}\,dx = \int \frac{(2x+1)-9}{\sqrt{1+x+x^2}}\,dx = \int \frac{(2x+1)\,dx}{\sqrt{1+x+x^2}} - 9\int \frac{dx}{\sqrt{1+x+x^2}}.$

Put $1+x+x^2 = z^2$ in the first integral; then $(2x+1)\,dx = 2z\,dz$.

Also $1+x+x^2 = \left(x^2+x\right)+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2.$

$$\therefore I = \int \frac{2z\,dz}{z} - 9\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} = 2\int dz - 9\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right| + c$$

$$= 2z - 9\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + c = \mathbf{2\sqrt{1+x+x^2} - 9\log\left|\left(x+\frac{1}{2}\right)+\sqrt{1+x+x^2}\right| + c}.$$

Example 9. *Evaluate:* $\int \frac{x}{2x^4-3x^2-2}dx.$ [C.U. B.Com.(H) 1995]

Solution:

$$\int \frac{x}{2x^4-3x^2-2}dx = \int \frac{\frac{1}{2}dz}{2z^2-3z-2} \qquad \Big| \text{ Put } x^2=z;\text{ then } 2x\,dx=dz \text{ or, } x\,dx=\frac{1}{2}dz$$

$$= \frac{1}{2}\int \frac{dz}{2\left(z^2-\frac{3}{2}z-1\right)} \qquad \left| z^2-\frac{3}{2}z-1 = z^2-2\cdot z\cdot\frac{3}{4}+\frac{9}{16}-\frac{9}{16}-1 = \left(z-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2 \right.$$

$$= \frac{1}{4}\int \frac{dz}{\left(z-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2} = \frac{1}{4}\times\frac{1}{2\cdot\frac{5}{4}}\log\left|\frac{\left(z-\frac{3}{4}\right)-\frac{5}{4}}{\left(z-\frac{3}{4}\right)+\frac{5}{4}}\right|+c$$

$$= \frac{1}{10}\log\left|\frac{z-2}{z+\frac{1}{2}}\right|+c = \mathbf{\frac{1}{10}\log\left|\frac{2(x^2-2)}{2x^2+1}\right|+c.}$$

Example 10. *Evaluate:* $\int \frac{dx}{(1+x)\sqrt{1-x^2}}.$ [C.U. B.Com.(H) 1994]

Solution:

$$\int \frac{dx}{(1+x)\sqrt{1-x^2}} \qquad \Big| \text{ Put } 1+x=\tfrac{1}{z};\text{ then } dx=-\tfrac{1}{z^2}dz,\; x=\tfrac{1}{z}-1=\tfrac{1-z}{z};\ \therefore\ 1-x^2=1-\tfrac{(1-z)^2}{z^2}=\tfrac{2z-1}{z^2}.$$

$$= \int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{2z-1}{z^2}}} = -\int \frac{dz}{\sqrt{2z-1}} = -\int \frac{u\,du}{u} \qquad \Big| \text{ Put } 2z-1=u^2,\text{ then } 2dz=2u\,du \text{ or, } dz=u\,du$$

$$= -\int du = -u+c = -\sqrt{2z-1}+c \quad \left[\because\ u^2=2z-1 \text{ gives } u=\sqrt{2z-1}\right]$$

$$= -\sqrt{2\cdot\frac{1}{1+x}-1}+c = -\sqrt{\frac{2-1-x}{1+x}}+c \quad \left[\because\ 1+x=\frac{1}{z};\ \therefore\ z=\frac{1}{1+x}\right]$$

$$= \mathbf{-\sqrt{\frac{1-x}{1+x}}+c.}$$

EXERCISES ON CHAPTER 8(III)

(Integration by Substitution and Standard Formulae)

A

1. Evaluate:

(a) $\int \frac{dx}{x^2-4}$; (b) $\int \frac{dx}{25-x^2}$; (c) $\int \frac{dx}{\sqrt{x^2-16}}$; (d) $\int \frac{dx}{\sqrt{x^2+25}}$.

2. Integrate each of the following functions:

(a) $\dfrac{3}{x^2-1}$; [C.U. B.Com.(H) 1993]
(b) $\dfrac{1}{x^2+2x-3}$;
(c) $\dfrac{1}{5+4x-x^2}$;
(d) $\dfrac{4}{5+8x-4x^2}$;
(e) $\dfrac{1}{x^2-3x+1}$.

3. Evaluate:

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+10}}$;
(b) $\displaystyle\int \frac{dx}{\sqrt{x^2-4x-5}}$;
(c) $\displaystyle\int \frac{dx}{\sqrt{2x^2-8x+26}}$;
(d) $\displaystyle\int \frac{dx}{x\left\{10+7\log x+(\log x)^2\right\}}$;
(e) $\displaystyle\int \frac{dx}{4x-x^2}$;
(f) $\displaystyle\int \frac{dx}{\sqrt{x^2-2ax}}$;
(g) $\displaystyle\int \frac{dx}{x^2+5x+1}$;
(h) $\displaystyle\int \frac{dx}{2x^2-4x-7}$;
(i) $\displaystyle\int \frac{dt}{2t^2+5t+1}$;
(j) $\displaystyle\int \frac{e^x\,dx}{e^{2x}-4e^x-5}$;
(k) $\displaystyle\int \frac{(x-2)\,dx}{\sqrt{x^2+4x+3}}$;
(l) $\displaystyle\int \frac{x\,dx}{x^2-3x-4}$; [C.U. B.Com.(H) 1994]
(m) $\displaystyle\int \frac{(x^2+3x+1)}{(x+1)(x+2)}dx$;
(n) $\displaystyle\int \frac{(x^2+7x+4)}{(x+3)(x+4)}$; [C.U. B.Com.(H) 2001]
(o) $\displaystyle\int \frac{dx}{(1+x)\sqrt{x^2+1}}$; [C.U. B.Com.(H) 2000]
(p) $\displaystyle\int \frac{dx}{(1-x)\sqrt{1+x^2}}$;
(q) $\displaystyle\int \frac{dx}{(1-x)\sqrt{1-x^2}}$;
(r) $\displaystyle\int \frac{dx}{x(10+7\log x)+(\log x)^2}$. [V.U. B.Com.(H) 2011]

[Hints: (r) $\displaystyle\int \frac{dx}{x(10+7\log x)+(\log x)^2} = \int \frac{dz}{10+7z+z^2}$.

(Put $\log x = z$; then $\frac{1}{x}dx = dz$. $10+7z+z^2 = z^2+2\cdot\frac{7}{2}z+\left(\frac{7}{2}\right)^2+10-\frac{49}{4} = \left(z+\frac{7}{2}\right)^2-\frac{9}{4} = \left(z+\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2$

$= \displaystyle\int \frac{dz}{\left(z+\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2} = \frac{1}{2\cdot\frac{3}{2}}\log\left|\frac{\left(z+\frac{7}{2}\right)-\frac{3}{2}}{\left(z+\frac{7}{2}\right)+\frac{3}{2}}\right|+c = \frac{1}{3}\log\left|\frac{z+2}{z+5}\right|+c.$]

B

Integrate the following w.r.t. the independent variable involved:

1. Evaluate:

(a) $\displaystyle\int x^3\left(1+x^2\right)^{1/3}dx$;
(b) $\displaystyle\int \frac{e^t-1}{e^t+1}dx$;
(c) $\displaystyle\int \frac{1}{e^{2x}+1}dx$;
(d) $\displaystyle\int \frac{x^2-1}{x^2}e^{x+1/x}dx$;
(e) $\displaystyle\int z^3\left(1+3z^4\right)^{1/2}dx$;
(f) $\displaystyle\int \frac{1}{\sqrt{5-6x}}dx$;
(g) $\displaystyle\int \frac{1}{x^2(3+4x)^2}dx$. [Put $(3+4x)/x = z$.]

2. Verify the following by evaluating the integrals:

(a) $\int \frac{dx}{x^2-16} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + c; \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c;$

(b) $\int \frac{e^x\,dx}{\sqrt{e^{2x}+1}} = \log\left|e^x + \sqrt{e^{2x}+1}\right| + c;$

(c) $\int \frac{dx}{\sqrt{x^2+10x+29}} = \log\left|(x+5) + \sqrt{x^2+10x+29}\right| + c;$

(d) $\int \frac{1}{4x^2+8x+3}\,dx = \frac{1}{4}\log\left|\frac{2x+1}{2x+3}\right| + c;$

(e) $\int \frac{x^2\,dx}{x^2-3x+2} = x + 4\log|x-2| - \log|x-1| + c;$ [C.U. B.Com.(H) 1997]

(f) $\int \frac{dx}{\sqrt{3x^2+2x+1}} = \frac{1}{\sqrt{3}}\log\left|x + \frac{1}{3} + \sqrt{x^2 + \frac{2}{3}x + \frac{1}{3}}\right| + c;$

(g) $\int \frac{(3x-2)\,dx}{1-6x-9x^2} = -\frac{1}{6}\log\left(1-6x-9x^2\right) + \frac{1}{2\sqrt{2}}\log\frac{3x+1-\sqrt{2}}{3x+1+\sqrt{2}} + c;$

(h) $\int \frac{x\,dx}{\sqrt{3x^2+2x+1}} = \frac{1}{3}\sqrt{3x^2+2x+1} - \frac{1}{3\sqrt{3}}\log\left\{x + \frac{1}{3} + \sqrt{x^2 + \frac{2}{3}x + \frac{1}{3}}\right\} + c;$

(i) $\int \frac{x^2\,dx}{x^6-6x^3+5} = \frac{1}{12}\log\frac{x^3-5}{x^3-1} + c;$

(j) $\int \sqrt{\frac{x+1}{x-1}}\,dx = \sqrt{x^2-1} + \log\left|x + \sqrt{x^2-1}\right| + c;$

(k) $\int \sqrt{\frac{3+x}{x}}\,dx = 3\log\left(\sqrt{x} + \sqrt{x+3}\right) + \sqrt{x(x+3)} + c;$

(l) $\int \frac{dx}{\sqrt{(x-a)(x-b)}} = 2\log\left(\sqrt{x-a} + \sqrt{x-b}\right) + c;$

(m) $\int \frac{dx}{\sqrt{x^2+2ax}} = \log\left(x + a + \sqrt{x^2+2ax}\right) + c;$

(n) $\int \frac{dx}{(1-x)\sqrt{1+x}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{1+x}+\sqrt{2}}{\sqrt{2}-\sqrt{1+x}} + c$ (Put $1+x=z^2$)

(o) $\int \frac{dx}{x\sqrt{x^2+1}} = \log\left|\frac{x}{1+\sqrt{x^2+1}}\right| + c;$

(p) $\int \frac{dx}{x\sqrt{1-x^2}} = -\log\left|\frac{1+\sqrt{1-x^2}}{x}\right| + c;$ [C.U. B.Com.(H) 1999]

(q) $\displaystyle\int \frac{dt}{\sqrt{2-3^t}}$. [V.U. B.Com.(H) 2010]

[Hints: Put $2-3^t=z^2$; then $-3^t\log 3dt=2zdt$ or, $dt=\dfrac{2z\,dz}{-3^t\log 3}=\dfrac{2z\,dz}{-(2-z^2)\log 3}$.

$\therefore \displaystyle\int\frac{dt}{\sqrt{2-3^t}}=\int\frac{2z\,dz}{-(2-z^2)\log 3\cdot z}=-\frac{2}{\log 3}\int\frac{dz}{\left(\sqrt{2}\right)^2-z^2}=-\frac{2}{\log 3}\times\frac{1}{2\sqrt{2}}\times\log\left|\frac{\sqrt{2}+z}{\sqrt{2}-z}\right|+c$, etc.]

ANSWERS

A

[Constant of integration c is to be added to each answer]

1. (a) $\frac{1}{4}\log\left|\frac{x-2}{x+2}\right|$;
   (b) $\frac{1}{10}\log\left|\frac{5+x}{5-x}\right|$;
   (c) $\log\left|x+\sqrt{x^2-16}\right|$;
   (d) $\log\left|x+\sqrt{x^2+25}\right|$.
2. (a) $\frac{3}{2}\log\left|\frac{x-1}{x+1}\right|$;
   (b) $\frac{1}{4}\log\left|\frac{x-1}{x+3}\right|$;
   (c) $\frac{1}{6}\log\left|\frac{x+1}{5-x}\right|$;
   (d) $\frac{1}{3}\log\left|\frac{1+2x}{5-2x}\right|$;
   (e) $\frac{1}{\sqrt{5}}\log\left|\frac{2x-3-\sqrt{3}}{2x-3+\sqrt{5}}\right|$.
3. (a) $\log\left|(x+1)+\sqrt{x^2+2x+10}\right|$;
   (b) $\log\left|(x-2)+\sqrt{x^2-4x-5}\right|$;
   (c) $\frac{1}{\sqrt{2}}\log\left|(x-2)+\sqrt{x^2-4x+13}\right|$;
   (d) $\frac{1}{3}\log\left|\frac{\log x+5}{\log x+2}\right|$;
   (e) $\frac{1}{4}\log\left|\frac{x}{4-x}\right|$;
   (f) $\log\left|(x-a)+\sqrt{x^2-2ax}\right|$;
   (g) $\frac{1}{\sqrt{21}}\log\left|\frac{2x-5-\sqrt{21}}{2x-5+\sqrt{21}}\right|$;
   (h) $\frac{\sqrt{2}}{12}\log\left|\frac{\sqrt{2}x-\sqrt{2}-3}{\sqrt{2}x-\sqrt{2}+3}\right|$;
   (i) $\frac{1}{\sqrt{17}}\log\left|\frac{4t+5-\sqrt{17}}{4t+5+\sqrt{17}}\right|$;
   (j) $\frac{1}{6}\log\left|\frac{e^x-5}{e^x+1}\right|$;
   (k) $\sqrt{x^2+4x+3}-8\log\left|\sqrt{x+1}+\sqrt{x+3}\right|$;
   (l) $\frac{1}{2}\log|x^2-3x-4|$
   $+\frac{3}{10}\log\left|\frac{x-4}{x+1}\right|$;
   (m) $x-\log\left|\frac{x+1}{x+2}\right|$;
   (n) $x-\log\left|\frac{x+3}{x+4}\right|$;
   (o) $-\frac{1}{\sqrt{2}}\log\left|\frac{1-x}{2(1+x)}+\frac{\sqrt{2(1+x^2)}}{2(x+1)}\right|4$;
   (p) $\frac{1}{\sqrt{2}}\log\left|\frac{1-x}{2(1+x)}+\frac{\sqrt{2(1+x^2)}}{2(x+1)}\right|$;
   (q) $\sqrt{\frac{1+x}{1-x}}$;
   (r) $\frac{1}{3}\log\left|\frac{\log x+2}{\log x+5}\right|$.

B

1. (a) $\frac{3}{2}\left[\frac{1}{7}\left(1+x^2\right)^{7/8}-\frac{1}{4}\left(1+x^2\right)^{4/3}\right]$;
   (b) $2\log_e\left(e^{t/2}+e^{-t/2}\right)$;
   (c) $-\log\left(1+e^{-2x}\right)$;
   (d) $e^{x+\frac{1}{x}}$;
   (e) $\frac{1}{18}\left(1+3x^4\right)^{3/2}$;
   (f) $-\frac{1}{3}(5-6x)^{1/2}$;
   (g) $\frac{8}{27}\log\left|\frac{3+4x}{x}\right|-\frac{3+8x}{9x(3+4x)}$.

Chapter 9

Integration by Parts and by Using Partial Fractions

9.1 Introduction

We shall first study *Integration by Parts* by using two Rules and a few important formulae and then discuss Integration by *using Partial Fractions*. We shall show applications of both the topics by giving various illustrations and worked-out examples.

I. Rule of Integration by Parts: $$\int u\,dv = uv - \int v\,du,$$

where u, v are functions of x.

If u and v are two functions of x, then

$$d(uv) = u\,dv + v\,du \text{ or, } \int d(uv) = \int u\,dv + \int v\,du \text{ or, } \int u\,dv = uv - \int v\,du.$$

This may be considered as the *Rule of Integration by Parts*. A few illustrations are given below:

Illustration 1. $\int \log x\,dx = x\log x - \int x\,d(\log x)$ $\quad$ [*here* $u = \log x, v = x$]

$$= x\log x - \int x\cdot\frac{1}{x}dx = x\log x - x + c = x(\log x - 1) + c$$ **[C.U. B.Com.(H) 1998]**

Illustration 2. $\int xe^x\,dx = \int x\,d(e^x)$ $\quad$ [*here* $u = x, v = e^x$]

$$= xe^x - \int e^x\,dx = xe^x - e^x + c = e^x(x-1) + c.$$

Illustration 3. $\int e^x(1+x)\log(xe^x)dx = \int \log(xe^x)\,d(xe^x)$ $\quad$ | *See that* $d(xe^x) = (e^x + xe^x)dx$; *we take* $u = \log(xe^x)$ *and* $v = xe^x$

$$= xe^x\log(xe^x) - \int xe^x\,d\{\log(xe^x)\}$$

$$= xe^x\log(xe^x) - \int xe^x\cdot\frac{1}{xe^x}d(xe^x)$$

$$= xe^x\log(xe^x) - xe^x + c.$$

9.1.1 Two Standard Formulae

I. $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c;$$

II. $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c.$$

Formula I. To evaluate $\int \sqrt{x^2+a^2}\,dx = I$ (say) we use Rule of Integration by Parts,

$$I = \int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int x\,d\left(\sqrt{x^2+a^2}\right) \quad \left[\text{here } u=\sqrt{x^2+a^2}, v=x\right]$$

$$= x\sqrt{x^2+a^2} - \int x\cdot\frac{x}{\sqrt{x^2+a^2}}\,dx = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}}.$$

$$\therefore\ 2I = x\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right| \text{ or, } I = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c.$$

Formula II. It can be obtained exactly in a similar manner.

Applications

Example 1. (i) $$\int \sqrt{x^2+16}\,dx = \frac{x\sqrt{x^2+16}}{2} + 8\log\left|x+\sqrt{x^2+16}\right| + c.$$

(ii) $$\int \sqrt{x^2-9}\,dx = \frac{x\sqrt{x^2-9}}{2} - \frac{9}{2}\log\left|x+\sqrt{x^2-9}\right| + c.$$

Example 2.

$$\int \sqrt{25x^2+16}\,dx = 5\int \sqrt{x^2+\frac{16}{25}}\,dx = 5\int \sqrt{x^2+\left(\frac{4}{5}\right)^2}\,dx$$

$$= 5\left[\frac{x\sqrt{x^2+\left(\frac{4}{5}\right)^2}}{2} + \frac{16}{25\times 2}\log\left|x+\sqrt{x^2+\left(\frac{4}{5}\right)^2}\right|\right] + c$$

$$= \frac{x\sqrt{25x^2+16}}{2} + \frac{8}{5}\log\left|\frac{5x+\sqrt{25x^2+16}}{5}\right| + c.$$

9.1.2 Another Form of the Rule of Integration by Parts

$$\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{d(u)}{dx}\cdot\int v\,dx\right\}dx$$

Proof. In $\int u\,dv = uv - \int v\,du$, put $u = f(x)$ and $dv = g(x)dx$ so that $v = \int g(x)dx$ and we obtain

$$\int f(x).g(x)dx = f(x)\int g(x)dx - \int\left\{\int g(x)dx\right\}d\{f(x)\}$$

$$= f(x)\int g(x)dx - \int\left\{f'(x)\cdot\int g(x)dx\right\}dx \quad [\because d\{f(x)\} = f'(x)dx]$$

Thus, the integral of the product of two functions

$$= \textbf{1st function} \times \textbf{Integral of the 2nd} - \int \{\textbf{(Derivative of 1st)} \times \textbf{Integral of 2nd}\}$$

Example 3. *Evaluate*:

(i) $\int xe^x\,dx$; [C.U. B.Com.(H) 1991] (ii) $\int x\log x\,dx$. [C.U. B.Com.(H) 2007]

Solution:

(i) $$\int xe^x\,dx = x\int e^x\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int e^x\,dx\right\}dx = xe^x - \int(1.e^x)dx + c = xe^x - e^x + c$$

$$= \boldsymbol{e^x(x-1)+c.}$$

(ii) $$\int x\log x\,dx = \int \log x\cdot x\,dx = \log x\int x\,dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int x\,dx\right\}dx$$

$$= \log x\cdot\frac{x^2}{2} - \int\left(\frac{1}{x}\times\frac{x^2}{2}\right)dx + c = \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx + c$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2} + c = \boldsymbol{\frac{x^2}{2}\left(\log x - \frac{1}{2}\right)+c.}$$

Note: As $\log x$ cannot be directly integrated (it can be differentiated), we have interchanged the two functions, i.e., we have considered x as the first function and $\log x$ as the second function in the above integral.

Example 4. *Evaluate*: (i) $\int \log x\,dx$; (ii) $\int x^n\log x\,dx\ (n \neq -1, x > 0)$.

Solution:

(i) $$\int \log x\,dx = \int \log x.1dx = \log x\int 1dx - \int\left\{\frac{d}{dx}(\log x).\int 1dx\right\}dx$$

$$= \log x\cdot x - \int\left(\frac{1}{x}\times x\right)dx + c = x\log x - \int 1dx + c = x\log x - x + c$$

$$= \boldsymbol{x(\log x - 1) + c.}$$

(ii) $\int x^n \log x\,dx = \int \log x \cdot x^n\,dx = \log x \int x^n\,dx - \int \left\{\frac{d(\log x)}{dx} \cdot \int x^n\,dx\right\} dx$

$= \log x \cdot \frac{x^{n+1}}{n+1} - \int \left(\frac{1}{x} \times \frac{x^{n+1}}{n+1}\right) dx + c = \frac{x^{n+1}}{n+1} \log x + \frac{1}{n+1} \int x^n\,dx + c$

$= \frac{x^{n+1}}{n+1} \log x + \frac{1}{n+1} \times \frac{x^{n+1}}{n+1} + c = \boldsymbol{\frac{x^{n+1}}{n+1}\left(\log x + \frac{1}{n+1}\right) + c.}$

Example 5. *Evaluate*:

(i) $\int x^2 e^{3x}\,dx;$ [C.U. B.Com.(H) 1997] (ii) $\int \left(x^2 - 2x + 5\right) e^{-x}\,dx.$ [C.U. B.Com.(H) 2000]

Solution:

(i) $\int x^2 e^{3x}\,dx = x^2 \int e^{3x}\,dx - \int \left\{\frac{d(x^2)}{dx} \cdot \int e^{3x}\,dx\right\} dx = x^2 \cdot \frac{e^{3x}}{3} - \int \left(2x \cdot \frac{e^{3x}}{3}\right) dx + c$

$= \frac{x^2}{3} e^{3x} - \frac{2}{3} \int x e^{3x}\,dx + c = \frac{x^2}{3} e^{3x} - \frac{2}{3}\left[x \int e^{3x}\,dx - \int \left\{\frac{d(x)}{dx} \cdot \int e^{3x}\,dx\right\} dx\right] + c$

$= \frac{x^2}{3} e^{3x} - \frac{2}{3}\left[x \cdot \frac{e^{3x}}{3} - \int \left(1 \cdot \frac{e^{3x}}{3}\right) dx\right] + c = \frac{x^2}{3} e^{3x} - \frac{2}{3}\left[\frac{x}{3} \cdot e^{3x} - \frac{1}{3} \cdot \frac{e^{3x}}{3}\right] + c$

$= \frac{x^2}{3} e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c = \boldsymbol{\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) + c.}$

(ii) $\int \left(x^2 - 2x + 5\right) e^{-x}\,dx = \left(x^2 - 2x + 5\right) \int e^{-x}\,dx - \int \left\{\frac{d}{dx}\left(x^2 - 2x + 5\right) \cdot \int e^{-x}\,dx\right\} dx$

$= -\left(x^2 - 2x + 5\right) e^{-x} - \int \left\{(2x - 2) \times \frac{e^{-x}}{-1}\right\} dx + c$

$= -\left(x^2 - 2x + 5\right) e^{-x} + 2\left[(x-1) \int e^{-x}\,dx - \int \left\{\frac{d}{dx}(x-1) \cdot \int e^{-x}\,dx\right\}\right] + c$

$= -\left(x^2 - 2x + 5\right) e^{-x} + 2\left[(x-1) \times \frac{e^{-x}}{-1} - \int \left\{1 \cdot \left(\frac{e^{-x}}{-1}\right)\right\} dx\right] + c$

$= -\left(x^2 - 2x + 5\right) e^{-x} - 2(x-1)e^{-x} + 2\left(\frac{e^{-x}}{-1}\right) + c$

$= -\left(x^2 - 2x + 5\right) e^{-x} - 2(x-1)e^{-x} - 2e^{-x} + c$

$= -e^{-x}\left\{x^2 - 2x + 5 + 2x - 2 + 2\right\} + c$

$= \boldsymbol{-e^{-x}\left(x^2 + 5\right) + c.}$

Example 6. *Evaluate*: $\int \left(\frac{x^4 - 1}{x^3}\right) \cdot e^{x + \frac{1}{x}}\,dx$ [C.U. B.Com.(H) 2001]

Solution:

$$\int\left(\frac{x^4-1}{x^3}\right)\cdot e^{x+\frac{1}{x}}\,dx=\int\frac{(x^2+1)(x^2-1)}{x\cdot x^2}e^{x+\frac{1}{x}}\,dx \quad \left|\text{Put } x+\frac{1}{x}=z;\text{ then }\left(1-\frac{1}{x^2}\right)dx=dz\right.$$

$$=\int\left(x+\frac{1}{x}\right)\cdot e^{x+\frac{1}{x}}\cdot\left(1-\frac{1}{x^2}\right)dx=\int z e^z\,dz$$

$$=z\int e^z\,dz-\int\left\{\frac{d(z)}{dx}\cdot\int e^z\,dz\right\}dz=ze^z-\int(1.e^z)dz+c$$

$$=ze^z-e^z+c=(z-1)e^z+c=\boldsymbol{\left(x+\frac{1}{x}-1\right)e^{x+\frac{1}{x}}+c.}$$

An Important Integral: $\boldsymbol{\int e^x[f(x)+f'(x)]dx=e^x f(x)+c.}$ [C.U. B.Com. 2009]

Proof.

$$\int e^x[f(x)+f'(x)]dx=\int e^x f(x)\,dx+\int e^x f'(x)\,dx$$

$$=f(x)\int e^x dx-\int\left[\frac{d}{dx}\{f(x)\}\cdot\int e^x\,dx\right]dx+\int e^x f'(x)\,dx$$

[integrating by parts, taking first function as $f(x)$ and second function e^x]

$$=f(x)e^x-\int f'(x)e^x\,dx+\int e^x f'(x)\,dx+c=\boldsymbol{e^x f(x)+c.}$$

Example 7. *Evaluate*:

(i) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx;$ [C.U. B.Com.(H) 2008] (ii) $\int\frac{e^x}{x}(1+x\log_e x)\,dx.$ [C.U. B.Com.(H) 1995]

Solution:

$$\text{(i)}\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\int e^x\frac{1}{x}dx-\int e^x\cdot\frac{1}{x^2}dx=\int\frac{1}{x}e^x\,dx-\int e^x\cdot\frac{1}{x^2}dx$$

$$=\frac{1}{x}\int e^x\,dx-\int\left\{\frac{d}{dx}\left(\frac{1}{x}\right)\cdot\int e^x\,dx\right\}dx-\int e^x\cdot\frac{1}{x^2}dx$$

[integrating the 1st integral by parts]

$$=\frac{1}{x}e^x-\int\left(-\frac{1}{x^2}\cdot e^x\right)dx-\int e^x\cdot\frac{1}{x^2}dx+c$$

$$=\frac{1}{x}e^x+\int e^x\cdot\frac{1}{x^2}dx-\int e^x\cdot\frac{1}{x^2}dx+c=\boldsymbol{\frac{1}{x}e^x+c.}$$

Otherwise. Using the result

$$\int e^x\{f(x)+f'(x)\}\,dx=e^x.f(x)+c,$$

we see that $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}.$

$$\therefore \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \boldsymbol{e^x \cdot \frac{1}{x} + c}.$$

(ii) $\displaystyle\int \frac{e^x}{x}(1+x\log_e x)\,dx = \int e^x\left(\frac{1}{x}+\log_e x\right)dx = \int e^x\left\{\log_e x + \frac{d}{dx}(\log_e x)\right\}dx = \boldsymbol{e^x \log_e x + c}.$

[*Otherwise.*

$$\int \frac{e^x}{x}(1+x\log_e x)\,dx = \int e^x \cdot \frac{1}{x}dx + \int e^x \log_e x\,dx$$

$$= e^x\int \frac{1}{x}dx - \int\left\{\frac{d(e^x)}{dx}\cdot\int\frac{1}{x}dx\right\}dx + \int e^x \log_e x\,dx$$

[integrating the first integral by parts]

$$= e^x \log_e x - \int e^x \log_e x\,dx + \int e^x \log_e x\,dx + c = \boldsymbol{e^x \log_e x + c}.]$$

Example 8. *Evaluate:*

(i) $\displaystyle\int \frac{xe^x}{(1+x)^2}dx;$ [C.U. B.Com.(H) 2001; V.U. B.Com.(H) 2011]

(ii) $\displaystyle\int \frac{\log x}{(1+\log x)^2}dx.$ [C.U. B.Com.(H) 2002]

Solution:

(i) $\displaystyle\int \frac{xe^x}{(1+x)^2}dx = \int \frac{(x+1-1)}{(1+x)^2}e^x\,dx = \int \frac{1}{1+x}e^x\,dx - \int \frac{e^x}{(1+x)^2}dx$

$$= \frac{1}{1+x}\int e^x\,dx - \int\left\{\frac{d}{dx}\left(\frac{1}{1+x}\right)\cdot\int e^x\,dx\right\}dx - \int\frac{e^x}{(1+x)^2}dx$$

[integrating 1st integral by parts]

$$= \frac{1}{1+x}e^x - \int -\frac{1}{(1+x)^2}e^x\,dx - \int \frac{e^x}{(1+x)^2}dx + c$$

$$= \frac{1}{1+x}e^x + \int\frac{e^x}{(1+x)^2}dx - \int\frac{e^x}{(1+x)^2}dx + c = \boldsymbol{\frac{1}{1+x}\cdot e^x + c}.$$

[*Otherwise.*

$$\int \frac{xe^x}{(1+x)^2}dx = \int e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\}dx = \boldsymbol{e^x \cdot \frac{1}{1+x} + c}\quad \left[\because \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}\right].]$$

(ii) $\int \frac{\log x}{(1+\log x)^2} dx = \int \frac{(1+\log x)-1}{(1+\log x)^2} dx = \int \frac{1}{1+\log x} dx - \int \frac{1}{(1+\log x)^2} dx$

$$= \frac{1}{1+\log x}\int 1dx - \int\left\{\frac{d}{dx}\left(\frac{1}{1+\log x}\right)\cdot\int 1dx\right\}dx - \int \frac{1}{(1+\log x)^2}dx$$

[integrating 1st integral by parts]

$$= \frac{x}{1+\log x} - \int\left\{-\frac{1}{(1+\log x)^2}\cdot\frac{1}{x}\times x\right\}dx - \int\frac{1}{(1+\log x)^2}dx + c$$

$$= \frac{x}{1+\log x} + \int\frac{1}{(1+\log x)^2}dx - \int\frac{1}{(1+\log x)^2}dx + c$$

$$= \mathbf{\frac{x}{1+\log x} + c.}$$

Example 9. *Evaluate:* $\int \log\left(x+\sqrt{x^2+a^2}\right)dx.$

Solution:

$$\int \log\left(x+\sqrt{x^2+a^2}\right)dx = \int \log\left(x+\sqrt{x^2+a^2}\right).1dx$$

$$= \log\left(x+\sqrt{x^2+a^2}\right).\int 1dx - \int\left\{\frac{d}{dx}\log\left(x+\sqrt{x^2+a^2}\right).\int 1dx\right\}dx$$

$$= \log\left(x+\sqrt{x^2+a^2}\right)\cdot x - \int\left\{\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)\cdot x\right\}dx + c$$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \int\left\{\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right)\cdot x\right\}dx + c$$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \int\left\{\frac{1}{x+\sqrt{x^2+a^2}}\times\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\times x\right\}dx + c$$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \int\frac{x\,dx}{\sqrt{x^2+a^2}} + c$$

Put $x^2+a^2 = z^2$; then
$2x\,dx = 2z\,dz$
or, $x\,dx = z\,dz$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - \int\frac{z\,dz}{z} + c$$

$$= x\log\left(x+\sqrt{x^2+a^2}\right) - z + c$$

$$= \mathbf{x\log\left(x+\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2} + c.}$$

Example 10. *Evaluate:* $\int \sqrt{4x^2-4x+10}\,dx.$

Solution:

Write: $4x^2-4x+10=4\left[x^2-x+\frac{5}{2}\right]$

$=4\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{5}{2}-\frac{1}{4}\right)\right]$

$=4\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2\right]$

$$\int\sqrt{4x^2-4x+10}\,dx=2\int\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2}\,dx$$

$$=2\int\sqrt{u^2+\left(\frac{3}{2}\right)^2}\,du,\ \text{where } u=x-\tfrac{1}{2} \text{ so that } du=dx$$

$$=2\left[\frac{u\sqrt{u^2+\left(\frac{3}{2}\right)^2}}{2}+\frac{9}{8}\log\left|u+\sqrt{u^2+\left(\frac{3}{2}\right)^2}\right|\right]+c$$

$$=\left(x-\frac{1}{2}\right)\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2}+\frac{9}{4}\log\left|x-\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2}\right|+c$$

$$=\frac{2x-1}{2}\cdot\frac{\sqrt{4x^2-4x+10}}{2}+\frac{9}{4}\log\left|\frac{2x-1}{2}+\frac{\sqrt{4x^2-4x+10}}{2}\right|+c$$

$$=\frac{2x-1}{4}\sqrt{4x^2-4x+10}+\frac{9}{4}\log\left|\frac{2x-1+\sqrt{4x^2-4x+10}}{2}\right|+c,$$

where c is the constant of integration.

General Rule. For Integrals of the form

$$\int\sqrt{ax^2+bx+c}\,dx.$$

Write:

$$ax^2+bx+c=a\left[(x\pm k)^2\pm l^2\right],$$

where k, l are constants and then proceed as in the above example 10.

Example 11. *Evaluate:* $\int(x+1)\sqrt{x^2-1}\,dx.$

Solution: We write:

$$x+1=\tfrac{1}{2}(2x)+1,\ \text{i.e.,}\ =\tfrac{1}{2}\ [\text{derivative of } \left(x^2-1\right)]+1.$$

$$\therefore\int(x+1)\sqrt{x^2-1}\,dx=\int\frac{1}{2}\cdot 2x\sqrt{x^2-1}\,dx+\int\sqrt{x^2-1}\,dx$$

$$=\frac{1}{2}\int 2x\sqrt{x^2-1}\,dx+\int\sqrt{x^2-1}\,dx.$$

Now

$$\int 2x\sqrt{x^2-1}\,dx=\int 2z^2dz \qquad \Big|\ \text{Put } x^2-1=z^2,\ 2x\,dx=2z\,dz$$

$$=\frac{2z^3}{3}=\frac{2}{3}\left(x^2-1\right)^{3/2}.$$

$$\therefore \int (x+1)\sqrt{x^2-1}\,dx = \frac{1}{2}\cdot\frac{2}{3}\left(x^2-1\right)^{3/2} + \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

$$= \frac{1}{3}\left(x^2-1\right)^{3/2} + \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c,$$

where c is the constant of integration.

EXERCISES ON CHAPTER 9(I)

(Integration by Parts)

Evaluate:

1. $\int xe^x\,dx.$

2. $\int \log x\,dx.$

3. $\int x^2\log x\,dx.$

4. $\int xe^{3x}\,dx.$

5. $\int x^2e^x\,dx.$

6. $\int x^2e^{2x}\,dx.$

7. $\int x^2e^{-x}\,dx.$

8. $\int (2x+3)e^{3x}\,dx.$

9. $\int x.3^x\,dx.$

10. $\int (\log x)^2\,dx.$

11. $\int \left(x^2+2x+3\right)e^{-x}\,dx.$ [C.U. B.Com.(H) 2000 Type]

12. $\int x(\log x)^2\,dx.$

13. $\int \frac{\log x}{x^3}\,dx.$ [C.U. B.Com.(H) 1994]

14. $\int x(\log x)^2\,dx.$

15. $\int \frac{\log x}{x^3}\,dx.$ [C.U. B.Com.(H) 1994]

16. $\int (x\log x)^2\,dx.$

17. $\int x^3e^{x^2}\,dx.$

18. $\int \log(2x+3)\,dx.$

19. $\int \log\left(x+\sqrt{x^2+4}\right)dx.$

20. $\int \log\left(x-\sqrt{x^2-1}\right)dx.$ [C.U. B.Com.(H) 2002]

21. $\int \frac{1}{x^5}e^{-1/x^2}\,dx.$

22. $\int \frac{x^4-1}{x^3}e^{x-\frac{1}{x}}\,dx.$ [C.U. B.Com.(H) 2001 Type]

23. $\int e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right)dx.$

24. $\int e^x\left(\log x+\frac{1}{x}\right)dx.$ [C.U. B.Com.(H) 1995 Type]

25. $\int \frac{xe^x}{(1+x)^2}\,dx.$ [C.U. B.Com.(H) 2001] [Hints: See worked-out Ex. 8.]

26. $\int \frac{(x+1)e^x}{(2+x)^2}\,dx.$

27. $\int e^x\frac{(x^2+1)\,dx}{(x+1)^2}.$

28. $\int \frac{x^4-1}{x^3}.e^{2\left(x+\frac{1}{x}\right)}\,dx.$ [C.U. B.Com.(H) 2003] [Hints: See worked-out Ex. 6.]

29. $\int \sqrt{x^2+4}\,dx.$

30. $\int \sqrt{x^2-9}\,dx.$ [Hints: Integrate by parts taking 2nd function as 1.]

31. $\int \sqrt{4x^2+4x+5}\,dx.$ [C.U. B.Com.(H) 1994]

32. $\int \sqrt{x^2+2ax}\,dx.$

33. $\int \sqrt{2x^2+3x+4}\,dx.$

34. $\int \dfrac{dx}{x+\sqrt{x^2-1}}.$

35. $\int \dfrac{dx}{x-\sqrt{x^2-1}}.$

36. $\int (x+1)\sqrt{x-1}\,dx.$

37. $\int (x-1)\sqrt{x^2-x+1}\,dx.$

38. $\int \dfrac{(x-2)e^x dx}{(x-1)^2}.$

[C.U. B.Com.(H) 2005]

[Hints: $\int \dfrac{(x-2)e^x dx}{(x-1)^2} = \int \dfrac{e^x}{x-1}dx - \int \dfrac{e^x}{(x-1)^2}dx.$
Now integrate 1st by-part.]

39. $\int \left\{\dfrac{1}{\log x} - \dfrac{1}{(\log x)^2}\right\} dx.$

[Hints: Find: $\int \left(\dfrac{1}{\log x}\cdot 1\right) dx$ by parts.]

ANSWERS

Constant of integration c is to be added to each answer

1. $e^x(x-1)$.
2. $x(\log x-1)$.
3. $\dfrac{x^3}{3}\left(\log x-\dfrac{1}{3}\right)$.
4. $\dfrac{e^{3x}}{3}\left(x-\dfrac{1}{3}\right)$.
5. $e^x\left(x^2-2x+2\right)$.
6. $\dfrac{e^{2x}}{4}\left(2x^2-2x+1\right)$.
7. $-e^{-x}\left(x^2+2x+2\right)$.
8. $\dfrac{1}{9}e^{3x}(6x+7)$.
9. $\dfrac{3^x}{\log_e 3}(x-\log_e 3)$.
10. $x(\log x)^2-2x(\log x-1)$.
11. $-e^{-x}\left(x^2+4x+7\right)$.
12. $\dfrac{1}{3}\left[(x+1)^3\log|x|-\dfrac{x^3}{3}-\dfrac{3}{2}x^2-3x-\log|x|\right]$.
13. $\dfrac{1}{2}\left(x^2+1\right)\log|1+x|-\dfrac{1}{4}x^2+\dfrac{1}{2}x$.
14. $\dfrac{x^2}{4}\left\{2(\log x)^2-2(\log x)+1\right\}$.
15. $-\dfrac{1}{4x^2}(2\log x+1)$.
16. $\dfrac{x^3}{27}\left[9(\log|x|)^2-6\log|x|+2\right]$.
17. $\dfrac{1}{2}e^{x^2}\left(x^2-1\right)$.
18. $\dfrac{1}{2}(2x+3)\log(2x+3)-x$.
19. $x\log\left(x+\sqrt{x^2+4}\right)-\sqrt{x^2+4}$.
20. $x\log\left(x-\sqrt{x^2-1}\right)+\sqrt{x^2-1}$.
21. $\dfrac{1}{2}e^{-1/x^2}\left(\dfrac{1}{x^2}+1\right)$.
22. $e^{x-\frac{1}{x}}\left(x-\dfrac{1}{x}-1\right)$.
23. $\dfrac{e^x}{x^2}$.
24. $e^x\log x$.
25. $\dfrac{e^x}{1+x}$.
26. $\dfrac{e^x}{2+x}$.
27. $e^x\left(\dfrac{x-1}{x+1}\right)$.
28. $\dfrac{1}{2}e^{2\left(x+\frac{1}{x}\right)}\left(x+\dfrac{1}{x}-\dfrac{1}{2}\right)$.
29. $\dfrac{x\sqrt{x^2+4}}{2}+2\log\left|x+\sqrt{x^2+4}\right|$.
30. $\dfrac{x\sqrt{x^2-9}}{2}-\dfrac{9}{2}\log\left|x+\sqrt{x^2-9}\right|$.
31. $\dfrac{(2x+1)}{4}\sqrt{4x^2+4x+5}$
$+\log\left|2x+1+\sqrt{4x^2+4x+5}\right|$.
32. $\dfrac{(x+a)\sqrt{x^2+2ax}}{2}$
$-\dfrac{a^2}{2}\log\left|x+a+\sqrt{x^2+2ax}\right|$.
33. $\sqrt{2}\left[\dfrac{4x+3}{8}\sqrt{x^2+\dfrac{3}{2}x+2}\right.$
$\left.+\dfrac{23}{32}\log\left|\dfrac{4x+3}{4}+\sqrt{x^2+\dfrac{3}{2}x+2}\right|\right]$.
34. $\dfrac{1}{2}x^2-\dfrac{x\sqrt{x^2-1}}{2}+\dfrac{1}{2}\log\left|x+\sqrt{x^2-1}\right|$.

35. $\frac{1}{2}x^2+\frac{x\sqrt{x^2-1}}{2}-\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|$.

36. $\frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}$.

37. $\frac{1}{3}\left(x^2-x+1\right)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}$
$-\frac{3}{16}\log\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right|$.

38. $\frac{e^x}{x-1}$.

39. $\frac{x}{\log x}$.

9.2 Integration by Partial Fractions (Elementary Problems)

If the integrand be a rational function of form $\frac{f(x)}{g(x)}$ in which

- $f(x)$, $g(x)$ are both polynomial functions,
- the degree of $f(x) <$ the degree of $g(x)$, and
- $g(x)$ is the product of non-repeated (or repeated) linear factors,

then we can break up $\frac{f(x)}{g(x)}$ into partial fractions and evaluate the integral. The method is shown below.

- If $\frac{f(x)}{g(x)}$ is of the form $\frac{px+q}{(x-a)(x-b)}$, we write:

$$\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}=\frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

$$\therefore\ px+q\equiv A(x-b)+B(x-a). \tag{1}$$

Putting $x=a$ and $x=b$ successively in identity (1), we can find A and B. (We can also find A and B by equating coefficients of x and constant terms from both sides.) Then we can write,

$$\int\frac{px+q}{(x-a)(x-b)}dx=A\int\frac{dx}{x-a}+B\int\frac{dx}{x-b}.$$

[In some cases, we can split into partial fractions by inspection also.]

If $f(x)$ and $g(x)$ have the same degree, then we can write,

$$\frac{f(x)}{g(x)}=1+\frac{A}{x-a}+\frac{B}{x-b},$$

if the quotient is 1, and then proceed as above.

- If $\frac{f(x)}{g(x)}$ is of the form $\frac{px^2+qx+r}{(x-a)(x-b)^2}$, we can write,

$$\frac{px^2+qx+r}{(x-a)(x-b)^2}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{(x-b)^2}=\frac{A(x-b)^2+B(x-a)(x-b)+C(x-a)}{(x-a)(x-b)^2}.$$

$$\therefore\ px^2+qx+r\equiv A(x-b)^2+B(x-a)(x-b)+C(x-a). \tag{2}$$

Putting $x=a$ and $x=b$ successively, we can find A and C. Again, equating coefficient of x^2 from both sides of (2), we can find B. Then we can write

$$\int\frac{px^2+qx+r}{(x-a)(x-b)^2}dx=A\int\frac{dx}{x-a}+B\int\frac{dx}{x-b}+C\int\frac{dx}{(x-b)^2}.$$

9.3 Illustrative Examples

Example 1. *Evaluate*:

(i) $\displaystyle\int \frac{x-1}{(x-3)(x+2)}dx$; [C.U. B.Com.(H) 1999] (ii) $\displaystyle\int \frac{5x+2}{(x-2)(x-3)}dx$. [C.U. B.Com.(H) 1994]

Solution: (i) Let
$$\frac{x-1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}.$$

Note: Denominator is the product of two non-repeated factors.

$$\therefore\ x-1 \equiv A(x+2)+B(x-3). \qquad (1)$$

Putting $x=3$, we get $3-1=A\cdot(3+2)+B\cdot 0=5A$ or, $A=2/5$.

Putting $x=-2$, we get $-2-1=A\cdot 0+B\cdot(-2-3)=-5B$ or, $B=3/5$.

$$\therefore\ \frac{x-1}{(x-3)(x+2)} = \frac{2/5}{x-3} + \frac{3/5}{x+2}.$$

$$\therefore \int \frac{x-1}{(x-3)(x+2)}dx = \frac{2}{5}\int \frac{dx}{x-3} + \frac{3}{5}\int \frac{dx}{x+2} = \frac{2}{5}\log|x-3| + \frac{3}{5}\log|x+2| + c,$$

where c is the constant of integration.

(ii) Let
$$\frac{5x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}.$$
$$\therefore\ 5x+2 \equiv A(x-3)+B(x-2). \qquad (1)$$

Putting $x=2$ and $x=3$ successively on both sides of identity (1), we get $5\cdot 2+2=A(2-3)+B\cdot 0$ and $5\cdot 3+2=A\cdot 0+B(3-2)$ or, $A=-12$ and $B=17$.

$$\therefore \int \frac{5x+2}{(x-2)(x-3)}dx = A\int \frac{dx}{x-2} + B\int \frac{dx}{x-3} = -12\log|x-2| + 17\log|x-3| + c,$$

where c is the constant of integration.

Example 2. *Evaluate*:

(i) $\displaystyle\int \frac{5x}{(2x+1)(3x+2)}dx$; [C.U. B.Com.(H) 1998] (ii) $\displaystyle\int \frac{x^2dx}{x^6-5x^3+6}$. [C.U. B.Com.(H) 2003]

Solution: (i) Let
$$\frac{5x}{(2x+1)(3x+2)} = \frac{A}{2x+1} + \frac{B}{3x+2} = \frac{A(3x+2)+B(2x+1)}{(2x+1)(3x+2)}$$
$$\therefore\ 5x \equiv A(3x+2)+B(2x+1). \qquad (1)$$

Putting $x=-\frac{1}{2}$, $5\times\left(-\frac{1}{2}\right) = A\left\{3\times\left(-\frac{1}{2}\right)+2\right\} + B\cdot 0$ $\quad[\because\ x=-\frac{1}{2}$ makes $2x+1=0.]$

$$\text{or, } -\frac{5}{2} = \frac{1}{2}A \text{ or, } A=-5.$$

Again, putting $x=-\frac{2}{3}$ in eq. (1), $5\times\left(-\frac{2}{3}\right)=A\cdot 0+B\left\{2\times\left(-\frac{2}{3}\right)+1\right\}$ or, $-\frac{10}{3}=-\frac{1}{3}B$ or, $B=10$.

$$\therefore \int \frac{5x}{(2x+1)(3x+2)}dx=A\int \frac{dx}{2x+1}+B\int \frac{dx}{3x+2}=\frac{A}{2}\int \frac{2dx}{2x+1}+\frac{B}{3}\int \frac{3dx}{3x+2}$$
$$=-\frac{5}{2}\log|2x+1|+\frac{10}{3}\log|3x+2|+c,$$

where c is the constant of integration.

(ii) $$\int \frac{x^2dx}{x^6-5x^3+6}=\int \frac{\frac{1}{3}dz}{z^2-5z+6}=\frac{1}{3}\int \frac{dz}{(z-3)(z-2)}$$ | Put $x^3=z$; then $3x^2dx=dz$ or, $x^2dx=\frac{1}{3}dz$.

$$=\frac{1}{3}\int\left(\frac{1}{z-3}-\frac{1}{z-2}\right)dz$$ [Splitting into partial fractions is shown below]

$$=\frac{1}{3}[\log|z-3|-\log|z-2|]+c=\frac{1}{3}\log\left|\frac{z-3}{z-2}\right|+c=\frac{1}{3}\log\left|\frac{x^3-3}{x^3-2}\right|+c.$$

Let $$\frac{1}{(z-3)(z-2)}=\frac{A}{z-3}+\frac{B}{z-2}=\frac{A(z-2)+B(z-3)}{(z-3)(z-2)};\text{ then } 1\equiv A(z-2)+B(z-3) \quad (1)$$

Putting $z=3$ in identity $(1)=A(3-2)=A$, i.e., $A=1$.
Putting $z=2$ in identity $(1)=B(2-3)=-B$, i.e., $B=-1$.

Example 3. *Evaluate:*

(i) $\int \frac{(x^2+5x+2)}{(x+2)(x+3)}dx$; [C.U. B.Com.(H) 2001] (ii) $\int \frac{(x-1)(x-5)}{(x-2)(x-4)}dx$. [C.U. B.Com.(H) 2002]

Solution: (i) Let

$$\frac{x^2+5x+2}{(x+2)(x+3)}=1+\frac{A}{x+2}+\frac{B}{x+3}=\frac{(x+2)(x+3)+A(x+3)+B(x+2)}{(x+2)(x+3)}.$$

$$\therefore\ x^2+5x+2\equiv(x+2)(x+3)+A(x+3)+B(x+2). \quad (1)$$

Putting $x=-2$, $(-2)^2+5\times(-2)+2=0+A(-2+3)+B\cdot 0$ or, $A=-4$.
Putting $x=-3$, $(-3)^2+5\times(-3)+2=0+A\cdot 0+B(-3+2)$ or, $B=4$.

$$\therefore \int \frac{(x^2+5x+2)}{(x+2)(x+3)}dx=\int 1dx+A\int \frac{dx}{x+2}+B\int \frac{dx}{x+3}$$
$$=x-4\log|x+2|+4\log|x+3|+c,$$ where c is the constant of integration
$$=x+4\log\left|\frac{x+3}{x+2}\right|+c.$$

(ii) Since the degree of the numerator is the same as that of the denominator, let

$$\frac{(x-1)(x-5)}{(x-2)(x-4)}=1+\frac{A}{x-2}+\frac{B}{x-4}=\frac{(x-2)(x-4)+A(x-4)+B(x-2)}{(x-2)(x-4)}.$$

$$\therefore\ (x-1)(x-5) \equiv (x-2)(x-4) + A(x-4) + B(x-2). \qquad (1)$$

Putting $x = 2$ in eq. (1), $1 \times (-3) = 0 + A(-2) + 0$ or, $2A = 3$ or, $A = 3/2$.
Putting $x = 4$ in eq. (1), $3 \times (-1) = 0 + 0 + B(2)$ or, $B = -3/2$.

$$\therefore \int \frac{(x-1)(x-5)}{(x-2)(x-4)}dx = \int \left(1 + \frac{A}{x-2} + \frac{B}{x-4}\right) dx = x + A\log|x-2| + B\log|x-4| + c$$
$$= x + \frac{3}{2}\log|x-2| - \frac{3}{2}\log|x-4| + c = \boldsymbol{x + \frac{3}{2}\log\left|\frac{x-2}{x-4}\right| + c}.$$

Example 4. *Evaluate:* $\displaystyle\int \frac{2x}{2+x-x^2}dx.$ [C.U. B.Com.(H) 1996]

Solution: $2 + x - x^2 = 2 + 2x - x - x^2 = 2(1+x) - x(1+x) = (1+x)(2-x)$.

Let
$$\frac{2x}{2+x-x^2} = \frac{2x}{(1+x)(2-x)} = \frac{A}{1+x} + \frac{B}{2-x} = \frac{A(2-x) + B(1+x)}{(1+x)(2-x)}.$$
$$\therefore\ 2x \equiv A(2-x) + B(1+x). \qquad (1)$$

Putting $x = -1$, $2 \times (-1) = A(2+1) = 3A$ or, $A = -2/3$.
Putting $x = 2$ in eq. (1), $2 \times 2 = A \cdot 0 + B(1+2)$ or, $B = 4/3$.

$$\therefore \int \frac{2x}{2+x-x^2}dx = A\int \frac{dx}{1+x} + B\int \frac{dx}{2-x} = -\frac{2}{3}\log|1+x| + \frac{4}{3} \times (-\log|2-x|) + c \quad \left[\because\ \frac{d(2-x)}{dx} = -1\right]$$
$$= \boldsymbol{-\frac{2}{3}\left[\log|1+x| + 2\log|2-x|\right] + c},$$

where c is the constant of integration.

Example 5. *Evaluate:* $\displaystyle\int \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}\ \left(b^2 > a^2\right).$

Solution: Put $x^2 = z$, then $x\,dx = \frac{1}{2}dz$ and

$$\int \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int \frac{1}{(z+a^2)(z+b^2)}dz.$$

Now, let
$$\frac{1}{(z+a^2)(z+b^2)} = \frac{A}{z+a^2} + \frac{B}{z+b^2}$$

Then
$$A\left(z+b^2\right) + B\left(z+a^2\right) = 1.$$

Put $z=-a^2$; to obtain $A=\dfrac{1}{b^2-a^2}$ and $z=-b^2$ obtain $B=\dfrac{1}{a^2-b^2}=-\dfrac{1}{b^2-a^2}$.

$$\therefore \int \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}\left(b^2>a^2\right)=\frac{1}{2}\cdot\frac{1}{b^2-a^2}\int\frac{dz}{z+a^2}-\frac{1}{2}\cdot\frac{1}{b^2-a^2}\int\frac{dz}{z+b^2}$$

$$=\frac{1}{2}\cdot\frac{1}{b^2-a^2}\left[\log|z+a^2|-\log|z+b^2|\right]+c$$

$$=\frac{1}{2}\cdot\frac{1}{b^2-a^2}\log\left|\frac{z+a^2}{z+b^2}\right|+c$$

$$=\boldsymbol{\frac{1}{2(b^2-a^2)}\log\left|\frac{x^2+a^2}{x^2+b^2}\right|+c.}$$

Example 6. *Evaluate:* $\displaystyle\int\frac{x^3dx}{(x-1)(x-2)(x-3)}$.

Solution: The degree of numerator is the same as that of denominator. In this case, we write,

$$\frac{x^3}{(x-1)(x-2)(x-3)}=1+\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

and $(x-1)(x-2)(x-3)+A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\equiv x^3$.

Put $x=1, x=2, x=3$ successively on both sides of the above identity and obtain $A=\frac{1}{2}$, $B=-8$, $C=\frac{27}{2}$.

$$\therefore\ I=\int dx+\frac{1}{2}\int\frac{dx}{x-1}-8\int\frac{dx}{x-2}+\frac{27}{2}\int\frac{dx}{x-3}=\boldsymbol{x+\frac{1}{2}\log|x-1|-8\log|x-2|+\frac{27}{2}\log|x-3|+c.}$$

Example 7. *Evaluate:* $\displaystyle\int\frac{3x^2+5x}{(x-1)(x+1)^2}dx$.

Solution: Let

$$\frac{3x^2+5x}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}.$$

$$\therefore\ 3x^2+5x=A(x+1)^2+B(x-1)(x+1)+C(x-1). \tag{1}$$

Putting $x=1$, $3\times1^2+5\times1=A(1+1)^2+0+0=4A$ or, $A=\dfrac{8}{4}=2$.

Putting $x=-1$, $3\times(-1)^2+5\times(-1)=A\cdot0+B\cdot0+C(-1-1)$ or, $-2C=-2$ or, $C=1$.

Equating coefficients of x^2 from both sides of eq. (1), we get

$$3=A+B \text{ or, } B=3-A=3-2=1.$$

$$\therefore\int\frac{3x^2+5x}{(x-1)(x+1)^2}dx=A\int\frac{dx}{x-1}+B\int\frac{dx}{x+1}+C\int\frac{dx}{(x+1)^2}$$

$$=2\log|x-1|+1.\log|x+1|+1\cdot\left(-\frac{1}{x+1}\right)+k$$

$$=\boldsymbol{2\log|x-1|+\log|x+1|-\frac{1}{x+1}+k,}$$

where k is the constant of integration.

EXERCISES ON CHAPTER 9(II)

(Integration by Partial Fractions)

A

Verify the following results:

1. $\displaystyle\int \frac{dx}{x^2-9} = \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + c.$

2. $\displaystyle\int \frac{dx}{x^2-x} = \log\left|\frac{x-1}{x}\right| + c.$

3. $\displaystyle\int \frac{2x+1}{x^2+x}\,dx = \log\left|x^2+x\right| + c.$

4. $\displaystyle\int \frac{2dx}{x(x-2)} = \log\left|\frac{x-2}{x}\right| + c.$

5. $\displaystyle\int \frac{dx}{(x-1)(x-2)} = \log\left|\frac{x-2}{x-1}\right| + c.$

6. $\displaystyle\int \frac{5dx}{(x-2)(x+3)} = \log\left|\frac{x-2}{x+3}\right| + c.$

7. $\displaystyle\int \frac{x\,dx}{(x-1)(2x-1)} = \log|x-1| - \frac{1}{2}\log|2x-1| + c.$

8. $\displaystyle\int \frac{x}{(2x+1)(x+1)} = \log|x+1| - \frac{1}{2}\log|2x+1| + c.$

9. $\displaystyle\int \frac{(2x+1)\,dx}{(x+1)(x-2)} = \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + c.$

10. $\displaystyle\int \frac{dx}{x(x-2)(x-4)} = \frac{1}{8}\log|x(x-4)| - \frac{1}{4}\log|x-2| + c.$

11. $\displaystyle\int \frac{dx}{(x-2)^2(x-3)} = \frac{1}{x-2} + \log\left|\frac{x-3}{x-2}\right| + c.$

12. $\displaystyle\int \frac{(3x-1)\,dx}{(x+2)(x-2)^2} = \frac{5}{16}\log|x-2| - \frac{7}{4(x-2)} - \frac{5}{16}\log|x+2| + c.$

13. $\displaystyle\int \frac{x^2}{(x-a)(x-b)}\,dx = x + \frac{1}{a-b}\left[a^2\log|x-a| - b^2\log|x-b|\right] + c.$

14. $\displaystyle\int \frac{4x-2}{x^3-x^2-2x}\,dx = \log\left|\frac{x^2-2x}{(x+1)^2}\right| + c.$

15. $\displaystyle\int \frac{x^3}{(x-1)(x-2)(x-3)}\,dx = x + \frac{1}{2}\left[\log|x-1| - 16\log|x-2| + 27\log|x-3|\right] + c.$

B

Evaluate the following integrals by using Partial Fractions:

1. $\int \frac{dx}{x(x+1)}$.

2. $\int \frac{2x-1}{x(x+5)}dx$.

3. $\int \frac{3x-1}{x^2-1}dx$.

4. $\int \frac{x}{(x-1)(x-2)}dx$.

5. $\int \frac{dx}{(x-a)(x-b)}$.

6. $\int \frac{x}{(x-a)(x-b)}dx$.

7. $\int \frac{x-1}{(x-3)(x+2)}dx$. [C.U. B.Com.(H) 1999]

8. $\int \frac{dt}{2t^2+3t+1}$.

9. $\int \frac{x^2 dx}{x^2-3x+2}$. [C.U. B.Com.(H) 1997]

10. $\int \frac{(x^2+5x+2)}{(x+2)(x+3)}dx$. [C.U. B.Com.(H) 2001]

11. $\int \frac{x}{2x^4-3x^2-2}dx$. [C.U. B.Com.(H) 1995]

12. $\int \frac{3x+5}{x^3-x^2-x+1}dx$.

13. $\int \frac{x^3}{(x-1)(x-2)}dx$.

14. $\int \frac{e^x\,dx}{e^{2x}-5e^x+6}$.

15. $\int \frac{(3x^2+5x)\,dx}{(x-1)(x+1)^2}$.

[**Hints**: 11. Put $x^2 = z$; then $x\,dx = \frac{1}{2}dz$ and integral $= \frac{1}{2}\int \frac{dz}{2z^2-3z-2}$ and $2z^2-3z-2 = 2z^2-4z+z-2 = 2z(z-2)+1(z-2)$, etc.

13. $(x-1)(x-2) = x^2-3x+2$. First divide x^3 by x^2-3x+2 and see that quotient is $(x+3)$ and remainder $= 7x-6$. Now write, $\frac{x^3}{(x-1)(x-2)} = x+3+\frac{7x-6}{(x-1)(x-2)}$ and proceed.]

ANSWERS

[The constant of integration, which is not written, is to be added to each answer]

B

1. $\log\left|\frac{x}{x+1}\right|$.

2. $\frac{1}{5}[\log|x|+9\log|x+5|]$.

3. $\log|x-1|+2\log|x+1|$.

4. $2\log|x-2|-\log|x-1|$.

5. $\frac{1}{a-b}\log\left|\frac{x-a}{x-b}\right|$.

6. $\frac{1}{a-b}[a\log|x-a|-b\log|x-b|]$.

7. $\frac{1}{5}[3\log|x+2|+2\log|x-3|]$.

8. $\log\left|\frac{2t+1}{t+1}\right|$.

9. $x+4\log|x-2|-\log|x-1|$.

10. $x+4\log\left|\frac{x+3}{x+2}\right|$.

11. $\frac{1}{10}\log\left|\frac{x^2-2}{2x^2+1}\right|$.

12. $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}$.

13. $\frac{1}{2}x^2+3x-\log|x-1|+8\log|x-2|$.

14. $\log\left|\frac{e^x-3}{e^x-2}\right|$.

15. $2\log|x-1|+\log|x+1|-\frac{1}{x+1}$.

QUICK REVISION

(All Problems Solved)

Below is a selected list of problems of Indefinite Integration. All types of problems are covered. The students will find them useful for a Quick Revision before the examinations. Here c represents the arbitrary constant of integration.

Standard Formula I

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, if $n \neq -1$;
2. $\int \dfrac{dx}{x} = \log|x| + c$;
3. $\int e^{mx}\,dx = \dfrac{e^{mx}}{m} + c$;
4. $\int e^x\,dx = e^x + c$;
5. $\int a^x\,dx = \dfrac{a^x}{\log a} + c\ (a > 0)$;
6. $\int a^{mx}\,dx = \dfrac{a^{mx}}{m\log a} + c$.

Use of Standard Formulae

Example 1. *Evaluate:* $\int \left(\dfrac{2a}{\sqrt{x}} - \dfrac{b}{x^2} + 3c\sqrt[3]{x^2}\right) dx$.

Solution:
$$I = \int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right) dx = 2a\int x^{-1/2}dx - b\int x^{-2}dx + 3c\int x^{2/3}dx$$
$$= 2a\frac{x^{1/2}}{1/2} - b\frac{x^{-2+1}}{-1} + 3c\frac{x^{\frac{2}{3}+1}}{5/3} + \text{Constant } c$$
$$= 4a\sqrt{x} + \frac{b}{x} + \frac{9c}{5}x^{5/3} + c.$$

Example 2. *Show that* $I = \int ba^{2x}dx = \dfrac{ba^{2x}}{2\log a} + c$.

[Hints: $\int a^{2x}dx = \dfrac{a^{2x}}{2\log a} + c$.]

Example 3. *Evaluate:* $I = \int \dfrac{x^4}{x-1}dx$.

Solution:
$$\frac{x^4}{x-1} = \frac{x^4-1+1}{x-1} = \frac{(x^2-1)(x^2+1)}{x-1} + \frac{1}{x-1} = (x+1)(x^2+1) + \frac{1}{x-1} = x^3+x^2+x+1+\frac{1}{x-1}.$$
$$\therefore\ I = \int x^3dx + \int x^2dx + \int x\,dx + \int dx + \int \frac{dx}{x-1} = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \log|x-1| + c.$$

Example 4. *Evaluate:* $I = \int \dfrac{(x^2+3)^2}{\sqrt{x}}dx$.

Solution:
$$I=\int\frac{(x^2+3)^2}{\sqrt{x}}dx=\int\frac{x^4+6x^2+9}{\sqrt{x}}dx=\int x^{4-\frac{1}{2}}dx+6\int x^{2-\frac{1}{2}}dx+9\int x^{-\frac{1}{2}}dx$$
$$=\frac{x^{\frac{7}{2}+1}}{9/2}+6\frac{x^{\frac{3}{2}+1}}{5/2}+9\frac{x^{-\frac{1}{2}+1}}{1/2}+c=\frac{2}{9}x^{9/2}+\frac{12}{5}x^{5/2}+18x^{1/2}+c$$
$$=\mathbf{\frac{2}{9}\sqrt{x^9}+\frac{12}{5}\sqrt{x^5}+18\sqrt{x}+c.}$$

Use of the Method of Substitution

Forms: 1. $\displaystyle\int(a+bx)^n\,dx=\frac{(a+bx)^{n+1}}{b(n+1)}+c\,(n\neq-1);$

2. $\displaystyle\int\sqrt{a+bx}\;dx=\frac{2(a+bx)^{3/2}}{3b}+c\,(n\neq-1);$

[Put $a+bx=z$ so that $dx=\frac{dz}{b}$]

3. $\displaystyle\int\sqrt{x^2\pm a^2}\;dx=\log\left|x+\sqrt{x^2\pm a^2}\right|+c.$

Example 5. *Evaluate*: $\displaystyle\int\sqrt{\frac{x}{a+x}}dx.$

Solution:
$$I=\int\sqrt{\frac{x}{a+x}}dx=\int\sqrt{\frac{x\times x}{(a+x)\,x}}dx=\int\frac{x}{\sqrt{x^2+ax}}dx=\frac{1}{2}\int\frac{2x}{\sqrt{x^2+ax}}dx$$
[remembering $\frac{d}{dx}\left(x^2+ax\right)=2x+a$]
$$=\frac{1}{2}\int\frac{2x+a-a}{\sqrt{x^2+ax}}dx=\frac{1}{2}\int\frac{2x+a}{\sqrt{x^2+ax}}dx-\frac{a}{2}\int\frac{dx}{\sqrt{x^2+ax}}=\frac{1}{2}I_1-\frac{a}{2}I_2\text{ (say).}$$

Now,
$$I_1=\int\frac{2x+a}{\sqrt{x^2+ax}}dx\qquad\Big|\text{Put }x^2+ax=z^2,\text{ then }(2x+a)\,dx=2z\,dz$$
$$=\int\frac{2z\,dz}{z}=2\int dz=2z=2\sqrt{x^2+ax}.$$
$$I_2=\int\frac{dx}{\sqrt{x^2+ax}}=\int\frac{dx}{\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}}=\log\left|\left(x+\frac{a}{2}\right)+\sqrt{x^2+ax}\right|.$$
$$\therefore\ I=\frac{1}{2}I_1-\frac{a}{2}I_2=\frac{1}{2}\cdot2\sqrt{x^2+ax}-\frac{a}{2}\log\left|\left(x+\frac{a}{2}\right)+\sqrt{x^2+ax}\right|$$
$$=\mathbf{\sqrt{x^2+ax}-\frac{a}{2}\log\left|x+\frac{a}{2}+\sqrt{x^2+ax}\right|+c.}$$

Example 6. *Evaluate*: $\displaystyle\int\frac{dx}{1+e^{x/2}}.$

Solution: $$I=\int \frac{dx}{1+e^{x/2}}=\int \frac{e^{-x/2}}{e^{-x/2}+1}dx \qquad \left|\ \begin{array}{l}\text{Put } e^{-x/2}+1=z,\\ -\frac{1}{2}e^{-x/2}dx=dz.\end{array}\right.$$
$$=\int \frac{-2dz}{z}=-2\log|z|+c=\mathbf{-2\log\left|e^{-x/2}+1\right|+c.}$$

Example 7. *Evaluate:* $\int \frac{2x\,dx}{\sqrt[3]{6-5x^2}}$.

Solution: $$I=\int \frac{2x\,dx}{\sqrt[3]{6-5x^2}}=2\int \frac{3z^2dz}{-10}\times\frac{1}{z} \qquad \left|\ \begin{array}{l}\text{Put } 6-5x^2=z^3;\\ \therefore\ -10x\,dx=3z^2dz,\\ \text{or, } x\,dx=-\frac{3}{10}z^2dz.\end{array}\right.$$
$$=\frac{-3}{5}\int z\,dz=\frac{-3}{5}\cdot\frac{z^2}{2}+c=\mathbf{\frac{-3}{10}\left(6-5x^2\right)^{2/3}+c.}$$

Use of Partial Fractions

Example 8. *Evaluate:* $\int \frac{dx}{x^2-a^2}$.

Solution: $$I=\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\int\frac{dx}{x-a}-\frac{1}{2a}\int\frac{dx}{x+a}$$
$$\therefore\ I=\frac{1}{2a}\log|x-a|-\frac{1}{2a}\log|x+a|+c=\mathbf{\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c.}$$

[$\because$ *by inspection, Integrand* $=\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$.

Check as above
$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c.]$$

Now try to solve problems like
$$\int\frac{dx}{9x^2-1}=\frac{1}{6}\log\left|\frac{3x-1}{3x+1}\right|+c$$
$$\text{and}\int\frac{dx}{4-9x^2}=\frac{1}{12}\log\left|\frac{2+3x}{2-3x}\right|+c.$$

Example 9. *Evaluate:* $I=\int\frac{x^2}{(x+1)(x+2)^2}dx$.

Solution: Integrand $=\frac{x^2}{(x+1)(x+2)^2}=\frac{A}{x+1}+\frac{B}{(x+2)^2}+\frac{C}{x+2}$

$\therefore\ A(x+2)^2+B(x+1)+C(x+2)(x+1)\equiv x^2$.

Equating coefficient of x^2, x and constant, $A+C=1$, $4A+B+3C=0$; $4A+B+2C=0$. Solving we obtain $C=0$, $B=-4$, $A=1$.

$$\therefore\ I=\int\frac{dx}{x+1}-4\int\frac{dx}{(x+2)^2}=\mathbf{\log|x+1|+\frac{4}{x+2}+c.}$$

Example 10. *Evaluate:* $\int \frac{dx}{x(x^4+1)}$. **[CA Foun. May 2003]**

Solution: Put $x^4+1=z$ so that $4x^3 dx = dz$.

$$\text{Given integral} = \int \frac{dz}{4x^4 z} = \int \frac{dz}{4z(z-1)} = \frac{1}{4}\int \left(\frac{1}{z-1} - \frac{1}{z}\right) dz \quad \text{[by inspection]}$$

$$= \frac{1}{4}[\log|z-1| - \log|z| + c] = \frac{1}{4}\log\left|\frac{z-1}{z}\right| + c = \mathbf{\frac{1}{4}\log\frac{x^4}{x^4+1} + c}.$$

Form:

1. $\int \frac{dx}{ax^2+bx+c}$, 2. $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

Write:

$$ax^2+bx+c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 \pm (\alpha)^2\right] \quad (\alpha = \text{constant}).$$

3. $\int \frac{px+q}{ax^2+bx+c}dx$, 4. $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Write $(px+q) = l$ (derivative of ax^2+bx+c) $\pm m$ (l, m are constants obtained by equating the coefficients of x and constant terms).

5. $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ [Put $px+q = \frac{1}{z}$.]

Example 11. *Evaluate:* $\int \frac{dx}{3x^2+4x-7}$.

Solution:

$$\int \frac{dx}{3x^2+4x-7} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{2}{3}\right)^2 - \left(\frac{5}{3}\right)^2}$$

$$\because \quad 3x^2+4x-7 = 3\left[x^2 + \frac{4}{3}x - \frac{7}{3}\right] = 3\left[\left(x+\frac{2}{3}\right)^2 - \frac{4}{9} - \frac{7}{3}\right] = 3\left[\left(x+\frac{2}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right].$$

$$= \frac{1}{3}\cdot\frac{1}{2\cdot\frac{5}{3}}\log\left|\frac{x+\frac{2}{3}-\frac{5}{3}}{x+\frac{2}{3}+\frac{5}{3}}\right| + c = \frac{1}{10}\log\left|\frac{x-1}{x+\frac{7}{3}}\right| + c = \mathbf{\frac{1}{10}\log\left|\frac{3x-3}{3x+7}\right| + c}.$$

Example 12. *Evalute:* $\int \frac{dx}{\sqrt{1+x+x^2}}$.

Solution:

$$\int \frac{dx}{\sqrt{1+x+x^2}} = \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} \quad \Big| \because x^2+x+1=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c = \mathbf{\log\left|\frac{2x+1}{2}+\sqrt{x^2+x+1}\right|+c.}$$

Example 13. *Evaluate:* $\int \frac{3x-1}{\sqrt{4x^2+9}}dx.$

Solution:

$$\int \frac{3x-1}{\sqrt{4x^2+9}}dx = \int \frac{\frac{3}{8}(8x)-1}{\sqrt{4x^2+9}}dx \quad \Big|$$

Let $3x-1=l(8x)+m$. Equating the coefficient of x and constant terms, $8l=3 \Rightarrow l=3/8$, $-1=m \Rightarrow m=-1$.

$$= \frac{3}{8}\int \frac{8x}{\sqrt{4x^2+9}}dx - \int \frac{dx}{2\sqrt{x^2+\left(\frac{3}{2}\right)^2}} = \frac{3}{8}\cdot 2\sqrt{4x^2+9}-\frac{1}{2}\log\left|x+\sqrt{x^2+\frac{9}{4}}\right|+c$$

$$= \mathbf{\frac{3}{4}\sqrt{4x^2+9}-\frac{1}{2}\log\left|x+\sqrt{x^2+\frac{9}{4}}\right|+c.}$$

[*Method of substitution* is shown below.
Put $4x^2+9=z^2$; $8x\,dx=2z\,dz$.

$$\therefore \int \frac{8x\,dx}{\sqrt{4x^2+9}} = \int \frac{2z\,dz}{z} = 2\int dz = 2z = 2\sqrt{4x^2+9}.]$$

Example 14. *Evaluate:* $\int \frac{x+3}{6x-x^2}dx.$

Solution:

$$\int \frac{x+3}{6x-x^2}dx$$

$$= \int \frac{-\frac{1}{2}(6-2x)+6}{6x-x^2}dx$$

$$= -\frac{1}{2}\int \frac{6-2x}{6x-x^2}dx + 6\int \frac{dx}{6x-x^2}$$

$$= -\frac{1}{2}\log\left|6x-x^2\right| + 6\int \frac{dx}{3^2-(x-3)^2}$$

$$= -\frac{1}{2}\log\left|6x-x^2\right| + 6\frac{1}{2\cdot 3}\log\left|\frac{3+x-3}{3-x+3}\right|+c$$

$$= \mathbf{-\frac{1}{2}\log\left|6x-x^2\right|+\log\left|\frac{x}{6-x}\right|+c.}$$

Let $x+3=l(6-2x)+m$. The $-2l=1 \Rightarrow l=-1/2$ and $6l+m=3 \Rightarrow -3+m=3 \Rightarrow m=6$.

$\therefore\ x+3=-\frac{1}{2}(6-2x)+6.$

Put $6x-x^2=z$, $\therefore\ (6-2x)dx=dz$

$\therefore \int \frac{6-2x}{6x-x^2}dx = \int \frac{dz}{z} = \log\left(6x-x^2\right).$

$6x-x^2 = -\left(x^2-6x\right) = -\left\{(x-3)^2-9\right\}$
$=(3)^2-(x-3)^2.$

Example 15. *To evaluate* $I=\int \frac{dx}{(x-1)\sqrt{x^2+1}}$, *put* $x-1=\frac{1}{z} \Rightarrow dx=-\frac{1}{z^2}dz.$

$$\therefore\ I=\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\left(\frac{1}{z}+1\right)^2+1}}=-\int \frac{dz}{\sqrt{(1+z)^2+z^2}}=-\int \frac{dz}{\sqrt{2z^2+2z+1}}$$

$$=-\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{z^2+z+\frac{1}{2}}}=-\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$=-\frac{1}{\sqrt{2}}\log\left|z+\frac{1}{2}+\sqrt{\left(z+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|+c$$

$$=-\frac{1}{\sqrt{2}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|+c$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{x+1}{2(x-1)}+\sqrt{\left(\frac{x+1}{2x-2}\right)^2+\frac{1}{4}}\right|+c$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{x+1}{2(x-1)}+\frac{\sqrt{2\left(x^2+1\right)}}{2(x-1)}\right|+c=-\frac{1}{\sqrt{2}}\log\left|\frac{x+1+\sqrt{2\left(x^2+1\right)}}{2(x-1)}\right|+c.$$

Try similarly: $\int \frac{dx}{(x-1)\sqrt{x^2-1}};\int \frac{dx}{(1-x)\sqrt{1-x^2}};\int \frac{dx}{(2-x)\sqrt{3x^2-2x+1}}.$

Rule of Integration by Parts

- $\int u\,d(v)=uv-\int v\,d(u).$

  Another form: $\int$ **First Function × Second Function = First Function** $\int$ **Second Function –** $\int$ **{Derivative of the First Function × Integral of the Second Function}. e.g.,**

- $\int \log x\,dx$. Taking $u=\log x,\ v=x$

$$=(\log x)(x)-\int x\,d(\log x)=x\log x-\int x\frac{1}{x}dx=x\log x-x+c.$$

- $\int xe^x\,dx$. Take x = first function and e^x = second function

$$=x\int e^x\,dx-\int\left\{\frac{d}{dx}(x)\int e^x\,dx\right\}dx=xe^x-\int(1\cdot e^x)dx$$
$$=xe^x-e^x+c.$$

- **Two important formulae:**

- $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c;$$
- $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c.$$

Example 16. *Evaluate:* $I = \int x\log(1+x)\,dx.$ [V.U. B.Com.(H) 2007]

Solution: We have

$$\begin{aligned}
I &= \int x\log(1+x)\,dx = \int \log(1+x)\,d\left(\frac{x^2}{2}\right) \quad \left[\text{of the form} \int u\,dv\right] \\
&= \{\log|1+x|\}\frac{x^2}{2} - \int \frac{x^2}{2} d\{\log(1+x)\}dx \\
&= \frac{x^2}{2}\log|1+x| - \frac{1}{2}\int \frac{x^2}{1+x}dx = \frac{x^2}{2}\log|1+x| - \frac{1}{2}\int \frac{x^2-1+1}{x+1}dx \\
&= \frac{x^2}{2}\log|1+x| - \frac{1}{2}\int \frac{(x^2-1)}{x+1}dx - \frac{1}{2}\int \frac{dx}{x+1} \\
&= \frac{x^2}{2}\log|1+x| - \frac{1}{2}\int (x-1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} \\
&= \frac{x^2}{2}\log|1+x| - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}x - \frac{1}{2}\log|x+1| + c.
\end{aligned}$$

Example 17. *Evaluate:* $\int e^x\{f(x)+f'(x)\}dx.$ [C.U. B.Com. 2009]

Solution: We have

$$\begin{aligned}
\int e^x\{f(x)+f'(x)\}dx &= \int e^x f(x)\,dx + \int e^x f'(x)\,dx \\
&= f(x)\int e^x\,dx - \int\left\{\frac{d}{dx}f(x)\int e^x\,dx\right\}dx + \int e^x f'(x)\,dx \\
&= e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx = e^x f(x) + c.
\end{aligned}$$

e.g., $\int e^x\left(\log x + \frac{1}{x}\right)dx = e^x \log x + c.$

Example 18. *Evaluate:* $\int \frac{xe^x}{(x+1)^2}dx.$ [C.U. B.Com. 2001]

Solution: We have

$$\int \frac{xe^x}{(x+1)^2}dx = \int e^x\left\{\frac{x+1-1}{(x+1)^2}\right\}dx = \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx = e^x\frac{1}{x+1}+c.$$

Example 19. *Evaluate:* $\int \sqrt{x^2+x+1}\,dx.$

Solution: We have

$$\int \sqrt{x^2+x+1}\,dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \int \sqrt{y^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dy \qquad \left|\text{Put } x+\tfrac{1}{2}=y,\ dx=dy\right.$$

$$= \frac{y\sqrt{y^2+\frac{3}{4}}}{2} + \frac{3}{2\cdot 4}\log\left|y+\sqrt{y^2+\frac{3}{4}}\right|+c$$

$$= \frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{2\cdot 4}}}{2} + \frac{3}{8}\log\left|y+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right|+c$$

$$= \frac{(2x+1)\sqrt{x^2+x+1}}{4} + \frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c.$$

Example 20. *Evaluate:*

$$I = \int (x+2)\sqrt{x^2+x+1}\,dx \qquad \left[\text{of the form } \int (px+q)\sqrt{ax^2+bx+c}\,dx\right].$$

Solution: We can write $\quad (x+2) = l\dfrac{d}{dx}\left(x^2+x+1\right)+m = l(2x+1)+m.$

Equating coefficient of x and constant terms, $2l=1$, i.e., $l=\frac{1}{2}$; $l+m=2$, i.e., $m=\frac{3}{2}$.

$$\therefore\ I = \int \left\{\frac{1}{2}(2x+1)+\frac{3}{2}\right\}\sqrt{x^2+x+1}\,dx$$

$$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \sqrt{x^2+x+1}\,dx.$$

For the first integral, put $x^2+x+1=z^2$ so that $(2x+1)\,dx = 2z\,dz$.

$$\text{First integral} = \frac{1}{2}\int (2z\,dz)z = \int z^2dz = \frac{z^3}{3} = \frac{(x^2+x+1)^{3/2}}{3}.$$

For the second integral see the *previous sum.*

$$\text{Given integral} = \frac{(x^2+x+1)^{3/2}}{3} + \frac{3}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right]+c.$$

Example 21. *Evaluate:* $I = \int \log\left(x - \sqrt{x^2-1}\right)\, dx.$ [C.U. B.Com.(H) 2002]

Solution: Taking $u = \log\left(x - \sqrt{x^2-1}\right)$, $v = x$ and using $\int u\,dv = uv - \int v\,du$, we get

$$\begin{aligned}
I &= x\log\left(x-\sqrt{x^2-1}\right) - \int x\frac{d}{dx}\log\left(x-\sqrt{x^2-1}\right)\cdot dx \\
&= x\log\left(x-\sqrt{x^2-1}\right) - \int x\frac{1}{x-\sqrt{x^2-1}}\left(1-\frac{x}{\sqrt{x^2-1}}\right)dx \\
&= x\log\left(x-\sqrt{x^2-1}\right) + \int \frac{x}{\sqrt{x^2-1}}dx \\
&= x\log\left(x-\sqrt{x^2-1}\right) + \int \frac{z\,dz}{z} \qquad [\text{Put } x^2-1=z^2 \Rightarrow x\,dx = z\,dz] \\
&= x\log\left(x-\sqrt{x^2-1}\right) + z + c \\
&= \mathbf{x\log\left(x-\sqrt{x^2-1}\right) + \sqrt{x^2-1} + c.}
\end{aligned}$$

Example 22. *Evaluate:* $\int \frac{\log x}{(1+\log x)^2}dx.$ [C.U. B.Com.(H) 2002]

Solution:

$$I = \int \frac{\log x}{(1+\log x)^2}dx = \int \frac{(1+\log x)-1}{(1+\log x)^2}dx = \int \frac{dx}{1+\log x} - \int \frac{dx}{(1+\log x)^2}.$$

$$\begin{aligned}
\text{First integral} &= \int \frac{dx}{1+\log x} = \frac{1}{1+\log x}x - \int \frac{d}{dx}\left(\frac{1}{1+\log x}\right)x\,dx \\
&= \frac{x}{1+\log x} - \int -\frac{1}{(1+\log x)^2}\cdot\frac{1}{x}\cdot x\,dx = \frac{x}{1+\log x} + \int \frac{dx}{(1+\log x)^2}.
\end{aligned}$$

$$\therefore\ I = \left\{\frac{x}{1+\log x} + \int \frac{dx}{(1+\log x)^2}\right\} - \int \frac{dx}{(1+\log x)^2} = \mathbf{\frac{x}{1+\log x}} + c.$$

Example 23. *Evaluate:* $\int \frac{(x-1)(x-5)}{(x-2)(x-4)}dx.$ [C.U. B.Com.(H) 2002]

Solution: We have

$$\begin{aligned}
\text{Integrand} &= \frac{x^2-6x+5}{(x-2)(x-4)} \quad \text{(both numerator and denominator of degree two)} \\
&= 1 + \frac{A}{x-2} + \frac{B}{x-4} = \frac{(x-2)(x-4)+A(x-4)+B(x-2)}{(x-2)(x-4)}
\end{aligned}$$

$$\therefore\ x^2-6x+5 = (x-2)(x-4)+A(x-4)+B(x-2). \tag{1}$$

This is an identity.

Equating coefficient of like powers of x, we get

$$-6 = -6 + A + B \text{ or, } A + B = 0 \text{ and } 5 = 8 - 4A - 2B, \text{ whence } B = -\frac{3}{2}, A = \frac{3}{2}.$$

[*Otherwise.* Putting $x = 2$ and $x = 4$ successively in (1), we get

$$4 - 12 + 5 = 0 - 2A + B \cdot 0 \text{ and } 16 - 24 + 5 = 0 + 0 + 2B \Rightarrow A = \frac{3}{2} \text{ and } B = -\frac{3}{2}]$$

$$\therefore\ I = \int dx + A\int \frac{dx}{x-2} + B\int \frac{dx}{x-4} = x + \frac{3}{2}\log|x-2| - \frac{3}{2}\log|x-4| + c = \boldsymbol{x + \frac{3}{2}\log\left|\frac{x-2}{x-4}\right| + c.}$$

REVISION EXERCISES ON CHAPTER 9(III)

(Miscellaneous Types)

Taken from various examination papers

1. $\int \frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$.
2. $\int \frac{dx}{\sqrt{3-2x}}$.
3. $\int \frac{2x-3}{\sqrt{4x-1}}\,dx$.
4. $\int xe^x\,dx$.
5. $\int \frac{(x^2+1)^2}{x^3}\,dx$.
6. $\int \frac{x\,dx}{\sqrt{x}+1}$.
7. $\int \frac{x-1}{\sqrt{2x^2-4x+5}}\,dx$.
8. $\int \frac{x^2+1}{x+1}\,dx$.
9. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$.
10. $\int \frac{dx}{x(1+\log x)^2}$.
11. $\int \frac{(x+2)^3}{x^6}\,dx$.
12. $\int \frac{x-2}{\sqrt{x^2-6x+2}}\,dx$.
13. $\int \frac{x\,dx}{x+1}$.
14. $\int \sqrt{x}\left(x^2+3x+4\right)dx$.
15. $\int \frac{dx}{x(1+\log x)^3}$.
16. $\int \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$.
17. $\int \frac{(x^2+3)^2}{\sqrt{x}}\,dx$.
18. $\int x\sqrt{x^2-1}\,dx$.
19. $\int \frac{x\,dx}{\sqrt{1+x^2}}$.
20. $\int \frac{(3x-2)^3}{\sqrt{x}}\,dx$.
21. $\int \frac{4x+3}{\sqrt{2x-1}}\,dx$.
22. $\int \frac{dx}{x\left\{10+7\log x+(\log x)^2\right\}}$.
23. $\int \frac{dx}{(x^2-a^2)(x^2-b^2)}$.
24. $\int \frac{x+1}{\sqrt{x-1}}\,dx$.
25. $\int \frac{(2x-3)\,dx}{\sqrt{2x^2-6x+1}}$.
26. $\int \frac{(x+1)^2}{\sqrt{x}}\,dx$.
27. $\int \frac{dx}{x^2-5x+6}$.
28. $\int \frac{(2x-3)^2}{x^{1/3}}\,dx$.
29. $\int \frac{x+2}{\sqrt{2x+1}}\,dx$.
30. $\int \frac{x-2}{\sqrt{x^2+4x+3}}\,dx$.
31. $\int \frac{dx}{2x^2-4x-7}$.
32. $\int \frac{dx}{x^2-3x+1}$.

33. $\int\left(1+\frac{2}{\sqrt{x}}+\frac{3}{x}\right)dx.$

34. $\int\frac{x-2}{\sqrt[3]{x^2-4x+5}}\,dx.$

35. $\int\frac{t^3dt}{\sqrt[3]{t^4+3}}.$ (Put $t^4+3=z^3$).

36. $\int(2x+3)\sqrt{x^2+3x-1}\,dx.$

37. $\int\frac{x^3}{x-1}dx.$

38. $\int\frac{dx}{\sqrt{2x+5}+\sqrt{2x-3}}.$

39. $\int\frac{2-x}{4x^2+4x-3}dx.$

40. $\int\frac{dx}{\sqrt{e^x+1}}.$

41. $\int\frac{\log x\,dx}{(x+1)^2}.$ [C.U. B.Com. 2004]

ANSWERS

[c denotes the constant of integration]

1. $\frac{2}{3}\left[(x+2)^{3/2}-(x+1)^{3/2}\right]+c.$
2. $-\sqrt{3-2x}+c.$
3. $\frac{1}{12}(4x-1)^{3/2}-\frac{5}{8}(4x-1)^{1/2}+c.$
4. $e^x(x+1)+c.$
5. $\frac{x^2}{2}+2\log|x|-\frac{1}{2x^2}+c.$
6. $\frac{2}{3}x^{3/2}+x+2\sqrt{x}-2\log\left(\sqrt{x}+1\right)+c.$
7. $\frac{1}{2}\sqrt{2x^2-4x+5}+c.$
8. $\frac{x^2}{2}-x+2\log|x+1|+c.$
9. $\frac{2}{3}(x+1)^{3/2}+\frac{2}{3}x^{3/2}+c.$
10. $-\frac{1}{(1+\log x)}+c.$
11. $-\frac{1}{x^5}\left(\frac{1}{2}x^3-2x^2-3x-\frac{8}{5}\right)+c.$
12. $\sqrt{x^2-6x+2}$ $+\log\left|x-3+\sqrt{x^2-6x+2}\right|+c.$
13. $x-\log|x+1|+c.$
14. $\frac{2}{7}x^{7/2}+\frac{6}{5}x^{5/2}+\frac{8}{3}x^{3/2}+c.$
15. $-\frac{1}{2}\cdot\frac{1}{(1+\log x)^2}+c.$
16. $\frac{1}{6}\left\{\frac{1}{b}(5x+1)^{3/2}-(x+1)^{3/2}\right\}+c.$
17. $\frac{2}{9}x^{9/2}+\frac{12}{5}x^{5/2}+18x^{1/2}+c.$
18. $\frac{1}{3}\left(x^2-1\right)^{3/2}+c.$
19. $\sqrt{1+x^2}+c.$
20. $\frac{54}{7}x^{7/2}-\frac{108}{5}x^{5/2}+24x^{3/2}-16x^{1/2}+c.$
21. $\frac{2}{3}(2x-1)^{3/2}+5(2x-1)^{1/2}+c.$
22. $\frac{1}{3}\log\left|\frac{5+\log x}{2+\log x}\right|+c.$
23. $\frac{1}{2(a^2-b^2)}\left[\frac{1}{a}\log\left|\frac{x-a}{x+a}\right|-\frac{1}{b}\log\left|\frac{x-b}{x+b}\right|\right]+c.$
24. $\frac{2}{3}(x-1)^{3/2}+4(x-1)^{1/2}+c.$
25. $\sqrt{2x^2-6x+1}+c.$
26. $\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+2x^{1/2}+c.$
27. $\log\left|\frac{x-3}{x-2}\right|+c.$
28. $\frac{3}{2}x^{5/3}-\frac{36}{5}x^{5/3}+\frac{27}{3}x^{2/3}+c.$
29. $\frac{1}{6}(2x+1)^{3/2}+\frac{3}{2}(2x+1)^{1/2}+c.$
30. $\sqrt{x^2+4x+3}$ $-4\log\left|x+2+\sqrt{x^2+4x+3}\right|+c.$
31. $\frac{\sqrt{2}}{12}\log\left|\frac{\sqrt{2}(x-1)-3}{\sqrt{2}(x-1)+3}\right|+c.$
32. $\frac{1}{\sqrt{5}}\log\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}+c.$
33. $x+4x^{1/2}+3\log_e|x|+c.$
34. $\frac{3}{4}\left(x^2-4x+5\right)^{2/3}+c.$
35. $\frac{3}{8}\left(t^4+3\right)^{2/3}+c.$
36. $\frac{2}{3}\left(x^2+3x-1\right)^{3/2}+c.$
37. $\frac{1}{3}x^3+\frac{1}{2}x^2+x+\log_e|x-1|$
38. $\frac{1}{24}\left\{(2x+5)^{3/2}-(2x+3)^{3/2}\right\}+c.$
39. $\frac{1}{16}\{3\log|2x-1|-7\log|2x+3|\}+c.$
40. $\log\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}+c.$
41. $-\frac{\log|x|}{x+1}+\log\left|\frac{x}{x+1}\right|+c.$

Chapter 10

Definite Integrals: Finding Areas and Simple Applications

10.1 Introduction

In the present chapter we first define definite integration as the limit of a sum and then with the help of the fundamental theorem of Integral Calculus we shall see how the concept of indefinite integration as inverse of differentiation helps us to evaluate the value of the definite integral. Another concept, namely definite integration as the measure of area under a curve, will then be discussed.

10.2 Definite Integral as the Limit of a Sum

We give a precise definition of definite integral:

Definition 1. *Let $f(x)$ be a bounded function defined in a closed interval $a \leq x \leq b$, where a and b are finite numbers. We divide this interval into n subdivisions—not necessarily all equal—by means of a set of the points $\{x_0(=a), x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}, x_n(=b)\}$.*

We call $\delta_1 = x_1 - x_0$, $\delta_2 = x_2 - x_1, \ldots, \delta_r = x_r - x_{r-1}, \ldots, \delta_n = x_n - x_{n-1}$.

We then take an arbitrary point c_1 in the first sub-interval δ_1; c_2, an arbitrary point in the second sub-interval δ_2, etc., $\cdots$; c_n, an arbitrary point in the nth sub-interval δ_n and find the functional values at these points. We then construct the sum

$$f(c_1)\delta_1 + f(c_2)\delta_2 + \cdots + f(c_r)\delta_r + \cdots + f(c_n)\delta_n = \sum_{r=1}^{n} f(c_r)\delta_r = S_n \text{ (say).}$$

S_n clearly depends on the choice of points of subdivision and also on the choice of the arbitrary points $c_1, c_2, \ldots, c_n$. When n increases indefinitely the lengths δ's tend to zero. If $\lim\limits_{n\to\infty} S_n$ tends to a definite limit S (say), where S is independent of the points of subdivision and also independent of the choice of c's, then we say that $f(x)$ is **integrable** over the interval $[a, b]$ and this limit S is called the *definite integral of $f(x)$ between the limits a and b and it is denoted by*

$$\int_a^b f(x)\,dx.$$

Thus, one may **remember**:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(c_r)\delta_r \textit{ (provided such a limit exists).}$$

Observations. When does such a limit exist? For the beginners it is sufficient to know that if $f(x)$ is continuous for every x in $a \le x \le b$, then such a limit always exists. In other words, a **continuous function $f(x)$ in $[a,b]$ is integrable.**

Once we know that the function is integrable, then for evaluation of $\lim_{n\to\infty} S_n$, we **need not take unequal lengths** for the subdivision and the choice of c's may also be made in a simplified manner so that it may be convenient for our calculation. In the present text, we shall consider only continuous functions (and hence they can be assumed to be always integrable) Therefore, in evaluation of definite integrals by the method of summation we shall usually take the length of each subdivision equal to h, say. Thus, the points of subdivision may be conveniently taken as a, $a+h$, $a+2h$, ..., $a+nh=b$. Then by definition,

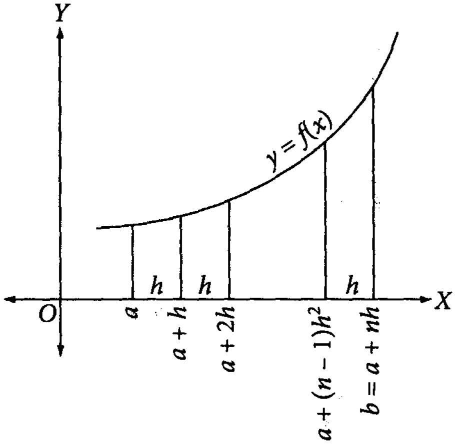

Fig. 10.1

- $\displaystyle\int_a^b f(x) = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$, **where $nh = b-a$.**
- $\displaystyle\int_0^1 f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh)$, **where $nh=1$;** $[\because\ a=0,\ b=1.]$

Either the **left end** or the **right end** point of each sub-interval may be taken for $c_1, c_2, \ldots, c_n$. The limit of S_n, as $n\to\infty$, now obtained will give the value of the definite integral, because by definition, any mode of subdivision and any arbitrary choice of c's will give the same limit, namely $S = \displaystyle\int_a^b f(x)\,dx$.

10.3 Illustrative Examples

Example 1. *Evaluate from definition*:

(i) $\displaystyle\int_0^1 2x^3dx$; [C.U. B.Com.(H) 1998] (ii) $\displaystyle\int_0^1 \left(x^2+2x\right)dx$. [C.U. B.Com.(H) 2007]

Solution: (i) Let $f(x)=2x^3$; then $f(rh)=2(rh)^3=2r^3h^3$. Here $a=0, b=1$. By definition,

$$\begin{aligned}
\int_0^1 2x^3dx &= \lim_{h\to 0} h\sum_{r=1}^{n} f(rh), \text{ where } nh=b-a=1-0=1\\
&= \lim_{h\to 0} h\sum_{r=1}^{n} 2r^3h^3 = \lim_{h\to 0} 2h\cdot h^3\sum_{r=1}^{n} r^3, \text{ where } nh=1\\
&= 2\lim_{h\to 0} h^4\left\{\frac{n(n+1)}{2}\right\}^2 \quad \left[\because\ \sum_{r=1}^{n} r^3 = 1^3+2^3+\cdots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2\right]\\
&= 2\lim_{h\to 0} h^4\cdot\frac{n^2(n+1)^2}{4} = \frac{2}{4}\cdot\lim_{h\to 0}\left\{(nh)^2\cdot(nh+h)^2\right\}\\
&= \frac{1}{2}\lim_{h\to 0}\left\{1\cdot(1+h)^2\right\} = \frac{1}{2}(1+0)^2 = \frac{1}{2} \quad [\because\ nh=1]
\end{aligned}$$

(ii) Let $f(x)=x^2+2x$. Here $a=0$, $b=1$ and $nh=b-a=1-0=1$.

$$\therefore\ f(rh)=(rh)^2+2(rh)=r^2h^2+2rh.$$

By definition,

$$\begin{aligned}
\int_0^1 \left(x^2+2x\right)dx &= \lim_{h\to 0} h\sum_{r=1}^{n} f(rh) = \lim_{h\to 0} h\sum_{r=1}^{n}\left(r^2h^2+2rh\right)\\
&= \lim_{h\to 0} h\left[h^2\sum_{r=1}^{n} r^2 + 2h\sum_{r=1}^{n} r\right], \text{ where } nh=1\\
&= \lim_{h\to 0}\left[h^3\cdot\frac{n(n+1)(2n+1)}{6}+2h^2\cdot\frac{n(n+1)}{2}\right]\\
&= \lim_{h\to 0}\left[\frac{nh(nh+h)(2nh+h)}{6}+(nh)(nh+h)\right]\\
&= \lim_{h\to 0}\left[\frac{1\cdot(1+h)(2\cdot 1+h)}{6}+1(1+h)\right]\\
&= \frac{(1+0)(2+0)}{6}+(1+0)=\frac{1}{3}+1=\mathbf{\frac{4}{3}}.
\end{aligned}$$

Example 2. *Evaluate from definition*:

(i) $\int_0^2 x^2 dx$; [C.U. B.Com.(H) 2001]

(ii) $\int_0^3 (2x+3)\,dx$; [C.U. B.Com.(H) 1999]

(iii) $\int_1^2 5x\,dx$. [V.U. B.Com.(H) 2007]

Solution: (i) $\int_0^2 x^2 dx$. Let $f(x)=x^2$; Here $a=0, b=2, nh=2-0=2$. By definition,

$$\begin{aligned}
\int_0^2 x^2 dx &= \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh)\\
&= \lim_{h\to 0} h\sum_{r=1}^{n} r^2h^2 = \lim_{h\to 0} h\cdot h^2\sum_{r=1}^{n} r^2, \text{ where } nh=2\\
&= \lim_{h\to 0} h^3\cdot\frac{n(n+1)(2n+1)}{6} = \lim_{h\to 0}\frac{nh(nh+h)(2nh+h)}{6}\\
&= \lim_{h\to 0}\frac{2\cdot(2+h)(2\cdot 2+h)}{6} = \frac{2(2+0)(4+0)}{6}=\mathbf{\frac{8}{3}}.
\end{aligned}$$

(ii) This problem is similar to (i). Here $nh=3-0=3$ and $f(rh)=2rh+3$.

(iii) $\int_1^2 5x\,dx$. Let $f(x)=5x$. Here $a=1$, $b=2$, $nh=2-1=1$. By definition,

$$\int_1^2 5x\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) = \lim_{h\to 0} h\sum_{r=1}^{n} f(1+rh)$$
$$= \lim_{h\to 0} h\sum_{r=1}^{n} 5(1+rh) \quad [\because f(x)=5x \Rightarrow f(1+rh)=5(1+rh)]$$
$$= \lim_{h\to 0} 5h\left[\sum_{r=1}^{n} 1 + h\sum_{r=1}^{n} r\right] = \lim_{h\to 0} 5h\left[n + h\cdot\frac{n(n+1)}{2}\right]$$
$$= \lim_{h\to 0} 5\left[nh + \frac{nh(nh+h)}{2}\right], \text{ where } nh=1$$
$$= \lim_{h\to 0} 5\left[1+\frac{1\cdot(1+h)}{2}\right] = 5\left[1+\frac{1+0}{2}\right] = \mathbf{\frac{15}{2}}.$$

Example 3. *To evaluate* $\int_a^b (5x+3)\,dx$, *by the definition of definite integral as the limit of a sum.*

Solution: Let $f(x)=5x+3$; then $f(a+rh)=5(a+rh)+3=5rh+5a+3$. By definition,

$$\int_a^b (5x+3)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) = \lim_{h\to 0} h\sum_{r=1}^{n}\{5rh+(5a+3)\}, \text{ where } nh=b-a$$
$$= \lim_{h\to 0} h\left[5h\sum_{r=1}^{n} r + (5a+3)\sum_{r=1}^{n} 1\right] = \lim_{h\to 0} h\left[5h\cdot\frac{n(n+1)}{2} + (5a+3)n\right]$$
$$= \lim_{h\to 0}\left[5\frac{nh(nh+h)}{2} + (5a+3)nh\right], \text{ where } nh=b-a$$
$$= \lim_{h\to 0}\left[\frac{5(b-a)(b-a+h)}{2} + (5a+3)(b-a)\right]$$
$$= \frac{5(b-a)(b-a+0)}{2} + (5a+3)(b-a)$$
$$= \frac{5}{2}(b-a)^2 + 5a(b-a) + 3(b-a)$$
$$= \frac{5}{2}(b-a)[b-a+2a] + 3(b-a)$$
$$= \mathbf{\frac{5}{2}\left(b^2-a^2\right) + 3(b-a).}$$

Example 4. *Evaluate:* $\int_3^6 x^2\,dx$, *by using definition as the limit of a sum.*

Solution: Here $f(x)=x^2$, $a=3$, $b=6$ and $nh=b-a=6-3=3$.

$$\therefore\ f(a+rh)=f(3+rh)=(3+rh)^2.$$

By definition,

$$\int_3^6 x^2 dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) = \lim_{h\to 0} h\sum_{r=1}^{n}(3+rh)^2 = \lim_{h\to 0} h\left[(3+h)^2+(3+2h)^2+\cdots+(3+nh)^2\right]$$

$$= \lim_{h\to 0} h\left[9n + h^2\left(1^2+2^2+3^2+\cdots+n^2\right)+6h(1+2+3+\cdots+n)\right]$$

[Now in elementary algebra we have the following formulae:

(i) $1+2+3+\cdots+n = \frac{n(n+1)}{2}$;

(ii) $1^2+2^2+3^3+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$]

$$= \lim_{h\to 0} h\left[9n + h^2\cdot\frac{n(n+1)(2n+1)}{6}+6h\frac{n(n+1)}{2}\right]$$

$$= \lim_{h\to 0}\left[9nh + \frac{nh(nh+h)(2nh+h)}{6}+3nh(nh+h)\right]$$

$$= \lim_{h\to 0}\left[9\times 3 + \frac{3(3+h)(6+h)}{6}+3\cdot 3(3+h)\right] \qquad (\because\ nh=3)$$

$$= 27 + \frac{3\cdot 3\cdot 6}{6}+9\cdot 3 = \mathbf{63}.$$

Example 5. *Find by the method of summation the value of* $\int_a^b e^{kx}\,dx$. [C.U. B.Com.(H) 2002]

Solution: Let $f(x)=e^{kx}$. Here the limits of integration are a and b.

$\therefore\ f(a+rh)=e^{k(a+rh)}$.

By the **Method of Summation**, i.e., by definition, we get

$$\int_a^b e^{kx}\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) = \lim_{h\to 0} h\sum_{r=1}^{n} e^{k(a+rh)}, \text{ where } nh=b-a$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} e^{ka}\cdot e^{krh} = e^{ka}\cdot\lim_{h\to 0} h\sum_{r=1}^{n} e^{krh}$$

$$= e^{ka}\cdot\lim_{h\to 0} h\left[e^{kh}+e^{2kh}+e^{3kh}+\cdots+e^{nkh}\right]$$

$$= e^{ka}\cdot\lim_{h\to 0} h\cdot\frac{e^{kh}\left\{\left(e^{kh}\right)^n-1\right\}}{e^{kh}-1} \quad [\because\ e^{kh}+e^{2kh}+\cdots \text{ is a GP with common ratio } e^{kh}.]$$

$$= e^{ka}\cdot\lim_{h\to 0} h\cdot e^{kh}\cdot\frac{e^{knh}-1}{e^{kh}-1} = e^{ka}\cdot\lim_{h\to 0} he^{kh}\cdot\frac{\left\{e^{k(b-a)}-1\right\}}{e^{kh}-1} \quad [\because\ nh=b-a]$$

$$= e^{ka}\cdot\left\{e^{k(b-a)}-1\right\}\lim_{h\to 0}\left[e^{kh}\cdot\frac{1}{\frac{e^{kh}-1}{kh}\times k}\right]$$

$$= \left\{e^{ka+k(b-a)}-e^{ka}\right\}\cdot\lim_{h\to 0} e^{kh}\times\frac{1}{k}\cdot\frac{1}{\lim\limits_{z\to 0}\frac{e^z-1}{z}}, \text{ where } z=kh\to 0 \text{ as } h\to 0$$

$$= \left(e^{kb}-e^{ka}\right)\frac{1}{k}\cdot e^0\cdot\frac{1}{1} = \frac{1}{k}\left(e^{kb}-e^{ka}\right).$$

10.4 Definite Integral: Evaluation Techniques

There is a theorem in Integral Calculus called *fundamental theorem of Integral Calculus* which states:

If $f(x)$ is integrable in $[a,b]$ and if there exists a function $\phi(x)$ such that

$$\frac{d}{dx}\{\phi(x)\} = f(x), \text{ for every } x \text{ in } a \le x \le b,$$

then *the definite integral of* $f(x)$ *from* $x=a$ *to* $x=b$ *is given by*

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

In fact, if $f(x)$ is continuous in $[a,b]$ (and hence integrable there), there exists a function $\phi(x)$ whose derivative $\phi'(x) = f(x)$; $\phi(x)$ is called a *primitive or an indefinite integral of* $f(x)$, denoted by

$$\int f(x)\,dx.$$

Therefore, to evaluate, say $\int_1^4 3x^2\,dx$, we proceed as follows:

(Note that $3x^2$ is a continuous function of x in $1 \le x \le 4$).

- We first evaluate the indefinite integral

$$\int 3x^2\,dx = x^3 + c = \phi(x), \text{ say.}$$

- Then we calculate

$$\phi(4) - \phi(1) = \left(4^3 + c\right) - \left(1^3 + c\right) = 4^3 - 1^3 = 63; \qquad \therefore \int_1^4 3x^2\,dx = 63.$$

It is usual to write,

$$\phi(4) - \phi(1) = [\phi(x)]_{x=1}^{x=4} \text{ or, } [\phi(x)]_1^4.$$

Illustrative Examples

Example 6. (i) $\displaystyle\int_a^b \frac{1}{x}\,dx = \left[\log|x|\right]_a^b = \log|b| - \log|a| = \mathbf{\log\left|\frac{b}{a}\right|}$. [B.U. B.Com.(H) 2008]

(ii) $\displaystyle\int_1^4 x^2\,dx = \left[\frac{x^3}{3} + c\right]_1^4 = \left(\frac{4^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right) = \mathbf{21}.$

The constant c of the indefinite integral need not be mentioned as it will not occur in the final step.

Example 7. *Evaluate*: (i) $\displaystyle\int_1^x \left(e^{3x} + 5x^4\right)dx$; (ii) $\displaystyle\int_1^{e^2} \frac{dx}{x}$.

Solution:

$$\text{(i)} \int_1^2 \left(e^{3x}+5x^4\right)dx = \int_1^2 e^{3x}\,dx + 5\int_1^2 x^4 dx = \left[\frac{e^{3x}}{3}\right]_1^2 + 5\left[\frac{x^5}{5}\right]_1^2 = \frac{1}{3}\left(e^6-e^3\right)+5\left(\frac{2^5-1^5}{5}\right)$$
$$= \mathbf{\frac{1}{3}\left(e^6-e^3\right)+31.}$$

$$\text{(ii)} \int_1^{e^2} \frac{dx}{x} = \left[\log x\right]_1^{e^2} = \log e^2 - \log 1 = 2\log e - 0 = 2\times 1 = \mathbf{2}.$$

Example 8. *Evaluate*:

(i) $\int_1^2 \left(1+\frac{2}{\sqrt{x}}+3x\right)dx.$

[C.U. B.Com.(H) 1994; N.B.U. B.Com.(H) 2007]

(ii) $\int_1^2 x\left(\sqrt{x}+\frac{2}{x}-\frac{5}{x^{3/2}}\right)dx.$

[C.U. B.Com.(H) 1996]

Solution:

$$\text{(i)} \int_1^2 \left(1+\frac{2}{\sqrt{x}}+3x\right)dx = \left[x+2\cdot\frac{x^{-(1/2)+1}}{-\frac{1}{2}+1}+\frac{3x^2}{2}\right]_1^2 = \left[x+4\sqrt{x}+\frac{3}{2}x^2\right]_1^2$$
$$= \left(2+4\sqrt{2}+\frac{3}{2}\times 4\right)-\left(1+4\cdot 1+\frac{3}{2}\times 1^2\right)$$
$$= 8+4\sqrt{2}-\frac{13}{2} = \mathbf{\frac{3}{2}+4\sqrt{2}}.$$

$$\text{(ii)} \int_1^2 x\left(\sqrt{x}+\frac{2}{x}-\frac{5}{x^{3/2}}\right)dx = \int_1^2 \left(x^{3/2}+2-5x^{-(1/2)}\right)dx = \left[\frac{x^{(3/2)+1}}{\frac{3}{2}+1}+2x-5\cdot\frac{x^{1/2}}{\frac{1}{2}}\right]_1^2$$
$$= \left[\frac{2}{5}x^{5/2}+2x-10\sqrt{x}\right]_1^2 = \left\{\frac{2}{5}(2)^{5/2}+2\cdot 2-10\sqrt{2}\right\}-\left(\frac{2}{5}\cdot 1+2\cdot 1-10\cdot 1\right)$$
$$= \frac{8\sqrt{2}}{5}+4-10\sqrt{2}+\frac{38}{5} = \frac{58}{5}-\frac{42\sqrt{2}}{5} = \mathbf{\frac{2}{5}\left(29-21\sqrt{2}\right)}.$$

Example 9. *Evaluate*:

(i) $\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}}.$ (ii) $\int_0^{\log 2} \frac{e^x}{1+e^x}\,dx.$ (iii) $\int_0^2 \frac{x\,dx}{(x+1)^2}.$

[C.U. B.Com.(H) 1993]

Solution: (i) Put $1+x^2=z^2$; then $2x\,dx=2z\,dz$ or, $x\,dx=z\,dz$. If $x=0$, $z=1$ and if $x=1$, $z=\sqrt{2}$.

$$\therefore \int_0^1 \frac{x\,dx}{\sqrt{1+x^2}} = \int_1^{\sqrt{2}} \frac{z\,dz}{z} = \int_1^{\sqrt{2}} dz = \left[z\right]_1^{\sqrt{2}} = \mathbf{\sqrt{2}-1.}$$

(ii) Put $1+e^x=z$; then $e^x\,dx=dz$. If $x=0$, then $z=1+e^0=1+1=2$ and if $x=\log 2$, then $z=1+e^{\log 2}=1+2=3$.

$$\therefore \int_0^{\log 2} \frac{e^x}{1+e^x}\,dx = \int_2^3 \frac{dz}{z} = \left[\log z\right]_2^3 = \log 3-\log 2 = \mathbf{\log\frac{3}{2}.}$$

(iii) $\displaystyle\int_0^2 \frac{x}{(x+1)^2} = \int_1^3 \frac{z-1}{z^2}dz = \int_1^3 \left(\frac{1}{z} - \frac{1}{z^2}\right) dz$ | Put $x+1=z$; then $x=z-1$ and $dx = dz$. If $x=0$, $z=1$ and if $x=2$, $z=3$.

$$= \left[\log z + \frac{1}{z}\right]_1^3 = \left(\log 3 + \frac{1}{3}\right) - (\log 1 + 1) = \mathbf{\log_e 3 - \frac{2}{3}}.$$

10.5 Method of Substitution

While we evaluate indefinite integral by the method of substitution, we can change the limits of integration in a definite integral in terms of the new variable introduced:

Example 10. (a) $\displaystyle\int_0^{16} \frac{x^{1/4}}{1+x^{1/4}} dx$; (b) $\displaystyle\int_1^2 \frac{x^2-1}{x^2} \cdot e^{x+\frac{1}{x}} dx$. [B.U. B.Com.(H) 2006]

Solution: (a) *First method.* Put $z = x^{1/4}$ or, $x = z^4$; then $dx = 4z^3 dz$.

$$\therefore \int \frac{x^{1/4}}{1+x^{1/4}} dx = \int \frac{z \cdot 4z^3 dz}{1+z} = 4\int \frac{z^4}{1+z} dz.$$

Now, when $x=0$, $z=0$; when $x=16$, $z=2$.

$$\begin{array}{r|l} z+1 & z^4 \quad (z^3 - z^2 + z - 1 \\ & \underline{z^4 + z^3} \\ & \quad -z^3 \\ & \quad \underline{-z^3 - z^2} \\ & \qquad z^2 \\ & \qquad \underline{z^2 + z} \\ & \qquad\quad -z \\ & \qquad\quad \underline{-z-1} \\ & \qquad\qquad 1 \end{array}$$

$$\therefore \int_0^{16} \frac{x^{1/4}}{1+x^{1/4}} dx = 4\int_0^2 \frac{z^4}{1+z} dz = 4\int_0^2 \left(z^3 - z^2 + z - 1 + \frac{1}{z+1}\right) dz$$

$$= 4\left\{\left[\frac{z^4}{4}\right]_0^2 - \left[\frac{z^3}{3}\right]_0^2 + \left[\frac{z^2}{2}\right]_0^2 - [z]_0^2 + [\log(z+1)]_0^2\right\}$$

$$= 4\left\{\frac{2^4}{4} - \frac{2^3}{3} + \frac{2^2}{2} - 2 + (\log 3 - \log 1)\right\}$$

$$= 4\left(\frac{4}{3} + \log 3\right) = \mathbf{\frac{16}{3} + \log 81.}$$

Second method (Evaluate as indefinite integral first)

$$\int \frac{x^{1/4}}{1+x^{1/4}} dx = 4\int \frac{z^4}{1+z} dz \text{ (putting } z = x^{1/4})$$

$$= 4\int \left(z^3 - z^2 + z - 1 + \frac{1}{z+1}\right) dz = 4\left[\frac{z^4}{4} - \frac{z^3}{3} + \frac{z^2}{2} - z + \log(z+1)\right] + c$$

$$= 4\left[\frac{x}{4} - \frac{x^{3/4}}{3} + \frac{x^{1/2}}{2} - x^{1/4} + \log\left(x^{1/4}+1\right)\right] + c \text{ (putting } z = x^{1/4})$$

$$\therefore \int_0^{16} \frac{x^{1/4} dx}{1+x^{1/4}} = 4\left[\frac{x}{4} - \frac{x^{3/4}}{3} + \frac{x^{1/2}}{2} - \frac{x^{1/4}}{1} + \log\left(x^{1/4}+1\right)\right]_0^{16}$$

$$= 4\left[\frac{16}{4} - \frac{8}{3} + \frac{4}{2} - 2 + \log(2+1) - \log 1\right] = \mathbf{\frac{16}{3} + \log 81} \text{ (as before).}$$

(b) Put $x+\frac{1}{x}=z$; then $\left(1-\frac{1}{x^2}\right)dx=dz$ or, $\frac{x^2-1}{x^2}dx=dz$; if $x=1, z=2$ and if $x=2$, then $z=\frac{5}{2}$ and $\int_1^2 \frac{(x^2-1)}{x^2}\cdot e^{x+\frac{1}{x}}\,dx=\int_2^{5/2} e^z\,dz=\left[e^z\right]_2^{5/2}=e^{5/2}-e^2=\mathbf{e^2(\sqrt{e}-1)}$.

Example 11. *Evaluate:* $I=\int_0^{1/2}\frac{dx}{\sqrt{3-2x}}$. [C.U. B.Com.(H) 1999]

Solution: Put $3-2x=z^2$ so that $dx=-z\,dz$. When $x=0$, $z=\sqrt{3}$ and when $x=1/2$, $z=\sqrt{2}$.

$$\therefore\ I=\int_{\sqrt{3}}^{\sqrt{2}}-\frac{z\,dz}{z}=-\int_{\sqrt{3}}^{\sqrt{2}}dz=-\left[z\right]_{\sqrt{3}}^{\sqrt{2}}=\mathbf{\sqrt{3}-\sqrt{2}}.$$

Example 12. *Evaluate:* $\int_0^1 xe^x\,dx$. [C.U. B.Com.(H) 1991]

Solution:

$$\int xe^x\,dx=x\int e^x\,dx-\int\left\{\frac{d(x)}{dx}\cdot\int e^x\,dx\right\}dx=xe^x-\int(1\cdot e^x)dx+c=xe^x-e^x+c=e^x(x-1)+c.$$

$$\therefore\ \int_0^1 xe^x\,dx=\left[e^x(x-1)\right]_0^1=0-(-1)=1 \text{ [omitting } c.]$$

Example 13. *Evaluate:* $\int_0^1\frac{dx}{\sqrt{x+1}-\sqrt{x}}$. [B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2011]

Solution:

$$\int_0^1\frac{\sqrt{x+1}+\sqrt{x}}{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x+1}-\sqrt{x}\right)}dx=\int_0^1\frac{\left(\sqrt{x+1}+\sqrt{x}\right)}{x+1-x}dx=\int_0^1\sqrt{x+1}\,dx+\int_0^1\sqrt{x}\,dx$$

$$=\frac{2}{3}\left[(x+1)^{3/2}\right]_0^1+\frac{2}{3}\left[x^{3/2}\right]_0^1=\frac{2}{3}\left(2^{3/2}-1\right)+\frac{2}{3}(1-0)=\mathbf{\frac{4\sqrt{2}}{3}}.$$

Example 14. *Evaluate:* $\int_0^3\frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$. [C.U. B.Com.(H) 1997]

Solution:

$$\int_0^3\frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}=\int_0^3\frac{x\left(\sqrt{5x+1}-\sqrt{x+1}\right)dx}{\left(\sqrt{5x+1}+\sqrt{x+1}\right)\left(\sqrt{5x+1}-\sqrt{x+1}\right)}$$

$$=\int_0^3\frac{x\left(\sqrt{5x+1}-\sqrt{x+1}\right)}{(5x+1)-(x+1)}dx=\int_0^3\frac{x\left(\sqrt{5x+1}-\sqrt{x+1}\right)}{4x}dx$$

$$=\frac{1}{4}\left[\int_0^3\sqrt{5x+1}\,dx-\int_0^3\sqrt{x+1}\,dx\right]$$

Put $5x+1=u^2$ and $x+1=v^2$; then $5dx=2u\,du$ and $dx=2v\,du$. If $x=0$, $u=1$, $v=1$ and if $x=3$, $u=4$ and $v=2$.

$$= \frac{1}{4}\left[\frac{1}{5}\int_1^4 u\cdot 2u\,du - \int_1^2 v\cdot 2v\,dv\right] = \frac{1}{4}\left\{\frac{2}{5}\times\left[\frac{u^3}{3}\right]_1^4 - 2\times\left[\frac{v^3}{3}\right]_1^2\right\}$$

$$= \frac{1}{30}(64-1) - \frac{1}{6}(8-1) = \frac{63}{30} - \frac{7}{6} = \frac{63-35}{30} = \frac{28}{30} = \mathbf{\frac{14}{15}}.$$

Example 15. *Evaluate*:

(a) $\displaystyle\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$. [C.U. B.Com.(H) 1994] (b) $\displaystyle\int_4^6 \frac{dx}{x^2-9}$. [B.U. B.Com.(H) 2002]

Solution: (a) Put $1+\log x = z$ so that $\dfrac{dx}{x} = dz$. When $x=1, z=1$ ($\because$ $\log 1 = 0$) and when $x = e^2, z = 1+2\log e = 3$ ($\because$ $\log e = 1$).

$$\text{Given integral} = \int_1^3 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_1^3 = -\frac{1}{3}+1 = \mathbf{\frac{2}{3}}.$$

(b) $\displaystyle\int_4^6 \frac{dx}{x^2-9} = \int_4^6 \frac{dx}{x^2-3^2} = \frac{1}{2\cdot 3}\left[\log\frac{x-3}{x+3}\right]_4^6 = \frac{1}{6}\left(\log\frac{3}{9} - \log\frac{1}{7}\right) = \mathbf{\frac{1}{6}\log\frac{7}{3}}.$

Example 16. *Show that* $\displaystyle\int_a^b \frac{\log x}{x}\,dx = \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$. [C.U. B.Com.(H) 2001]

Solution:

$$\int \frac{\log x}{x}\,dx = \int \log x\cdot\frac{1}{x}\,dx = \int z\,dz \qquad \Big|\ \text{Put } \log x = z;\text{ then } \frac{1}{x}\,dx = dz$$

$$= \frac{z^2}{2} + c = \frac{1}{2}(\log x)^2 + c.$$

$$\therefore \int_a^b \frac{\log x}{x}\,dx = \frac{1}{2}\left[(\log x)^2\right]_a^b = \frac{1}{2}\left[(\log b)^2 - (\log a)^2\right] = \frac{1}{2}(\log b + \log a)(\log b - \log a)$$

$$= \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)\ \textbf{(Proved)}.$$

Example 17. *Evaluate:* $\displaystyle\int_1^3 \frac{dx}{\sqrt{(4+5x^2)^3}}$.

Solution: Put $x = \dfrac{1}{z}$ so that $dx = -\dfrac{1}{z^2}dz$. When $x=1, z=1$; when $x=3, z=\dfrac{1}{3}$.

$$\text{Given integral} = \int_1^{1/3} -\frac{1}{z^2}\frac{dz}{\sqrt{\left(4+\frac{5}{z^2}\right)^3}} = \int_1^{1/3} -\frac{1}{z^2}\times\frac{z^3\,dz}{(4z^2+5)^{3/2}} = -\int_1^{1/3}\frac{z\,dz}{(4z^2+5)^{3/2}}.$$

Again, put $4z^2+5 = t^2$ so that $z\,dz = \dfrac{t\,dt}{4}$. When $z=1$, $t^2 = 9$, $t=3$.

When $z=\frac{1}{3}, t^2=\frac{49}{9}, t=\frac{7}{3}$.

$$\therefore \text{ Integral becomes} -\int_3^{7/3}\frac{1}{4}\cdot\frac{t\,dt}{t^3}=-\frac{1}{4}\int_3^{7/3}\frac{dt}{t^2}=\frac{1}{4}\left[\frac{1}{t}\right]_3^{7/3}=\frac{1}{4}\left(\frac{3}{7}-\frac{1}{3}\right)=\mathbf{\frac{1}{42}}.$$

Example 18. *Evaluate*: $\int_0^1 \frac{xe^x}{(x+1)^2}\,dx$. [C.U. B.Com.(H) 1996]

Solution:

$$\int\frac{xe^x}{(x+1)^2}\,dx=\int\frac{(\overline{x+1}-1)}{(x+1)^2}e^x\,dx=\int\frac{e^x}{x+1}\,dx-\int\frac{e^x}{(x+1)^2}\,dx$$
$$=\frac{1}{x+1}\int e^x\,dx-\int\left\{\frac{d}{dx}\left(\frac{1}{x+1}\right)\cdot\int e^x\,dx\right\}dx-\int\frac{e^x}{(x+1)^2}\,dx$$
$$=\frac{1}{x+1}e^x+\int\frac{e^x}{(x+1)^2}\,dx-\int\frac{e^x}{(x+1)^2}\,dx+c=\frac{1}{x+1}e^x+c.$$

$$\therefore \int_0^1\frac{xe^x}{(x+1)^2}\,dx=\left[\frac{1}{x+1}e^x\right]_0^1=\frac{e}{2}-1=\mathbf{\frac{e-2}{2}}.$$

Example 19. $\int_0^1 x\log(1+2x)\,dx$. [C.U. B.Com.(H) 1999]

Solution:

$$\int_0^1 x\log(1+2x)\,dx=\int_1^3\frac{(z-1)}{2}\log z\cdot\frac{1}{2}dz$$

Put $1+2x=z$; then $dx=\frac{1}{2}dz$ and $x=\frac{z-1}{2}$.
If $x=0, z=1$ and if $x=1, z=3$.

$$=\frac{1}{4}\int_1^3(z-1)\log z\,dz=\frac{1}{4}\left[\log z\left(\frac{z^2}{2}-z\right)-\int\left(\frac{z^2}{2}-z\right)\times\frac{1}{z}dz\right]_{z=1}^{z=3}$$
$$=\frac{1}{4}\left[\left(\frac{z^2}{2}-z\right)\log z-\int\left(\frac{1}{2}z-1\right)dz\right]_1^3=\frac{1}{4}\left[\left(\frac{z^2}{2}-z\right)\log z-\left(\frac{z^2}{4}-z\right)\right]_{z=1}^{z=3}$$
$$=\frac{1}{4}\left[\left\{\left(\frac{9}{2}-3\right)\log 3-\left(\frac{9}{4}-3\right)\right\}-\left\{\left(\frac{1}{2}-1\right)\log 1-\left(\frac{1}{4}-1\right)\right\}\right]$$
$$=\frac{1}{4}\left[\frac{3}{2}\log 3+\frac{3}{4}-\frac{3}{4}\right]=\mathbf{\frac{3}{8}\log 3}.$$

Example 20. *Evaluate*:

(i) $\int_2^3\frac{x^5}{x^4-1}\,dx$; [C.U. B.Com.(H) 1996] (ii) $\int_0^1\frac{1}{2x^2+3x+1}\,dx$. [C.U. B.Com.(H) 1999]

Solution:

(i) $\displaystyle\int_2^3 \frac{x^5}{x^4-1}\,dx = \int_4^9 \frac{z^2\cdot\frac{1}{2}dz}{z^2-1}$ | Put $x^2 = z$; then $x\,dx = \frac{1}{2}dz$. If $x = 2, z = 4$ and if $x = 3, z = 9$. Also $x^4 = z^2$.

$$= \frac{1}{2}\int_4^9 \frac{z^2}{z^2-1}dz = \frac{1}{2}\left[\int_4^9 \left(1+\frac{1}{z^2-1}\right)dz\right] = \frac{1}{2}\left[z+\frac{1}{2}\log\left(\frac{z-1}{z+1}\right)\right]_{z=4}^{z=9}$$

$$= \frac{1}{2}\left[9+\frac{1}{2}\log\left(\frac{9-1}{9+1}\right)-\left(4+\frac{1}{2}\log\frac{4-1}{4+1}\right)\right] = \frac{1}{2}\left[5+\frac{1}{2}\left(\log\frac{4}{5}-\log\frac{3}{5}\right)\right]$$

$$= \frac{1}{2}\left[5+\frac{1}{2}\log\left(\frac{4}{5}\times\frac{5}{3}\right)\right] = \mathbf{\frac{1}{2}\left[5+\frac{1}{2}\log\frac{4}{3}\right]}.$$

(ii) $2x^2+3x+1 = 2x^2+2x+x+1 = 2x(x+1)+1(x+1) = (x+1)(2x+1)$

$$\int_0^1 \frac{1}{2x^2+3x+1}dx = \int_0^1 \frac{1}{(x+1)(2x+1)}dx = \int_0^1 \left(\frac{2}{2x+1}-\frac{1}{x+1}\right)dx \text{ [by inspection]}$$

[For splitting integrand into partial fractions see worked-out Ex. 2 in Section 9.2.]

$$= [\log(2x+1)-\log(x+1)]_0^1 = (\log 3-\log 2)-(\log 1-\log 1) = \mathbf{\log\frac{3}{2}}.$$

Example 21. *Evaluate:* $\displaystyle\int_0^1 \frac{3x^2+7x}{(x+1)(x+2)(x+3)}dx.$

Solution: Split of integrand by partial fractions is shown below:

$$\frac{3x^2+7x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3} = \frac{-2}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}$$

| $A(x+2)(x+3)+B(x+1)(x+3)$ $+C(x+1)(x+2) \equiv 3x^2+7x$
Put $x=-2$, $B=2$
Put $x=-1$, $A=-2$
Put $x=-3$, $C=3$.

$$\therefore \int_0^1 \frac{3x^2+7x}{(x+1)(x+2)(x+3)}dx = -2\int_0^1 \frac{dx}{x+1}+2\int_0^1 \frac{dx}{x+2}+3\int_0^1 \frac{dx}{x+3}$$

$$= -2[\log(x+1)]_0^1+2[\log(x+2)]_0^1+3[\log(x+3)]_0^1$$

$$= -2[\log 2-\log 1]+2[\log 3-\log 2]+3[\log 4-\log 3]$$

$$= -4\log 2+3\log 4+2\log 3-3\log 3$$

$$= 2\log 2-\log 3 = \mathbf{\log\frac{4}{3}}.$$

Example 22. *Prove that* $\displaystyle\int_0^1 \frac{x}{x+\sqrt{x^2+1}}dx = \frac{2}{3}\left(\sqrt{2}-1\right).$ [CA PE-I, Nov. 2002]

Solution:

$$\int_0^1 \frac{x}{x+\sqrt{x^2+1}}\,dx = \int_0^1 \frac{x\left(x-\sqrt{x^2+1}\right)}{\left(x+\sqrt{x^2+1}\right)\left(x-\sqrt{x^2+1}\right)}\,dx = \int_0^1 \frac{\left(x^2-x\sqrt{x^2+1}\right)}{x^2-(x^2+1)}\,dx$$

$$= -\int_0^1 \left(x^2-x\sqrt{x^2+1}\right)dx = -\int_0^1 x^2dx + \int_0^1 x\sqrt{x^2+1}\,dx$$

[For the second integral, put $x^2+1=z^2$ so that $x\,dx = z\,dz$]

$$= -\int_0^1 x^2dx + \int_1^{\sqrt{2}} z^2dz = -\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{z^3}{3}\right]_1^{\sqrt{2}}$$

$$= -\frac{1}{3}+\frac{2^{3/2}}{3}-\frac{1}{3} = -\frac{2}{3}+\frac{2\sqrt{2}}{3} = \frac{2}{3}\left(\sqrt{2}-1\right) \textbf{ (Proved.)}$$

Example 23. $\int_3^6 2^{\sqrt{3x+7}}\,dx.$ [CA PE-I, Nov. 2002]

Solution: Put $3x+7=z^2$; then $3dx = 2z\,dz$; for $x=3, z=4$; for $x=6, z=5$.

$$\therefore \text{ given integral} = \int_4^5 2^z \cdot \frac{2}{3}z\,dz = \frac{2}{3}\int_4^5 z\cdot 2^z\,dz \Big| \text{ (Use integration by parts; 1st function } z\text{)}$$

$$= \frac{2}{3}\left[z\frac{2^z}{\log 2}\right]_4^5 - \frac{2}{3}\int_4^5 1\cdot\frac{2^z}{\log 2}dz = \frac{2}{3}\left[\frac{5.2^5}{\log 2}-\frac{4.2^4}{\log 2}\right] - \frac{2}{3\log 2}\left[\frac{2^z}{\log 2}\right]_4^5$$

$$= \frac{2}{3}\cdot\frac{2^4}{\log 2}(10-4) - \frac{2}{3(\log 2)^2}[2^5-2^4] = \frac{32}{3\log 2}\times 6 - \frac{2}{3(\log 2)^2}(32-16)$$

$$= \mathbf{\frac{64}{\log 2} - \frac{32}{3(\log 2)^2}}.$$

EXERCISES ON CHAPTER 10(I)

(Definite Integral)

A

1. Use the definition of definite integral as the limit of a sum to evaluate the following:

(a) $\int_0^1 2x\,dx$; [V.U. B.Com.(H) 2011]

(b) $\int_0^1 3x^2dx$; [V.U. B.Com.(H) 2011]

(c) $\int_0^1 x^2dx$; [V.U. B.Com.(H) 2008]

(d) $\int_1^2 x\,dx$; [V.U. B.Com.(H) 2010]

(e) $\int_1^2 x^3dx$;

(f) $\int_0^1 \left(1+2x+3x^2\right)dx$; [C.U. B.Com.(H) 1992]

(g) $\int_0^1 \left(ax^2+bx+c\right)dx$;

(h) $\int_0^1 e^x\,dx$;

(i) $\int_0^1 \left(3x^2+5\right)dx$; [C.U. B.Com.(H) 2000]

(j) $\int_3^4 (3x+7)\,dx$;

(k) $\int_a^b 5\,dx$;

(l) $\int_a^b e^{3x}\,dx$;

(m) $\int_0^1 3x^2\,dx$; [C.U. B.Com.(H) 1990]

(n) $\int_0^3 (2x+3)\,dx$. [C.U. B.Com.(H) 1999]

(o) $\int_1^2 4x^3\,dx$.

2. Define definite integral as the limit of a sum.

3. Evaluate the following integrals by the method of summation:

(a) $\int_0^1 2x\,dx$;

(b) $\int_0^1 2x^3\,dx$; [C.U. B.Com.(H) 1998]

(c) $\int_0^3 6x\,dx$;

(d) $\int_0^1 e^{-x}\,dx$;

(e) $\int_1^4 3x^2\,dx$;

(f) $\int_1^2 (5x+7)\,dx$; [C.U. B.Com.(H) 1994]

(g) $\int_{-1}^2 x^3\,dx$.

Remember: $1^3+2^3+3^3+\cdots+n^3=\left\{\dfrac{n(n+1)}{2}\right\}^2$; $1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

B

In each of the following cases evaluate the definite integral with the help of Fundamental Theorem of Integral Calculus and obtain the values given against each:

1. $\int_0^1 7x^3\,dx=\dfrac{7}{4}$. [C.U. B.Com. 2009]

2. $\int_0^1 \sqrt{x}\,dx=\dfrac{2}{3}$.

3. $\int_0^{\log x} e^x\,dx=x-1$. [B.U. B.Com.(H) 2007]

4. $\int_1^2 \left(3x+5x^7\right)dx=163\dfrac{7}{8}$.

5. $\int_1^2 \dfrac{2x\,dx}{1+x^2}=\log\dfrac{5}{2}$.

6. (a) $\int_0^2 \dfrac{x\,dx}{x^2+4}=\dfrac{1}{2}\log 2$.

(b) $\int_0^2 \dfrac{x^2}{x^3+1}\,dx=\dfrac{2}{3}\log 3$.

7. $\int_2^3 \dfrac{2x+6}{x^2+6x-9}\,dx=\log\dfrac{18}{7}$.

8. $\int_0^{1/3} \sqrt{1-3x}\,dx=\dfrac{2}{9}$.

9. $\int_0^1 x^3\sqrt{1+3x^4}\,dx=\dfrac{7}{18}$.

10. $\int_0^7 \dfrac{dx}{(1+x)^{1/3}}=\dfrac{9}{2}$.

11. $\int_0^1 2e^{-x^2}\cdot x\,dx=1-\dfrac{1}{e}$.

12. $\int_e^{e^2} \dfrac{dx}{x\log_e x}=\log_e 2$.

13. $\displaystyle\int_1^{e^2} \frac{dx}{x(1+\log x)^2} = \frac{2}{3}.$

14. $\displaystyle\int_1^2 \frac{x^2-1}{x^2} e^{x+\frac{1}{x}}\, dx = e^{5/2} - e^2.$

15. $\displaystyle\int_{-2}^{-1} \frac{dx}{(11+5x)^3} = \frac{7}{12}.$

16. $\displaystyle\int_1^3 \frac{dx}{\sqrt{(4+5x^2)^3}} = \frac{1}{42}$ (Put $x = 1/z$).

17. $\displaystyle\int_1^5 \frac{dx}{\sqrt{2x-1}} = 2.$

18. $\displaystyle\int_0^a \left(\sqrt{a}-\sqrt{x}\right)^2 dx = \frac{a^2}{6}.$

19. $\displaystyle\int_{-1}^2 \frac{x^2 dx}{x+2} = -\frac{9}{2} + 8\log 2.$

20. $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}} = \frac{14}{15}.$

[C.U. B.Com.(H) 1997]

21. $\displaystyle\int_4^8 \frac{dx}{(x-1)\sqrt{x^2-1}} = \sqrt{\frac{5}{3}} - \frac{3}{\sqrt{7}}.$

22. $\displaystyle\int_1^2 \frac{x-3}{x^3+x^2}\, dx = 4\log\frac{4}{3} - \frac{3}{2}.$

23. $\displaystyle\int_0^1 \frac{3x^2+7x}{(x+1)(x+2)(x+3)}\, dx = \log\frac{4}{3}.$

24. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{2} - 1.$

25. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1.$

26. $\displaystyle\int_1^2 \left(3x^2+4x^3\right) dx = 22.$

27. $\displaystyle\int_0^{\log 2} \frac{e^x}{1+e^x}\, dx = \log\frac{3}{2}.$ [V.U. B.Com.(H) 2008]

28. $\displaystyle\int_0^1 x\log(1+2x)\,dx = \frac{3}{8}\log 3.$

29. $\displaystyle\int_2^3 \frac{x^5 dx}{x^4-1} = \frac{1}{2}\left(5+\log\frac{2}{\sqrt{3}}\right).$

[C.U. B.Com.(H) 1996]

30. $\displaystyle\int_1^e \frac{1+\log x}{x}\, dx = \frac{3}{2}.$

31. $\displaystyle\int_1^2 (\log x)^2 dx = 2(\log 2)^2 - 4\log 2 + 1.$

32. $\displaystyle\int_7^{23} \frac{dx}{(x-2)\sqrt{x+2}} = \frac{1}{2}\log\frac{15}{7}$

[Put $x+2 = z^2$; $x = 7$ gives $z = 3$; $x = 23$ gives $z = 5$. Taking positive z only]

33. $\displaystyle\int_1^e \frac{(x+1)(x+\log x)}{x}\, dx = \frac{1}{2}(e^2+2e).$

[N.B.U. B.Com.(H) 2006]

[Hints: Put $x + \log x = z$; then

$$\left(1+\frac{1}{x}\right)dx = dz \text{ or, } \left(\frac{x+1}{x}\right)dx = dz.$$

$$\text{Integral} = \int_1^{e+1} z\,dz = \left[\frac{z^2}{2}\right]_1^{e+1} = \frac{1}{2}\left[(e+1)^2 - 1^2\right]$$

$$= \frac{1}{2}(e^2+2e)]$$

C

Evaluate the following integrals:

1. $\int_{-1}^{1} (3x^2+2x+1)\,dx.$

2. $\int_{1}^{2} \frac{(1+x)^2}{x}\,dx.$ [C.U. B.Com.(H) 1994]

3. $\int_{2}^{3} \frac{(1-x)^3}{x^2}\,dx.$

4. $\int_{1}^{2} \left(1+\frac{2}{\sqrt{x}}+3x\right)dx.$ [C.U. B.Com.(H) 1994]

5. (a) $\int_{2}^{4} \frac{x}{1+x^2}\,dx;$ [C.U. B.Com. 2008]

   (b) $\int_{1}^{e} \frac{dx}{x(1+\log x)}.$ [V.U. B.Com.(H) 2007]

6. $\int_{0}^{1} \frac{(x+1)(x-4)}{\sqrt{x}}\,dx.$

7. $\int_{0}^{2} \frac{x+2}{x+1}\,dx.$

8. $\int_{4}^{6} \frac{dx}{x^2-9}.$

9. $\int_{0}^{1} \frac{2x+5}{x^2+5x+9}\,dx.$

10. $\int_{0}^{4} \sqrt{3x+4}\,dx.$

11. $\int_{0}^{1} x^3\sqrt{1+3x^4}\,dx.$

12. $\int_{0}^{1/2} \frac{dt}{\sqrt{2-3t}}.$ [C.U. B.Com.(H) 1995]

13. $\int_{0}^{2} \frac{dx}{\sqrt{1+x}-\sqrt{x}}.$ [C.U. B.Com.(H) 1993]

14. $\int_{0}^{1} \frac{dx}{\sqrt{1+x}+\sqrt{x}}.$ [C.U. B.Com.(H) 1995]

15. $\int_{0}^{1} xe^x\,dx.$

16. $\int_{1}^{e^2} \log x\,dx.$

17. $\int_{1}^{2} x\log x\,dx.$

18. $\int_{1}^{2} x^2\log x\,dx.$

19. $\int_{1}^{\sqrt{e}} x\log x\,dx.$

20. $\int_{1}^{e} \frac{dx}{x(1+\log x)^2}.$ [C.U. B.Com.(H) 1997]

21. $\int_{1}^{e^2} \frac{dx}{x(1+\log x)}.$ [V.U. B.Com.(H) 2010]

22. $\int_{1}^{2} \frac{dx}{x(1+\log x)^2}.$ [B.U. B.Com.(H) 2003]

23. $\int_{0}^{1} \frac{dt}{2t^2+3t+1}.$ [C.U. B.Com.(H) 1994]

24. $\int_{0}^{1} x^2e^x\,dx.$ [N.B.U. B.Com.(H) 2007]

25. $\int_{0}^{1} \frac{x+1}{(x^2+2x+3)^2}\,dx.$

26. $\int_{0}^{2} \frac{x^4\,dx}{(4+x^5)^{3/2}}.$

27. $\int_{0}^{1} x(1-x)^5\,dx.$

28. $\int_{8}^{15} \frac{dx}{(x-3)\sqrt{x+1}}.$

29. $\int_0^1 x^2 e^{3x}\, dx.$ [C.U. B.Com.(H) 1997]

30. $\int_0^2 \frac{x^2}{\sqrt{1+x^2}}\, dx.$

31. A curve passes through the origin and the slope of the curve at (x, y) is $\frac{x+1}{y+1}$. Find the equation of the curve. [B.U. B.Com.(H) 2008]

[Hints: $\frac{dy}{dx} = \text{slope} = \frac{x+1}{y+1}$ or, $(y+1)dy = (x+1)dx$. Integrating, $\int (y+1)dy = \int (x+1)dx$ or, $\frac{y^2}{2} + y = \frac{x^2}{2} + x + c$.

If this curve passes through the origin, then $0 = 0 + c$ or, $c = 0$. The equation is $\frac{y^2}{2} + y = \frac{x^2}{2} + x$.]

ANSWERS

A

1. (a) 1.
(b) 1.
(c) $\frac{8}{3}$.
(d) $\frac{3}{2}$.
(e) $\frac{15}{4}$.
(f) 3.
(g) $\frac{a}{3} + \frac{b}{2} + c$.
(h) $e - 1$;
(i) 6..
(j) $\frac{35}{2}$.
(k) $5(b - a)$.
(l) $\frac{1}{3}\left(e^{3b} - e^{3a}\right)$.
(m) 1.
(n) 18.
(o) 15.

3. (a) 1;
(b) $\frac{1}{2}$;
(c) 27;
(d) $\frac{e-1}{e}$;
(e) 63;
(f) $\frac{29}{2}$;
(g) $\frac{15}{4}$.

C

1. 4.
2. $\log_e 2 + \frac{7}{2}$.
3. $3\log_e \frac{2}{3} + \frac{2}{3}$.
4. $4\sqrt{2} + \frac{3}{2}$.
5. (a) $\frac{1}{2}\log\frac{17}{5}$;
(b) $\log 2$.
6. $-\frac{48}{5}$.
7. $2 + \log_e 3$.
8. $\frac{1}{6}\log\frac{7}{3}$.
9. $\log_e \frac{5}{3}$.
10. $\frac{112}{9}$.
11. $\frac{7}{18}$.
12. $\frac{\sqrt{2}}{3}$.
13. $\frac{2}{3}\left(3\sqrt{3} + 2\sqrt{2} - 1\right)$.
14. $\frac{4}{3}\left(\sqrt{2} - 1\right)$.
15. 1.
16. $e^2 + 1$..
17. $\log 4 - \frac{3}{4}$.
18. $\frac{8}{3}\log - \frac{7}{9}$.
19. $\frac{2-e}{4}$.
20. $\frac{1}{2}$.
21. $\log 3$.
22. $\frac{\log 2}{1 + \log 2}$.
23. $\log\frac{3}{2}$.
24. $e - 2$.
25. $\frac{1}{6}$.
26. $\frac{2}{15}$.
27. $\frac{1}{42}$.
28. $\frac{1}{2}\log_e \frac{5}{3}$.
29. $\frac{1}{27}\left(5e^3 - 2\right)$.
30. $\sqrt{5} - \frac{1}{2}\log\left(\sqrt{5} + 2\right)$.
31. $(x - y)(x + y + 2) = 0$.

10.6 Evaluation of Limit of a Sum by Using Definite Integral

By definition, we have

$$\int_0^1 f(x)\,dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(rh),$$

where $nh = 1$ or, $n = \frac{1}{h}$; as $h \to 0$, $n \to \infty$,

$$\text{i.e., } \lim_{h \to 0} h \sum_{r=1}^{n} f(rh) = \int_0^1 f(x)\,dx \text{ or, } \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$$

Remark. *To write the integral* $\int_0^1 f(x)\,dx$ *from the limit on the left side, we replace* $rh\left(\text{or } \frac{r}{n}\right)$ *by* x, $h\left(\text{or } \frac{1}{n}\right)$ *by* dx *and take the lower limit as* 0 *and upper limit as* 1.

Rule. *To find the limit of the sum of a series, we first write the series in the form*

$$\lim_{n\to x} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \text{ or, } \lim_{h\to 0} h\sum_{r=1}^{n} f(rh),$$

where $nh = 1$, *and then using the definition of definite integral, we get* $\int_0^1 f(x)\,dx$, *whose value is the required limit.* [*See worked-out Exs* 1 *and* 2.]

Example 1. *Evaluate the limit:* $\lim\limits_{n\to\infty} \dfrac{1^5+2^5+\cdots+n^5}{n^6}$.

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^5+2^5+\cdots+n^5}{n^6} &= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1^5+2^5+\cdots+n^5}{n^5}\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5+\left(\frac{2}{n}\right)^5+\cdots+\left(\frac{n}{n}\right)^5\right] \quad \left|\ \text{Put } \frac{1}{n}=h;\ \text{then as } n\to\infty,\ h\to 0 \text{ and } nh=1\right] \\
&= \lim_{h\to 0} h\left[(1.h)^5+(2.h)^5+\cdots+(n.h)^5\right] \\
&= \lim_{h\to 0} h\sum_{r=1}^{n}(rh)^5, \text{ where } nh=1 \\
&= \int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6} - 0 = \mathbf{\frac{1}{6}}.
\end{aligned}$$

Example 2. *Evaluate:*

(i) $\lim\limits_{n\to\infty}\left[\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+\cdots+\dfrac{1}{n+n}\right]$ [C.U. B.Com.(H) 2008]

(ii) $\lim\limits_{n\to\infty}\left[\dfrac{1}{n}+\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{2n}\right]$. [WBHS 1990]

Solution:

(i)
$$\begin{aligned}
\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right] &= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right] \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}} = \lim_{h\to 0} h\sum_{r=1}^{n}\frac{1}{1+rh}, \quad \left|\ \text{where } \frac{1}{n}=h \text{ or, } nh=1;\ \text{as } n\to\infty, h\to 0\right. \\
&= \int_0^1 \frac{1}{1+x}\,dx \ [\text{by definition}] \\
&= [\log(1+x)]_0^1 = \log 2 - \log 1 = \mathbf{\log 2}\ [\because \log 1 = 0].
\end{aligned}$$

(ii) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$

$=0+\log 2$ [$\because 1/n\to 0$ as $n\to\infty$ and the second limit is the same as (i).]

$=\mathbf{\log 2.}$

Example 3. *Evaluate*: $\lim_{n\to\infty}\left[\frac{1}{1+2n}+\frac{1}{2+2n}+\cdots+\frac{1}{n+2n}\right]$. [C.U. B.Com.(H) 2009]

Solution:

$$\lim_{n\to\infty}\left[\frac{1}{1+2n}+\frac{1}{2+2n}+\cdots+\frac{1}{n+2n}\right]=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\frac{1}{n}+2}+\frac{1}{\frac{2}{n}+2}+\cdots+\frac{1}{\frac{n}{n}+2}\right] \text{ [taking } 1/n \text{ common]}$$

$$=\lim_{h\to 0}\left[\frac{1}{1h+2}+\frac{1}{2h+2}+\cdots+\frac{1}{nh+2}\right] \Big| \text{ Putting } \frac{1}{n}=h \text{ or, } nh=1 \text{ as } n\to\infty, h\to 0.$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{rh+2}=\int_0^1\frac{dx}{x+2}=\Big[\log(x+2)\Big]_0^1=\log 3-\log 2=\mathbf{\log\frac{3}{2}}.$$

Example 4. *Show that* $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{1}{2n}\right]=\frac{1}{3}\log 2.$ [C.U. B.Com.(H) 2001]

Solution:

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{1}{2n}\right]=\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{n^3+n^3}\right] \left[\because \tfrac{1}{2n}=\tfrac{n^2}{2n^3}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\left[\frac{1^2}{1+\frac{1^3}{n^3}}+\frac{2^2}{1+\frac{2^3}{n^3}}+\cdots+\frac{n^2}{1+\frac{n^3}{n^3}}\right] \text{ [Taking } 1/n^3 \text{ common]}$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{\frac{1^2}{n^2}}{1+\frac{1^3}{n^3}}+\frac{\frac{2^2}{n^2}}{1+\frac{2^3}{n^3}}+\cdots+\frac{\frac{n^2}{n^2}}{1+\frac{n^3}{n^3}}\right] \Big| \text{ dividing numerator of each term by } n^2$$

$$=\lim_{h\to 0}h\left[\frac{1^2h^2}{1+1^3h^3}+\frac{2^2h^2}{1+2^3h^3}+\cdots+\frac{n^2h^2}{1+n^3h^3}\right] \Big| \text{ putting } 1/n=h,\ nh=1, \text{ as } n\to\infty,\ h\to 0.$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{r^2h^2}{1+r^3h^3}=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{(rh)^2}{1+(rh)^3}=\int_0^1\frac{x^2}{1+x^3}\,dx=\frac{1}{3}\int_0^1\frac{3x^2}{1+x^3}\,dx$$

$$=\frac{1}{3}\Big[\log\left(1+x^3\right)\Big]_0^1=\frac{1}{3}(\log 2-\log 1)=\mathbf{\frac{1}{3}\log 2.} \qquad [\because \log 1=0]$$

Example 5. *Show that* $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\cdots+\left(1+\frac{n}{n}\right)\right\}^{1/n}=\frac{4}{e}.$

Solution: Let

$$u=\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\cdots+\left(1+\frac{n}{n}\right)\right\}^{1/n}.\ \therefore\ \log u=\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right).$$

$$\therefore \lim_{n\to\infty} \log u = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right) = \lim_{h\to 0} h \sum_{r=1}^{n} \log(1+rh), \quad \text{where } 1/n = h \text{ or, } nh = 1; \text{ as } n\to\infty, h\to 0$$

$$= \int_0^1 \log(1+x)\,dx = \left[\log(1+x)\cdot x - \int \frac{1}{1+x}\cdot x\,dx\right]_{x=0}^{x=1}$$

$$= \left[x\log(1+x) - \int \frac{1+x-1}{1+x}\,dx\right]_{x=0}^{x=1} = \left[x\log(1+x) - x + \log(1+x)\right]_{x=0}^{x=1}$$

$$= 2\log 2 - 1 = \log 4 - \log e = \log\frac{4}{e}$$

or, $\log\left[\lim\limits_{\substack{n\to\infty\\(h\to 0)}} u\right] = \log\frac{4}{e}$, i.e., $\lim\limits_{n\to\infty} u = \frac{4}{e}$ **(Proved).**

Remarks. We have used the fact $\lim\limits_{n\to\infty} \log u = \log\left[\lim\limits_{n\to\infty} u\right]$.

EXERCISES ON CHAPTER 10(II)

(Evaluation of Limit of a Sum by using Definite Integral)

Evaluate each of the following limits:

1. $\lim\limits_{n\to\infty}\left[\left(\frac{1}{n+1}\right)+\left(\frac{1}{n+2}\right)+\left(\frac{1}{n+3}\right)+\cdots+\left(\frac{1}{2n}\right)\right]$.

2. $\lim\limits_{n\to\infty}\left[\left(\frac{1}{n+c}\right)+\left(\frac{1}{n+2c}\right)+\left(\frac{1}{n+3c}\right)+\cdots+\left(\frac{1}{n+nc}\right)\right]$.

3. (a) $\lim\limits_{n\to\infty}\left[\frac{1^6+2^6+3^6+\cdots+n^6}{n^7}\right]$; (b) $\lim\limits_{n\to\infty}\left[\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}\right]$.

4. $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$.

5. $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$.

6. $\lim\limits_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+\cdots+\frac{n^2}{2n^3}\right]$.

ANSWERS

1. $\log 2$.
2. $\frac{1}{c}\log(1+c)$.
3. (a) $\frac{1}{7}$; (b) $\frac{1}{p+1}$.
4. $\frac{\pi}{4}$.
5. $\log 2$.
6. $\frac{1}{3}\log 2$.

10.7 Definite Integral as an Area under a Curve [B.U. B.Com.(H) 2008]

Let $y = f(x)$ be a continuous function of x and suppose we can represent it by the curve GB (assumed to be continuous). See Fig. 10.2.

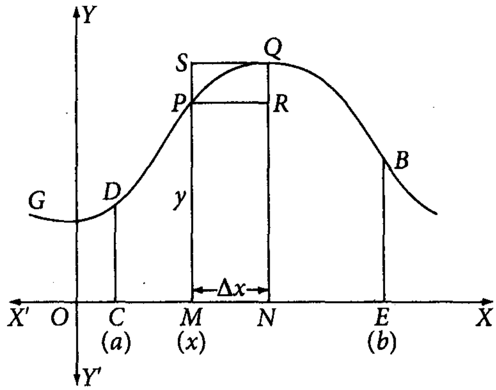

Fig. 10.2

Let CD be a fixed ordinate (at a point $x = a$).

Let MP be a variable ordinate (at any point x).

Let the measure of the area $CMPD$ be A. Suppose, x takes an increment Δx, and accordingly A takes an increment $\Delta A\{=\text{area } MNQP\}$.

We complete rectangles $MNRP$ and $MNQS$. From the figure we observe that

$$\text{area } MNRP < \text{area } MNQP < \text{area } MNQS,$$
$$\text{i.e., } MP \cdot \Delta x < \Delta A < NQ \cdot \Delta x,$$
$$\text{i.e., } MP < \Delta A/\Delta x < NQ$$

[In Fig. 10.2, we see that $MP < NQ$. If $MP > NQ$, we are to reverse the inequality signs.]

Now let $\Delta x \to 0$. Then since the ordinate $MP = y$ and $NQ \to MP$ as $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = y \text{ or, } \frac{dA}{dx} = y.$$

Or, using differentials, we can write $dA = y\,dx$.

It then follows

$$A = \int y\,dx = \int f(x)\,dx = \phi(x) + \text{constant } c, \text{ where} \frac{d}{dx}\{\phi(x)\} = f(x).$$

To determine the constant c, we see that

$$A = 0, \text{ if } x = a, \text{ which gives } c = -\phi(a) \text{ and hence } A = \phi(x) - \phi(a).$$

Next to find the area $CEBQPD$ (area bounded by the curve $y = f(x)$, the X-axis, the two ordinates at $x = a$ and at $x = b$) we take b for x and obtain

$$\text{area } CEBQPD = \phi(b) - \phi(a).$$

This value is the same as the difference of the values of $\int y\,dx$ for $x = a$ and $x = b$. We represent this difference by the symbol

$$\int_a^b y\,dx \text{ or, } \int_a^b f(x)\,dx.$$

(Read: the definite integral of $f(x)$ from $x = a$ to $x = b$).

Conclusion 1. *The symbol* $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$ *gives the measure of the area bounded by the curve* $y = f(x)$, *the X-axis and the two ordinates of the curve at* $x = a$ *and at* $x = b$.

Conclusion 2. *If there exists a function* $\phi(x)$ *such that* $\frac{d}{dx}\phi(x) = f(x)$, *then* $\int f(x)\,dx$ *(indefinite integral)* = $\phi(x) + c$ *and it can be remembered that the area given above*

$$= \int_a^b f(x)\,dx = \Big[\phi(x) + c\Big]_{x=a}^{x=b} = \{\phi(b) + c\} - \{\phi(a) + c\} = \phi(b) - \phi(a).$$

Note: This definition of definite integral as an area presupposes that the vertical lines at $x = a$ and $x = b$ bound a finite area, i.e., the curve does not rise or fall to infinity and does not cross the X-axis and a, b are finite.

Example 1. *To find the area bounded by the parabola* $y = x^2$, *the X-axis and the ordinates at* $x = 2$ *and* $x = 4$.

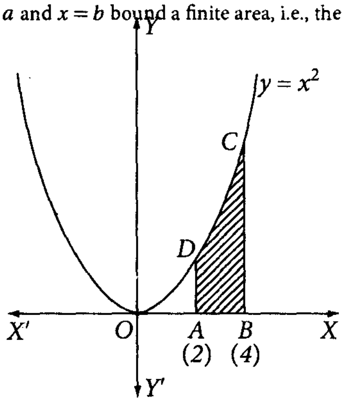

Fig. 10.3

Solution: In Fig. 10.3, we have shown the graph of $y = x^2$ (A parabola whose vertex is at the origin, concavity upwards, Y-axis is the axis of the parabola).

$$\text{The required area } ABCD = \int_2^4 y\,dx = \int_2^4 x^2\,dx = \left[\frac{x^3}{3}\right]_2^4$$

$$= \frac{4^3}{3} - \frac{2^3}{3} = \frac{56}{3} = \mathbf{18\frac{2}{3}} \textbf{ sq units of area.}$$

Example 2. *Find the area bounded by the parabola* $y^2 = 4ax$ *and its double ordinate* $x = b$. [C.U. B.Com.(H) 2001]

Solution: In Fig. 10.4, we have drawn a rough sketch of the parabola $y^2 = 4ax$. The line $x = b$ cuts the parabola at P and Q so that PQ is a double ordinate.

Required area = 2 × area of the region $OBPO$

(This is so because the parabola $y^2 = 4ax$ is symmetrical about the X-axis)

$$= 2\int_0^b y\,dx = 2\int_0^b \sqrt{4a}\,x^{1/2}\,dx$$

$$= 4\sqrt{a}\int_0^b x^{1/2}\,dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^b$$

$$= 4\sqrt{a} \times \frac{2}{3}\left[b^{3/2}\right] = \frac{8}{3}b\sqrt{ab} \text{ sq units.}$$

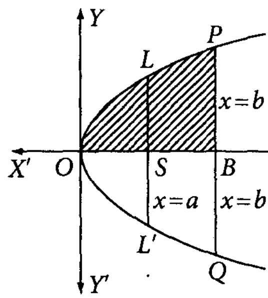

Fig. 10.4

Note: In particular, the area bounded by the parabola $y^2 = 4ax$ and its latus rectum (double ordinate bounded by $x = a$) is $\frac{8}{3}a\sqrt{a}\sqrt{a} = \frac{8}{3}a^2$ sq units.

10.7.1 Area bounded by a curve $x = \phi\ (y)$, the y-axis and the two abscissae $y = c$ and $y = d$

It is easy to verify that the area of the region bounded by a continuous curve $x = \phi(y)$, the Y-axis and the two horizontal lines $y = c$ and $y = d$ is given by Fig. 10.5, i.e.,

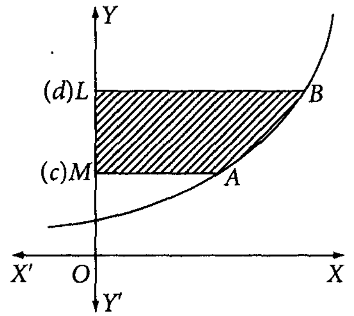

Fig. 10.5

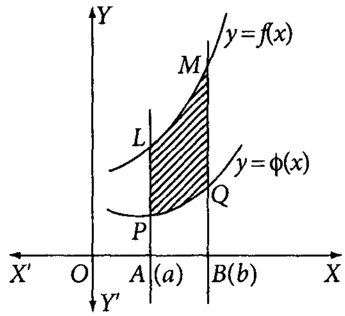

Fig. 10.6

$$\text{Area } \boldsymbol{ABLM} = \int_c^d \boldsymbol{x\,dy} = \int_c^d \boldsymbol{\phi(y)\,dy}.$$

10.7.2 Area included between two curves

The area of the region $PQML$ between two curves $y = f(x)$ and $y = \phi(x)$, bounded by $x = a$ and $x = b$ is given by Fig. 10.6.

$$\therefore \text{ area } PQML = \text{area } ABML - \text{area } ABQP = \int_a^b \boldsymbol{f(x)\,dx} - \int_a^b \boldsymbol{\phi(x)\,dx}.$$

Remarks. If two curves $y = f(x)$ and $y = \phi(x)$ intersect at the points (a, b) and (c, d), then the area of the region between the two curves is

$$\int_a^c f(x)\,dx - \int_a^c \phi(x)\,dx = \int_a^c \{f(x) - \phi(x)\}\,dx = \int_a^c (y_1 - y_2)\,dx \text{ (say)}.$$

Example 3. *Find the area included between $y^2 = 9x$ and $y = x$. (Draft figure necessary).*

[C.U. B.Com.(H) 2003; B.U. B.Com.(H) 2008 Type]

Solution: We have

$$y^2 = 9x \tag{1}$$

$$\text{and } \ y = x. \tag{2}$$

Solving equations (1) and (2), we get $x^2 - 9x = 0$ or, $x(x-9) = 0$. $\therefore\ x = 0, 9$; $\therefore\ y = 0, 9$.

With these ideas we draw the Fig. 10.7.

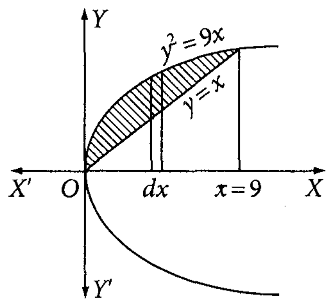

Fig. 10.7

The required area

$$= \int_0^9 y\,dx \text{ [where } y^2 = 9x] - \int_0^9 y\,dx \text{ [where } y = x]$$

$$= \int_0^9 \sqrt{9x}\,dx - \int_0^9 x\,dx = 3\int_0^9 x^{1/2}\,dx - \left[\frac{x^2}{2}\right]_0^9$$

$$= 3 \times \frac{2}{3}\left[x^{3/2}\right]_0^9 - \frac{1}{2}[9^2 - 0] = 2\left[9\sqrt{9} - 0\right] - \frac{81}{2}$$

$$= 2 \times 9 \times 3 - \frac{81}{2} = 54 - 40.5 = 13.5 \text{ or } \mathbf{\frac{27}{2}} \textbf{ sq units.}$$

Example 4. *Find the area of the region bounded by two parabolas $y = x^2$ and $x = y^2$.*

Solution: The two parabolas [Fig. 10.8] intersect at $(0, 0)$ and $(1, 1)$ obtained by solving the two equations $y = x^2$ and $x = y^2$.

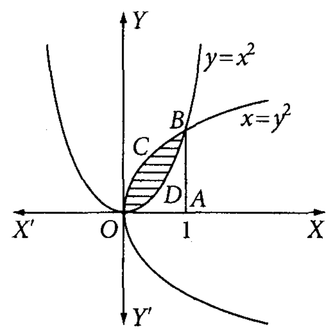

Fig. 10.8

Required area = area $OABCO$ − area $OABDO$ [See Fig. 10.8]

$$= \left[\int_0^1 y\,dx, \text{ where } y = \sqrt{x}\right] - \left[\int_0^1 y\,dx, \text{ where } y = x^2\right]$$

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx = \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \mathbf{\frac{1}{3}} \textbf{ sq units of area.}$$

Example 5. *Find the area bounded by the parabolas $y^2 = 16x$ and $x^2 = 16y$.* [C.U. B.Com.(H) 2007]

Solution: The two parabolas are

$$y^2 = 16x \tag{1}$$
$$\text{and } x^2 = 16y. \tag{2}$$

Solving equations (1) and (2), we have to find the values of x. Eliminating y,

$$\left(\frac{x^2}{16}\right)^2 = 16x \quad \text{or,} \quad \frac{x^4}{(16)^2} = 16x$$
$$\text{or,} \quad x^4 - (16)^3 x = 0$$
$$\text{or,} \quad x\left(x^3 - 16^3\right) = 0.$$

$\therefore\ x = 0, 16$. Also $y = 0, 16$. The equations of two parabolas (1) and (2) intersect at $(0, 0)$ and $(16, 16)$. [See

Fig. 10.8 and draw a figure.] Hence, the required area bounded by the equations of parabolas (1) and (2)

$$= \int_0^{16} y\,dx[\text{where } y^2 = 16x] - \int_0^{16} y\,dx[\text{where } x^2 = 16y]$$

$$= \int_0^{16} \sqrt{16x}\,dx - \int_0^{16} \frac{x^2}{16}\,dx = 4\int_0^{16} x^{1/2}\,dx - \frac{1}{16}\int_0^{16} x^2 dx$$

$$= 4\cdot\frac{2}{3}\left[x^{3/2}\right]_0^{16} - \frac{1}{16}\cdot\frac{1}{3}\left[x^3\right]_0^{16} = \frac{8}{3}\times 4^3 - \frac{1}{48}\times(16)^3$$

$$= \frac{8\times 64}{3} - \frac{16\times 16}{3} = \frac{16\times 16}{3}(2-1) = \frac{\mathbf{256}}{\mathbf{3}} \textbf{ sq units.}$$

Example 6. *Find the area bounded by $y = 3(x-1)(5-x)$ and the X-axis.*

Solution: In order to sketch the graph of the given function we notice that $y = 0$ when $x = 1$, $x = 5$, i.e., the curve cuts the X-axis at these two points. At $x = 3$, y is maximum.

The portion of the curve is symmetrical about the line $x = 3$.

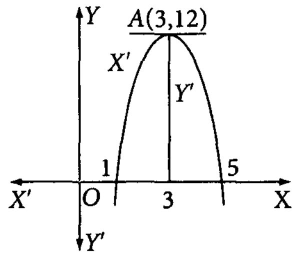

Fig. 10.9

$$\text{Required area} = 2\int_1^3 y\,dx = 2\int_1^3 \left(-3x^2 + 18x - 15\right)dx$$

$$= 2\left[-3\frac{x^3}{3} + 18\frac{x^2}{2} - 15x\right]_1^3 = 2\times 16$$

$= \textbf{32 square units.}$

EXERCISES ON CHAPTER 10(III)

(Definite Integral as an Area under a Curve)

A

In each of the following cases sketch a graph (roughly) and indicate the region whose area is required:

1. Show that the area bounded by $y = x^2$, the X-axis and the ordinates at $x = 0$ and $x = 3$ is 9 sq units.

2. Prove that the area bounded by $x = y^2 + 1$, the Y-axis and the lines $y = 3$, $y = 6$ is 66 sq units.

3. Find by integration, the area of the trapezoid bounded by the line $x + y = 10$, the X-axis and the ordinates at $x = 1$ and $x = 8$. Verify your result by finding the area as half the product of the sum of the two parallel sides and the altitude.

4. Find the area bounded by the given curve, the X-axis and the given ordinates:

(a) $y = x^3$; $x = 0$ and $x = 4$;

(b) $y = 9 - x^2$, $x = 0$ and $x = 3$; [V.U. B.Com.(H) 2010]

(c) $y = x^3 + 3x^2 + 2x$; $x = -3$ and $x = 3$;

(d) $y = x^2 + x + 1$; $x = 2$ and $x = 3$;

(e) $xy = c^3$; $x = a$ and $x = b$;

(f) $y = 2x + \frac{1}{x^2}$; $x = 1$ and $x = 4$;

(g) $y = \dfrac{10}{\sqrt{x+4}}$; $x = 0$ and $x = 5$;

(h) $2x - 3y - 6 = 0$; $x = 4, x = 6$.

[C.U. B.Com.(H) 1996]

5. Find the area bounded by the given curve, the Y-axis and the horizontal lines whose equations are given:

   (a) $y^2 = 4x$; $y = 0, y = 4$;

   (b) $y = 4 - x^2$; $y = 0, y = 3$;

   (c) $x = 9y - y^3$; $y = 0, y = 3$.

6. Find the area included between two parabolas:

   (a) $y^2 = 2px$ and $x^2 = 2py$;

   (b) $y^2 = ax$ and $x^2 = by$.

7. Let $f(x) = \begin{cases} x^3 - 2x^{5/2} + 3x + 1, & \text{when } x < 1 \\ x^5 + 2x^2, & \text{when } x \geq 1. \end{cases}$

   Find $\displaystyle\int_0^2 f(x)\,dx$.

   [Hints: $\displaystyle\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 \left(x^3 - 2x^{5/2} + 3x + 1\right) dx + \int_1^2 \left(x^5 + 2x^2\right) dx$.]

8. Sketch the graph of $2y = x^2(x+2)(7-2x)$. Obtain the area of the region bounded by it, the X-axis and $x = 1, x = 3$.

9. The slope of a curve at (x, y) is $2x - 3$. The curve passes through the point $(3, 2)$. Find the area bounded by the curve, the X-axis and $x = 2$, $x = 4$.

10. Find the area of the region bounded by $x = 0$, $y = 0$ and the curve $y = 10 + 12x - x^3$ and the ordinate corresponding to the maximum point of the curve.

11. Find the area enclosed between the curve $y^2 = x$, the X-axis and the ordinates $x = 1$ and $x = 9$.

    [C.U. B.Com.(H) 1992]

    [Hints: Area $= \displaystyle\int_1^9 y\,dx = \int_1^9 \sqrt{x}\,dx = \frac{2}{3}[x^{3/2}]_1^9 = \frac{52}{3}$ sq units.]

12. Find the area enclosed between the parabola $y^2 = 8x$ and the straight line $y = x$.

    [C.U. B.Com.(H) 1991]

    [Hints: Draw the parabola opening along the positive X-axis and the line $y = x$ passing through the origin. They intersect, where $y^2 = 8x$ and $y = x$ have common points namely, where $y^2 = 8x$, i.e., at $x = 0$, $x = 8$.

    Required area $= \displaystyle\int_0^8 \sqrt{8}\sqrt{x}\,dx - \int_0^8 x\,dx = \frac{128}{3} - 32 = \frac{32}{3}$ sq units.]

13. Find the area of the region bounded by the curve $16y = x^2$ and the line $y = x$. [C.U. B.Com.(H) 1997]

14. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $y = x$ (A rough sketch of the region is to be drawn). [B.U. B.Com.(H) 2008]

[Hints: The area of the shaded region

$$= \int_a^b (y_1 - y_2)\,dx, \text{ where } y_1 = x, y_2 = \frac{x^2}{4}$$

$$= \int_0^4 \left(x - \frac{x^2}{4}\right) dx \quad \left|\ \begin{array}{l} x^2 = 4y = 4x \Rightarrow x = 0, 4 \\ \Rightarrow a = 0, b = 4 \end{array}\right.$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3 \times 4}\right]_0^4 = \left(\frac{16}{2} - \frac{64}{12}\right) - 0 = \frac{8}{3} \text{ sq units}]$$

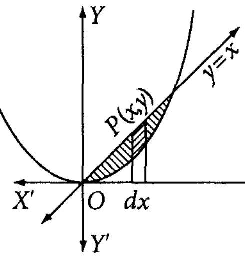

Fig. 10.10

B

1. What is the geometrical meaning of $\int_a^b f(x)\,dx$?

2. Find by integration the area bounded by the straight lines $y = 2x$, $y = 0$, $x = 4$ and $x = 6$.

3. Find by integration the area of the triangle formed by the straight lines $3x - 5y = 0$, $x = 5$ and $y = 0$.

4. (a) Find by integration the area bounded by the curve $y = x^2$, the X-axis and the ordinates $x = 2$ and $x = 5$.

   (b) Find the area enclosed by the curve $x^2 = y$, the Y-axis and the lines $y = 1$ and $y = 9$. [C.U. B.Com.(H) 1995]

   (c) Find the area bounded by $2y = x^2, y = 0$ and $x = 3$.

5. (a) Find the area of the region bounded by the curve $2y = x^2$, the X-axis and the lines $x = 1$ and $x = 3$.

   (b) Make a rough sketch of the graph of $y = x^2$; compute the area under the curve between $x = 4$ and $x = 6$ and above the X-axis.

6. (a) Find the area bounded by the curve $y^2 = 4x$, the X-axis and the ordinates $x = 1$ and $x = 4$. [B.U. B.Com.(H) 2006]

   (b) Find the area bounded by the curve $y^2 = 9x$, the X-axis and the ordinates $x = 1$ and $x = 4$. [C.U. B.Com.(H) 1994]

7. Find the area bounded by the parabola $y^2 = 16x$, the X-axis and the ordinates $x = 4$. [C.U. B.Com.(H) 1990]

8. Find the area of the segment bounded by $y^2 = 4ax$ and its latus rectum.

9. (a) Find the area bounded by the parabola $y = 16(x - 1)(4 - x)$ and the X-axis. [C.U. B.Com.(H) 1999]

   (b) Draw a rough sketch of the curve $y = 4 + 3x - x^2$ for values of x between -1 and 4. Find the area of the figure bounded by the curve and the X-axis.

10. Find the area of the segment bounded by $y^2 = 8x$ and its latus rectum.

11. A plane area is bounded by the curve $y = x(5-x)$ and the X-axis. Find its area.

[B.U. B.Com.(H) 2003]

12. (a) Find the area bounded by $y = 3(x-1)(5-x)$ and the X-axis.
    (b) Find the area bounded by the curve $y = 9 - x^2$, the X-axis and $x = 0$ and $x = 3$.

[C.U. B.Com.(H) 1998]

[**Hints**: The parabola is $x^2 = -(y-9)$ whose vertex is at $(0,9)$ and axis is along the negative Y-axis. Required area = $\int_0^3 y\,dx$ = etc.]

13. (a) Find the area of the plane portion bounded by $y^2 = 16x$ and $y = x$.
    (b) Find the area included between $y^2 = 9x$ and $y = x$.
    (c) Find the area of the region bounded by the curve $16y = x^2$ and the line $y = x$.

[C.U. B.Com.(H) 2006]

    (d) Find the area bounded by $y^2 = 8x$ and $y = x$. [C.U. B.Com.(H) 2008]

14. Find the area enclosed by the curves $y = 2x$ and $y = x^2$.

15. Find the area of the region common to the curves $y^2 = 2x$ and $x^2 = 2y$.

16. (a) Shade the area enclosed by the parabolas $y^2 = 8x$ and $x^2 = y$ and use the method of integration to find the area so enclosed.
    (b) Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. [C.U. B.Com. 2005]
    (c) Find the area bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$. [C.U. B.Com.(H)(P) 2010]

[**Hints**: See worked-out Exs 4 and 5 of **Section 10.7.1**.]

17. Find the area included between the curve $x = 9y - y^2$ (parabola) and the lines $y = 0$ and $y = 3$.

[C.U. B.Com.(H) 2002]

[**Hints**: Required area = $\int_{y=0}^{y=3} x\,dy = \int_0^3 \left(9y - y^2\right)dy = \left[9\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{y=3} = \frac{81}{2} - 9 = \frac{63}{2} = 31.5$ sq units.]

ANSWERS

A

3. 38.5 sq units.
4. (a) 64 sq units;
   (b) 18 sq units;
   (c) 54 sq units;
   (d) $9\frac{5}{6}$ sq units;
   (e) $c^2 \log\frac{b}{a}$ sq units;
   (f) $15\frac{3}{4}$ sq units;
   (g) 20 sq units;
   (h) $\frac{8}{3}$ sq units.
5. (a) $\frac{16}{3}$ sq units;
   (b) $4\frac{2}{3}$ sq units;
   (c) $31\frac{1}{2}$ sq units.
6. (a) $\frac{4p^2}{3}$;
   (b) $\frac{1}{3}ab$.
8. $42\frac{4}{15}$.
9. $4\frac{2}{3}$.
10. 40.
13. $\frac{128}{3}$ sq units.
14. $\frac{8}{3}$ sq units.

B

2. 20 sq units.
3. 7.5 sq units.
4. (a) 39 sq units;
   (b) $\frac{52}{3}$ sq units;
   (c) 4.5 sq units.
5. (a) $\frac{13}{3}$ sq units;
   (b) $\frac{152}{3}$ sq units.
6. (a) $\frac{28}{3}$ sq units.
   (b) 14 sq units.
7. $\frac{64}{3}$ sq units.
8. $\frac{8a^2}{3}$.
9. (a) 72 sq units;
   (b) $\frac{125}{6}$ sq units.
10. $\frac{32}{3}$ sq units.
11. $\frac{125}{6}$ sq units.
12. (a) 32 sq units;
    (b) 18 sq units.
13. (a) $42\frac{2}{3}$ sq units;
    (b) $\frac{27}{2}$ sq units;
    (c) $\frac{128}{3}$ sq units.
    (d) $\frac{128}{3}$ sq units.
14. $\frac{4}{3}$ sq units.
15. $\frac{4}{3}$ sq units.
16. (a) $\frac{8}{3}$ sq units;
    (b) $\frac{16a^2}{3}$ sq units;
    (c) $\frac{16}{3}$ sq units.
17. 31.5 sq units.

10.8 Simple Applications of Integration

As *Integration* is the reverse of *Differentiation,* we can find a function whose *derivative* is known. Thus, if $f'(x)$, i.e., derivative of a function $y = f(x)$ is given, we can find $f(x)$ by integrating $f'(x)$ w.r.t. x, i.e., $f(x) = \int f'(x)dx + C$, where C is the constant of integration and C can be determined if the value of $f(x)$ is given at some point x of its domain.

Illustration 1. *If $f'(x) = 3x^2 + 2x + 1$ and $f(0) = 4$, find $f(x)$.*

We have $f'(x) = 3x^2 + 2x + 1$, where $f'(x)$ is the derivative of $f(x)$ w.r.t. x.
Integrating both sides w.r.t. x, we get

$$f(x) = \int (3x^2 + 2x + 1)\,dx = \frac{3x^3}{3} + 2\cdot\frac{x^2}{2} + x + C,$$

$$\text{i.e.,}\quad f(x) = x^3 + x^2 + x + C. \tag{1}$$

At $x = 0$, $f(0) = 0^3 + 0^2 + 0 + C$ or, $4 = C$, i.e., $C = 4$.
$\therefore$ from (1), we have $\boldsymbol{f(x) = x^3 + x^2 + x + 4.}$

I. To find the equation of the curve which passes through a given point and whose slope at any point (x, y) on it is known.

Example 1. *A curve passes through the point $(5, 3)$ and product of its slope and ordinate at any point (x, y) is equal to its abscissa. Find the equation of the curve.* [C.U. B.Com.(H) 2000]

Solution: Slope at any point $(x, y) = \frac{dy}{dx}$ and ordinate $= y$, abscissa $= x$.

By the given condition, $\frac{dy}{dx}\cdot y = x$, or, $x\,dx = y\,dy$.
Integrating both sides, we get

$$\frac{x^2}{2} = \frac{y^2}{2} + c. \tag{1}$$

where c is a constant. Given that the curve (1) passes through the points $(5,3)$.

$\therefore \frac{5^2}{2}=\frac{3^2}{2}+c$, or, $c=\frac{25}{2}-\frac{9}{2}=\frac{16}{2}=8$.

Hence, from (1), *the required equation of the curve is* $\frac{x^2}{2}=\frac{y^2}{2}+8$ or, $x^2=y^2+16$, or, $\boldsymbol{x^2-y^2=16}$.

II. To find the Total Revenue Function and the Demand Function when the MR Function is given

Let R be the *total revenue* when the output is x. Then the MR is given by

$$\text{MR}=\frac{dR}{dx} \quad [\because \text{ MR is the derivative of total revenue w.r.t. sales (in units)}]$$

$$\text{i.e.,} \quad \frac{dR}{dx}=\text{MR}$$

If MR is given, then integrating both sides w.r.t. x, we get

$$R=\int(\text{MR})\,dx+C, \tag{1}$$

where C is the constant of integration.

Now we know that the total revenue R is zero when output x is zero. Therefore, C can be determined from the condition that $R=0$ when $x=0$.

If R be the *total revenue for d units sold,* then we can write $\boldsymbol{R=\int_0^d(\text{MR})\,dx}$.

If p be the price per unit, then $R=px$ or, $p=\frac{R}{x}$, where R is known from (1).

Thus, we get p as a function of x, called *demand function*.

Example 2. *The MR function when output is x units is given by $MR=6-2x-3x^2$. Find the total revenue function and the demand function.*

Solution: We have $\frac{dR}{dx}=\text{MR}=6-2x-3x^2$.

Integrating both sides w.r.t. x, we get

$$R=\int\left(6-2x-3x^2\right)dx+C=6x-x^2-x^3+C, \text{ i.e., } R=6x-x^2-x^3+C, \tag{1}$$

where C is the constant of integration.

Now $R=0$ when $x=0$; $\quad\therefore\quad 0=6\cdot0-0-0+C$ or, $\boldsymbol{C=0}$.

Hence, from (1), we get $\boldsymbol{R=6x-x^2-x^3}$, which is the required *total revenue function.* The *demand function* is given by

$$p=\frac{R}{x}=\frac{6x-x^2-x^3}{x}=6-x-x^2, \quad \text{i.e., } \boldsymbol{p=6-x-x^2}.$$

Example 3. *The value of the MR function (given in thousands of* rupees) *for a particular commodity is $R'(x)=4+e^{-0.03x}$, where x denotes the number of units sold. Determine the total revenue from the sale of* 100 *units of the commodity.* [*Given*: $e^{-3}=0.05$]

Solution: We have $\dfrac{dR}{dx} = \text{MR} = R'(x) = 4 + e^{-0.03x}$, where R is the total revenue when x units are sold.

Integrating both sides w.r.t. x, we get

$$R = \int \left(4 + e^{-0.03x}\right) dx + C.$$

If R_{100} be the total revenue when 100 units are sold, then

$$R_{100} = \int_0^{100} \left(4 + e^{-0.03x}\right) dx = \left[4x + \frac{e^{-0.03x}}{-0.03}\right]_0^{100}$$

$$= 4 \times 100 - \frac{1}{0.03} e^{-0.03 \times 100} - \left(4 \times 0 - \frac{1}{0.03} e^0\right) = 400 - \frac{1}{0.03}(e^{-3} - e^0)$$

$$= 400 - \frac{100}{3}(0.05 - 1) = 400 - \frac{100}{3} \times (-0.95) = 400 + \frac{95}{3}$$

$$= 400 + 31.667 \text{ (approx.)} = 431.667 = ₹431667.$$

III. To find the Cost Function when Marginal Cost is given

Let C be the total cost of production when output is x units. Then the marginal cost (MC) is given by

$$\text{MC} = \frac{dC}{dx} \quad [\because \text{ MC is the derivative of total cost w.r.t. output}]$$

i.e., $$\frac{dC}{dx} = \text{MC}.$$

If MC, i.e., marginal cost is given, then integrating both sides w.r.t. x, we get

$$C = \int (\text{MC})\, dx + k, \tag{1}$$

where k is the constant of integration. Now we know that the total cost C is zero when output x is zero. Therefore, k can be determined from the condition that $C = 0$ when $x = 0$.

If C be the *total cost for d units produced,* then we can write

$$\mathbf{C} = \int_0^d (\text{MC})\, dx.$$

Then *average cost* (AC) *is given by* AC = $\dfrac{C}{x}$ *when C is known from* (1).

Example 4. *The marginal cost function* (MC) *when output is x units is given by* $\text{MC} = x^2 - 2x + 5$. *Find the total cost function and the average cost, if the fixed cost is* 30.

Solution: We have $\dfrac{dC}{dx} = \text{MC} = x^2 - 2x + 5.$

Integrating both sides w.r.t. x, we get

$$C = \int (x^2 - 2x + 5)\, dx + k = \frac{1}{3}x^3 - x^2 + 5x + k \text{ i.e., } C = \frac{1}{3}x^3 - x^2 + 5x + k, \tag{1}$$

where k is the constant of integration.

Now, since the fixed cost is 30, $\therefore\ C=30$, when $x=0$.

$$\therefore\quad 30=0-0+0+k \text{ or, } k=30.$$

Hence, from (1), we get $\boldsymbol{C=\frac{1}{3}x^3-x^2+5x+30}$, which is the required **total cost function.**

Average Cost (AC) is given by $\text{AC}=\frac{C}{x}=\boldsymbol{\frac{1}{3}x^2-x+5+\frac{30}{x}}$.

Example 5. *Find the total cost of producing* 2000 *pens if the marginal cost* $C'(x)$ *(in rupees per unit) is given by* $C'(x)=\frac{x}{2000}+3.50$, *when output is* x.

Solution: We have

$$\frac{dC}{dx}=C'(x)=\frac{x}{2000}+3.50 \quad [\because \text{ MC is the derivative of total cost } C(x) \text{ w.r.t. } x.]$$

Integrating both sides w.r.t. x,

$$C=\int\left(\frac{x}{2000}+3.50\right)dx+k, \text{ where } k \text{ is the constant of integration.}$$

If C_{2000} be the total cost when 2000 pens (2000 units) are produced, then

$$C_{2000}=\int_0^{2000}\left(\frac{x}{2000}+3.50\right)dx=\frac{1}{2000}\int_0^{2000}x\,dx+3.50\int_0^{2000}dx=\frac{1}{2000}\left[\frac{1}{2}x^2\right]_0^{2000}+3.50\left[x\right]_0^{2000}$$

$$=\frac{1}{4000}(2000)^2+3.50(2000)=1000+7000=\text{₹}\mathbf{8000}.$$

IV. To find Total Inventory Carrying Cost when Unit Holding Cost is H_c and Inventory on Hand $I(x)$ are given

Let I_x be the total inventory carrying cost when unit holding cost is H_c and x is the time of holding. Then

$$\frac{dI_x}{dx}=H_c\,I(x).$$

Integrating both sides w.r.t. x, we get

$$I_x=H_c\int I(x)\,dx+k.$$

If I_T be the total inventory carrying cost for the time period T, then

$$I_T=H_c\int_0^T I(x)dx.$$

Example 6. *A company receives a shipment of* 400 *scooters every month* (30 *days*). *From past experience it is known that the inventory on hand depends on the number of days x from the last shipment and is given by* $I(x)=400-0.05x^2$ *and the daily holding cost for a scooter is* ₹0.50. *Calculate the total cost for maintaining inventory for* 30 *days.*

Solution: If I_x be the total cost for maintaining inventory for x days and H_c the daily holding cost per scooter, then

$$\frac{dI_x}{dx} = H_c I(x) = 0.50 \cdot I(x) \quad [\because H_c = 0.50]$$

Integrating both sides w.r.t. x, we get

$$I_x = 0.50 \int I(x)dx + k = 0.5 \int \left(400 - 0.05x^2\right) dx + k,$$

where k is the constant of integration.

If I_{30} be the total cost for maintaining inventory for 30 days, then

$$I_{30} = 0.50 \int_0^{30} \left(400 - 0.05x^2\right) dx = 0.50 \left[400x - \frac{5}{100} \cdot \frac{x^3}{3}\right]_0^{30}$$

$$= 0.50 \left[400 \times 30 - \frac{1}{20} \times \frac{(30)^3}{3}\right]$$

$$= 0.50[12000 - 450] = 0.50 \times 11550 = ₹5775.$$

V. To find the Amount and Present Value of an Annuity of Equal Payments P in which Interest Rate is Compounded Continuously

1. Let A be the amount of an annuity of equal annual payments P when the interest rate $r\%$ is compounded continuously for n years. Then A is given by

$$A = \int_0^n Pe^{rt} dt.$$

2. Let V be the present value of an annuity of equal payments P when interest rate $r\%$ is compounded continuously for n years. Then V is given by

$$V = \int_0^n Pe^{-rt} dt.$$

Example 7. *An account fetches interest at* 5% *per annum compounded continuously. An individual deposits* ₹1,000 *each year in the account. How much will be the amount in the account after* 5 *years? It is given that* $e^{0.25} = 1.284$.

Solution: If A be the required amount lying in the account after 5 years, then

$$A = \int_0^5 Pe^{rt} dt.$$

Here $P = 1000$ and $r = \frac{5}{100} = 0.05$.

$$\therefore A = \int_0^5 1000 \cdot e^{0.05t} dt = 1000 \left[\frac{e^{0.05t}}{0.05}\right]_0^5 = \frac{1000}{0.05}\left[e^{0.05 \times 5} - e^0\right] = 20000(e^{0.25} - 1)$$

$$= 20000(1.284 - 1) = 20000 \times 0.284 = ₹5680.$$

Example 8. *A Motor Cycle is purchased on installment basis such that ₹2000 is to be paid each year for* 5 *years. If interest is charged at* 5% *per annum compounded continuously, what would be the cash down price of the Motor Cycle?* [*Given:* $e^{0.25} = 1.284$.]

Solution: If V be the required present value, then V is given by $V = \int_0^n Pe^{-rt}\,dt$.

Here $P = 2000, n = 5, r = \dfrac{5}{100} = 0.05$.

$$\therefore\ V = \int_0^5 2000 \cdot e^{-0.05t}\,dt = 2000\left[\frac{e^{-0.05t}}{-0.05}\right]_0^5 = \frac{2000}{-0.05}\left[e^{-0.05\times 5} - e^0\right] = \frac{2000}{0.05}\left(1 - e^{-0.25}\right)$$

$$= 40000\left(1 - \frac{1}{e^{0.25}}\right) = 40000\left(1 - \frac{1}{1.284}\right) = 40000 \times \frac{0.284}{1.284} = \frac{40000 \times 284}{1284}$$

$= ₹8847.35$.

EXERCISES ON CHAPTER 10(IV)

(Simple Applications of Integration)

1. If $\dfrac{dp}{dt} = \dfrac{1}{2p}$ and $p = 1$, when $t = 0$, find t, when $p = 3$. [C.U. B.Com. 1995]

   [Hints: $dt = 2p\,dp$. Integrating, $t + c = p^2$. Given: $p = 1$ when $t = 0$; $\therefore\ c = 1 \Rightarrow t + 1 = p^2$. If $p = 3$, $t = 8$.]

2. Find the equation of the curve whose slope at any point (x, y) on it is $2y$ and which passes through the point $(2, 3)$. [C.U. B.Com. 1994]

3. The slope at any point (x, y) of a curve is $\dfrac{y}{x}$ and it passes through the point $(2, 3)$. Find the equation of the curve. [C.U. B.Com. 1996]

   [Hints: $\dfrac{dy}{dx}$ = slope at $(x, y) = \dfrac{y}{x}$, or, $\dfrac{dy}{y} = \dfrac{dx}{x}$. Integrating, $\log|y| = \log|x| + \log c \Rightarrow |y| = c|x|$.]

4. Find the equation to a curve whose slope is $\left(-\dfrac{x}{y}\right)$ at any point (x, y) and which passes through the point $(2, 3)$. [C.U. B.Com. 1998]

5. The MR function when output is x units is given by MR $= 10 - 2x - 3x^2$. Find the total revenue function and the demand function.

6. The value of the MR function (given in thousand of rupees) for a particular commodity is $R'(x) = 5 + e^{-0.04x}$, where x denotes the number of units sold. Determine the total revenue from the sale of 100 units of a commodity. (Given: $e^{-4} = 0.018$)

7. A firm has a MR function given by MR $= \dfrac{a}{(x+b)^2} - c$, when x is the output and a, b, c are constants. Show that the total revenue function is $\dfrac{ax}{b(x+b)} - cx$ and the demand function is $p = \dfrac{a}{b(x+b)} - c$.

8. The marginal cost function (MC) when output is x units is given by MC $= x^2 - 10x + 99$. Find the total cost function and the average cost, if the fixed cost is 45.

9. (a) If the marginal cost function for a product is $MC = 15x^2 + 6x + 4$ and the fixed cost is ₹200, find the total cost and average cost functions.

   (b) Determine the cost of producing 200 cars, if the marginal cost (in rupees per unit) is given by $MC(x) = \frac{15}{2}x^2 - 4x + 8000$. What is the cost of production for one car? [CA Final May 2000]

   [Hints: Cost of producing 200 cars $= \int_0^{200} MC(x)\,dx =$ etc.]

   (c) Calculate the cost of producing a certain type of 10 calculators, if the marginal cost (in rupees per unit) is $C(x) = 0.3x^2 - 2.4x + 30$. [CA PE-I Nov. 2002]

   [Hints: Required cost of producing 10 calculators $= \int_0^{10} C(x)\,dx =$ etc.]

   (d) The cost of producing 20 floppies, if the marginal cost (in rupees) is (a) ₹250, (b) ₹270, (c) ₹290, (d) ₹310. Indicate the correct answer with proper explanations. [CA Final May 2003]

10. (a) Find the total cost of producing 600 pens, if the marginal cost $C'(x)$ is given by $C'(x) = \frac{x}{600} + 2.50$, when the output is x.

    (b) Determine the cost of 300 video cassette players, if the marginal cost (in rupees per unit) is given by $C(x) = \frac{1}{3}x^2 - 2x + 700$. [CA Final Nov. 1998]

11. The marginal cost function of a firm is $2 + 3e^x$, where x is the output. Find the total cost and total average cost functions, if the fixed cost is ₹500.

12. A firm receives a shipment of 600 Motor Cycles every month (30 days). From past experience it was found that the inventory on hand is related to the number of days x since the last shipment by $f(x) = 600 - 0.04x^2$. The daily holding cost for one Motor Cycle is ₹0.40. Determine the total cost for maintaining inventory for 30 days.

13. An account fetches interest at 10% per annum compounded continuously. A man deposits ₹600 each year in the account. How much will be the amount in the account after 6 years? It is given that $e^{0.6} = 1.822$.

14. A scooter is purchased on instalment basis such that ₹1500 is to be paid each year for 5 years. If interest is charged at 8% per annum compounded continuously, what would be the cash down price of the scooter? (Given: $e^{-0.4} = 0.67$).

    [Hints. $$V = \int_0^n Pe^{-rt}\,dt = \int_0^5 1500e^{-0.8t}\,dt = 1500\left[\frac{e^{-0.8t}}{-0.8}\right]_0^5 = \frac{1500}{-0.8}\left[e^{-0.4} - 1\right]$$
    $$= \frac{1500 \times 100}{8}\left(1 - e^{-0.4}\right) = 750 \times 25(1 - 0.67) = 6187.5]$$

ANSWERS

1. $t = 8$.
2. $2x - 4 = \log \frac{|y|}{3}$.
3. $2y = \pm 3x$.
4. $x^2 + y^2 = 13$.
5. $100x - x^2 - x^3, 10 - x - x^2$.
6. ₹524550.
8. $\frac{1}{3}x^3 - 5x^2 + 99x + 45; \frac{1}{3}x^2 - 5x + 99 + \frac{45}{x}$.
9. (a) $5x^3 + 3x^2 + 4x + 200; 5x^2 + 3x + 4 + \frac{200}{x}$;
   (b) ₹21500000; ₹107600;
   (c) ₹250;
   (d) ₹200, i.e., (c) is correct.
10. (a) ₹1800;
    (b) ₹3120000.
11. $2x + 3e^x + 500; 2 + \frac{3}{x}e^x + \frac{500}{x}$.
12. ₹7056.9; ₹4932.
14. ₹6187.50.

ADVANCED BUSINESS MATHEMATICS

STATISTICS

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Chapter 1

Introduction: Definition of Statistics: Importance and Scope, Types, Sources and Collection of Data

1.1 Introduction: Origin and Development of Statistics

The word Statistics seems to have been derived from the Latin word *Status* meaning Political State. The Italian word *Statista* or the German word *Statistik* also bears similar meaning. In ancient times, the governments used to collect information regarding the population and property or wealth of the State. These information were necessary to achieve political purposes of rulers — to assess the manpower and to introduce new taxes or levies. In course of time the character of quantitative information extended, and details regarding births, deaths, marriages, etc. were necessary. In modern times collection of data and analyzing them are not limited to any particular sphere of human activity. Quantitative information are not only necessary to run the Government or semi-government organizations like Corporations, etc. but they are the backbone of the study of Agriculture, Biology, Electronics, Medicine, Political Science, Economics, Psychology, Sociology, Business and Commerce — in fact, any branch of study, where numerical figures (or data as we call them) are involved. With the introduction of Theory of Probability since mid-seventeenth century Statistics has been placed on a very sound mathematical footing. The present treatise, however, will not go deep into such theoretical development of Statistics. Our main purpose will be to acquaint our readers with the definition of Statistics and the systematic method of collection and summarization of data, analyzing them and interpreting them to help business decisions.

1.1.1 Definition of Statistics: Statistics as Statistical Data and Statistics as Statistical Methods

By Statistical Data we mean numerical statement of facts while Statistical Methods will refer to complete information of the principles and techniques used in collecting and analyzing such data.

The word **Statistics** has been defined by several Statisticians in two different but at the same time two allied senses:

(a) Plural sense — Statistics as Statistical Data and

(b) Singular sense — Statistics as Statistical Methods.

Statistics as Statistical Data

In this sense Statistics connects numerical statement of facts. To be more precise, it means that Statistics implies the statement of those classified facts that represent the conditions of the people in a state — especially those facts which can be stated in numbers or in any other tabulated and classified arrangement. This being too concise the more exhaustive definition on Prof. Horace Secrist may be quoted here:

"By Statistics is meant the aggregate of facts, affected to a marked extent by a multiplicity of causes, numerically expressed, enumerated or estimated, according to reasonable standards of accuracy, collected in a systematic manner for a pre-determined purpose and placed in relation to each other."

When the definition of **Horace Secrist** is elucidated it comes to mean:

Statistics as aggregate of facts always means that a single fact or isolated facts or unrelated figures do not constitute Statistics. Thus, the case of a single death, however illustrious the person may be, a single street accident or the value of imports of one kind of article, however valuable, do not come under the purview of Statistics.

Then Horace Secrist refers to Statistics as affected to a marked extent by a multiplicity of causes. This is also very remarkable. *For example,* Statistical data in relation to number of deaths at different ages in a particular country are sure to be affected, among other factors, by the climatic condition of the country, by the habits, customs, nature of occupation, illiteracy and ignorance of hygenetic rules, etc. of the people inhabiting that country.

Then Horace Secrist lays down that Statistics must be numerically expressed. This condition is self-evident. *For example,* statistical data in respect of rice production in West Bengal have no concern with the quality of rice but it occupies itself with the quantity of rice production in the State which must be expressed in numbers of quintals, years, etc. and must be presented in tabular form.

Two other characteristics of Statistics as emphasised by Horace Secrist are — reasonable standard of accuracy and systematic collection for pre-determined purpose. These two characteristics are present in the very nature of Statistics. Statistical data must be precise and accurate. But when such precise and accurate data are not forthcoming, one must remain content with their reasonably accurate values. *For example,* statistics of lengths of different highways in different states of Union of India when recorded may reasonably ignore centimetres and metres and that will not detract from the statistical data any of their importance. Then, as regards the collection of data for a pre-determined purpose, it can be said that a prior knowledge of the purpose for which an investigation is conducted certainly facilitates the collection of data. Without this pre-determined purpose, mere collection of data will serve no end and is likely to hinder ultimate analysis with the help of collected data.

Statistics as Statistical Methods

Statistical methods literally mean methods that are applied for collection, analysis and interpretation of numerical observations. By the efficient application of such methods (or tools), useful deductions are made and statistical relationships governing the data are suitably formulated. These methods also facilitate emphasizing the underlying pattern and trend of such data and also their proper interpretation.

It is for this reason that **Saligman** has defined Statistics as "the science which deals with the methods of collecting, classifying, presenting, analyzing, comparing and interpreting numerical data collected to throw light on any sphere of enquiry."

Croxton and **Cowden** defined **Statistics** more concisely as "the science which deals with the collection, analysis and interpretation of numerical data."

1.2 Importance and Scope of Statistics

In ancient days, Statistics was regarded as the Science of **Statecraft** and was used by Governments (or Rulers) to collect data relating to population, property or wealth, military strength, crimes, etc. Such collections, were aimed at assessing manpower to safeguard the country and for introducing new taxes and other duties. With the passage of time and concept of Welfare State, the scope of Statistics has widened to social and economic phenomena. Again, with developments in statistical techniques, statistics is now used not only for mere collecting data, but also for their handling, analysis and drawing valid inferences from them. It is now widely used in almost all spheres of life. Statistics is used in all sciences — Social, Physical and Natural, and also in various diversified fields like Agriculture, Industry, Sociology, Economics, Education, Business Management, Psychology, Accountancy, Insurance Planning, etc. etc.

Statistics has now assumed such a dimension that statistical techniques are becoming more and more indispensable in our everyday human activity. As a subject, Statistics has also acquired a tremendous progress and even an elementary knowledge of Statistical Methods has become a part of general education in the curricula of most of the countries. Importance of Statistics is clearly stated in the following words of Carrol D. Wright of USA.

"To a very striking degree, our culture has become a Statistical Culture. Even a person who may never have heard of an index number is affected by those index numbers which describe the cost of living. It is impossible to understand Psychology, Sociology, Economics, Finance or Physical Science without some general idea of the meaning of an average, of variation of concomitance, of sampling of how to interpret charts and tables."

According to **H. G. Wells:** "Statistical thinking will one day be as necessary for effective citizenship as the ability to read and write."

Importance and Scope of Mathematics in Business and Business Decisions

Mathematics is an integral part of education of students in Business, Economics, Statistics and Social Sciences. Mathematical Principles are now used in all spheres of making business decisions.

Formulae of Compound interests may be used to determine growth in investment and depreciation in machinery of a company. *Annuities* are used in creating *Sinking Fund* to replace machinery in future or to fix an EMI in repayment of loans.

Linear (or curvilinear) trend in Business may be represented by a line (or a curve). This line (or curve) may be fitted to the past (given) few years business data by the *principle of least squares involving Maxima and Minima of Mathematics.* This fitted line (or curve) may be used for forecasting in making business decisions on production, knowing increase or decrease in sales. Graphs and charts are used to depict production, sales, profits and earning per share in the Annual Report of a company. These help shareholders to discuss the Profit and Loss Accounts in the Annual General Meeting about the growth of the company and for taking new decisions in producing various new products. Differential Calculus is used to find MC and MR (i.e., marginal cost and marginal revenue) when cost (C) and revenue (R) are known. Integral Calculus is used to find C and R when MC and MR are known.

Linear Programming is used to maximize profit and minimize cost of a company on producing and selling various items, the decision variables being men, materials, machines and lands involved in the objective function of the Linear Programming Problem (LPP).

1.3 Application of Statistics to Business and Industry

With the gradual industrialization and expansion of the business world, businessmen find Statistics an indispensable tool. Now-a-days the success of a particular business or industry very much depends on the accuracy and precision of statistical analysis. Wrong forecasting due to inaccurate statistical analysis may lead to disaster. Before taking a new venture or for the purpose of improvement of an existing one the Business Executives must have a large number of quantitative facts (e.g., price of raw materials, price and demand of similar products prevailing in the present market, various taxes to be paid, labour conditions, taste of the consumers, sales records, etc). All these facts are to be analyzed statistically before stepping in for a new enterprise or before fixing the price of a commodity.

Statistical methods are now used for exploring possibilities to advertising campaigns, for adjustment of production methods and as an aid to establish standards. Business activity follows a definite trend — boom periods being followed by periods of depression. Statistical techniques determine such business cycle and help in forecasting future markets. Market research and market surveys by statistical sampling methods are now extremely useful for any businessman. In industry, Statistics is widely used in quality control. In production engineering, to find whether the product conforms to specifications, statistical tools like inspection plans, control charts, etc. are of great use. Wide applications of Statistics can be found in Insurance, where the premium rates are fixed on the basis of mortality, average length of life, possibilities of investments, etc.

Statistical Techniques Commonly Used in Business Activities

(i) Various statistical measures like average, dispersion, skewness, correlation are necessary to bring out the characteristics of the available data.

(ii) Time Series Analysis helps in isolating various components (trend, seasonal and cyclical). They are useful for forecasting and consequent planning.

(iii) Regression analysis establishes relationship between production and sales, demand and per capita income, input and output, etc. They are useful for prediction.

(iv) Sampling methods (like random sampling or stratified sampling) are used for conducting business and market surveys or for checking the accuracy of records.

(v) Statistical Quality Control is used to find whether the manufactured goods conform to required specifications. Sampling Inspection in this connection is indispensable in any manufacturing concern.

(vi) For the calculation of mortality rates Vital Statistics and Demography are useful. These rates and calculation of probabilities of death are used in insurance for the determination of premium rates.

(vii) Even for appointment of personnels the efficiency of candidates are statistically determined by using Test Scores.

(viii) Index numbers (of price, of production, of cost of living, etc.) are constructed by the use of statistical formulae. They are extremely useful in Business and Commerce, and also helpful to the Government for making economic decisions.

1.4 Limitations of Statistics and also Its Characteristics

(i) Statistics is not suited to the study of qualitative phenomenon. Statistics is applicable to the study of those objects of enquiry which are capable of quantitative measurement. As such qualitative objects like honesty, poverty, culture, etc. are not capable of statistical analysis unless one reduces qualitative expressions to precise quantitative terms. Intelligence of a group of candidates can be studied on the basis of their Test Scores.

(ii) Statistics does not study individuals. Statistics can be used only to analyze an aggregate of objects. No specific recognition is given to the individual items of a series. We study group characteristics through statistical analysis.

(iii) Statistical decisions are true only on an average and also the average is to be taken for a large number of observations. For a few cases in succession the decision may not be true.

(iv) Statistical decisions are to be made carefully by the experts. The use of statistical tools by untrained persons may lead to false conclusions. Statistics are like clay out of which one can make a god or a devil as one pleases. Misuse of Statistics has, in fact, created some distrust on the subject. That is why we often hear comments like 'Statistics can prove anything in this universe;' or, 'There are three types of lies — lies, damned lies and Statistics.' Our final observations are: The Science of Statistics is the most useful servant but only of great value to those who understand its proper use.

From the discussions given above the reader can easily summarize what we call Characteristics of Statistical Analysis:

(a) In Statistics all information are to be expressed in quantitative terms. Even in the study of a quality like intelligence of a group of students we require scores or marks secured in a test.

(b) Statistics deals with a collection of facts, not an individual happening.

(c) Statistical data are collected with a definite object in mind, i.e., there must be a definite field of enquiry.

(d) In every field of enquiry there are a large number of factors, each of which contributes to the final data collected. So Statistics may be affected by a multiplicity of causes.

(e) Statistics is not an exact science. Conclusions are usually derived from samples and hence exactness cannot be guaranteed.

(f) Statistics should be so related that cause-and-effect relationship can be established.

(g) A statistical enquiry passes through four stages:

COLLECTION OF DATA

↓

CLASSIFICATION AND TABULATION OF THE DATA

↓

ANALYSIS OF THE DATA

↓

INTERPRETATION OF THE DATA

1.5 A Few Terms Commonly Used in Statistics

Data

A collection of observations expressed in numerical figures obtained by measuring or counting.

Population

To a statistician the word *population* is simply a useful means of denoting the totality of the set of objects under consideration. **Population** (or Universe) is the aggregate or totality of statistical data forming a subject of investigation, e.g.,

- the population of books in the National Library,
- the population of the heights of Indians,
- the population of Nationalized Banks in India, etc.

A Sample is a portion of the population which is examined with a view to estimating the characteristics of the population, i.e.,

- to assess the quality of a bag of rice, we examine only a portion of it. The portion selected from the bag is called a *sample*, while the whole quantity of rice in the bag is the population,
- to estimate the proportion of defective articles in a large consignment, only a portion (i.e., a few of them) is selected and examined. The portion selected is a sample.

When information is collected in respect of every individual item, the enquiry is said to be done by Complete Enumeration or Census; e.g., during the Census of Population (which is done every ten years in India), information in respect of each individual person residing in India is collected. This method gives information for each and every unit of the population with greater accuracy. But this method involves huge amount of money, long time and much effort. In most cases of statistical inquiry, because of limitations of time and cost, only a portion (i.e., a sample) of the available source of information is examined, and the data collected from them. This process of partial enumeration is known as **Sample Survey.** The results and findings are then generalized and made applicable to the whole field of inquiry. This is known as **Sampling.** The process of sampling is based on two important principles of Statistics:

1. **Law of Statistical Regularity,** which states that a moderately large number of items selected at random from a given population exhibits nearly the same composition and characteristics of the population.
2. **Law of Inertia of Large Numbers,** which states that, other things remaining the same, the larger the size of the sample, the more accurate is the result obtained. This is nothing but a corollary to the Law of Statistical Regularity.

Characteristic

A quality possessed by an individual person, object or item of a population is called a *characteristic* of the individual. Height of a person, Age, Nationality are examples of characteristics.

Variable and Attribute

A characteristic may be measurable or non-measurable. Measurable characteristics are those which are expressible numerically in terms of some units. A measurable characteristic is also called a **Variable or a Variate,** e.g., Age, Height, Income are variables.

A non-measurable characteristic is a qualitative object, e.g., Religion, Nationality, Occupation. We call such characteristics **Attributes.**

A variable takes different values and these values can be measured numerically in suitable units. An attribute cannot be measured numerically, but can be classified under different heads or categories (e.g., persons classified as Single, Married, Widowed, Divorced).

Continuous Variable and Discrete Variable

A variable which can assume for its value any real quantity within a specified interval is a continuous variable. In measuring height of a group of people we may observe that the height of an individual may have any value between 147.3 cm and 182.8 cm. Clearly, the height can be measured to arbitrary degree of accuracy. We say that height is a continuous variable. Similarly, the weight of an object is also a continuous variable which can be recorded to any degree of approximation we desire. Other *examples* are temperature, time, size of agricultural holding of a family, etc.

A discrete variable can, however, take only isolated values. In most cases a discrete variable takes only integral values, e.g., the number of workers in a factory, the number of defectives produced, the prices of articles, readings on a taxi-meter, bus fares in a certain route, the number of births in a certain number of years, the number of telephone calls on different days, etc.

EXERCISES ON CHAPTER 1(I)

1. What are the two senses in which the word Statistics is generally used? [C.U. B.Com. 1993]
2. What are the different statistical techniques used in Commerce, Business and Industry? [C.U. B.Com. 1995]
3. What are the characteristics of Statistics? What are its limitations? [C.U. B.Com. 1993]
4. Explain with examples the distinction between:

(a) an attribute and a variable,

(b) a continuous variable and a discrete variable.

Classify the following characteristics as either an attribute or variable; if variable, mention whether continuous or discrete:

Family size; family income (p.m.); mother tongue of a student; size of agricultural holding of a family; division obtained by a student at the Higher Secondary Examination.

1.6 Types and Sources of Data

The raw material (i.e., numerical data) of Statistics ordinates from the operation of counting (or enumeration) or measurement.

The person who conducts statistical enquiry, i.e., counts or measures the characteristic under study is called *investigator*. The process of counting or measurement and systematic recording of results is called *collection of data*.

The first task of a Statistician is to collect and assemble his data. He may prepare the data himself or borrow them from other sources (government, semi-government or non-official records). We may call these two types of data — one **primary** and the other **secondary**. For collection of primary data we may use complete Enumeration or Sampling. In the present chapter we shall study the whole procedure of collection of such data. We may observe here that utmost care and importance should be placed in the collection of these data — in fact, they are the raw materials and all subsequent analysis will be based on these materials. If these raw materials are not reliable, then the conclusions based on them will lead to useless and false conclusions. So reliability of raw data, as we call them, should be our first and foremost consideration.

1.6.1 Primary and Secondary Data

Primary Data are collected for a specific purpose directly from the field of enquiry — in this sense these data are original in nature. The primary data are published by authorities who themselves are responsible for their collection.

Usually trade associations collect data from their member concerns, Government organizations collect data from its subordinate offices. They are considered **primary data.** But individuals or any organization also can collect primary data from the actual field of enquiry by appointing trained investigators.

Examples: Sources of Primary and Secondary Data

Primary Data	*Secondary Data*
1. Reserve Bank of India Bulletin (Monthly) [issued by the Reserve Bank of India, Bombay]	1. The Statistical Abstract of India (Annual) [Central Statistical Organization (CSO), New Delhi]
2. Indian Textile Bulletin (Monthly) [issued by the Textile Commission, Govt of India]	2. Annual Statement of the Foreign Trade of India [DGCIS, Kolkata]
3. Annual Report of the Railway Board [Ministry of Railways, Govt of India]	3. Monthly Abstract Statistics [CSO Delhi]
4. Indian Coal Statistics (Annual) [Chief Inspector of Mines, Ministry of Labour, Govt of India]	4. International Labour Review (Monthly) [ILO, Geneva]
5. Jute Bulletin (Monthly) [Indian Central Jute Committee]	

Secondary Data are numerical information which have been previously collected as primary data by some agency for a specific purpose but are now compiled from that source for use in a different connection. In fact, data collected by some agency when used by another or collected for one purpose when used for another, may be termed **secondary data.** The same data is primary for its collecting authority but secondary to another agency who uses them.

On Secondary Data

The chief sources of secondary data are:

(a) Publications of Central and State Governments, or Foreign Governments and international bodies like ILO, UNO, UNESCO, WHO, etc.

(b) Publications of various Chambers of Commerce, Trade Associations, Cooperative Societies, etc.

(c) Reports of Committees and Commissions of Enquiry appointed time to time for specific purposes of Enquiry.

(d) Reports and Research Papers published by Research Scholars, Labour and Trade Unions, etc.

One must be very careful before using secondary data because of its many limitations. Secondary data may contain the following errors:

(i) Transcribing Errors (i.e., errors occurring due to wrong transcription of the primary data).

(ii) Estimating Errors (the data published in many secondary data are mere estimates and not facts).

(iii) Error due to Bias; sometimes fictitious figures are put in secondary data.

Before using secondary data the user should scrutinize the following points and then decide how far such data will be useful for his purpose:

- the scope and object of enquiry for which the data were originally collected;
- the methods of collection;
- the time and area covered;
- the reliability of the authority who collected the original data;
- terms used and unit of measurements considered while the data were collected.

Relative Advantages of Primary Data

(a) Primary data contain more detailed information and some information are suppressed or condensed in secondary data.

(b) Secondary data may contain errors like transcribing errors, etc. while primary data cannot have such errors.

(c) In the primary data precise definition of the terms used and the scope of the data are mentioned.

(d) Primary data often include the method of procedure or any approximation used so that one can know its limitations but secondary data usually lack such information.

Observations: In spite of these advantages of primary data, it is suggested that secondary data should be used particularly when a large number of related items are needed. One main reason for accepting secondary data is that it is much less expensive.

1.7 Collection of Primary Data

An investigator may collect primary data: either

- by direct personal observation, or
- by indirect oral investigation, or
- by sending questionnaires by mail, or
- by sending schedules through paid investigators.

Comments on These Methods

(a) A research worker may collect data by direct experimentation. An investigator may himself meet persons who can supply his requisite information. The results obtained are quite reliable for further analysis but it involves time and money.

(b) In this case indirect sources are tapped for collecting information, e.g., Enquiry Commissions obtain information through persons who are likely to have the requisite information. The reliability of these data very much depends on the integrity of the persons selected.

(c) A set of questions is prepared keeping in mind the object of enquiry; these questions are sent by mail to selected persons with a request for their replies by return mail. Though this method can cover large areas and is also comparatively cheaper but the responses are, in most cases, not satisfactory and as such expected amount of data are seldom available.

(d) 'Schedules sent through paid investigators' method is most widely used, particularly in market research. Schedules are prepared; paid investigators are trained; they carry with them the schedules and meet people concerned. The investigators fill up the schedules on the spot on the basis of the replies they receive from the informants. The data are thus collected.

(e) This method is popular and yields better results. Much, however, depend on the tactfulness and personality of the investigators. This method is adopted during the decennial census of the population in our country.

Two Procedures for Collection of Data

By Complete Enumeration (or Census)

By Sample Survey

When information is collected in respect of every individual person or item of a given population, we say that the inquiry has been done by complete enumeration or census. This process is called *Census Survey.*

But in most cases of statistical inquiry, because of limitations of time and money, a portion only of the population is examined and the data are collected therefrom. This process of partial enumeration is known as *sample survey.* The results and findings are, however, made applicable to the whole field of inquiry. This is known as *Sampling.*

Illustration 1. *During the census of population of a country (in India census is taken after every ten years) each and every individual person residing in that country is examined and the data are collected. This is an example of complete enumeration.*

A census of manufacturing industries in India must include each and every manufacturing industry and none is excluded. Usually, a complete enumeration of these industries are made with regard to the wages, attendance and absenteeism, hours of work, labour disputes and time lost, accidents and compensation, etc. in respect of the employees, etc.

In verifying the accuracy of entries in Books of Accounts, only a small number of entries are checked (Test Audit). This then is an example of Sample Survey. Other fields, where sample survey is used are estimation of the crop yield of a country, counting the number of trees in a forest, etc.

A Comparative Study of the Two Methods

- In the census method the entire population is investigated and hence a very large number of investigators (or enumerators) are to be employed. Considerable time are necessary for collection of data and processing the large quantity of information collected. Thus, census method involves considerable time, money and labour. But on the positive side this method is expected to yield accurate information (provided, of course, enumerators do their work honestly).

  In fact, if intensive investigations are needed, complete enumeration method cannot be avoided (however expensive it may be). Again, if the field of survey is small, this method should be used.
- Sample survey method, on the other hand, has the following advantages; that is why sampling (particularly, random sampling techniques) is getting more popular in recent times:
  (a) In sample survey only a part of the population is to be investigated and hence it takes less time, less money and less labour.
  (b) In census method a large group of investigators are employed — a thorough and intensive training of such a large group of people can hardly be possible. But in sample survey method only a small group of estimators are employed — they are given specialized training. Hence, though little costlier, the output in this method is much more.
  (c) Sample survey even with a limited budget can cover more geographical area.
  (d) In sample survey, sampling error cannot be avoided, but in census there is no question of sampling errors (since the whole population is investigated).

  However, large magnitudes of non-sampling errors are involved in the census method — their magnitudes remain undetermined and they greatly influence the accuracy of the results.
  (e) Sampling errors can be theoretically calculated and hence, their magnitudes are known but non-sampling errors do not follow any law of probability and hence their magnitudes cannot be precisely determined.
  (f) Sometimes complete enumeration is not feasible or practicable. For instance, to examine the quality of rice in a particular consignment, it is not possible to investigate every grain of rice — one must take a small portion of the contents and thereby decide the quality of the whole population. It is true that suitable method of sampling should be used so that the conclusion about the sample can be extended to the whole population.

Collection of Statistical Data: Different Stages

Statistical Enquiry: We use this term to mean some statistical investigation wherein relevant information is collected, analyzed and interpreted by the application of statistical techniques. Statistical enquiry consists of the following stages:

— Planning.

— Framing questionnaire and devising other forms to be used for the enquiry.

— Actual collection of data.

— Editing the information collected through questionnaire.

— Analysis of data and interpretation of results.

— Preparation of report.

Short Explanations of Different Stages

Planning: Before making a venture for the actual investigation, planning is the most essential step. While making the planning the following points are to be considered:

- Object and scope of enquiry.
- Source of data — primary or secondary [If time and money are available, efforts should be made for collecting primary data].
- Type of enquiry should be decided upon — complete enumeration or sample survey.
- Units in terms of which requisite information are to be collected should be clearly defined (Such units are called *Statistical Units*).
- Degree of accuracy [Maximum accuracy subject to limitations and available resources should be aimed at].

Questionnaire: In a statistical enquiry the necessary information are generally collected in a printed sheet in the form of a questionnaire. Such a sheet contains a set of questions which the investigators are supposed to ask the informants and note down their answers against each question.

The framing of the questions requires skill and foresight.

The success of the enquiry depends very much on thoughtful drafting of the questionnaire. In setting up the questions the following points are to be considered:

- The schedule of questions must not be lengthy. Many questions may arise but not all of them can be included in the questionnaire; with a lengthy questionnaire the respondents become bored and may not disclose accurate information. Only essential points are to be included in the questionnaire.
- Questions must be simple, brief and unambiguous — otherwise, accurate information may not be available.

  If necessary, several complicated questions are to be broken up into smaller parts so that they can be easily answered by the respondents. Questions should be so framed so as to elicit only one of the two possible answers — YES or NO.
- Personal questions (income, property, etc.) are to be avoided, if possible, or to be put in indirect form. People are reluctant to disclose such facts. No questions should be asked as to hurt the feelings or sentiments (religious or otherwise) of the informants.
- Questions should be of objective nature — there should be less scope for matters of opinion.
- The units in which the information are to be collected should be precisely mentioned; e.g.,

  State your age: ......years ......months. Annual Output: ₹ ......
- The arrangement of questions should be made in such a way that an easy and systematic flow of answers may come out.

  After the preparation of a questionnaire it is desirable to make a PILOT SURVEY — to try on a few individuals. It will be helpful in finding out the shortcomings of the questionnaire and hence necessary modifications can be made before the actual survey.

Collection of Data:

- Questionnaires may be sent to the respondents by post and request may be made to send the completed form also by post (necessary postage being enclosed). This method has many advantages:

  It is least expensive and vast areas can be covered. Moreover time required for collection of the data will be comparatively short and the respondents feel more free to answer questions which they are ordinarily reluctant to answer before investigators.

  But this mailing system of the questionnaire has the greatest disadvantage — LOW PROPORTION

OF RESPONSE and consequently the information may not represent the representative view. Another minor point is that due to misunderstanding of the meanings of some questions by the respondents, correct information may not always be available.

- Another method is to employ suitable persons as investigators and to train them for collection of data. These investigators then interview the informants, ask questions as per the questionnaire and note down their answers. This method is commonly used. Though reliable information are expected through this method but it requires huge organizational efforts including necessary fund.

Editing the Returns: Soon after the actual collection of data starts, arrangements should be made to receive the completed forms and scrutinize them. Some of the completed forms may contain inconsistencies or omissions. These forms are to be referred back for correction.

When all returns have been received and checked, the information are classified and tabulated. If necessary, the work of tabulation and sorting may be done through specially devised machines like electric tabulators. However, when the study does not involve a large number of returns, manual sorting and tabulations are used.

Analysis of Data and Their Interpretations: The main object of statistical analysis is to abstract significant facts from a large collection of data. With the help of the data and using statistical theories various statistical measures are computed. They help to make statistical conclusions about the present position and also to forecast the future.

Preparation of Report: It is usual to publish a report after the completion of the enquiry. The report should contain a complete description of all the stages of enquiry, definitions of terms used, coverage degree of reliability of the data and final results of the enquiry. Such a report is helpful for planning future enquiries on related problems.

EXERCISES ON CHAPTER 8(II)

1. Explain the terms: Primary data and Secondary data. Give some illustrations.
   [B.U. B.Com.(H) 2002; V.U. B.Com.(H) 2007]

2. Describe various methods of collecting primary data and comment on their relative advantages.
   [C.U. B.Com.(H) 1991; Guwahati U. 1996; Bangalore U. 1997]

3. Define 'secondary data'. State the chief sources and point out the precautions necessary before using them. [Madras U. B.Com. 1997; Kanpur U. B.Com. 1997]

4. Statistical data are usually of two types — (i) Primary, (ii) Secondary. Explain in brief. State the various methods of collecting primary data and the chief sources of secondary data (three of each).

5. (a) Distinguish between a census and a sample enquiry, and briefly discuss their comparative advantages. [CA 1992; B.U. B.Com. 1996; V.U. B.Com.(H) 2010]
   (b) Distinguish between Population and Sample, with the help of an example. [C.U. B.Com. 1996]
   (c) Explain the term Questionnaire. [V.U. B.Com.(H) 2007]

6. What is meant by Statistical Enquiry? Describe different stages of a statistical enquiry.

7. (a) What is a questionnaire as used as statistical enquiry? Describe the important considerations in the framing of a questionnaire. What special measures are necessary in case of enquiries by mail? [V.U. B.Com.(H) 2010]

(b) What is the difference between census survey and sample survey? Which one of them is more suitable when the available data is very large? Give reasons.

8. (a) Describe the characteristics of a good questionnaire. [D.U. B.Com. 1996; V.U. B.Com.(H) 2010]

   (b) Write short notes on Technique of Drafting Questionnaires.

   (c) Explain the Law of Statistical Regularity.

9. Write short notes on the following: (a) Collection of data, (b) Various stages of sample survey.

10. Distinguish between Complete Enumeration and Sample Survey. How far is the latter more advantageous than the former and why? Name different types of sampling in Statistics.

11. (a) Enumerate the various steps in conducting a sample survey. [K.U. B.Com. 1994]

    (b) Explain the following statistical terms: (i) Primary data, (ii) Population, (iii) Sample survey, (iv) Questionnaire.

12. What are the methods by which primary data can be collected? Write brief account of each of them pointing out their merits and demerits. [M.U. B.Com. 1999]

Chapter 2

Summarization of Data: Classification and Tabulation; Frequency Distribution

2.1 Introduction

In Statistics we often use a term called *Characteristic.* An attribute (or a quality) possessed by an individual person, object or item of a population is called a *characteristic of the individual.* Height, age, nationality are examples of characteristics.

A characteristic may be measurable or non-measurable.

Measurable characteristics are those which are expressible numerically in terms of some units. A characteristic of this type is also called a *variable,* or sometimes a *variate.* Age, height, income, etc. are examples of this type of characteristics.

Non-measurable characteristics cannot be expressed numerically. Nationality, occupation, etc. are examples of non-measurable characteristics.

Classification

All statistical investigations are carried on with a definite purpose in view and the collection of data, described in the previous chapter, is only the first step in this direction. To make the collected data really useful, they must be *classified or grouped under appropriate heads and systematically arranged according to some common characteristic possessed by the individuals, about whom the data have been collected.*

The process of arranging or bringing together all the enumerated individuals/items under separate heads or classes according to some common characteristics possessed by them is known as Classification. It facilitates comparison and drawing of inferences by dropping unnecessary details and very clearly showing different points of agreement and disagreement.

During the population census, apart from the number of members in each family, various other information, e.g., age, sex, occupation, nationality, etc. of all the people in the country are collected. The total population of the country are then classified:

- according to sex, into Males and Females.
- according to marital status, into Single (never married) and Married.
- according to the place of residence, into Rural and Urban.
- according to livelihood categories, into Agricultural classes, Production other than cultivation, Commerce, Transport and Other services.
- according to age, into 0–5 years, 6–10 years, 11–15 years and 16–20 years, etc.
- according to residence in States, into West Bengal, Assam and Bihar; etc.

In the classifications illustrated above, the common characteristics are sex, marital status, place of residence, livelihood category, age and residence in States.

When the items/individuals are classified according to some common non-measurable characteristics possessed by them, they are said to form a Statistical Class and when they are classified according to some common measurable characteristics possessed by them, they are said to form a Statistical Group.

2.2 Types of Classification

We may broadly consider four types of classification:

(i) *On a Qualitative Basis:* Classification according to some non-measurable characteristics, such as sex, nationality, occupation, etc. are examples of this type. Such classifications may also be termed classification by attributes.

Illustration 1. *Total population of the country may be classified according to nationality — into Indian, Chinese, Ceylonese, English, Dutch, etc.; according to mothertongue — into Bengali, Hindi, Gujarati, Malayalam, Punjabi, etc.; similarly, insurance may be classified into life insurance, fire insurance, marine insurance, accident insurance, etc. all of which are examples of classification by attributes.*

(ii) *On a Quantitative Basis:* The classification of population according to age, industries according to the number of persons employed, banks according to the amount of paid-up capital and reserves, etc. are included under this type. Here the basis of classification is some measurable quantity. Such classifications may also be termed classification by variables.

Illustration 2. *The age of a person is measurable in years, say* 31 *years, and is thus a variable and the population may be classified by the variable age. Similarly, prices, wages, barometer readings, etc. are examples of variables.*

(iii) *On a Time (Chronological or Temporal) Basis:* In this type, the enumerated individuals or items are classified according to the time at which they were measured. Thus, the production of a factory may be shown by weeks or months or quarters. Similarly, ships first registered at Indian Ports may be classified according to their year of registration. Statistical data arranged chronologically constitute what is called a *Time Series.*

(iv) *On a Geographical Basis:* The classification of the total population of a country according to states or districts falls under this type. The total value of imports of merchandize may, similarly, be classified according to the place or country from which imports have been made.

2.3 Presentation of Statistical Data

There are three different ways, in which statistical data may be presented: (i) *Textual Presentation,* (ii) *Tabular Presentation,* and (iii) *Graphical Presentation.*

Textual Presentation: In this way of presentation numerical data are presented in a descriptive form.

Two illustrations presenting information in a textual form are given below:

Illustration 1. *In* 2005 *out of a total of* 2000 *workers in a factory* 1550 *were members of a trade union. The number of women workers employed was* 250, *out of which* 200 *did not belong to any trade union.*

In 2010 *the number of union workers was* 1725 *of which* 1600 *were men. The number of non-union workers was* 380 *among which* 155 *were women.*

Illustration 2. *Numerical data with regard to industrial diseases and deaths therefrom in Great Britain during the years* 1995–99 *and* 2000–04 *are given in a descriptive form:*

During the quinquennium, 1995–99, there were in Great Britain 1775 cases of industrial diseases made up of 677 cases of lead poisoning, 111 of other poisoning, 144 of anthrax and 843 of gassing. The number of deaths reported was 20%. Of the cases for all the four diseases taken together, that for lead poisoning was 135, for other poisoning 25, and that for anthrax was 30.

During the next quinquennium, 2000–04, the total number of cases reported was 2807. But lead poisoning cases reported fell by 351 and anthrax cases by 35. Other poisoning cases increased by 748 between the two periods. The number of deaths reported decreased by 45 for lead poisoning, but decreased only by 2 for anthrax from the pre-war to the post-war quinquennium. In the later period, 52 deaths were reported for poisoning other than lead poisoning. The total number of deaths reported in 2000–04 including those from gassing was 64 greater than in 1995–99.

The disadvantages to be faced with such a mode of presentation are:

- that it is a lengthy text,
- that there has been much of repetition in words,
- that comparison between the corresponding figures in the two periods is difficult, and
- that it is difficult to grasp, from a lengthy text the important points if there be a number of them, making it all the more difficult to arrive at any conclusion.

Tabular Presentation: Tabulation may be defined to be the orderly and systematic presentation of numerical data in rows and columns designed to clarify the problem under consideration and to facilitate comparison between the figures.

Tabular presentation or Tabulation is also a form of presentation of quantitative data in a condensed form so that the numerical figures are capable of easy reception through the eyes.

The numerical descriptions of Exs 1 and 2 given above have been condensed in the form of Tables shown below:

It is clear from the illustrations 1–4 that the lengthy text have been condensed into a small table, while facilitating better comparison between the corresponding figures in the two time periods. Much of repetition in words has been avoided with the help of the table. The tabular form of presentation of data is thus much superior in use to a text statement.

The tables provide examples of time series, where classification on both qualitative and quantitative bases occurs. Textual presentation in Illustrations 1 and 2 is given in Tabular forms in Illustrations 3 and 4.

Illustration 3. *Number of workers in a factory classified according to sex and their membership of the Union.*

TABLE 2.1: NUMBER OF WORKERS IN A FACTORY

Sl. No.	Characteristic	2005		2010	
		No. of Male Workers	No. of Female Workers	No. of Male Workers	No. of Female Workers
(1)	(2)	(3)	(4)	(5)	(6)
1	Trade Union Members	1500	50	1600	125
2	Non-Trade Union Members	250	200	225	155
	Total	1750	250	1825	280

Illustration 4.

TABLE 2.2: DEATHS FROM INDUSTRIAL DISEASES IN GREAT BRITAIN							
		2003–07			2008–12		
Sl. No.	Types of Disease	No. of Cases	No. of Deaths	% of Deaths	No. of Cases	No. of Deaths	% of Deaths
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	Lead Poisoning	677	135	7.6	326	90	3.2
2	Other Poisoning	111	25	1.4	859	52	1.9
3	Anthrax	144	30	1.7	109	28	1.0
4	Gassing	843	165	9.3	1513	249	8.8
	Total	1775	355	20.0	2807	419	14.9

Workings: 20% of 1775 = 355 and $355-(135+25+30)=165$.

$$\frac{135}{1775}\times 100=7.6,\quad \frac{25}{1775}\times 100=1.4,$$

$$\frac{30}{1775}\times 100=1.7,\quad \frac{165}{1775}\times 100=9.3,\text{ etc.}$$

A **table** consists of *four parts*: (i) **Title**, (ii) **Stub**, (iii) **Caption** and (iv) **Body**.

Title: A title of a table is self-explanatory and clearly conveys in as few words as possible, the contents of the table; it is usually put at the head of the table concerned.

Stub: In a table, the left-hand column, which contains the headings of the rows, with its heading is called *Stub.*

Captions: These are the headings of the columns other than the Stub.

Body: The whole portion of a table excepting its Title, Stub and Captions and consisting usually of the figures of the table is called *Body* of the table.

Note: It is usual to indicate at the end of the table the source from where the information have been collected.

There is no ideal method of tabulation. Skill in tabulation is generally attained by years of experience. Nevertheless, the following general rules may be borne in mind while tabulating:

- A table must contain a title. This title should be clear and convey, in as few words as possible, the contents of the table. When several tables are used, each should be numbered, e.g., **Table 1**, **Table 2**, etc. to facilitate future references. For our purpose we have also given references to chapters. Thus, **Table 2.1** refers to **Table 1** of Ch. 2.

  The columns should, similarly, have definite and at the same time, sufficiently comprehensive headings. The columns should also be numbered, if their number in a table is 4 or more.
- Units of measurement must be clearly shown. The units are generally shown at the top of columns, if the different columns show figures in different units; or sometimes at the top right-hand corner of the table, if all the measurements are in the same unit.
- The table should be well-balanced in length and breadth. If the table is unduly long, try a separate coarse grouping. If, however, this causes any serious loss of information, some breaks or extra spaces should be left at equal distances apart (compare **Table 2.9**).
- Columns of figures which are directly comparable should be kept as close as possible.

- Light and heavy rulings may be used to distinguish the sub-columns and main columns. Double rulings may also be used instead of heavy ones.
- The totals of columns may be shown at the bottom of the table between heavy rulings. Sometimes, totals of rows are also useful and should accordingly be shown.
- Tables should not usually be burdened with unnecessary details. If necessary, such details may be shown in separate smaller tables.
- In case, the data are procured from some sources, references must be made of the sources, at the right-hand corner below the tables.
- Where doubts exist in any figure, this should be explained at the bottom of the table with a suitable explanatory note.

Graphical Presentation: The third form of presentation of quantitative data is made by graphs and charts.

2.4 Types of Tabulation

Tabulation may either be — (i) *Simple* or (ii) *Complex.*

Simple Tabulation: In this type the number or measurement of the items are placed below the headings showing the characteristics.

Illustration 1.

TABLE 2.3: PROGRESS OF COOPERATIVE SOCIETIES IN INDIA

Year	No. of Societies	No. of Members
2000–01	1,81,189	1,54,83,159
2001–02	1,85,650	1,57,83,571
2002–03	1,89,436	1,63,78,364
2003–04	1,98,598	1,73,24,744
2004–05	2,19,288	1,82,98,645
2005–06	2,40,395	1,76,21,978

In this table the characteristics of the progress of cooperative societies as measured by the number of societies and the number of members, have been shown year by year.

This is an example of a time series in which quantitative basis of classification is also present.

Complex Tabulation: In this type, each numerical figure in the table is the value of the measurement having the characteristics shown both by the column and the row headings.

Illustration 2.

TABLE 2.4: INDIA'S BALANCE OF PAYMENTS (CURRENT ACCOUNT) FOR THE YEAR 2010 (Core of ₹)

Area ↓ Balance of Payment →	Receipts	Payments	Balance
Starling Area	503.8	447.7	+ 56.1
Dollar Area	178.8	137.4	+ 41.4
Other Areas	205.6	262.8	– 57.2
Total	888.2	847.9	+ 40.3

In this table, the figure 503.8 (for example) which represents 503.8 crore of ₹, i.e., ₹503,80,00,000 is the value of India's current account receipts (indicated by the column heading) during 2010 from the sterling area (indicated by the row heading).

The complex tabulation shown in **Table 2.4** may also be called a *two-fold tabulation;* because the table shows on one side classification according to Receipts, Payments and Balance, and on the other side classification according to areas, viz., Sterling area, Dollar area and other areas. Classification here, has been made on geographical basis combined with quantitative basis.

Similarly, complex tabulation may show three-fold, four-fold, etc. tabulations.

2.5 Frequency Distributions

Variables: Continuous and Discrete

A Variable is a symbol, such as x, y, h, n, X, Y, etc. which can assume any of a prescribed set of values (called the *domain* of the variable).

If a variable can assume any value (any real number for our purpose) within two given values, we call it a continuous variable; otherwise the variable is a discrete variable. Discrete variables assume isolated values (say, whole numbers only). Continuous variables may be measured to an arbitrary degree of accuracy.

Illustration 1. *The number n of children in a family which can assume values like* 2, 3, 5, *etc. but not* 2.5 *or* 3.42 *which is an example of a discrete variable.*

Illustration 2. *The height h of an individual, which can be* 123 cm *or* 123.5 cm *or* 123.489 cm *or* 123.4885 cm *depending on the accuracy of measurement desired, is a continuous variable. Another example is the weight of an object which may be recorded in* grams, *tenths of* grams *or thousandths of* grams.

We may come across discrete variables like: marks obtained by a sample of students in an examination; reading of a taxi meter; number of births/deaths; number of accidents; number of defects found in a sample of production, etc.

Data which can be described by a discrete variable are called *discrete data;* those which can be described by a continuous variable are called *continuous data.* In general, measurements give rise to continuous data while enumerations or countings give rise to discrete data.

Raw Data

Statistical data may originally appear in a form (See **Table 2.5**), where the collected data are not organized numerically. We call them *raw data.*

Illustration 3.

TABLE 2.5: MARKS IN MATHEMATICS OF 50 STUDENTS (SELECTED AT RANDOM FROM AMONG CANDIDATES APPEARING IN B.COM. EXAMINATION)

37	38	40	36	38	37	36	40	50
47	41	46	38	31	33	48	37	52
32	50	40	50	47	41	50	43	
26	45	52	45	41	44	39	16	
21	30	38	32	48	47	41	45	
41	51	37	26	40	38	46	32	

This representation of the data does not furnish useful information to a statistician. A better way may be to express the figures in ascending or descending order of magnitude, commonly termed as ARRAY.

Illustration 4.

TABLE 2.6: ARRAY OF MARKS GIVEN IN TABLE 2.5 (ARRANGED IN ORDER OF ASCENDING MAGNITUDES)

16	32	37	38	40	41	46	48	52
21	32	37	38	40	43	46	50	52
26	32	37	38	41	44	47	50	
26	33	37	39	41	45	47	50	
30	36	38	40	41	45	47	50	
31	36	38	40	41	45	48	51	

Range

From the array it is easy to find what we call Range. Range of a given data is the difference (Largest Measure – Smallest Measure).

In **Table 2.6**, the largest measure is 52 and the smallest one is 16; hence the range = 52 – 16 = 36.

Though *Array* has certain advantages over *Raw Data* still this does not reduce the bulk of the data and when large masses of data are given the array is also cumbersome.

Frequency: Tally Sheet

Frequency of a value of variable is the number of times it occurs in a given series of observations. In Table 2.6, 38 occurs 5 times — we say that frequency of the value 38 is 5.

A Tally sheet may be used to calculate the frequencies from the raw data. A tally mark (/) is put against the value when it occurs in the raw data. Having occurred four times, the fifth occurrence is represented by putting a Cross tally mark (\) on the first four tally marks, see **Table 2.7**. This technique facilitates the counting of tally marks at the end.

Illustration 5.

TABLE 2.7

Marks (x)	Tally Marks	Frequencies (f)	Marks (x)	Tally Marks	Frequencies (f)
16	/	1	40	////	4
21	/	1	41	~~////~~	5
26	//	2	43	/	1
30	/	1	44	/	1
31	/	1	45	///	3
32	///	3	46	//	2
33	/	1	47	///	3
36	//	2	48	//	2
37	////	4	50	////	4
38	~~////~~	5	51	/	1
39	/	1	52	//	2
				Total Frequency	50

Such a representation of the data is known as the Frequency Distribution. Here marks of students may be taken as a discrete variable x and the number of students against each value of the variable is the frequency of the value of the variable. (The word frequency thus refers to the information how frequently a value of a variable occurs.)

Such representation of data, though better than Array, does not condense the data very much.

Grouped Frequency Distribution

When large masses of raw data are to be summarized and the identity of individual observation or the order in which observations arise is not relevant for the analysis, we distribute the data into *Classes* or *Categories* and determine the number of individuals belonging to each class, called the **Class frequency.**

A tabular arrangement of raw data by classes, where the corresponding class frequencies are indicated (tally marks should, in general, be shown in such presentation), is known as a Grouped Frequency Distribution (or sometimes, simply, Frequency Distribution). The name Grouped Frequency Distribution is given because the whole range of observations has been divided into smaller groups and the frequency of each such group is taken into consideration.

In the data of **Table 2.7** the Range is 36 (nearly, 40). We take 8 classes or groups — first group containing marks ranging from 16 to 20, the next group 21 to 25 and so on, the last group of 51–55. We have then a Grouped Frequency Distribution as shown in **Table 2.8**. Data organized and summarized as in the frequency **Table 2.8** are often called *Grouped Data*. Though it destroys much of the details of raw data, it helps us to give a clear overall picture in a condensed form.

Class Interval or Class

A symbol defining a class (such as 16–20, 21–25, 26–30, etc.) is called a *class interval* (or we may refer them simply as *a class*). There are 8 classes in **Table 2.8**.

Illustration 6.

TABLE 2.8: FREQUENCY DISTRIBUTION OF MARKS OF 50 STUDENTS IN MATHEMATICS

Sl. No.	Marks	Number of Students
1	16–20	1
2	21–25	1
3	26–30	3
4	31–35	5
5	36–40	16
6	41–45	10
7	46–50	11
8	51–55	3
Total		50

A Few Points to Remember before the Construction of Classes

- The classes should be clearly defined and should not lead to any ambiguity.
- The classes should be exhaustive. (Each value of the raw data should be included.)
- The classes should be mutually exclusive and non-overlapping.
- The classes should preferably be of equal size or width. In special cases classes of unequal size may have to be used.

- The number of classes should not be too large or too small. Normally, it should not exceed 20 but should not be less than 5, depending on the number of observations in the raw data.

Class Limits

In the construction of a grouped frequency distribution the class intervals must be defined by a pair of numbers such that the upper end of one class does not coincide with the lower end of the next class (See 16–20, then 21–25, etc.).

The two numbers used to specify the limits of a class interval are called *class limits* — the smaller number is called the *lower class limit* and the larger one is the *upper class limit.* We note that both the class limits may coincide with the observed data. Class limits are used solely for the construction of grouped frequency distribution from raw data.

Class Mark

The Class Mark is the middle point of the class interval (and hence also called *Mid-value* of the Class):

$$\text{Class Mark} = \tfrac{1}{2}\,(\text{Upper Class Limit} + \text{Lower Class Limit})$$

[See **Table 2.8**: The class interval 16–20 has lower limit 16 and the upper limit 20; mid-value is 18.]

Remember that class mark is used as a representative value of the class interval (useful in discussions on Mean, Standard deviation, etc.).

Class Boundaries

When we deal with a continuous variable all data are recorded nearest to a certain unit. Let us consider the distribution height of individuals in centimetres. If we decide to record height to the nearest centimetre and if then we have a class interval like 125–130, then all heights between 124.5 cm and 125.5 cm will be recorded as 125 cm. Similarly, all heights between 129.5 cm and 130.5 cm will be recorded as 130 cm. The extreme values (indicated by the number 124.5 and 130.5) such that any observed value (any real number) included between them may be taken within the class interval are called *Class boundaries* or *True class limits* — the smaller number is the lower class boundary and the larger number is the upper class boundary.

In practice, remember:

Lower Class Boundary = Lower Class Limit −0.5 (if observations are recorded to the nearest integral unit) but
= Lower Class Limit −0.05 (if observations are recorded to the nearest tenth of a unit)

Upper Class Boundary = Upper Class Limit +0.5 or Upper Class Limit +0.05 in the two respective cases mentioned in 1.

For observations taken to the nearest whole number, suppose the classes are 126–130, 131–135, etc. See **Table 2.9.**

$$\left.\begin{aligned}\text{Lower Class Boundary} &= 126 - 0.5 = 125.5\\ \text{Upper Class Boundary} &= 130 + 0.5 = 130.5\end{aligned}\right\}\text{for the class 126–130.}$$

For observations taken to the nearest tenth (i.e., correct to one place of decimal) suppose the class intervals are: 126.5–130.5, 130.6–134.6, etc.

$$\left.\begin{aligned}\text{Then the lower class boundary} &= 126.5 - 0.05 = 126.45\\ \text{and the upper class boundary} &= 130.5 + 0.05 = 130.55\end{aligned}\right\}\text{for the class 126.5–130.5.}$$

We give below a table, where class limits, class boundaries and class marks are indicated along with the class intervals and their frequencies.

Illustration 7.

TABLE 2.9: HEIGHTS OF 100 STUDENTS

Class Intervals	Frequency	Class Limits		Class Boundaries		Class
Height (in cm)	No. of Students	Lower	Upper	Lower	Upper	Marks
126–130	5	126	130	125.5	130.5	128
131–135	18	131	135	130.5	135.5	133
136–140	42	136	140	135.5	140.5	138
141–145	27	141	145	140.5	145.5	143
146–150	8	146	150	145.5	150.5	148
Total	100					

Note: The upper boundary of any class coincides with the lower boundary of the next class; but the upper limit of any class may be different from the lower limit of the next class.

Width (or Size) of a Class

Width of a class = Upper class boundary − Lower class boundary, e.g., the width of the class 126 − 130 is 130.5 − 125.5 = 5.

In the construction of a frequency distribution it is preferable to have classes of equal width. Classes of unequal width are also in use (when some of the observations are isolated and far apart from the rest) specially to avoid empty classes (classes with frequency zero).

Remember that if all class intervals of a frequency distribution have equal widths, this common width

= difference between two successive lower class limits (or two successive upper class limits)

or = difference between two successive lower class boundaries (or upper class boundaries)

or = difference between two successive class marks.

[See Table 2.9]

A Few More Terms

Frequency Density: We have observed that in a frequency distribution the class intervals may or may not be of equal width. If the classes be a varying width, the different class frequencies will not be comparable. Comparable figures can be obtained by dividing the frequency of the class by its width. We call this ratio *Frequency Density.*

Thus, **Frequency Density of a class** = Frequency of the class ÷ Width of the class.

Frequency density is used in the construction of histogram, when the classes are of unequal width.

$$\textbf{Percentage Frequency of a class} = \frac{\text{Frequency of the class}}{\text{Total frequency}} \times 100\%.$$

Relative Frequency of a class is the frequency of the class divided by the total frequency of all classes and is generally expressed as a percentage.

For example, the relative frequency of the class 126–130 in **Table 2.9** is 5/100 = 5%. Note that the sum of the relative frequencies of all classes is clearly 100% or 1.

If the frequencies in **Table 2.9** are replaced by corresponding relative frequencies, the resulting table is called a *relative frequency distribution* or a *percentage distribution.*

We may have occasion to use the expressions like open class interval to mean that the class has either no upper class limit or no lower class limit. *For example,* we may refer to age groups of individuals, the class '60 years and over' or '25 years and below' — these are open class intervals.

Systematic Procedure for Construction of a Frequency Distribution from Raw Data

Obtain the largest and smallest numbers in the raw data and thus find the Range.

Divide the Range into a convenient number of class intervals (of equal size preferably). The number of class intervals should not exceed 20 but not less than 5 either.

Class intervals are so chosen that the class marks (or mid-values) may, if possible, coincide with observed data or most of the observations may lie near class marks. But class boundaries should not coincide with actually observed data.

Determine the number of observations falling into each class interval, i.e., find the class frequencies by means of a Tally Sheet.

The frequency table is now constructed showing the class intervals and their frequencies.

2.6 Different Ways of Presenting Frequency Distributions

We give below a few specimens of frequency distributions for illustrating the different ways in which they are presented.

Illustration 1.

TABLE 2.10: FREQUENCY DISTRIBUTION OF MONTHLY WAGES EARNED BY THE WORKERS OF A FACTORY

Monthly Wages in ₹	No. of Workers
110–120	25
120–130	40
130–140	65
140–150	85
150–160	80
160–170	70
170–180	35
180–190	20

Here, the upper limit of one class is the lower limit of the next class and an upper limit or a lower limit occurs in two classes; but the explicit understanding is that only the items measuring less than the upper limit of a class are included in that class while the upper limit of a class is to be considered an item of the next class. Thus, 120, 130, 140, etc. are respectively the items of the 2nd, 3rd, 4th, etc. classes. The limits look like overlapping ones, but actually they are non-overlapping.

Illustration 2.

TABLE 2.11: FREQUENCY DISTRIBUTION OF HEAD CIRCUMFERENCE OF 1071 INDIAN BOYS 16 TO 19 YEARS OF AGE

Head Circumference (cm)	Number of Boys
50 but less than 51	4
51 but less than 52	23
52 but less than 53	59
53 but less than 54	108
54 but less than 55	224
55 but less than 56	257
56 but less than 57	230
57 but less than 58	110
58 but less than 59	38
59 but less than 60	16
60 but less than 61	2
Total	1071

Here, the upper limit of a class does not belong to the class, but is an item of the next class. Clearly, 51 belongs to the second class, 52 to the third and so on. The limits are non-overlapping. Sometimes, in place of the words, 'but less than' or 'and less than' the words 'and under' are used; thus we get '50 and under 51', '51 and under 52', etc.

Illustration 3.

TABLE 2.12: FREQUENCY DISTRIBUTION OF MONTHLY INCOMES (IN ₹) OF MIDDLE-CLASS FAMILIES IN A CERTAIN AREA OF KOLKATA

Monthly Income in (₹)	0–	50–	100–	150–	200–	300–	500–	750–	1000–	1500–2000	Total
Number of Families	18	62	185	261	364	299	170	65	48	38	1510

Here, the first class is the same as '0 but less than 50', the second class the same as '50 but less than 100', and so on, so that the upper limit of a class does not belong to the class, but is, as is clearly shown, an item of the next class. The limits, here, are non-overlapping.

Note that the class intervals are unequal. In the cases when there are great fluctuations in data, when there is a sharp rise or fall in the frequency over a small interval, unequal class intervals will be more useful in the construction of frequency distributions than class intervals of equal widths.

Illustration 4.

TABLE 2.13: FREQUENCY DISTRIBUTION OF ANNUAL INCOME OF A GROUP OF MIDDLE-CLASS FAMILIES IN A CERTAIN AREA OF CALCUTTA

Annual Income (₹'000)	No. of Families
Above 1.2 and not exceeding 1.8	206
Above 1.8 and not exceeding 2.4	325
Above 2.4 and not exceeding 3.0	475
Above 3.0 and not exceeding 3.6	556
Above 3.6 and not exceeding 4.2	762
Above 4.2 and not exceeding 4.8	603
Above 4.8 and not exceeding 5.4	490
Above 5.4 and not exceeding 6.0	350
Above 6.0 and not exceeding 6.6	220
Above 6.6 and not exceeding 7.2	108
Total	4095

Here, the first class is 'Above ₹1200 and not exceeding ₹1800'. Clearly, the upper limit of a class belongs to that class, but the lower limit does not belong to the class. The limits are thus non-overlapping. Sometimes, in place of the words Above ⋯ and not exceeding ⋯' the words 'Exceeding ⋯ and not exceeding ⋯' or 'More than ⋯ and not more than' or clauses bearing like meanings are used.

Illustration 5.

TABLE 2.14: FREQUENCY DISTRIBUTION OF OUTPUT OF 180 WORKERS

Output (units per worker)	No. of Workers
500–09	8
510–19	18
520–29	23
530–39	37
540–49	47
550–59	26
560–69	16
570–79	5
Total	180

The frequency distribution is of a discrete variable. The classes are 500–509, 510–519, etc. The limits are non-overlapping.

Illustration 6. *Grouped Frequency Distribution with Open Ends.*

TABLE 2.15: FREQUENCY DISTRIBUTION OF MARKS OBTAINED IN COMMERCIAL MATHEMATICS BY 350 STUDENTS IN AN ANNUAL EXAMINATION	
Marks Obtained	**No. of Students**
Below 13	6
14–23	21
24–33	54
34–43	71
44–53	89
54–63	62
64–73	38
74 and above	9
Total	350

In the case of grouped distributions with open-end class at one extremity or at both extremities, it is difficult to choose the values of the variable which are representative of the open classes, i.e., the mid-values of those classes. If the closed classes in the frequency distribution are of equal widths, it is most usual in such cases to take the widths of the open classes to be the same as the common width and determine the mid-value.

The reason why grouped distributions are sometimes left with open end or ends is that there are only a small number of items which are scattered over a long interval, and the frequencies falling within class intervals at these ends with common width are few and far between.

Illustration 7. *Percentage Frequency Distribution:*

TABLE 2.16: PERCENTAGE FREQUENCY DISTRIBUTION FROM THE DISTRIBUTION OF HEIGHTS OF 200 STUDENTS		
Class Intervals (Heights in cm)	**Frequency**	**Percentage Frequency**
126–130	2	1.0
131–135	9	4.5
136–140	16	8.0
141–145	26	13.0
146–150	33	16.5
151–155	41	20.5
156–160	36	18.0
161–165	21	10.5
166–170	11	5.5
171–175	3	1.5
176–180	2	1.0
Total	200	100.0

2.7 Cumulative Frequency Distribution

There is another type of frequency distribution, viz., Cumulative Frequency Distribution, in which as the name suggests, the frequencies are cumulated. This is prepared from a grouped frequency distribution, showing the end values, i.e., the class boundaries by adding each frequency to the total of the previous ones, or those following it. The former is termed cumulative frequencies from below, or less than cumulative frequencies and the latter cumulative frequencies from above, or more than cumulative frequencies.

We now refer to **Table 2.17**, where a cumulative frequency distribution of the data of **Table 2.16** is given.

Method of Compilation of the Cumulative Frequency Table

After writing down the class boundaries, from **Table 2.16**, in the left-hand column, it is argued, for less than column as: number of men of heights less than 125.5 cm is nil; hence 0 against 125.5, number of men of heights less than 130.5 cm is 2: hence 2 against 130.5; number of men of heights less than 135.5 cm is 2 + 9 = 11; hence 11 against 135.5; ...

Again, for more than column, the frequencies are to be cumulated from bottom. It is argued as: number of men of heights 180.5 cm or more is nil; hence 0 against 180.5 ; number of men of heights 175.5 cm or more is 2; hence 2 against 175.5; number of men of heights 170.5 cm or more is 3 + 2 = 5; hence 5 against 170.5; ...

The cumulative frequency distribution from below is useful for answering questions like: How many students have heights less than 150.5 cm. The answer is the frequency under column (2) of **Table 2.17** against the class boundary 150.5, viz., 86. Similarly, the number of students who have heights 145.5 cm or more is the corresponding frequency under column (3) of **Table 2.17**, viz., 147, and so on.

Illustration 1.

TABLE 2.17: CUMULATIVE FREQUENCY DISTRIBUTION OF HEIGHTS OF 200 STUDENTS (Data of Table 2.16)

Class Boundaries (Heights in cm)	Cumulative Frequency	
	Less than (from below)	More than (from above)
(1)	(2)	(3)
125.5	0	200
130.5	2	198
135.5	11	189
140.5	27	173
145.5	53	147
150.5	86	114
155.5	127	73
160.5	163	37
165.5	184	16
170.5	195	5
175.5	198	2
180.5	200	0

2.8 Some Illustrative Examples

Example 1. *Draw up a blank table to show the number of students sex-wise, attending for the Higher Secondary (HS) as well as First Year, Second Year and Third Year classes of Bachelor Degree courses in the faculties of Arts, Science and Commerce of a college in a certain session.*

Solution:

TABLE 2.18: NUMBER OF STUDENTS APPEARING IN DIFFERENT CLASSES OF A COLLEGE ACCORDING TO SEX AND FACULTY IN A CERTAIN SESSION

Class → / Sex → / Faculty			Bachelor Degree Courses						Total
	H.S.		First Year		Second Year		Third Year		
	Male	Female	Male	Female	Male	Female	Male	Female	
1. Arts 2. Science 3. Commerce									
Total									

Example 2. *Present the following information in a suitable tabular form, supplying the figures not directly given:*

In 1990, *out of a total of* 5000 *workers in a factory* 3900 *were members of a trade union. The number of women workers employed was* 650, *out of which* 500 *did not belong to any trade union.*

In 1991, *the number of Union workers was* 4200 *of which* 3400 *were men. The number of non-Union workers was* 1400, *among which* 450 *were women.* [V.U. B.Com. 1994]

Solution:

TABLE 2.19: NUMBER OF WORKERS IN THE FACTORY ACCORDING TO SEX AND MEMBERSHIP OF TRADE UNION FOR THE YEARS 1990 AND 1991

Year → / Sex → / Trade Union Membership	1990			1991		
	No. of Male Workers (1)	No. of Female Workers (2)	Total Cols. (1) & (2)	No. of Male Workers (3)	No. of Female Workers (4)	Total Cols. (3) & (4)
Trade Union Members	3750	150	3900	3400	800	4200
Non-Trade Union Workers	600	500	1100	950	450	1400
Total	4350	650	5000	4350	1250	5600

Example 3. *In a sample study about coffee-habit in two towns A and B the following information were received:*

Town A: Females were 40%, *total coffee drinkers were* 45% *and male coffee drinkers were* 20%.

Town B: Males were 55%, *male non-coffee drinkers were* 30% *and female coffee drinkers were* 15%.

Present the above data in a tabular form. [CA May 1997; C.U. B.Com. 2001]

Solution: Since data are given in percentages, let the total number of persons be 100 in each of town *A* and town *B*.

TABLE 2.20: SAMPLE STUDY ABOUT COFFEE HABITS IN TWO TOWNS A AND B

Town → / Sex → / Drinker	A			B		
	Male	**Female**	**Total**	**Male**	**Female**	**Total**
Coffee	20	25	45	25	15	40
Non-coffee	40	15	55	30	30	60
Total	60	40	100	55	45	100

Example 4. *Represent the following information in a suitable tabular form with proper rulings and headings:*

The Annual Report of the Ishapore Public Library reveals the following points regarding the reading-habits of its members:

Out of the total 3713 *books issued to the members in the month of June* 2010, 2100 *were books on fiction. There were* 467 *members of the library during the period and they were classified into five classes, A, B, C, D and E. The number of members belonging to the first four classes were respectively* 15, 176, 98 *and* 129; *and the number of books on fiction issued to them were* 103, 1187, 647 *and* 58 *respectively. Number of books, other than textbooks and fictions, issued to these four classes of members were respectively* 4, 390, 217 *and* 341. *Textbooks were issued only to members belonging to the classes C, D and E, and the number of textbooks issued to them were respectively* 3317 *and* 160.

During the same period 1246 *periodicals were issued. These included* 396 *technical journals of which* 36 *were issued to members of class B,* 45 *to class D and* 315 *to class E.*

To members of the classes B, C, D and E the number of other journals issued were 419, 26, 231 *and* 99 *respectively.*

The report, however, showed an increase by 3.9% *in the number of books issued over last month, though there was a corresponding decrease by* 6.1% *in the number of periodicals and journals issued to members.*

Solution:

TABLE 2.21: STATISTICS SHOWING READING HABITS OF MEMBERS OF ISHAPORE PUBLIC LIBRARY, JUNE 2010

Classes	No. of Members	No. of Books Issued				Periodicals and Journals		
		Books on Fiction	Text-books	Other Books	Total	Technical Journals	Other Journals	Total
A	15	103	...	4	107	...	75	75
B	176	1187	...	390	1577	36	419	455
C	98	647	3	217	867	...	26	26
D	129	58	317	341	716	45	231	276
E	49	105	160	181	446	315	99	414
Total	467	2100	480	1133	3713	396	850	1246
% increase (+)					+3.9			
% decrease (−)								−6.1

Example 5. *Marks obtained by* 50 *boys of a class are as under:*

34	54	10	21	51	52	12	43	48	36
48	22	39	26	34	19	10	17	47	38
13	30	30	60	59	15	7	18	40	49
40	51	55	32	41	22	30	35	53	25
14	18	19	40	43	4	17	45	25	43

Construct a frequency table with class intervals 0–9, 10–19, 20–29, *and so on.*

Solution:

TABLE 2.22: MARKS OBTAINED BY 50 BOYS OF A CLASS

Class Intervals	Tally Marks	Frequency
0–9	//	2
10–19	$\cancel{////}$ $\cancel{////}$ //	12
20–29	$\cancel{////}$ /	6
30–39	$\cancel{////}$ $\cancel{////}$	10
40–49	$\cancel{////}$ $\cancel{////}$ //	12
50–59	$\cancel{////}$ //	7
60–69	/	1
Total		50

Example 6. *Monthly wages in* ₹, *received by* 30 *workers in a certain factory are as follows:*

310	320	325	354	370	335	300	397	331	375
315	390	350	386	359	360	380	323	342	327
305	318	337	367	392	340	363	385	367	393

Draw a frequency distribution table, classified on the basis of wages, with class interval of 10. *Also obtain the percentage frequency in each class interval.* [C.U. B.Com. 1996 Type]

Solution: Here by 'class interval of 10', we mean class interval of length 10. From the given data, we see that maximum wage = ₹397 and minimum wage = ₹300 (note that here Range = ₹97).

Since the width (or length) of the class is to be 10, we can take the class intervals as 300–309, 310–319, ..., 390–399.

TABLE 2.23: MONTHLY WAGES (IN ₹) RECEIVED BY 30 WORKERS IN A CERTAIN FACTORY

Class Intervals	Tally Marks	Frequency	Percentage Frequency
300–309	//	2	$\frac{2}{30} \times 100 = \frac{20}{3}$
310–319	///	3	$\frac{3}{30} \times 100 = 10$
320–329	////	4	$\frac{4}{30} \times 100 = \frac{40}{3}$
330–339	///	3	10
340–349	//	2	$\frac{20}{3}$
350–359	///	3	10
360–369	////	4	$\frac{40}{3}$
370–379	//	2	$\frac{20}{3}$
380–389	///	3	10
390–399	////	4	$\frac{40}{3}$
Total	—	30	100

Example 7. *From the following data, calculate the percentage of workers getting wages between ₹22 and ₹58:*

Wages (in ₹)	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	Total
No. of Workers	20	45	85	160	70	55	35	30	500

[C.U. B.Com.(H) 2002]

Solution: We see that the class intervals are all of equal lengths, each being 10.

∴ number of workers getting wages up to ₹22 $= 20 + 45 + \frac{85}{10} \times 2 = 65 + 17 = 82$

and the number of workers getting wages up to ₹58 $= 20 + 45 + 85 + 160 + 70 + \frac{55}{10} \times 8 = 380 + 44 = 424$.

∴ the number of workers getting wages between ₹22 and ₹58 $= 424 - 82 = 342$.

Hence, *the required percentage of workers getting wages between ₹22 and ₹58* $= \frac{342}{500} \times 100 = \mathbf{68.4}$.

Otherwise: First construct a cumulative frequency table (less than 'type') and then find the numbers of workers who are getting wages up to ₹22 and ₹58 by the method of Simple Interpolation. Thus, 82 and 424 will be obtained.

Example 8. *Age at death of* 50 *persons of a town are given below:*

36	48	50	45	49	31	50	48	43	42
37	32	40	39	41	47	45	39	43	47
38	39	37	40	32	52	56	31	54	36
51	46	41	55	58	31	42	53	32	44
53	36	60	59	41	53	58	36	38	60

(a) *Arrange the data in frequency distribution in* 10 *class intervals; and*

(b) *Obtain the percentage frequency in each class interval.* [C.U. B.Com. 1998]

(c) *Also find the class boundaries and cumulative frequencies from below and from above.*

Solution: From the given data, we find that

the highest age = 60, and the lowest age = 31 and hence, range = 29.

Since we have to arrange the given data in 10 class intervals,

the width of each class = $(60-31) \div 10 = 2.9 = 3$ (approximately).

We can take the class intervals as 31–33, 34–36, ..., 58–60.

TABLE 2.24: AGE AT DEATH OF 50 PERSONS OF A TOWN

Class Intervals	Tally Marks	Frequency	Percentage Frequency
31–33	~~////~~ /	6	$\frac{6}{50} \times 100 = 12$
34–36	////	4	$\frac{4}{50} \times 100 = 8$
37–39	~~////~~ //	7	$\frac{7}{50} \times 100 = 14$
40–42	~~////~~ //	7	14
43–45	~~////~~	5	$\frac{5}{50} \times 100 = 10$
46–48	~~////~~	5	10
49–51	////	4	8
52–54	~~////~~	5	10
55–57	//	2	$\frac{2}{50} \times 100 = 4$
58–60	~~////~~	5	10
Total	—	50	100

TABLE 2.25: CUMULATIVE FREQUENCY DISTRIBUTION

Age (Years) (Class boundary points)	Cumulative Frequency Less than (from below)	Cumulative Frequency More than (from above)
30.5	0	50
33.5	6	44
36.5	10	40
39.5	17	33
42.5	24	26
45.5	29	21
48.5	34	16
51.5	38	12
54.5	43	7
57.5	45	5
60.5	50	0

In the **Table 2.24**, the gap between any two consecutive classes is 1 and $\frac{1}{2} \times 1 = 0.5$. Therefore, the class boundaries are 30.5–33.5, 33.5–36.5, ..., 57.5–60.5 and the class boundary points are 30.5, 33.5, 36.5, ..., 60.5.

EXERCISES ON CHAPTER 2

Theory

1. (a) Define Classification. What part does it play in Statistics? State the different methods of classification of statistical data. [C.U. B.Com. 2008; D.U. B.Com. 1996]
   (b) Name the different methods of presentation of statistical data. [C.U. B.Com. 1996]
   (c) Write short notes on Primary and Secondary data. [B.U. B.Com.(H) 2002]
2. (a) Discuss the function and importance of Tabulation in a scheme of statistical investigation. What precautions should be taken in tabulation of data?
   (b) Explain the term Frequency Distribution.
3. (a) Discuss briefly the purpose served by tabulation. State the requirements of a good statistical table. [D.U. B.Com. 1990; Bangalore U. B.Com. 1993]
   (b) What is meant by Tabulation? [V.U. B.Com.(H) 2010]
4. Discuss the different steps in the construction of a frequency distribution of raw data. [D.U. B.Com. 1990]
5. What do you mean by a cumulative frequency distribution? Point out its special advantages and uses.
6. (a) Define a Statistical Table and state the essentials of a good table.
   (b) What are the different parts of Statistical Table? [C.U. B.Com. 1991; V.U. B.Com. 1997]
7. Explain with example the exclusive method and the inclusive method of determining limits of class intervals. [D.U. B.Com. 1991]
8. Explain with example both *less than* and *more than* types of cumulative frequency distribution.

Problems

1. (a) Prepare a blank table to show the number of candidates sex-wise, appearing for the Pre-University, First Year, Second Year and Third Year of Bachelor Degree examinations of a university in the faculties of Arts, Science and Commerce in a certain year.
   (b) Draw a blank table showing the distribution of students in a college with three faculties — Science, Commerce and Humanities for first year, second year and third year pass classes only for the year 2000-2001. [C.U. B.Com. 1999]
2. Prepare a blank table to show the exports of three companies A, B, C to the five countries UK, USA, Russia, France and West Germany, in each of the years 2000–04.
3. Prepare a blank table to show the distribution of population of the various States and Union Territories of India, according to sex and literacy.
4. (a) Present the following information in a suitable tabular form supplying the figures not directly given:

   In 1995, out of a total of 4000 workers in a factory 3300 were members of a trade union. The number of women workers employed was 500 out of which 400 did not belong to any Union.

   In 1994, the number of workers in union was 3450 of which 3200 were men. The number of non-union workers were 760 of which 330 were women. [C.U. B.Com. 2000 Type]

(b) In 1994, out of a total of 3600 workers in a factory 2050 were members of trade union. The number of women workers employed were 1200 of which 650 did not belong to any union.

In 1999, the number of workers in the union was 2600 of which 1800 were men. The number of non-union workers was 1900 of which 1200 were women. Present the information in a suitable table. [C.U. B.Com. 2000]

[**Hints:** See worked-out Ex. 2 in **Section 2.8** (Statistics).]

5. There are two families A and B whose monthly average expenses are classified under five heads, viz., (a) House rent, (b) Household, (c) Education, (d) Medical and (e) Miscellaneous. Family A expends ₹150 as monthly house rent whereas B expends ₹100 only. Family A expends double the amount under the head (b) as B expends on account of (a). Expenditures under (c) are ₹40 and ₹50, where A expends more than B. Medical expenses of family B is ₹5 more than that of A. Total expenses under the head (e) of the two families taken together is ₹50. Medical expenses of A is half its expenses under (c). Household expenses of family B is six times its expenses on account of (d). Total expenses of family A is ₹460.

Present the above information for comparison in a net tabular form.

6. Draw up a blank table to show the number of employees in a large commercial firm, classified according to:

(a) Sex: male and female;

(b) Three age-groups: below 30, 30 and above but below 45, 45 and above; and

(c) Four income-groups: below ₹400, ₹400–750, ₹750–1000, above ₹1000.

7. Design blank table, with proper title, headings and sub-headings, to present data relating to the distribution of the employees of a factory classified according to:

(a) Sex: Male and Female;

(b) Age in years: Below 25, between 25 and 34, between 35 and 44, and 45 and above;

(c) Category: Skilled and Unskilled; and

(d) Wage in ₹: Less than 500, between 500 and 750, and above 750.

[**Hints:** The distribution of the employees of a factory classified according to Sex, Age, Category and Wages in ₹.]

Age in Years	Below 25				25–34				35–44				45 and above				Total
Category	Skilled		Unskilled		Skilled		Unskilled		Skilled		Unskilled		Skilled		Unskilled		
Wages in ₹ / Sex	M	F	M	F	M	F	M	F	M	F	M	F	M	F	M	F	
1. Less than 500																	
2. 500–750																	
3. Above 750																	
Total																	

8. Represent the statistical information contained in the following passage in a suitable tabular form:

   The cropped area of vegetables (excluding potatoes) grown for human consumption in the U.K. rose in 2005–06 and was the highest since 2000–01. The cropped area increased to 509,000 acres, some 11,000 acres more than in 2004–05. The area of root-vegetables increased by 8,100 acres to 62,400 acres, carrots alone increasing by 5,700 acres to 33,200 acres. The area of cabbage rose slightly, thus halting the steady decline since 1997–98; the cropped area was 75,700 acres with 74,800 acres in 2004–05. The cropped area of cauliflower and broccoli was 33,400 acres, 2,400 acres less than in 2004–05. Peas (harvested dry) decreased by about 121,800 acres to 9,800 acres, but a larger area of beans, mainly broad beans and green peas were grown. The area of broad beans increased by 2,600 acres to 7,300 acres and the area of green peas for canning and quick freezing rose by 7,000 acres to 50,400 acres.

9. (a) Present the following information in a tabular form and suggest a suitable title:

   The production of 10.95 lac tons of rice in Maharashtra in 2002–03 was the lowest in the period since 1995–96. In 2003–04, however, it has shown a spectacular recovery and reached the level of 15.14 lac tons. During 2003–04 wheat and bajri output decreased. The production of bajri which was 5.50 lac tons in 2002–03 declined to 4.51 lac tons in 2003–04. The production of wheat also decreased from 4.63 lac tons in 2002–03 to 3.43 lac tons in 2003–04. The area under pulses has shown a decreasing trend and the production was less by 22,000 tons in 2003–04 than the production of 8.89 lac tons in 2002–03.

   (b) In a trip organized by a college there were 100 persons, the average cost works out to be ₹15.60 per head. There were 80 students each of whom pays ₹16. Members of the teaching staff were charged at a higher rate. The number of servants was 6 (all males) and they were not charged. The number of ladies was 20% of the total of which two were lady staff members.

   Tabulate the above information in proper tabular form. [CA Nov. 1998]

   [**Hints:**

Types of Participant	Sex		Total	Contribution	Total
	Male	Female		per head	Contribution (₹)
1. Students	62	18	80	16	1280
2. Teaching Staff	12	2	14	20	280
3. Non-teaching Staff	6	0	6	0	0
Total	80	20	100		1560

   Working: ₹15.60 × 100 = ₹1560; ₹16 × 80 = ₹1280; ∴ ₹1560 – ₹1280 = ₹280.]

10. Write down the class boundaries and class marks of the following distribution:

Class interval	237–239	240–242	243–245	246–248
Frequency	2	8	14	19

11. (a) The distribution of marks in an examination was as under for the candidates appearing thereat:

35	76	49	25	67	34	43	51	38	21	62	16	37
55	48	72	20	40	38	56	25	39	58	33	68	59
44	32	41	60	52	23	37	42	47	27	38	50	36
44	41	20	18	54	39	51	47	35	39	62	30	49

Construct a frequency table from the above data taking class intervals of 5 marks beginning with the interval 11–15.

(b) Following are the daily wages (in ₹) of 30 workers in a factory:

60	45	41	32	47	45	50	37	53	17
26	39	59	68	44	12	30	25	36	18
40	62	46	29	32	54	41	14	32	30

Make a frequency distribution taking class intervals of ₹10 and find the mean of the distribution. [C.U. B.Com. 1996]

[Hints: See worked-out Ex. 5 in Section 2.8.]

12. (a) Given below are the records of maximum temperature (in centigrade) at 47 selected stations in India during 24 hours preceding 8.30 a.m. (IST) on the 19th June, 2012:

39	39	39	28	35	32	18	37	18	35
33	36	42	39	22	35	28	37	35	30
39	24	36	22	35	27	34	33	33	
38	29	33	35	28	35	35	27	35	
40	33	39	41	37	35	37	23	37	

Construct a frequency distribution taking the lowest class interval as 16–20.

(b) If the class marks of a frequency distribution be 5.5, 15.5, 25.5, 35.5 and 45.5, find the class limits of the distribution.

13. (a) Marks obtained by 50 boys of a class are:

32	45	20	39	52	15	55	30	18	72
10	45	34	50	28	42	71	17	40	35
12	39	44	58	16	21	61	39	21	10
48	63	25	15	51	68	47	15	10	59
34	55	28	14	31	47	19	40	49	58

Construct a frequency table with class intervals 10–19, 20–29, 30–39 and so on.

(b) The monthly salaries of 20 employees are as follows (in ₹):

130	62	145	95	116	100	103	71	76	151
142	110	98	85	80	122	132	118	125	95

Form a frequency distribution with 5 classes of equal intervals.

[C.U. B.Com. 2002]

[Hints:

Classes	62–79	80–97	98–115	116–133	134–151	Total
Frequency	3	4	4	6	3	20

]

(c) The weights (in kg) of 50 persons are given below:

53	46	47	55	48	43	52	64	65	56
55	51	65	67	74	61	62	71	55	67
64	76	60	44	64	69	42	63	46	78
52	66	56	73	46	61	72	67	75	77
63	72	53	55	72	58	43	58	48	64

Construct a frequency distribution table in class interval of length 5 kg.

14. (a) The distribution of marks in an examination was as under for the candidates appearing thereat:

43	41	62	47	49	53	51	45	66	23	72
31	62	26	34	76	49	25	67	34	43	51
38	21	11	16	18	32	48	72	20	25	28
38	12	31	19	45	53	40	13			

Construct a frequency distribution with a class interval of 10 marks.

(b) From the following data of marks obtained by the students, form a frequency distribution table of eight class intervals by tally marks:

17	81	60	25	50	33	53	48	57	81
65	58	28	37	42	71	78	89	43	47
11	58	26	23	82	73	22	44	31	58
14	75	16	83	24	36	35	47	40	76
39	37	27	49	56	77	81	20	45	19

15. Given below are the marks obtained by a batch of 84 students at the IA Examination, 2011 of the Calcutta University:

34	43	32	57	35	71	65	10	52	19	48	17	24	43
65	40	54	62	44	0	13	18	49	57	21	64	71	32
21	52	40	35	57	43	45	44	55	39	37	19	14	45
17	51	35	27	47	22	0	22	15	0	23	35	0	31
21	52	48	0	22	12	12	15	40	39	30	42	27	17
4	19	0	30	6	19	31	25	33	22	51	68	42	66

Construct a frequency distribution with a class interval of 10 marks.

16. The weights in pounds of 50 persons are given below:

160	155	178	90	101	105	124	118	126	176
135	157	134	99	112	115	104	108	178	159
175	147	129	117	128	151	140	169	120	165
98	107	119	93	170	144	154	164	174	160
95	105	113	123	138	168	172	149	179	97

Arrange the above in a frequency distribution consisting of 9 class intervals.

Draw up the cumulative frequency distributions both from below and from above, and also, the percentage frequency distribution from the distribution so constructed.

17. The following table gives the scholastic aptitude scores of the 50 departmental students of a certain department in a certain university:

345	530	556	354	590	472	475	610	586	523
395	515	479	494	420	691	520	465	468	545
563	444	629	440	485	624	582	570	578	505
505	604	490	445	605	523	575	420	605	527
402	406	730	506	516	461	440	585	420	384

Construct a frequency distribution table with appropriate class limits and class boundaries. (Take the length of the class equal to 30 units.)

18. The following are the marks obtained by 50 boys:

7	18	37	53	24	39	41	23	64	67
68	40	93	43	11	27	68	72	19	12
21	19	32	75	52	84	15	11	23	19
52	29	92	79	45	81	63	36	21	33
53	8	41	14	26	26	33	49	40	19

(a) Construct a frequency distribution with a class interval of 10. Also obtain cumulative frequency distribution, (b) from above and (c) from below.

19. Monthly incomes of 40 workers in a factory in ₹ are as under:

120	121.2	122.5	100.2	101.5	102	103	130	139	142.4
150	157	160	100	105.6	108	175	165	166	189
146	100	102	100.5	165	139	188	192	190.2	155
130	142.4	123	124	185	189	170	180	160.2	157

(a) Construct a frequency table with class intervals in the fashion 100 and under 110, 110 and under 120; 120 and under 130, and so on. Also obtain cumulative frequency distribution, (b) from above and (c) from below.

20. Construct a blank table showing the total population of India (2012 census) classified according to

(a) Sex;

(b) Age-groups (years) 0–4, 5–14, 15–24, 25–34, 35–44, 45–54, 55–64, 65–74, 75 and over;

(c) Civil condition — Married, Unmarried, Widowed or Divorced.

21. Present in a tabular form with suitable title, captions, etc. the information contained in the following:

    In 1995, out of a total of 1750 workers in a factory 1200 workers were members of a Trade Union. The number of women employed was 200 of which 175 did not belong to a Trade Union. In 2000, the number of union workers increased to 1580 of which 1290 were men. On the other hand, the number of non-union workers fell down to 208 of which 180 were men. In 2005, there were on the pay rolls of the factory, 1800 employees who belonged to a Trade Union and 50 who did not belong to a Trade Union. Of all the employees in 2005, 300 were women of whom only 8 did not belong to a Trade Union.

22. Draw up a blank table to show the number of wholly unemployed, temporarily stopped and the total unemployed persons, each class being divided into males and females, for the following industries:

    (a) Textile, (b) Tobacco, (c) Footwear, (d) Furniture and fixture, (e) Paper and paper products, (f) Leather and leather products, (g) Chemicals, (h) Engineering and (i) Transport equipments.

Chapter 3

Diagrammatic Representation of Statistical Data

3.1 Diagrammatic Representation — Their Advantages and Disadvantages

We have so far discussed how statistical data are collected, condensed and finally presented in a tabular form. Another method of presenting statistical facts and explaining them in a forceful manner is with the help of Graphs, Charts and Diagrams, i.e., through diagrams.

Advantages of Diagrammatic Representations

- Through such representations statistical facts become more apparent and appealing to the eyes.
- A chart can sometimes clarify a complex problem and often reveal hidden facts which cannot be otherwise easily detected.
- Graphs of statistical data bring out clearly the relative importance of different figures — the trend or tendency of the values of the variables involved can be easily studied thereby making the forecasting easier.
- Graphs are helpful in finding out the relation between two or more sets of data.
- Interpolation of values of the variables can be made.
- A graph is often an important means of detecting mistakes in the computation of data.

Disadvantages of Diagrammatic Representations

- First, the charts do not show details. In a table, we may show information covering a large number of items or on related topics by providing additional rows and columns. This is not possible in a chart. Because, in that case, the main purpose of diagrammatic representation (viz., presentation of a few results so as to be easily acceptable to eyes and remembered) is lost.
- Secondly, graphical form of representation reveals only the approximate position. In a table we may show exact figures to the nearest unit. But a chart shows only the overall position.
- Thirdly, graphs and charts require much more time to construct; but the desired information can be very quickly conveyed by arranging the figures in the form of a table.

  However, these defects are more than compensated by a good graph or a nicely drawn chart. We, however, remark that graphs and charts should be very accurately drawn. Otherwise, they may create a wrong impression and consequently false conclusions.

3.2 Types of Charts and Diagrams

There are various types of charts and diagrams to choose from. Some of the more common types used are described below:

- Graphs or Line Charts;
- Bar Charts or Bar Graphs;
- Ratio Charts or Logarithmic Charts;
- Pie Charts;
- Histogram, Frequency Polygon and Ogive.

3.2.1 Graphs or Line Charts

This is the most widely used method of presenting statistical data, especially in business or any other sphere, where data are collected over long periods of time. It is also very simple to construct.

For drawing line charts, we take two straight lines, called *Axes*, which cut at right angles at a point, called the *Origin*. The vertical axis is called the *Y-axis* and the horizontal axis is called the *X-axis*. There is a convention to assume the right-hand side of the X-axis and the upper side of the Y-axis as positive and the opposites as negative.

Suppose now that we have two sets of data — the values of one set giving the values of the independent variable x and the values of the other set giving the values of the dependent variable y. We now conveniently select the scales of the data, i.e., we fix up a relationship between the length of each axis and the value of the variable represented thereby. It is usual to represent the values of the independent variable on the X-axis and those of the dependent variable on the Y-axis. If one of the variables is time, we shall always represent its values on the X-axis. It may be noted here that the scales on the two axes need not be the same. For convenience, squared papers or graph-papers are very often used. Scales are then shown by a relationship between the side of a square on the graph-paper and the value of the item it represents. Points are now plotted according to the magnitude of an item in one set of data and that of the corresponding item in the other set of data — the scales selected being taken into consideration. These points are then joined by broken lines or a continuous curve which give the graphical representation or graph of the two sets of data. The figure obtained is called a *Line Chart* or a *Graph* or *simply a Curve*.

We give below a few examples.

Example 1. *The monthly productions for* 2011 *of Hind Motor Cars in India are given below:*

January 175 cars,	February 185 cars,	March 160 cars,	April 110 cars,
May 120 cars,	June 80 cars,	July 85 cars,	August 88 cars,
September 120 cars,	October 110 cars,	November 155 cars,	December 170 cars

Represent the production figures by a line chart.

The above data has been represented by a Line Chart (Fig. 3.1). We have represented the months along the X-axis and the production of cars along the Y-axis.

Scales are: 1 side of a small square along the X-axis $\equiv$ 1 month,

1 side of a small square along the Y-axis $\equiv$ 10 cars.

The origin is taken as January 31 on the X-axis and 0 (zero) production on the Y-axis. In January 175 cars were produced, we take 17.5 squares along the Y-axis and plot the point corresponding to this production. In this way other points are plotted and then joined by broken lines.

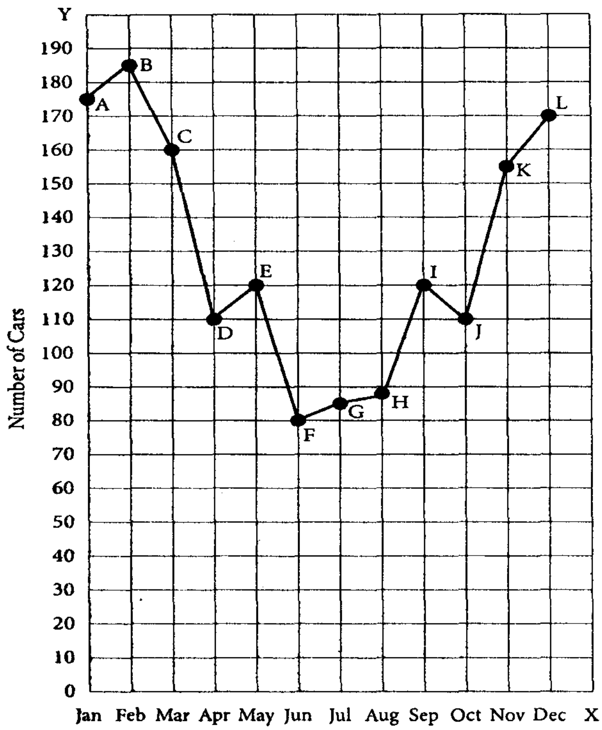


Fig. 3.1: Line Chart representing the monthly production of Hind Motor Cars in 2011.

Important Observations

When the data of a time series are plotted on the graph, the curve is called a *Historigram* (not Histogram, discussed later). Thus, the above graph is, therefore, a Histogram. We give below another example of a Histogram. In case of time series the vertical line should not be used as zero line or base line.

Example 2. *The daily profits and losses for the first fortnight of a month of a business concern are given below:*

Days of the month	Profit (in ₹)	Loss (in ₹)
1	500	
2	600	
3		300
4		100
5	700	
6	900	
7	1000	
8	800	
9	400	
10		400
11		200
12		50
13	500	
14	700	
15	1000	

Represent the above data by a Line Chart.

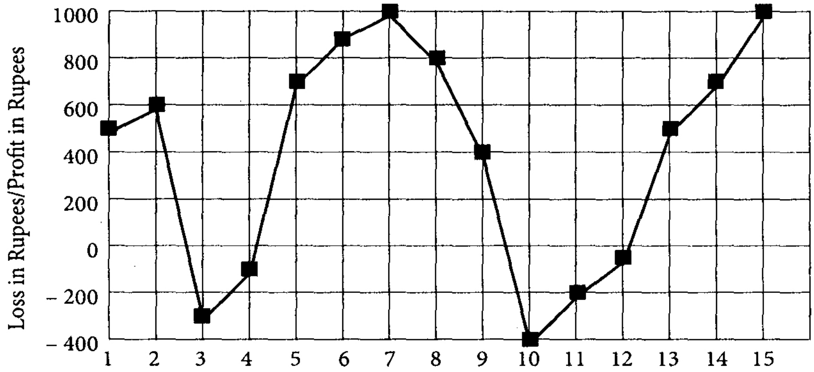


Fig. 3.2: Daily Profit and Loss for the first fortnight of a month.

We represent 'Days' along X-axis, 'Profits' along the positive Y-axis and 'Losses' along the negative Y-axis according to the following scales:

1 side of a small square on the graph-paper along the X-axis $\equiv$ 1 day,

1 side of a small square on the graph-paper along the positive Y-axis $\equiv$ ₹200 profit,

1 side of the small square on the graph-paper along the negative Y-axis $\equiv$ ₹200 loss.

Note that the zero line is the horizontal X-axis. Profits are shown above the zero line and Losses are shown below the zero line.

Fig. 3.2 represents the required graph.

Inclusion of Zero in the Vertical Scale

Example 3. Fig. 3.3 *represents the graph of the data in the following table. Scales are clearly shown in the figure:*

TABLE 3.1: PRODUCTION OF PIG IRON AND FERRO-ALLOYS IN WEST BENGAL DURING 2004–05

Months	Production (in Tons)	
	2004	2005
January	64,980	73,728
February	71,005	83,715
March	64,138	95,262
April	73,175	86,826
May	73,815	94,105
June	67,479	78,795
July	74,606	97,082
August	79,740	94,830
September	77,627	79,615
October	79,428	79,539
November	90,205	77,380
December	99,541	82,295

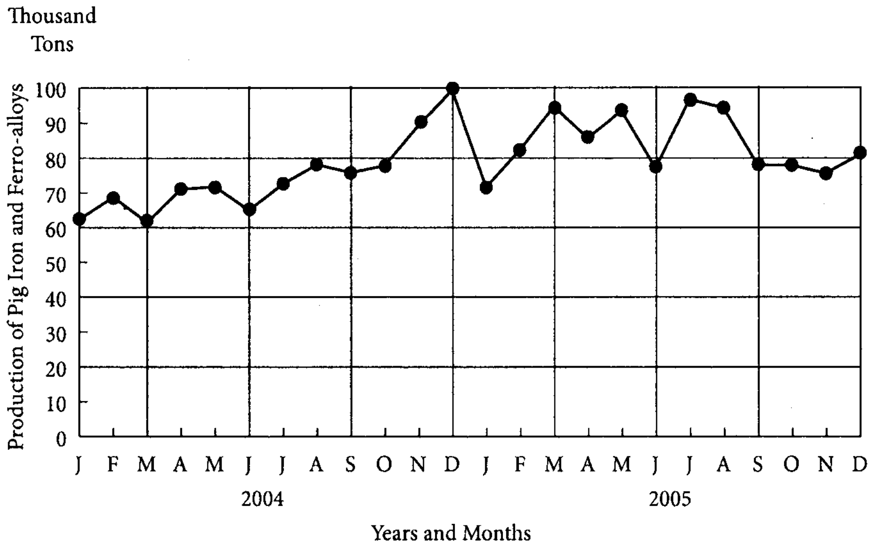


Fig. 3.3: Production of Pig Iron and Ferro-alloys in West Bengal during 2004-05.

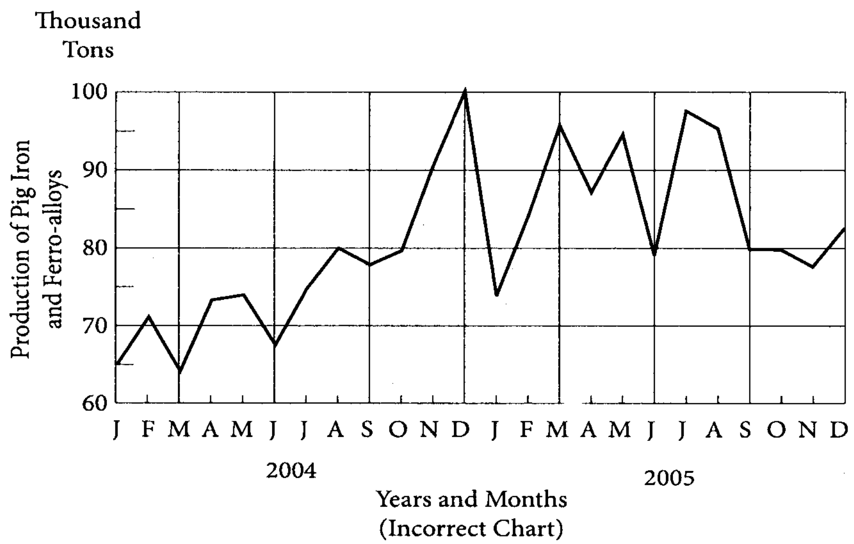


Fig. 3.4: Production of Pig Iron and Ferro-alloys in West Bengal during 2004-05.

Observations

- The inclusion of zero in the vertical scale, as has been done in Fig. 3.3 is very important. Most chart-makers neglect to include the zero point, as has been in Fig. 3.4. The vertical scale in Fig. 3.4 starts from 60 thousand tons. Consequently, this chart gives a misleading impression of the production of Pig Iron and Ferro-alloys. For instance, production in December, 2004 appears to be 8 times that in January 2004, which is not at all so. In fact, it will be seen from the **Table 3.1** that the figures were actually 64,980 tons in January and 99,541 tons in December, the latter being only $1\frac{1}{2}$ times the former approximately.

  Fig. 3.4 should not, therefore, be used to indicate the actual magnitude of Pig Iron and Ferro-alloys production.

- Sometimes, one wishes to emphasis the movements of the curve and is less interested in the magnitude of the data. Thus where it is desired not to include the complete scale, the fact may be indicated by breaks **Figs 3.5** and 3.6. In any case, the zero of the vertical scale must be shown.

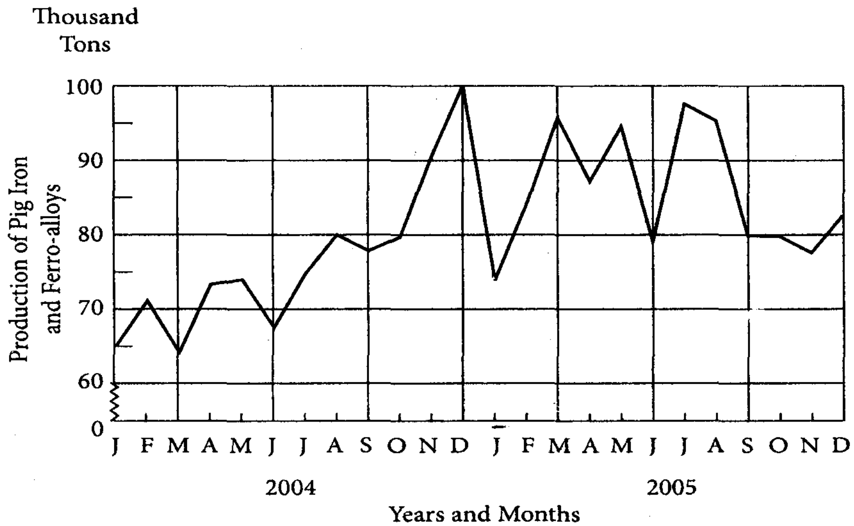


Fig. 3.5: Production of Pig Iron and Ferro-alloys in West Bengal during 2004-05.

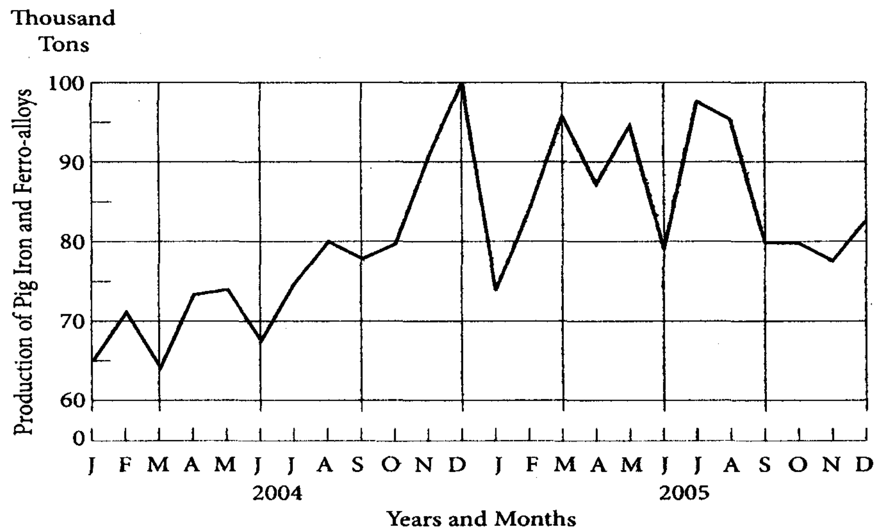


Fig. 3.6: Production of Pig Iron and Ferro-alloys in West Bengal during 2004-05.

Two or More Charts in the Same Graph

It is not necessary that one diagram will contain only one curve. Two or more curves may also be drawn on the same sheet, representing two or more related series of values. Such curves are helpful for comparing the related series. Too many curves should not, however, be drawn on the same chart, because in that case the plotting may overlap and the simplicity in presenting the data is lost. The different lines in this case should be drawn so as to be easily distinguishable.

They should be differentiated from each other by solid, dotted, dashed and the like lines and properly designated. The placing of designation along the curves is desirable [Fig. 3.7].

Example 4. Fig. 3.7 *shows the plotted data contained in the following table:*

TABLE 3.2: COMPENSATION PAID FOR ACCIDENTS TO WORKMEN IN INDIA UNDER THE WORKMEN'S COMPENSATION ACT, 1923 (in Thousand ₹)

Year	Amount paid for			Total
	Death	Permanent Disability	Temporary Disability	
1996	1179	1210	937	3326
1997	1581	1615	1024	4220
1998	1871	2026	1320	5217
1999	2003	2265	1439	5707
2000	2208	2293	1397	5908
2001	2008	2288	1342	5638
2002	2341	2315	1482	6138
2003	2474	2443	1508	6425
2004	2159	2190	1618	5967
2005	2041	1760	1382	5183

Note: See that the points have been plotted in the chart at the middle of each interval representing the years. This is an alternative way of plotting of points on the vertical lines at equal distances apart, which may indicate the periods, as shown in the previous figures.

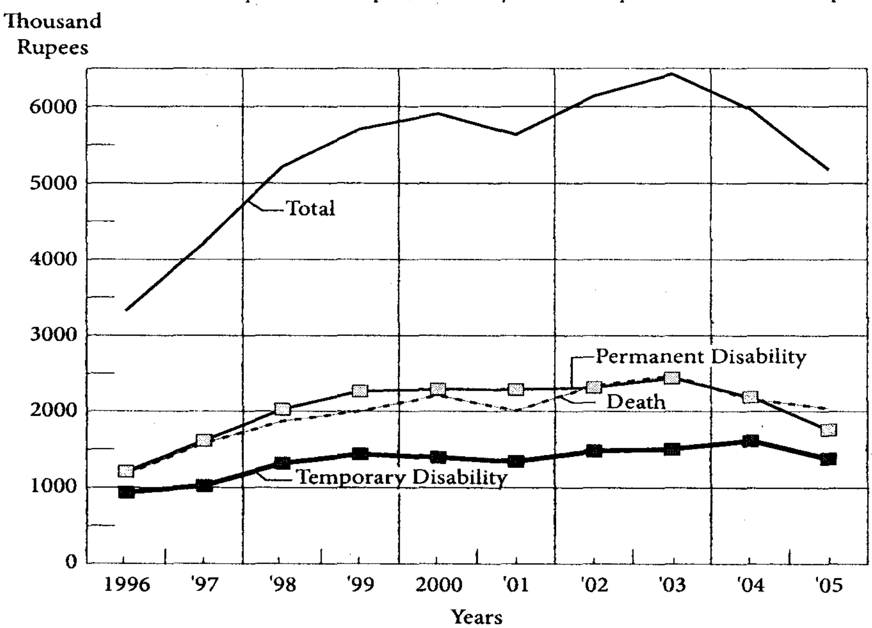


Fig. 3.7: Compensation paid for accidents to workmen in India under the Workmen's Compensation Act, 1923.

Graphical Interpolation

When the graphs of continuously varying variables are drawn we may find the values of the variables at any intermediate stage from the graph itself. The process is known as *Interpolation* by *Graphical Methods.* We illustrate the process by the following example:

Example 5. *The table below gives the height and weight of a man at different ages:*

TABLE 3.3									
Age (years)	4	6	8	10	12	14	16	18	20
Height (in cm)	75	90	105	125	140	150	175	185	195
Weight (in kg)	20	25	35	40	45	55	60	65	70

Draw two line charts on the same graph to represent the above data. Assuming that the height and weight of the man to vary continuously, find by graphical interpolation the height and weight of the man when he was 9 years old.

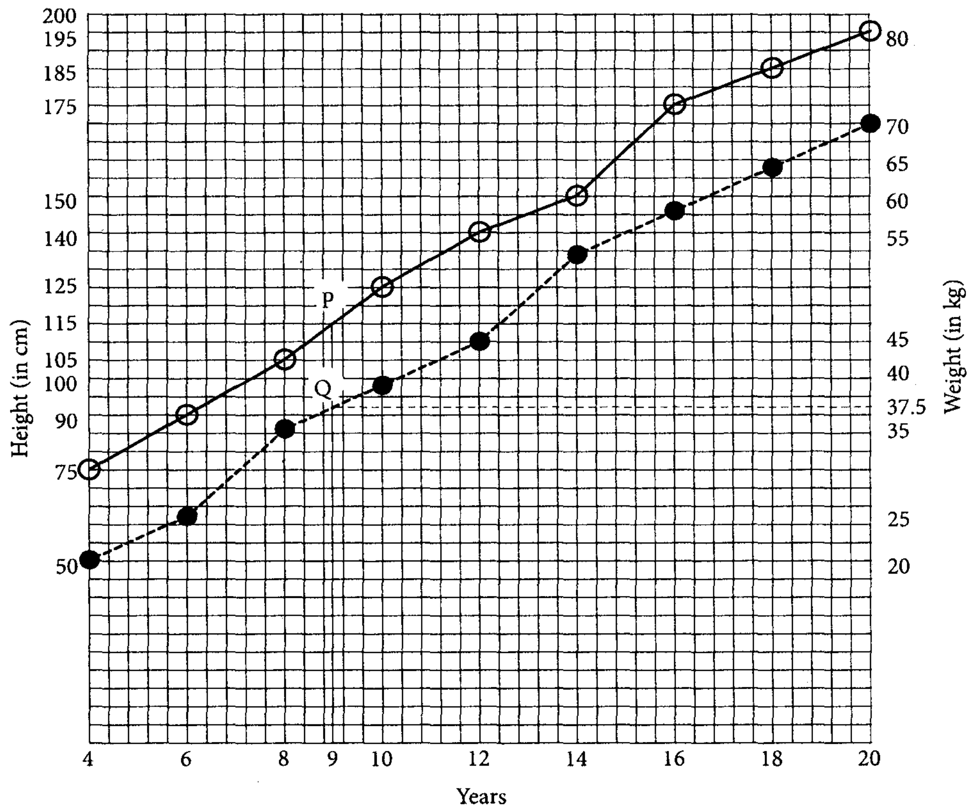


Fig. 3.8: Height and Weight of a man at different ages.

Fig. 3.8 gives the two line charts — one for the height and the other for the weight. We have represented the Ages along X-axis.

Scales: 5 sides of the squares on the graph-paper along X-axis $\equiv$ 2 years
10 sides of the square on the graph-paper along Y-axis $\equiv$ 50 cm
10 sides of the squares on the graph-paper along Y-axis $\equiv$ 20 kg

The height and the weight of the man when he was 9 years old are given by the points P and Q respectively. Now read off from the graph-paper the values corresponding to these points on the Y-axis. We see that the required height = 115 cm and the required weight = 37.5 kg.

Bar Charts or Bar Graphs

Bar Charts (or Bar Graphs) are another most frequently employed diagrammatic representation of statistical data. They are particularly useful for comparing the values of a variable classified qualitatively (i.e., classified according to some non-measurable characteristic possessed by them). They are also used in presenting statistical facts involving time-factor (i.e., in time-series).

Bar Charts consist of a number of rectangular areas usually made up with deep black ink, having the appearance of solid bars. The bars originate from a horizontal base line and their lengths are proportional to the values they represent. The bars should be of uniform width and equally spaced. The Bar Charts are called *Vertical Bar Charts* (or *Column Charts*), if the bars are placed vertically. On the other hand, we have the Horizontal Bar Charts, where the bars are placed horizontally. Column Charts (and NOT Horizontal Bar Charts) are used in time-series.

Fig. 3.9 illustrates a column chart and Fig. 3.14 illustrates a horizontal bar chart.

Simple Column Charts

In a column chart the space between consecutive columns should generally be one-half of the width of each column. These charts are generally used for depicting the values of a variable over a period of time. Such charts are also used for data classified quantitatively, e.g., the population figures classified by age-groups, the number of coal-mines classified according to the number of persons employed, etc. The values of the variable over a period of time or the frequencies, in the case of data classified according to groups, are represented by the heights of the columns.

Example 6. (i) *The monthly production of Hind Motor Cars for the first six months of the year* 2005 *is given below:*

January — 180	February — 240	March — 210
April — 250	May — 200	June — 160

Represent the production figures by Bar Charts.

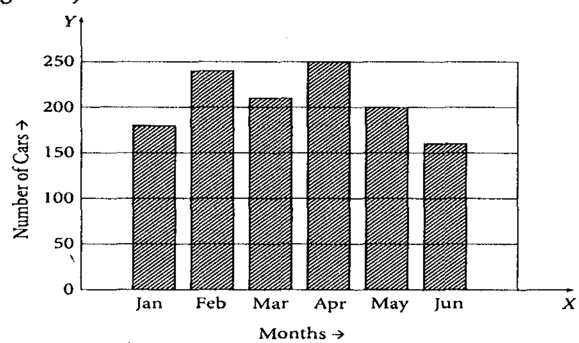


Fig. 3.9(a): Bar Chart showing the monthly production of Hind Motor Cars.

We represent 'Months' along the X-axis OX and the 'production of Cars' along the Y-axis OY. The above data have been represented by Bar Chart. [See Fig. 3.9(a)].

Scale: 1 side of a small square along the Y-axis OY represents 5 cars. [Students are advised to use graph-paper. Small square of a graph paper is not shown in Fig. 3.9(a)]

(ii) *Below is a table giving the daily average number of workers employed in the factories in India.*

TABLE 3.4: DAILY AVERAGE NUMBER OF WORKERS EMPLOYED IN THE FACTORIES IN INDIA			
Year	Number of Workers	Year	Number of Workers
1996	22,74,689	2001	25,67,458
1997	23,60,201	2002	25,28,026
1998	24,33,966	2003	25,89,757
1999	25,04,399	2004	26,90,403
2000	25,36,544	2005	28,82,309

Fig. 3.9(b) gives a simple column chart showing the number of factory workers in different years as given by the above table. The heights of the bars are proportional to the number of workers. Taking, say, 0.3″ to represent 5 lac of workers we see that in 1996 the number of workers is 22.746 lac$^+$; so we take a column of height $\frac{0.3 \times 22.746}{5} = 1.36''$ (approx.) or $(\frac{1}{5} \times 22.746 = 4.55)$, i.e., 4.5$^+$ divisions. Similarly, other columns are to be drawn.

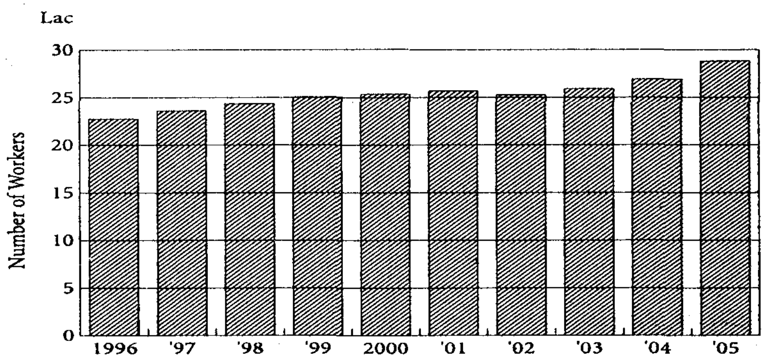


Fig. 3.9(b): Daily average number of workers employed in the factories in India.

Example 7. (i) *The profits and losses of a business concern for the years* 2001–2005 *are given below:*

TABLE 3.5		
Year	Profit (in ₹)	Loss (in ₹)
2001	3000	
2002	4000	
2003	2500	
2004		2000
2005	6000	

Represent the above data by a Bar Graph.

We have represented the above data by a column chart in Fig. 3.10(a). Taking 0.2″ of the vertical line to represent a profit or loss of ₹500 we see that in 2001 there is a profit of ₹3000. Hence, the height of the column corresponding to the year 2001 = (0.2 × 6) = 1.2″ or 6 divisions, and so on for other cases. The bar representing the loss has been shown below the zero line or base line. Such bars drawn both above and below the base line, to represent profits and losses, excesses and deficits, etc. are called *deviation bars.*

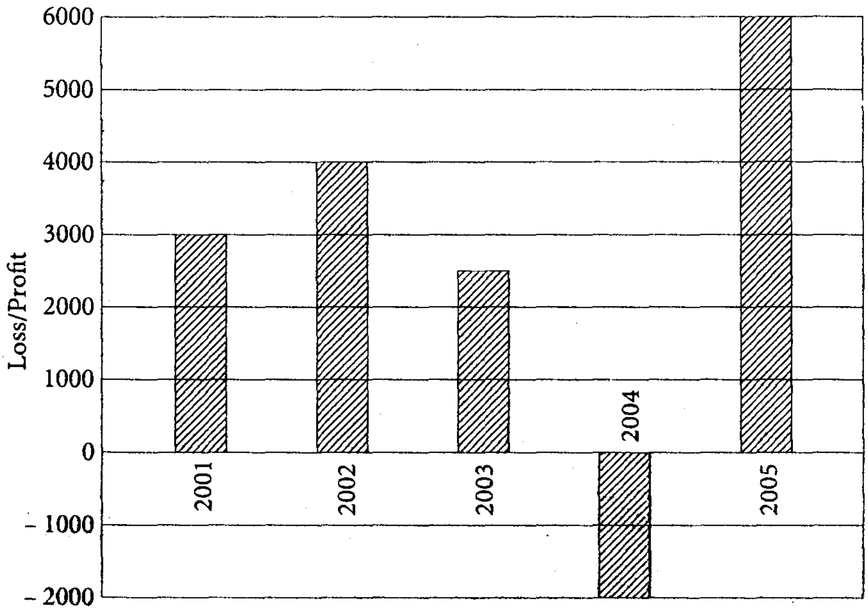


Fig. 3.10(a): Bar Chart showing the profits and losses of a business concern.

The above data can also be represented by a line chart. This is left as an exercise to the students.

(ii) *Another example of Deviation Bars (deviated from the zero line) is given by* Fig. 3.10(b). *The corresponding table is given below:*

TABLE 3.6: INDIA'S TRADE WITH A FOREIGN COUNTRY
(Value in crore of ₹)

Year	Exports	Imports	Balance	
			+	–
2001	126	149		23
2002	148	141	7	
2003	176	146	30	
2004	168	160	8	
2005	187	208		21

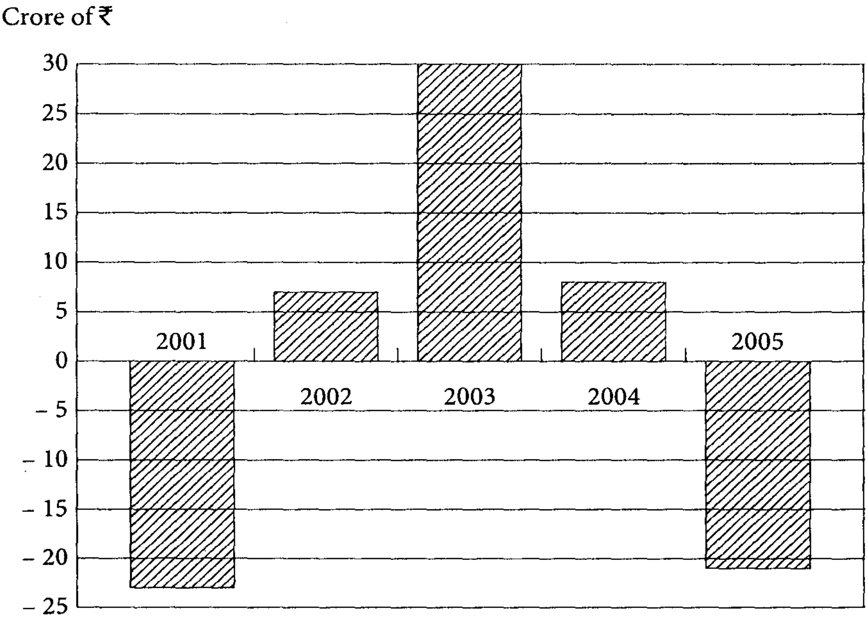


Fig. 3.10(b): Deviation Bars showing India's Trade with a foreign country.

Multiple Column Charts (Compound Column Charts)

These charts depict more than one type of data at a time. In these diagrams we may have 2, 3 or 4 or even more bars constructed at the same time side by side to represent 2, 3 or 4 series of values for comparison. For convenience the columns are differently shaded.

Example 8. *The following figures relate the value of sugar manufactured in the states of the Union of India in a certain year:*

States	Value in ₹
U.P.	57,53,87,000
Bihar	32,48,32,000
Other States	15,12,65,000

Represent the above data by a multiple column chart.

We have represented the above data by a triple column chart in Fig. 3.11.

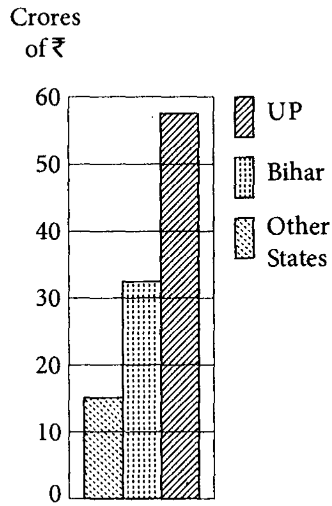


Fig. 3.11: Triple bar chart showing the manufacture of sugar in different states.

Example 9. *Fig. 3.12 is also another compound column chart depicting the data of the following table:*

TABLE 3.7: AVERAGE MONTHLY CASH EARNINGS OF TEA-PLANTATION WORKERS (SETTLED LABOURERS) IN ASSAM

Year	Monthly Cash Earning (₹ P. hundred)	
	Men	Women
1996–97	14.31	11.02
1997–98	17.54	14.60
1998–99	18.56	14.22
1999–2000	21.10	15.33
2000–01	19.80	16.90
2001–02	21.64	19.35
2002–03	20.52	18.13
2003–04	34.66	29.64
2004–05	39.45	34.41

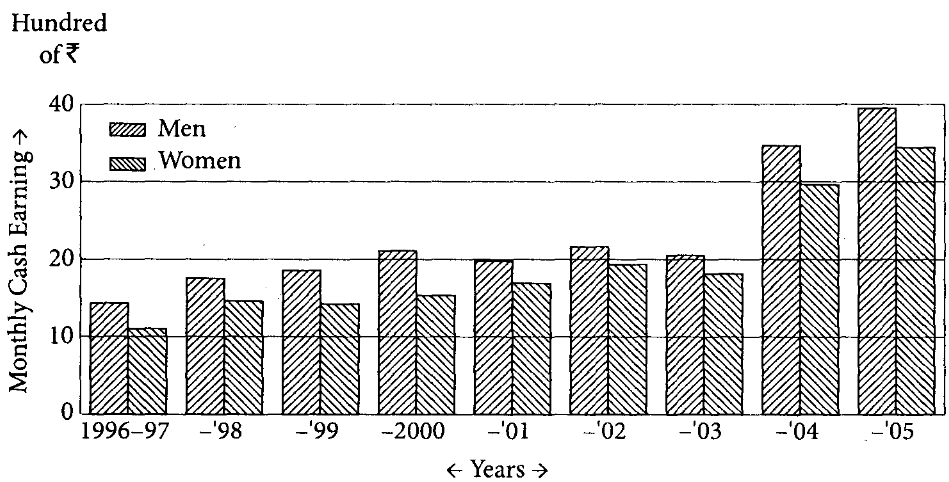


Fig. 3.12: Average monthly cash earnings of tea-plantation workers (settled labourers) in Assam.

Component Bar Charts

In these, each bar is subdivided into certain parts. Fig. 3.13(b) represents such a component bar chart. There are two bars — one of them represents the total cost and its component parts (viz., Cost of Direct Materials, Direct Labour, Direct Expenses, Overhead Charges) for the year 1999 and the other depicting the same items for the year 2000. The component parts are indicated by different hatchings and the total cost by the complete bars.

Example 10. (i) *Following are the heads of incomes of a Railway Company during* 1999 *and* 2000:

	1999 (in crore of ₹)	2000 (in crore of ₹)
Coaching	26	31
Goods	40	39
Others	4.50	3.50

Represent the above data by a Bar Chart.

We see that the given data consists of three components for two consecutive years. In order that one may make a comparison of the incomes under different heads it will be appropriate in this case to draw a component bar chart. This has been done in the adjoining diagram.

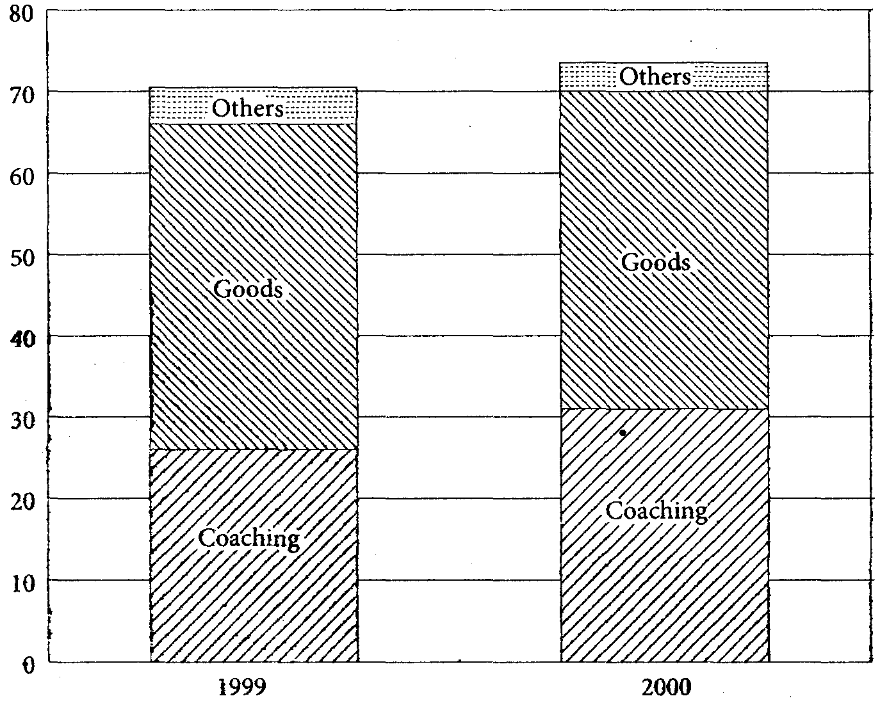


Fig. 3.13(a): Component bar chart.

(ii) *The following table gives the total cost in ₹ and its component parts in two consecutive years:*

TABLE 3.8

	1999	2000
Direct Materials	50,000	60,000
Direct Labour	55,000	70,000
Direct Expenses	15,000	18,000
Overhead Charges	25,000	32,000
Total Cost	1,45,000	1,80,000

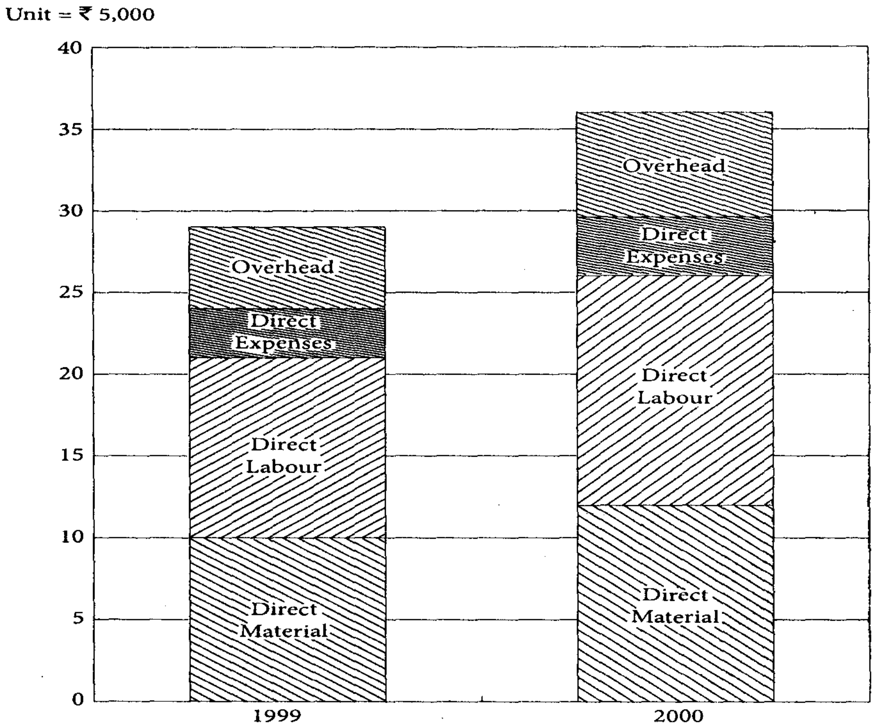


Fig. 3.13(b): Component bar chart.

Fig. 3.13(b) gives the component bar charts of the above data. Scales are clear from the figure. See that how effectively the chart visualizes a heap of figures.

Horizontal Bar Charts

Like the column charts they are also very simple to draw. The bars originate from a vertical base line (Fig. 3.14) to the left and extend to the right. The horizontal lengths of the bars representing the values of the variable are read from a few vertical scale lines.

Example 11. *The horizontal bar chart corresponding to the following table is represented by* Fig. 3.14.

TABLE 3.9: NET AREA UNDER IRRIGATION IN INDIA DURING 2004–05

Modes of Irrigation	Net Area Irrigated (thousand acres)
By Canals	
(i) Government	19,356
(ii) Private	2,863
By Tanks	9,889
By Wells	16,562
By Other Sources	5,587
Total	54,257

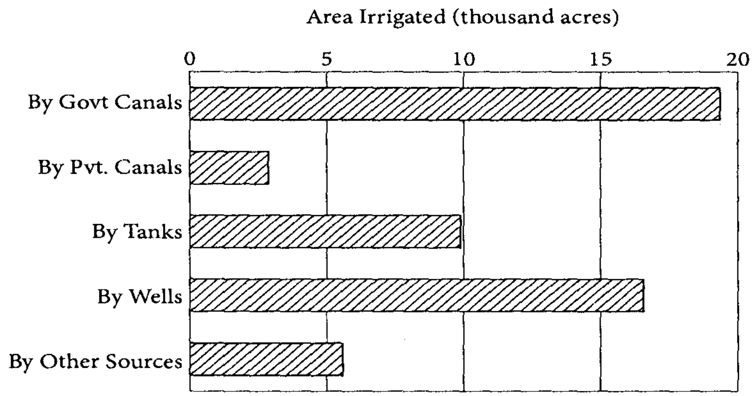


Fig. 3.14: Net Area under Irrigation in India during 2004–05 (Data of Table 3.9).

Sometimes, numerical values (Fig. 3.15) are inserted at the extreme right ends of the bars. The attributes are listed to the left of the bars.

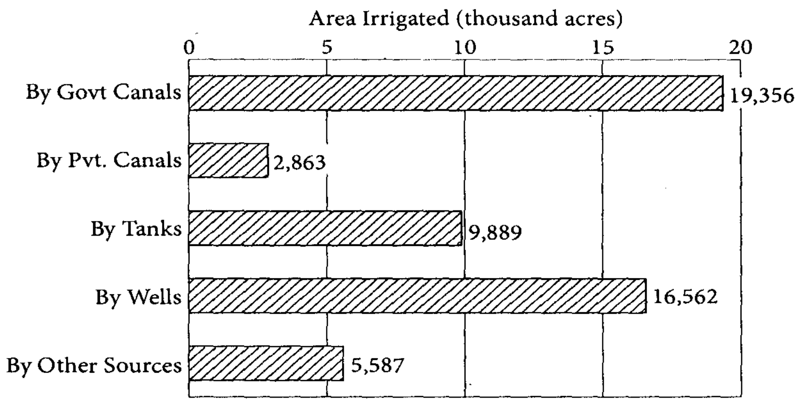


Fig. 3.15: Net Area under Irrigation in India during 2004-05 (Data of Table 3.9).

If a bar is unusually long as compared to the others, it may be broken at the end and a numerical figure indicating its value is inserted in the broken portion (Fig. 3.16).

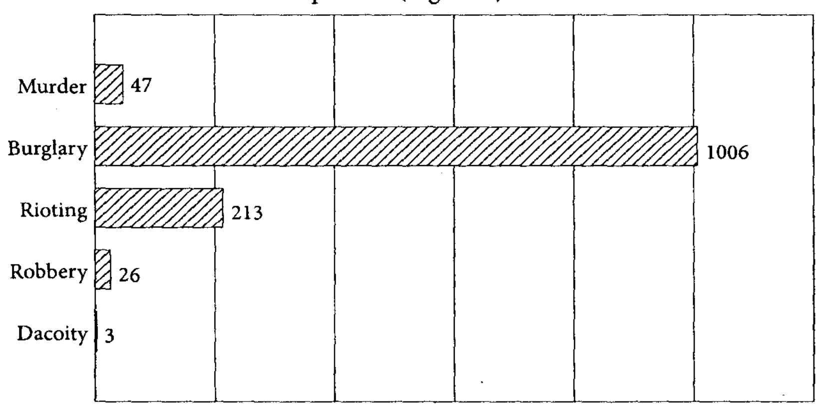


Fig. 3.16: Incidences of Crimes in Kolkata (Year 2005).

As in the case of column charts we may also have multiple horizontal bar charts (Fig. 3.17) drawn with reference to the Table 3.10 given below.

TABLE 3.10: NUMBER OF TEACHERS IN EDUCATIONAL INSTITUTIONS IN CERTAIN STATES OF INDIA (2004–05)

States of India	Number of Teachers	
	Men	Women
West Bengal	96,740	10,886
Uttar Pradesh	1,23,155	15,312
Andhra Pradesh	64,517	12,996
Bihar	78,176	5,679
Bombay	1,11,686	29,233
Madras	1,00,503	42,644
Punjab	31,718	7,240

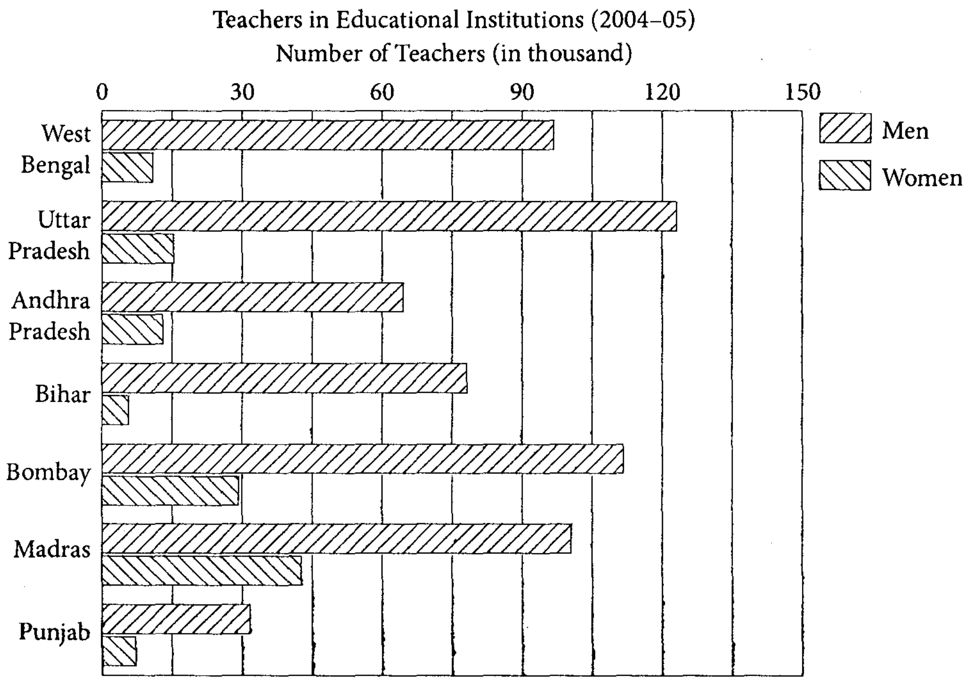


Fig. 3.17: Number of Teachers in Educational Institutions in certain States of India (Data Table 3.10).

Ratio Charts or Logarithmic Charts

The graphs previously described and illustrated give absolute changes in the values represented by the vertical axis. Take, for example, the line chart of Fig. 3.3, where productions have been shown on the vertical axis. In this figure a change in the amount of production by, say, 10,000 tonnes, is represented by the same length in the vertical scale, whether the production rises from 65,000 tonnes to 75,000 tonnes, or from 89,000 tonnes to 99,000 tonnes. Such scales, therefore, give equal lengths in the graph for equal lengths in the values of a variable. They are called **Arithmetic Scales** (or **Natural Scales**).

But sometimes we are interested to measure relative changes in the values of the dependent variable (not Absolute Changes). We clarify the idea with the help of the following example:

Suppose the production of Iron in four years are as shown below:

(1)		(2)	
2000	2001	2002	2003
65,000	75,000	89,000	99,000
tonnes	tonnes	tonnes	tonnes

In both (1) and (2) absolute changes in production = 10,000 tonnes.

But in (1) the percentage of increase = $\frac{10000}{65000} \times 100\% = 15.4\%$

and in (2) the percentage of increase = $\frac{10000}{89000} \times 100\% = 11.2\%$.

If we draw graphs in the conventional Natural Scale, we cannot visualize such relative (or percentage) changes. In order to compare such relative changes over a period of time we use a special form of graph called **Semi-Logarithmic Graph.** In this graph the horizontal axis (where usually time is represented) is marked off as in the case of our usual Arithmetic Scale. But the vertical axis should be scaled off in such a manner that equal lengths on this scale show equal ratios of change. Such a scale may be prepared by marking the divisions proportional to the logarithms of natural numbers.

Suppose we would like to prepare a semi-logarithmic graph-paper and we wish to represent the following numbers: 1, 10, 20, 30, 40, 50, 100, 1000.

Their logarithms are: 0, 1.00, 1.30, 1.47, 1.60, 1.69, 2, 3.

Thus, we may take 1 at the origin, 10 at a height 1 (unit of length) above X-axis, 20 at a height 1.30 units of length above the X-axis, and so on. It will be observed that the lines parallel to the horizontal axis will be unevenly spaced, but that the spacing will be similar at intervals 10, 100, 1000, 10,000, etc.

A complete group of such lines in each interval is known as a Cycle. Since, $\log \frac{y_2}{y_1} = \log y_2 - \log y_1$, the distance between the lines $y = y_1$ and $y = y_2$ on such a chart will really indicate the value of the ratio $\frac{y_2}{y_1}$ (not the difference $y_2 - y_1$). We usually take the top and bottom limiting lines as the powers of ten when the semi-logarithmic chart is used. It should be noted that there is no zero point on the vertical semi-logarithmic scale since log 0 is not defined.

Fig. 3.18 represents the production of Gypsum in India (as given by the table below) by using a semi-logarithmic graph-paper. We have taken 10 (thousand tons) at the origin; hence, 79 (thousand tons) should be 0.89 unit of length above the origin since log 79 = 1.89 and log 10 = 1. Similarly, other points have been plotted.

TABLE 3.11: PRODUCTION OF GYPSUM ORE IN INDIA

Year	Quantity ('000 tons)	Year	Quantity ('000 tons)	Year	Quantity ('000 tons)
1997	79	2000	204	2003	612
1998	140	2001	411	2004	618
1999	206	2002	586	2005	850

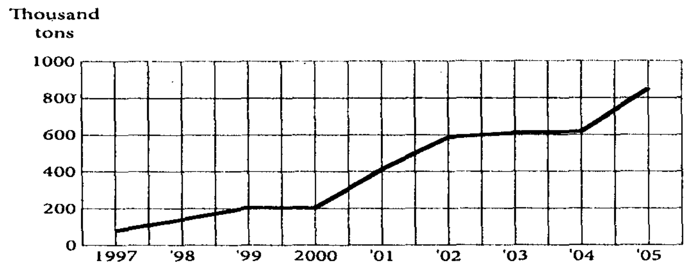


Fig. 3.18: Production of Gypsum ore in India during 1997–2005.

If both the scales (vertical and horizontal) are logarithmic, it is called *Double Logarithmic Chart.* They are seldom used in business concerns.

Advantages of Ratio Scale

- Relative changes are best represented. The slope or steepness of the lines on a Logarithmic chart indicates the rates of change. The steeper the rise or fall of these lines, the greater is the ratio of increase or decrease. The relative rate of change of two sets of figures can then be easily compared by comparing the slopes of the lines.
- Extrapolation is possible.
- When a large range of values (say, from 1000 to 20,00,000) has to be accommodated within a comparatively small space the Ratio Chart is useful.

Disadvantages of Ratio Scale

- Absolute changes cannot be easily compared on a ratio chart.
- Zero and negative values cannot be represented.
- It requires a little knowledge of logarithms.

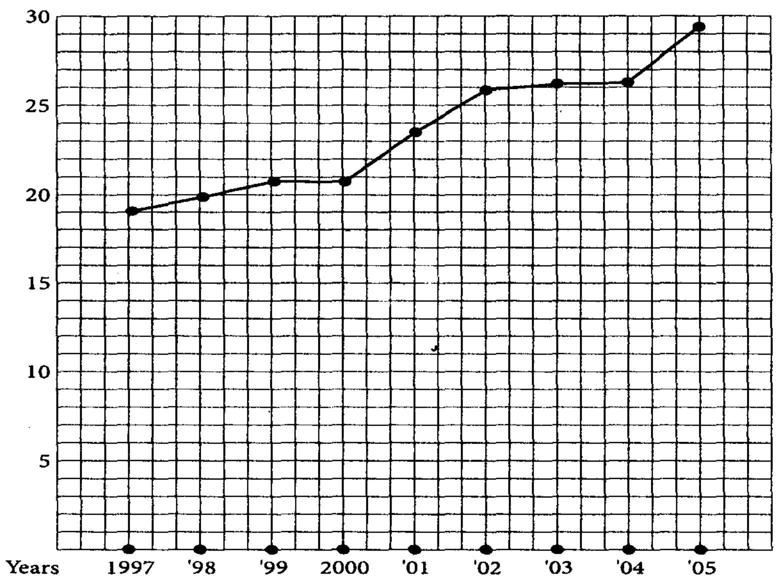


Fig. 3.19: Logarithmic Graph on a Natural Scale.

Instead of making a special graph-paper on which Y-axis is graduated according to the logarithms of numbers we may draw a logarithmic chart by plotting the values of logarithms of the values of y on an ordinary graph-paper (i.e., a graph-paper, where arithmetic scales are taken).

In Fig. 3.19 we have taken an ordinary graph-paper. On the vertical axis we have represented the logarithms of the numbers. The data are the same as in the Fig. 3.18.

Pie Charts

The Pie Chart is a very useful pictorial device for visualizing the weight of different items in a composite quantity. In fact, like component bar charts, Pie Charts can effectively display the comparison between the various components or between a part and the whole.

A Pie Chart consists of a circle subdivided into sectors by radii in such a way that the areas of the sectors are proportional to the values of the component items under investigation — the whole circle, of course, representing the whole of the data under investigation.

It should be remembered that the areas of sectors are proportional to the angles at the centre. So we may also consider a Pie Chart to consist of a circle, broken up radially into sectors, in such a way that the angles of the sectors at the centre would be proportional to the various components of the given aggregate.

In order to draw a Pie Chart, we first express the different components of the given data as percentages of the whole. Now the total angle at the centre being 360°, it will represent the whole, i.e., 100%. Therefore, 3.6° will represent 1% of the whole. Consequently, if x be the percentage of a certain component, the angle of the corresponding sector at the centre $= x \times 3.6°$. (Conversely, if a certain component is represented by a certain sector, we measure the angle subtended by it at the centre from which we can find the percentage of that component w.r.t. the whole.) In this way, the central angles of the sectors corresponding to the different components are determined.

A circle drawn with a convenient radius is then divided into different sectors with those central angles. The different sectors thus obtained are finally shaded differently.

Example 12. *Draw a Pie Chart to represent the following data relating to the production cost of manufacture: Cost of Materials:* ₹38,400; *Cost of Labour:* ₹30,720; *Direct Expenses of Manufacture:* ₹11,520; *Factory Overhead Expenses:* ₹15,360.

Solution: We first express each item as a percentage of the total cost, viz., ₹96,000.

TABLE 3.12: CALCULATIONS FOR PIE CHART

	Percentage	Central Angles
(1) Cost of Materials	$\frac{38400}{96000} \times 100\% = 40\%$	$3.6° \times 40 = 144°$
(2) Cost of Labour	$\frac{30720}{96000} \times 100\% = 32\%$	$3.6° \times 32 = 115.2°$
(3) Direct Expenses	$\frac{11520}{96000} \times 100\% = 12\%$	$3.6° \times 12 = 43.2°$
(4) Factory Overhead	$\frac{15360}{96000} \times 100\% = 16\%$	$3.6° \times 16 = 57.6°$
Total	100%	360°

A circle of convenient radius is now drawn and the above angles are marked out at the centre of the circle. 4 radii will then divide the whole circle into four required sectors.

The different sectors are generally differently shaded. Fig. 3.20 gives the Pie Chart required.

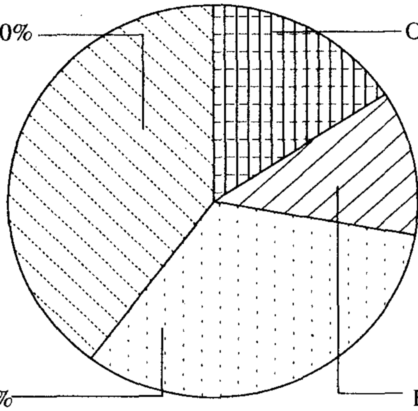


Fig. 3.20: Pie Chart

Example 13. *Represent the following data by Circular or Pie Chart: Revenue of the Government of India:*

Source	₹(in crore)
Customs	160
Excise	500
Income tax	330
Corporation tax	110
Other sources	100
Total	1,200

[V.U. B.Com. 1996; N.B.U. B.Com. 1994; C.U. B.Com. 1995]

Solution:

TABLE 3.13: CALCULATIONS FOR PIE CHART

Source	Percentage	Central Angles
Customs	$\frac{160}{1200} \times 100\% = \frac{40}{3}\% = 13.33\%$	$\frac{40}{3} \times 3.6° = 48°$
Excise	$\frac{500}{1200} \times 100\% = \frac{125}{3}\% = 41.67\%$	$\frac{125}{3} \times 3.6° = 150°$
Income tax	$\frac{330}{1200} \times 100\% = \frac{55}{2}\% = 27.5\%$	$\frac{55}{2} \times 3.6° = 99°$
Corporation tax	$\frac{110}{1200} \times 100\% = \frac{55}{6}\% = 9.17\%$	$\frac{55}{6} \times 3.6° = 33°$
Other sources	$\frac{100}{1200} \times 100\% = \frac{25}{3}\% = 8.33\%$	$\frac{25}{3} \times 3.6° = 30°$
Total	100%	360°

A circle of convenient radius is first drawn and then it is subdivided into sectors by drawing central angles 48°, 150°, 99°, 33° and 30°. The sectors are then differently shaded to obtain the required Pie Chart. Fig. 3.21 shows the Pie Chart of the given data.

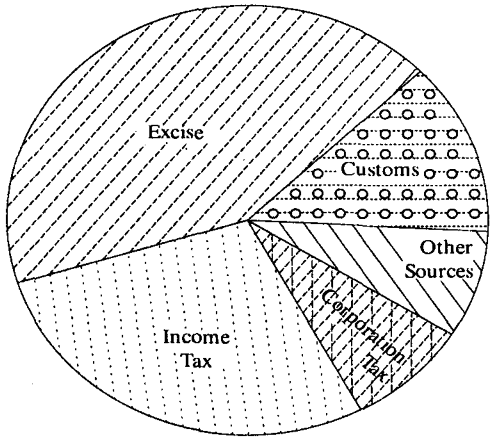


Fig. 3.21: Pie Chart showing Revenue of the Govt of India

Note: (i) The calculations of angles at the centre can be avoided if the circumference of the circle be divided into 100 equal parts.
(ii) The given data can also be represented by a component bar chart as shown in Exs 10 and 13.

Example 14. *The following is an extract of the Expenditure of the Government of India on different counts. (in lac of ₹):*

Direct Demands on Revenue	3,400
Civil Administration	14,960
Defense Services	28,560
Contribution and Grants-in-aid to States	5,440
Other Items	15,640
Total Expenditure	68,000

We propose to represent the above data by a Pie Chart. The five components of the total expenditure are respectively 5%, 22%, 42%, 8%, 23% of the whole. We have divided the circle [See Fig. 3.22(a)] into 100 equal parts and the components can then be easily shown by five sectors.

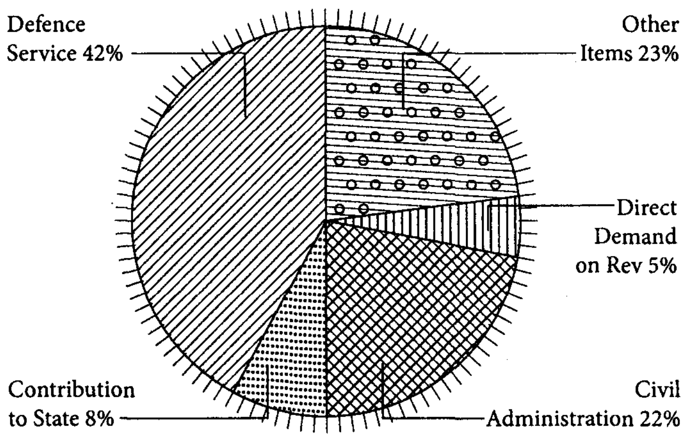


Fig. 3.22(a): Pie Chart

A graphical representation of different percentage compositions of a composite data is called a *percentage diagram*. Examples of such diagrams are: (i) Pie Charts [Fig. 3.22(a) and Fig. 3.21], (ii) Percentage Bar Charts [Fig. 3.22(b)].

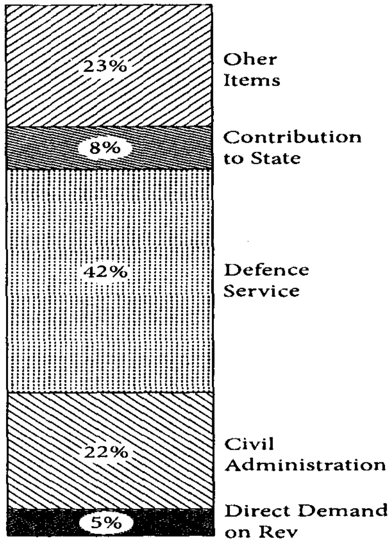


Fig. 3.22(b): Percentage Bar Chart.

Example 15. *In* Fig. 3.22(b) *we have represented the data given in example* 13, *by a percentage Bar Chart. The height of the bar has been proportionally divided to represent expenditures on different accounts. To make effective visualization we have differently shaded the components of the bar.*

Histogram, Frequency Polygon and Ogive

Histograms and Frequency Polygons

Histograms are the most commonly used diagrams to depict the frequency distributions of both continuous and discontinuous types, graphically.

A Histogram comprises a series of continuous vertical rectangles. The values of the variable are plotted along the horizontal scale and the frequencies along the vertical scale. The width of each rectangle extends over the boundaries of each class, plotted along the horizontal axis and a vertical rectangle is erected over each of these class-intervals, the area of the rectangle being proportional to the frequency in the corresponding class. The total area under the Histogram is, thus, proportional to the total frequency, i.e., to the total number of observations.

In order that the area of a rectangle drawn may be proportional to the frequency of the corresponding class, the heights of the rectangle should be proportional to the frequency density of the class.

It may be noted that

$$\text{Frequency Density} = \frac{\text{Class Frequency}}{\text{Width of the Class-interval}}.$$

In particular, if the class-intervals are all of the equal length, the heights of the rectangles may be taken as proportional to the class-frequencies, so that the heights, then, represent diagrammatically the corresponding class-frequencies. Evidently, in doing so, the heights are taken, in fact, proportional to the frequency densities. In case the class-intervals are of unequal lengths, i.e., they are either all unequal or some are equal and some unequal, the heights of the rectangles should be taken proportional to the corresponding frequency-densities and never to the frequencies.

Example 16. *We consider the following frequency distribution:*

TABLE 3.14: MONTHLY INCOME OF 738 FAMILIES	
Monthly Income (in ₹)	**Number of Families**
0–50	80
50–100	210
100–150	170
150–200	120
200–250	65
250–300	48
300–350	30
350–400	15
Total	738

The Histogram corresponding to the above frequency distribution has been shown in Fig. 3.23. Here the class-intervals are equal. See that the heights of the rectangles do represent the corresponding class-frequencies.

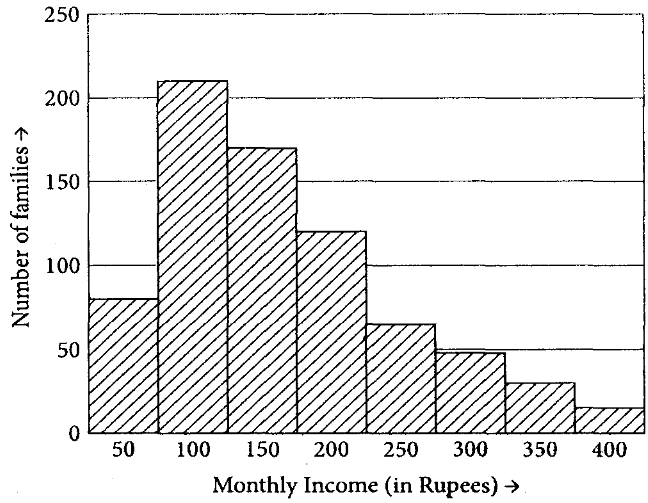


Fig. 3.23: Histogram representing the frequency distribution of 738 families with regard to their monthly income.

Fig. 3.24(a) represents the Histogram of a discrete frequency distribution, viz., the weekly wages of 430 workers of a factory. Fig. 3.24(b) represents, on the other hand, the Histogram of a continuous frequency distribution, viz., the heights of 200 students. The tables for the two figures are also given below. Scales used may be read off from the figures.

It should be noted that if there be gaps in a grouped frequency distribution, those gaps are to be removed by taking the class-boundaries, before going to construct the corresponding Histogram, and in these cases, where the gaps appear in a grouped frequency distribution, class-boundaries and not the class limits are to be plotted along the horizontal axis.

Example 17. *We consider the following frequency distribution:*

TABLE 3.15: DISTRIBUTION OF WEEKLY WAGES OF 430 WORKERS

Weekly Wages (in ₹)	Number of Workers, i.e., Frequency
20–30	30
30–40	50
40–50	80
50–60	100
60–70	70
70–80	60
80–90	40
Total	430

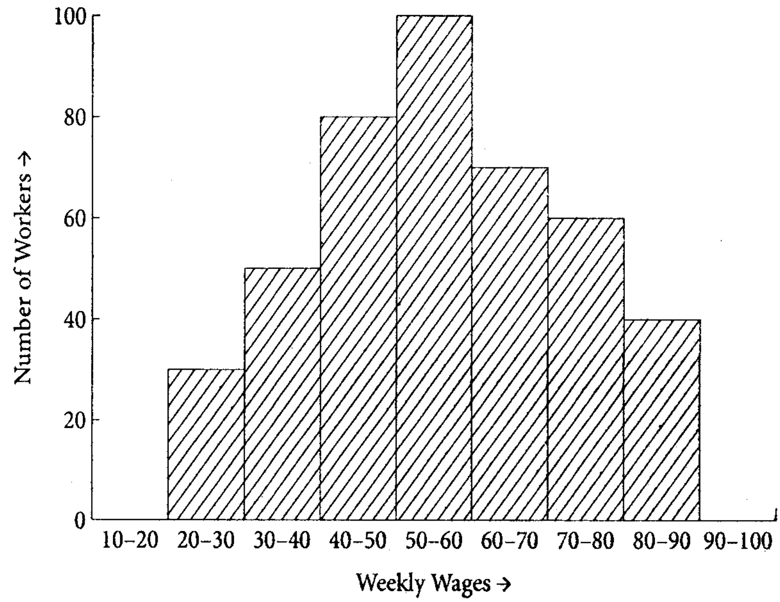


Fig. 3.24(a): Histogram of weekly wages of 430 workers.

Example 18.

TABLE 3.16: FREQUENCY DISTRIBUTION OF HEIGHTS OF STUDENTS

Class-intervals (cm)	Class-boundaries	Frequency
126–130	125.5–130.5	2
131–135	130.5–135.5	9
136–140	135.5–140.5	16
141–145	140.5–145.5	26
146–150	145.5–150.5	33
151–155	150.5–155.5	41
156–160	155.5–160.5	36
161–165	160.5–165.5	21
166–170	165.5–170.5	11
171–175	170.5–175.5	3
176–180	175.5–180.5	2
Total	—	200

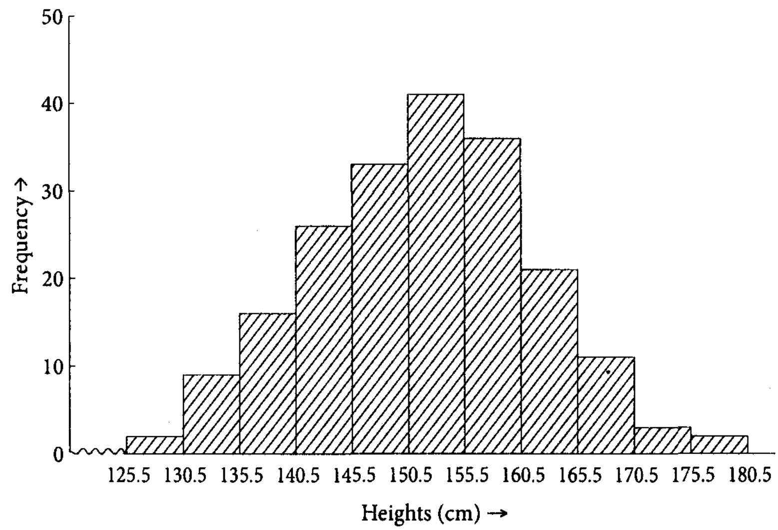


Fig. 3.24(b): Histogram of the heights of 200 students.

Example 19. *Draw the Histogram of the following frequency distribution and use it to find the total number of wage earners in the age-group* 22–38 *years*:

Age-group	14–16	17–19	20–24	25–29	30–35	36–40	41–44
No. of Wage Earners	45	60	140	100	90	50	20

[C.U. B.Com. 2002 Type]

Solution: Since the class limits (which are not class-boundaries in this case) are given and the class-intervals are not all of equal widths, we first find the class-boundaries and the frequency-densities of the corresponding classes.

The class-boundaries are 13.5–16.5, 16.5–19.5, 19.5–24.5, 24.5–29.5, 29.5–35.5, 35.5–40.5, 40.5–44.5.

The frequency densities are

$$\frac{45}{3}, \frac{60}{3}, \frac{140}{5}, \frac{100}{5}, \frac{90}{6}, \frac{50}{5}, \frac{20}{4}, \quad \text{i.e.,} \quad 15, 20, 28, 20, 15, 10, 5.$$

The histogram of the given data has been drawn in the following figure:

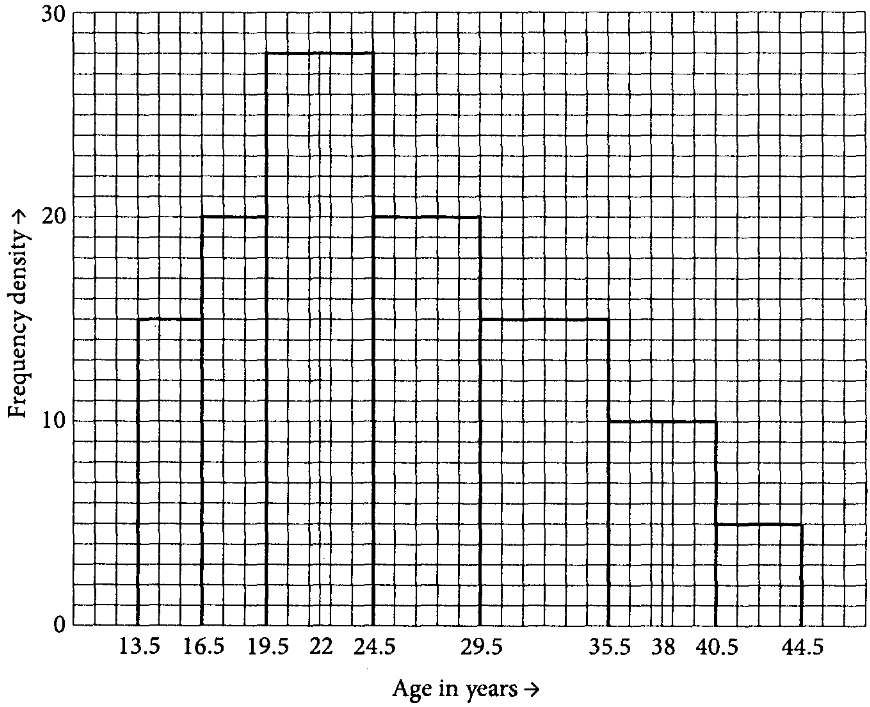


Fig. 3.25: Histogram of age-groups of 505 wage earners.

To find the total number of wage earners in the age-group 22–28 years, we notice that 22 is the midpoint of 19.--24.5 and 38 is the midpoint of 35.5–40.5.

Hence, the required number of wage earners in the age-group 22–38

$$= \tfrac{1}{2} \times 140 + 100 + 90 + \tfrac{1}{2} \times 50 = 285.$$

Frequency Polygon: In the case of an ungrouped frequency distribution of a discrete variable, the values of the variable and the corresponding frequencies are plotted respectively along the horizontal and the vertical axis, on suitable scales on an ordinary graph-paper. The points so obtained are joined by broken lines and the polygon is completed by joining the two extremities of the rectilinear figure so obtained, usually to the two points on the horizontal axis corresponding to the values of the variable, just after and before the last and the first one respectively, the two values having clearly zero frequency each. The polygon so constructed is called a *frequency polygon.*

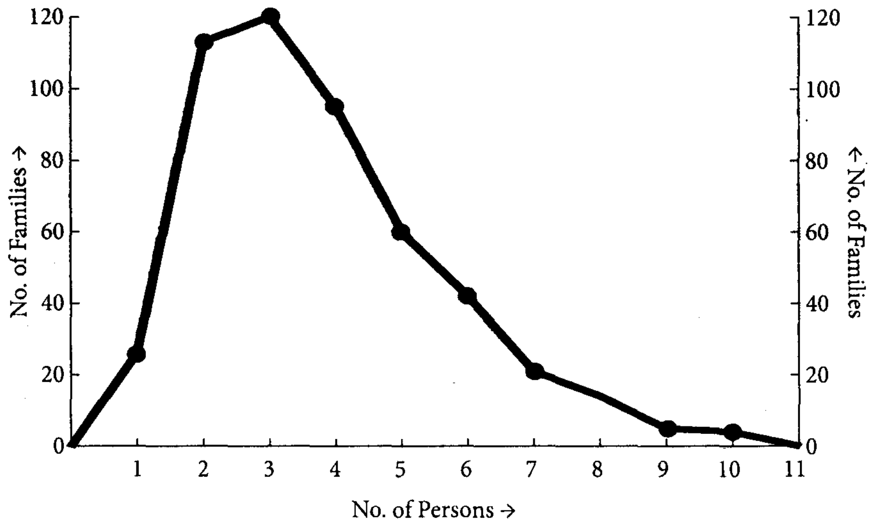


Fig. 3.26: Frequency Polygon for the data given below.

In the case of a grouped frequency distribution, a frequency polygon is a line chart of class-frequencies plotted against midpoints (or class marks) of the corresponding class-intervals. When all the classes have a common width, it is usually constructed from histogram of the grouped frequency distribution by connecting midpoints of the top of the consecutive rectangles. It is customary to join the two end-points of the polygon so obtained to the next lower and higher class marks having zero frequency each.

Fig. 3.26 is the frequency polygon for the data of an ungrouped frequency distribution of a discrete variable, viz., number of persons per family given below:

Number of Persons per Family	1	2	3	4	5	6	7	8	9	10
Number of Families	26	113	120	95	60	42	21	14	5	4

The polygon has been completed by joining its two ends with the two points on the horizontal axis, one representing the value of the variable just before the first one, viz., 0, and the other representing the value of the variable just after the last one, viz., 11.

Example 20. *Construct the frequency polygon for the following frequency distribution:*

Weights (in kg)	41–45	46–50	51–55	56–60	61–65	66–70	71–75	76–80
No. of Men	4	5	9	6	11	5	7	3

Solution: In the given frequency distribution, the class-boundaries of the corresponding class-intervals are 40.5–45.5, 45.5–50.5, ..., 75.5–80.5. The next lower and higher class-intervals having zero frequency each are 36–40 and 81–85, and the corresponding class-boundaries are 35.5–40.5 and 80.5–85.5.

To construct the frequency polygon, we have to draw a Line Chart of class-frequencies 0, 4, 5, 9, 6, 11, 5, 7, 3, 0 plotted against the midpoints of the corresponding class-boundaries (or class-intervals). The required frequency polygon for the given data has been drawn in the adjoining Fig. 3.27.

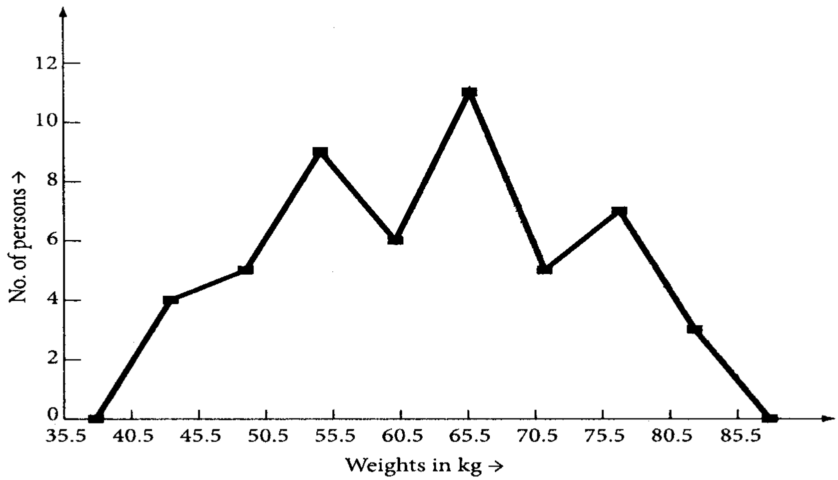


Fig. 3.27: Frequency Polygon for the data given above.

Cumulative Frequency Polygon and Ogive

If the cumulative frequencies are plotted against the class-boundaries and the successive points are joined by straight lines we get what is known as a *Cumulative Frequency Polygon.* There are two types of these polygons:

- In one type, the cumulative frequencies from below are plotted against the upper class-boundaries; and
- In the second type, the cumulative frequencies from above are plotted against the corresponding lower class-boundaries.

The former is known as *less than cumulative frequency polygon,* because the ordinate of any point on the curve indicates the frequency of all values less than or equal to the corresponding value of the variable represented by the abscissa of the point. Similarly, the second type is known as the *more than Cumulative frequency polygon.*

TABLE 3.17: FREQUENCY DISTRIBUTION OF HEIGHTS OF STUDENTS

Class-intervals (cm)	Class-boundaries	Frequency	Cumulative Frequency	
			From below	From above
126–130	125.5–130.5	2	2	200
131–135	130.5–135.5	9	11	198
136–140	135.5–140.5	16	27	189
141–145	140.5–145.5	26	53	173
146–150	145.5–150.5	33	86	147
151–155	150.5–155.5	41	127	114
156–160	155.5–160.5	36	163	73
161–165	160.5–165.5	21	184	37
166–170	165.5–170.5	11	195	16
171–175	170.5–175.5	3	198	5
176–180	175.5–180.5	2	200	2
Total	—	200	...	...

The two types of cumulative frequency polygons, constructed from the data of **Table 3.17**, are shown in Fig. 3.28.

It will be noticed from Fig. 3.28 that the two polygons cut at a point whose ordinate is 100, i.e., half the total frequency, and the abscissa is 152.21 cm, which is the median of the distribution. Even when only one polygon is drawn the median can be determined by locating the abscissa of the point on the polygon whose cumulative frequency is $N/2$. Similarly, the abscissae of the points of the *Less than Polygon* corresponding to cumulative frequencies $N/4$ and $3N/4$ give the quartiles Q_1 and Q_3 respectively.

Just as a frequency curve is obtained by smoothing a histogram, the result of smoothing a cumulative frequency polygon is a curve, known as *Ogive*. The name ogive is sometimes given to the cumulative frequency polygons and the two are considered as identical.

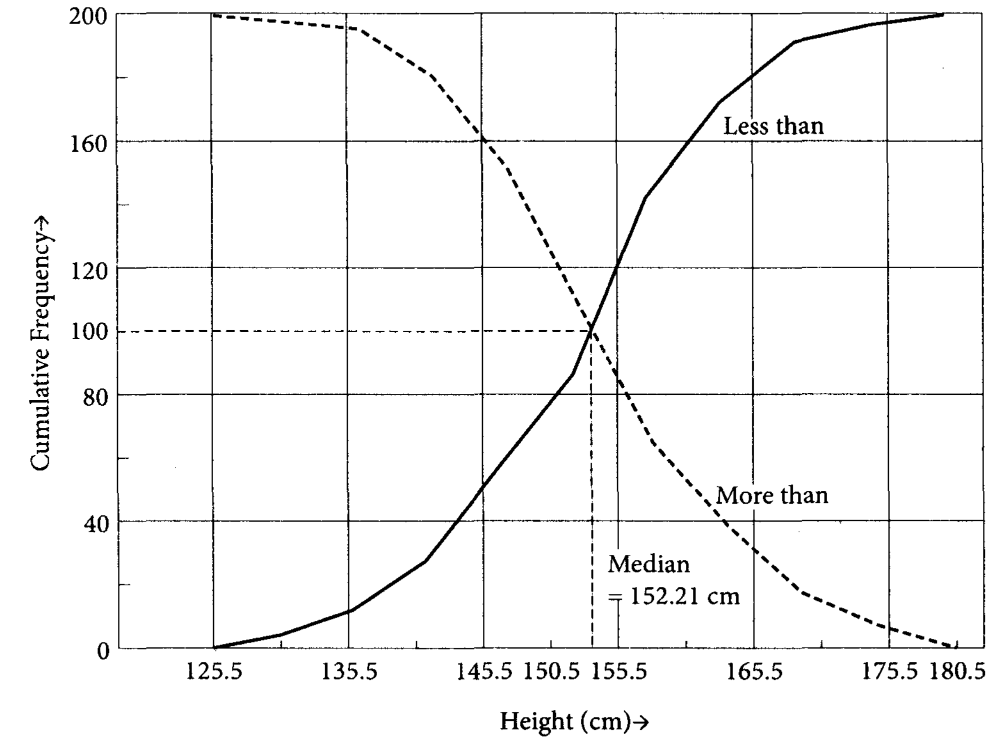


Fig. 3.28: Cumulative frequency polygons of heights of students.

Example 21. *Draw ogives from the following frequency distribution and hence compute the median of the distribution:*

Profit (₹lac)	50–99	100–149	150–199	200–249	250–299	300–349
No. of Companies	7	12	18	27	20	19

[C.U. B.Com. 2007]

Solution: The given frequency distribution is *discontinuous*. We convert it into a continuous distribution by considering class-boundaries as follows:

TABLE 3.18: CUMULATIVE FREQUENCY DISTRIBUTION			
Class-interval	Class-boundaries	Cumulative Frequency (c.f.)	
		Less than	*More than*
50–99	49.5	0	103
100–149	99.5	7	96
150–199	149.5	19	84
200–249	199.5	37	66
250–299	249.5	64	39
300–349	299.5	84	19
	349.5	103	0

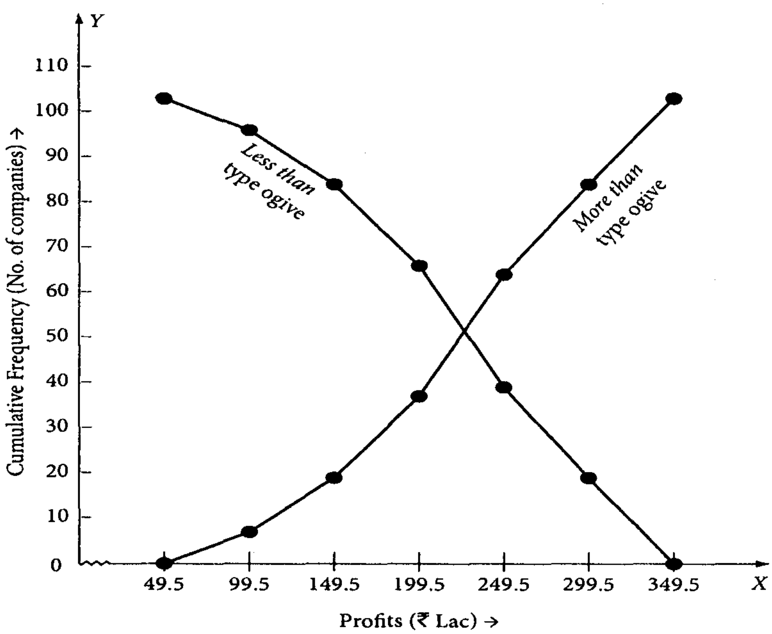


Fig. 3.29: *Less than* and *More than* type ogives.

The *cumulative frequencies* (both *less than* and *more than* types) are plotted on a graph-paper against the *corresponding class-boundaries,* taking profit (in ₹lac), i.e., class-boundaries along the horizontal axis *OX* and no. of companies (i.e., c.f.) along the vertical axis *OY*. The two ogives (*less than* and *more than* types) are drawn in Fig. 3.29.

Example 22. *Draw a suitable diagram from the following data:*

Year	Sales ('000 ₹)	Gross Profit ('000 ₹)	Net Profit ('000 ₹)
2002	120	40	20
2003	135	45	30
2004	140	55	35
2005	150	60	40

Solution: To show Sales, Gross Profit and Net Profit in each of the years 2002–2005, it will be suitable to draw multiple bar diagrams for each year. We represent years along the horizontal line OX and Sales, Gross Profit and Net Profit along the vertical line OY in the same scale.

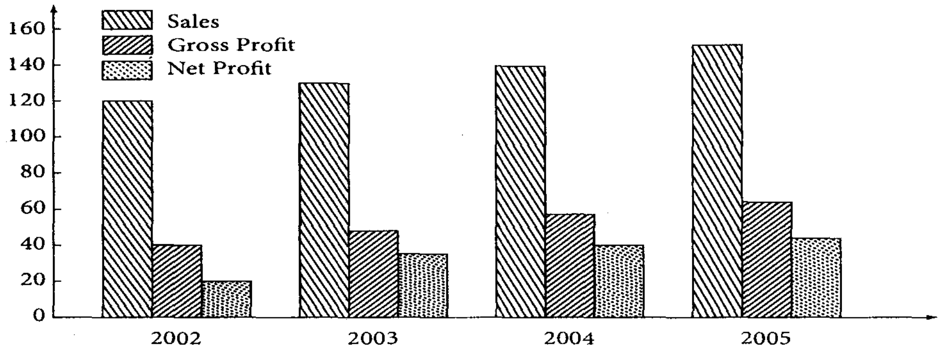


Fig. 3.30: Multiple bar diagram showing Sales, Gross Profit and Net Profit in the years 2002–2005.

Example 23. *The budget of two families are given below. Represent the data by percentage rectangular diagram:*

Items of Expenditure	Family A	Family B
Food	1600	1200
Clothing	800	320
Rent.	600	480
Power and Fuel	200	160
Miscellaneous	800	240

[D.U. B.Com. 1991]

Solution: We first convert the given data (figures) into percentages of the total for both Family A and Family B, and then draw the rectangular diagram (the component bar charts). As total percentage is 100 for both the families A and B, we take widths of the two rectangles in the ratio 4000 : 2400 = 5: 3.

TABLE 3.19: PERCENTAGE CALCULATIONS

Items of Expenditure	Family A		Family B	
	₹	Percentage	₹	Percentage
Food	1600	$\frac{1600}{4000} \times 100 = 40$	1200	$\frac{1200}{2400} \times 100 = 50.00$
Clothing	800	$\frac{800}{4000} \times 100 = 20$	320	$\frac{320}{2400} \times 100 = 13.33$
Rent	600	$\frac{600}{4000} \times 100 = 15$	480	$\frac{480}{2400} \times 100 = 20.00$
Power and Fuel	200	$\frac{200}{4000} \times 100 = 5$	160	$\frac{160}{2400} \times 100 = 6.67$
Miscellaneous	800	$\frac{800}{4000} \times 100 = 20$	240	$\frac{240}{2400} \times 100 = 10.00$
Total	4000	100	2400	100

We represent Family *A* and Family *B* along the horizontal line *OX* and expenditure on different items along the vertical line *OY*. Percentage rectangular diagram showing the budgets of two families *A* and *B* is drawn in Fig. 3.31.

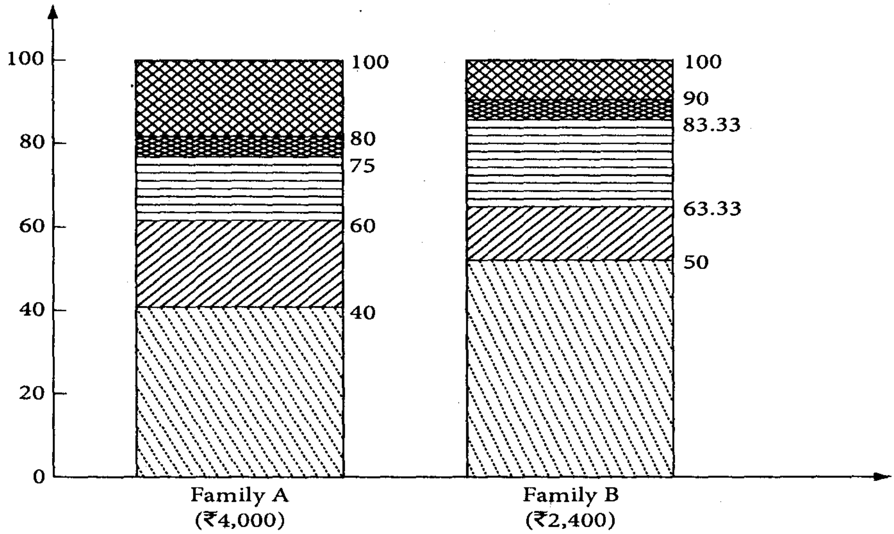


Fig. 3.31: Percentage rectangular diagram showing the budgets of two families A and B.

EXERCISES ON CHAPTER 3

Theory

1. What is graphical representation? Discuss its advantages.
2. How is a set of quantitative data presented by graphs and charts? Name at least four such graphical representations with examples.
3. Define and illustrate: Histogram, Pie Chart, Bar Chart, Line Chart.
4. What do you understand by Bar Chart? How is it drawn?
5. Define a Histogram and describe how it is constructed.
6. (a) What is a Pie Chart? [V.U. B.Com.(H) 2008]
   (b) Discuss the types of data which are usually represented by Pie Diagram. State how they are drawn.
7. Explain the following with illustrations:
   (a) Pie Chart, (b) Column Chart, (c) Logarithmic Graphs.
8. Explain briefly what is meant by Bar Charts, Pie Charts and Histograms, and describe briefly how they are useful in business analysis. Illustrate also any one of these methods of graphical representation. [C.U. B.Com.(H) 1991]
9. Write the advantages and disadvantages of diagrammatic representation of data. [C.U. B.Com. 2006]
10. What is ogive? [V.U. B.Com.(H) 2008]

Problems

1. (a) The monthly productions of Hind Motor Cars for the first six months of the year 1977 are given below:

   January — 150, February — 200, March — 180, April — 230, May — 210, June — 140.

   Represent the production figures by a Line Chart.

   (b) The monthly productions of Maruti Udyog Limited for the first six months of the year 1985 are given below:

   January — 250, February — 300, March — 340, April — 320, May — 270, June — 240.

   Represent the production figures by a Bar Chart.

2. Draw a Bar Chart for the number of students of a college:

   B.Com. 1st year class — 600, B.Com. 2nd year class — 500, B.Com. 3rd year class — 350.

3. Draw a Pie Chart to represent the following data relating to the production cost of a manufacturer:

   Cost of material — ₹20,000, Cost of labour — ₹18,500, Other expenses — ₹11,500.

4. Draw a Histogram to represent the following frequency distribution:

   (a)

Daily Wages (in ₹)	Number of Workers
5–10	15
10–15	25
15–20	30
20–25	20
25–30	10

   (b)

Income (in ₹):	11–15	16–20	21–25	26–30	31–35	36–40
Frequency:	4	13	22	33	12	6

5. (a) Construct a Frequency Polygon from the following table:

Weight in kg	No. of Students
30–40	400
40–50	500
50–60	700
60–70	300
70–80	100

   (b) A frequency distribution of marks obtained by 200 students in a competitive examination is given below:

Marks	20–30	30–40	40–50	50–60	60–70	70–80	Total
Number of Students	10	40	75	20	25	30	200

   Exhibit the distribution by a Histogram and hence, draw the Frequency Polygon.

6. The monthly production of scooters in India is as given below:

January — 550	February — 500	March — 625	April — 525
May — 650	June — 600	July — 475	August — 575
September — 425	October — 400	November — 700	December — 625

Represent the above data by Bar Graph.

7. (a) The daily profits and losses of a business concern for the first 10 days of a month are given in the following table:

Days of the Month	1	2	3	4	5	6	7	8	9	10
Profit in ₹	400	800	600	900	200	200			800	1000
Loss in ₹							400	100		

Represent the above data by a Line Chart, choosing suitable scales.

(b) The daily selling price of gold in India (each 10 gm) from the 10th to 19th October in the year 2007 were as follows:

Days of October	10	11	12	13	14	15	16	17	18	19
Prices (in ₹)	8600	8750	9200	9600	9600	10200	9500	8800	9750	10000

Draw a Line Chart to represent the above data.

8. The monthly production of cycle in India is as under:

January — 5720	February — 4900	March — 6110	April — 5930
May — 6040	June — 4610	July — 3060	August — 4700
September — 5605	October — 3275	November — 6850	December — 6130

Represent the above data by (a) a Line Chart and (b) a Bar Graph.

9. Draw a Bar Chart for the number of students of a college:

Pre-University Class	200
B.Com. 1st-year Class	1,000
B.Com. 2nd-year Class	600
B.Com. 3rd-year Class	400

10. Prepare a Bar Chart from the following data: India's Foreign Debt as at 31st March, 2006:

Source of Borrowing	Amount of Loan in crore (₹)
Federal Republic of Germany	350
Japan	160
United Kingdom	480
U.S.A.	1200
I.B.R.D. (International Bank)	260
I.D.A.	450

11. Draw the graph of the following:

Year	1998	1999	2000	2001	2002	2003	2004	2005
Yield (in million tonnes)	12.8	13.9	12.8	13.9	13.4	6.5	2.9	14.8

[**Hints**: Draw a Line Chart.]

12. Draw a Pie Chart from the following data:

(a) Revenue of the Central Govt.

Customs	160 crore
Excise	500 "
Income tax	330 "
Corporation tax	110 "
Other sources	100 "
Total	1200 crore

[C.U. B.Com. 1995; N.B.U. B.Com. 1994; V.U. B.Com.(H) 2008]

(b) Revenue of the Central Govt.

Customs	160 crore
Excise	450 "
Income tax	380 "
Corporation tax	110 "
Other sources	200 "
Total	1300 crore

[C.U. B.Com. 2001; V.U. B.Com.(H) 2008]

(c) Construct a Pie Chart for the following data:
Principal Exporting Countries of cotton ('000 bales) 1985–86:

USA	India	Egypt	Brazil	Argentina
6367	2999	1688	650	202

[V.U. B.Com.(H) 2011]

13. (a) Draw a Pie Chart to represent the following data relating to the production cost of a manufacturer:

Cost of materials	₹18,360
Cost of labour	₹16,524
Direct expenses	₹ 3,672
Overhead	₹ 7,344

(b) Draw a Pie Chart from the following data:
Revenue earning of the Central Government:

Excise duty	₹3500 crore
Income tax	₹2500 "
Customs duty	₹1500 "
Corporation tax	₹1000 "
Other sources	₹1500 "

[C.U. B.Com. 1999]

(c) Draw a Pie Chart to represent the following data:

Revenue of a Government

Income tax	₹560 crore
Sales tax	₹420 "
Customs	₹170 "
Excise	₹150 "
Others	₹200 "

[C.U. B.Com. 2005]

14. (a) The cost of production of a certain commodity in a factory is given below. Represent the data in a Pie Diagram:

Items	(in thousand of ₹)	Items	(in thousand of ₹)
Raw-material	555	Packing	30
Labour	730	Factory Overhead	56
Transport	150	Advertisement	29

(b) Represent the following data by a circular or Pie Chart:

Item	Food	Clothing	Fuel	Rent	Education	Misc.	Total
Expenses	3400	960	200	780	1500	1080	7920

15. From the following table, draw a Ratio Chart on a graph-paper:

Year	1997	1998	1999	2000	2001	2002	2003	2004
Units Produced	2	4	8	16	32	64	128	256

16. (a) Draw a Pie Chart to represent the following data on the proposed outlay of the Fifth Five-Year Plan:

Fifth Five-Year Plan	₹(in crore)
Agriculture	12,000
Irrigation and Power	5,000
Industries and Minerals	8,000
Education	9,000
Roads and Communication	6,000
Total outlay	40,000

[C.U. B.Com. 1997 Type]

(b) Draw a Pie Chart to represent the following data on the proposed outlay during a Five-Year Plan of a Government:

Fifth Five-Year Plan	₹(in crore)
Agriculture	8000
Irrigation and Power	7000
Industries and Minerals	4000
Education	5500
Roads and Communication	2500

[V.U. B.Com. 1994]

(c) Represent the following data by a Pie Chart:

Items	Expenditure of a Govt. in 1995–96 ₹(in crore)
Agriculture	8000
Industries and Minerals	7000
Irrigation and Powers	4000
Communications	5500
Miscellaneous	2500

[C.U. B.Com. 2003]

17. (a) The total costs and its components in two different months are given below:

Items	January	February
Direct Material	700	600
Direct Labour	800	700
Direct Expenses	100	70
Overhead	200	300
Total	1,800	1,670

Draw component Bar Charts showing the total cost and its components.

(b) In an examination 70% passed in English, 75% passed in Mathematics and 50% passed in both the subjects while 40 students failed in both the subjects. Draw a Pie Chart to represent the number of students,

(a) who passed in both,
(b) who passed in English only,
(c) who passed in Mathematics only, and
(d) who failed in both. [N.B.U. B.Com. 1992]

[**Hints**: Calculations for Pie Chart are shown below:]

Item	Percentage	Central Angles
Passed in both subjects	50%	$50 \times 3.6° = 180°$
Passed only in English	$70 - 50 = 20\%$	$20 \times 3.6° = 72°$
Passed only in Mathematics	$75 - 50 = 25\%$	$25 \times 3.6° = 90°$
Failed in both subjects	$100 - (50 + 20 + 25) = 5\%$	$5 \times 3.6° = 18°$
Total	100%	360°

18. Draw a Histogram to represent graphically the following Frequency Distribution:

Frequency Distribution of Income of 2775 workers

Income in ₹	Frequency of Workers
70–80	375
80–90	400
90–100	650
100–110	575
110–120	350
120–130	425
Total	2775

19. (a) Draw a Histogram to represent the following Frequency Distribution:

Frequency Distribution of the weights of 2,000 students

Weight in lb	Number of Students
90–100	500
100–110	700
110–120	300
120–130	400
130–140	100
Total	2,000

(b) Draw a Histogram of the following distribution:

Age (in years)	20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59
No. of Workers	50	70	100	180	150	120	70	59

[C.U. B.Com. 2000]

20. (a) Draw a Histogram to represent the following frequency distribution:

Output (units per worker)	500-510	510-520	520-530	530-540	540-550	550-560	560-570	570-580
Number of Workers	8	18	23	37	47	26	16	5

(b) The weekly wages of 430 workers are given below:

Weekly Wages (₹)	20–30	30–40	40–50	50–60	60–70	70–80	80–90
Number of Workers	30	50	80	100	70	60	40

Draw the Histogram and Frequency Polygon for the distribution.

21. (a) Draw a Histogram representing the following frequency distribution of 1000 students by weight: Frequency Distribution of the Weights of 1000 Students.

Weight in lb	Number of Students
80 and under 90	85
90 and under 100	300
100 and under 110	215
110 and under 120	150
120 and under 130	50
130 and under 140	200
Total	1000

(b) Draw the Histogram of the following distribution and use it to find the total number of Wage Earners in the age-group 19–32 years:

Age-group	14–15	16–17	18–20	21–24	25–29	30–34	35–39
No. of Wage Earners	60	140	150	110	110	100	90

[**Hints**: Find frequency densities and then proceed as in worked-out Ex. 19.]

22. Draw the Histogram to represent the following frequency distribution:

(a)

Weekly Wages (in ₹)	300–350	350–400	400–450	450–500	500–550	550–600	600–650	650–700
No. of Men	115	340	220	175	80	35	20	15

[C.U. B.Com. 1994]

(b)

Income (in ₹)	101–110	111–120	121–130	131–140	141–150	151–160	161–170
No. of Workers	225	375	500	725	650	350	175

[N.B.U. B.Com. 1996]

23. Draw (a) Histogram and (b) Cumulative Frequency Polygons (both Less than and More than) from the following distribution:

Weekly Wages Distribution of 250 Workers	
Weekly Wages (in ₹)	**Number of Workers**
30–31	2
32–33	9
34–35	25
36–37	30
38–39	49
40–41	62
42–43	39
44–45	20
46–47	11
48–49	3
Total	250

24. Draw a Grouped Bar Chart with the following data:

World Production of Raw Cotton during 1996–98		
Country	Production ('000 bales of 478 lb each)	
	1996–97	1997–98
U.S.A.	13,027	10,900
China	6,000	6,500
U.S.S.R.	6,200	5,800
India	4,180	4,300
Pakistan	1,323	1,350
Brazil	1,340	1,340
Egypt	1,498	1,870

25. Draw Histogram and Frequency Polygon to present the following data:

Income (in ₹)	Number of Individuals
100–149	21
150–199	32
200–249	52
250–299	105
300–349	62
350–399	43
400–449	18
450–499	9
Total	342

26. Construct a Frequency Polygon from the following table:

Output of Workers	
Output (units per worker)	Number of Workers
500–509	8
510–519	18
520–529	23
530–539	37
540–549	47
550–559	26
560–569	16
570–579	5

27. Draw a Cumulative Frequency Diagram (less than type) from the following data:

Monthly Wages (in ₹)	125–175	175–225	225–275	275–325	325–375	375–425	425–475
Number of Workers	6	10	25	35	12	10	4

28. Draw Cumulative Frequency Diagram (less than type) of the following frequency distribution and hence determine the median:

Monthly Wages (in ₹)	Number of Workers
12.5–17.5	2
17.5–22.5	22
22.5–27.5	10
27.5–32.5	14
32.5–37.5	3
37.5–42.5	4
42.5–47.5	6
47.5–52.5	1
52.5–57.5	1
Total	63

29. (a) Draw Ogives (both *less than* and *more than* types) from the following frequency distribution:

Wages (in ₹)	200–209	210–219	220–229	230–239	240–249	250–259
Number of Labours	10	20	22	20	25	3

[C.U. B.Com. 1998]

(b) Draw Ogives (both "less than" and "more than") for the following distribution:

Wages (in ₹)	50-59	60-69	70-79	80-89	90-99	100-109	110-119
Number of Employees	5	10	15	8	6	4	2

[C.U. B.Com. 2006]

[Hints: See worked-out Ex. 21.]

30. Draw the Cumulative Frequency Diagram of the following frequency distribution and hence determine the median weight of an apple:

Weight in gm	Frequency
110–119	5
120–129	7
130–139	12
140–149	20
150–159	16
160–169	10
170–179	7
180–189	3
Total	80

31. The trends in Revenue and Expenditure of the Central Govt. for the years 1995–2001 are given in the table below:

	1995–96	1996–97	1997–98	1998–99	1999–2000	2000–01
Revenue (₹ in crore)	4800	5600	6800	6700	7400	8200
Expenditure (₹ in crore)	4400	4700	6300	6800	7600	8300

Draw Multiple Bar Charts to represent the Revenue and Expenditure of the Government since 1995. Read off the surplus or deficit from the figures so drawn.

[**Hints**: For each financial year draw a pair of column charts — one representing Expenditure and the other representing Revenue.]

32. Draw a Pie Chart representing the following data on the proposed outlay of a Five-Year Plan of a Government (₹in crore):

Agriculture — 12,000, Industries and Minerals — 9000, Irrigation and Power — 5000, Education — 8000, Communications — 4000, Miscellaneous — 2000. [C.U. B.Com. 1991]

33. Draw a Histogram of the following frequency distribution and find the proportion of firms with annual sales greater than ₹70,000:

Annual Sales (₹ '000)	0–20	20–50	50–100	100–250	250–500	500–1000
No. of Firms	20	50	69	30	25	19

[C.U. B.Com. 1998]

34. Draw a Pie Chart of the following data:

Type of Commodity	Family Expenditure (₹)
Food	3000
Clothes	1250
House Rent	2000
Education	1100
Savings	750
Misc.	900

[C.U. B.Com. 2008]

[**Hints**: Total family expenditure = ₹9000. We express each type of family expenditure as percentage of total family expenditure as shown below:

Type of Commodity	Percentage Expenditure	Central Angles
Food	$\frac{3000}{9000} \times 100 = \frac{100}{3}\%$	$\frac{100}{3} \times 3.6° = 120°$
Clothes	$\frac{1250}{9000} \times 100 = \frac{125}{9}\%$	$\frac{125}{9} \times 3.6° = 50°$
House Rent	$\frac{2000}{9000} \times 100 = \frac{200}{9}\%$	$\frac{200}{9} \times 3.6° = 80°$
Education	$\frac{1100}{9000} \times 100 = \frac{110}{9}\%$	$\frac{110}{9} \times 3.6° = 44°$
Savings	$\frac{750}{9000} \times 100 = \frac{25}{3}\%$	$\frac{25}{3} \times 3.6° = 30°$
Misc.	$\frac{900}{9000} \times 100 = 10\%$	$10 \times 3.6° = 36°$
Total	100%	360°

Now draw the *Pie Chart* of the given data.]

Chapter 4

Measures of Central Tendency: Mean, Median and Mode

4.1 Introduction

We may condense statistical data to a large extent by classification and tabulation. For interpretation of statistical data mere tabulation is not enough, especially when comparison of two or more series of data is needed. In the present chapter our object will be to have some mathematical measures which will characterize the whole series of tabulated data.

4.2 Central Tendency of Data

In many frequency distributions, the tabulated values show small frequencies at the beginning and at the end, and very high frequency at the middle of the distribution. This indicates that the typical values of the variable lie near the central part of the distribution and other values cluster around these central values. This behaviour of the data about the concentration of the values in the central part of the distribution is called *central tendency of the data*. We shall measure this central tendency with the help of mathematical quantities. A central value which 'enables us to comprehend in a single effort the significance of the whole' is known as *Statistical Average* or simply *Average*. In fact, an average of a statistical series is the value of the variable which is the representative of the entire distribution and, therefore, gives a measure of central tendency.

Measures of Central Tendency

There are three common measures of central tendency: (1) **Mean,** (2) **Median,** (3) **Mode.**

The most common and useful measure of central tendency is, however, the Mean. In the following articles the method of calculation of various measures of central tendency will be discussed. In all such discussions we need a very useful notation known as *Summation Notation* which we explain before proceeding further.

4.3 Summation Notation (Σ)

The symbol Σ (read: Sigma) means *summation.*

If $x_1, x_2, x_3, \ldots, x_n$ be the n values of a variable x, then their sum $= x_1+x_2+\cdots+x_n$ is shortly written as

$$\sum_{i=1}^{n} x_i \text{ or } \Sigma x_i \text{ or simply } \Sigma x \text{ (sometimes, } \Sigma x_1\text{).}$$

Similarly, the sum $w_1x_1 + w_2x_2 + \cdots + w_nx_n$ is denoted by

$$\sum_{i=1}^{n} w_i x_i \text{ or } \Sigma w_i x_i \text{ or simply } \Sigma wx \text{ (or sometimes, } \Sigma w_1 x_1).$$

Some Important Results

- $\sum_{i=1}^{n}(x_i \pm y_i) = \sum_{i=1}^{n} x_i \pm \sum_{i=1}^{n} y_i.$
- $\sum_{i=1}^{n} A = \underbrace{A + A + \cdots + A}_{n \text{ terms}} = nA$ (A is a constant).
- $\sum_{i=1}^{n} Ax_i = Ax_1 + Ax_2 + \cdots + Ax_n = A(x_1 + x_2 + \cdots + x_n) = A\sum_{i=1}^{n} x_i.$

4.4 Mean: Arithmetic Mean ($\bar{x}$)

There are three types of mean: (1) **Arithmetic Mean (AM)**, (2) **Geometric Mean (GM)**, and (3) **Harmonic Mean (HM)**.

Of the three means, **Arithmetic Mean** is most commonly used. In fact, if no specific mention be made, by Mean we shall always refer to Arithmetic Mean (AM) and calculate accordingly.

Simple Arithmetic Mean

Definition 1. *The Arithmetic Mean ($\bar{x}$) of a given series of n values, say, $x_1, x_2, \ldots, x_n$ is defined as the sum of these values divided by their total number; thus*

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{\Sigma x}{n}, \quad \text{i.e.,} \quad \boldsymbol{\bar{x} = \frac{\Sigma x}{n}}.$$

Thus, **Arithmetic Mean** (or simply **Mean**) $= \boldsymbol{\frac{\Sigma x}{n}}$.

Example 1. (a) *Find the Arithmetic Mean of* 3, 6, 24 *and* 48.

(b) *The mean of the numbers x_1, x_2, x_3, x_4 and x_5 is $\bar{x}$. If a constant k be added to each of them, prove that the mean of the new set of numbers is $\bar{x} + k$.*

Solution: (a) The required AM $(\bar{x}) = \frac{3 + 6 + 24 + 48}{4} = \frac{81}{4} = 20.25.$

(b) $\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$. If k is added to each of the numbers, then

$$\text{Mean of the new numbers} = \frac{(x_1 + k) + (x_2 + k) + (x_3 + k) + (x_4 + k) + (x_5 + k)}{5} = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + 5k}{5}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} + \frac{5k}{5} = \bar{x} + k.$$

Weighted Arithmetic Mean

Definition 2. *If $x_1, x_2, \ldots, x_n$ be n values of a variable x and if $f_1, f_2, \ldots, f_n$ be their respective weights or their respective frequencies, then the weighted arithmetic mean $\bar{x}$ is defined by*

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\Sigma fx}{\Sigma f} = \frac{\Sigma fx}{N},$$

where $N = \Sigma f$ = total frequency.

Thus, **Weighted Arithmetic Mean or simply Mean $(\bar{x}) = \frac{\Sigma fx}{N}$, where $N = \Sigma f$.**

We see that in this case along with each value of the variable we have given due weight or importance to the number of times it occurs.

If the weights $f_1, f_2, \ldots, f_n$ are all equal, then the weighted arithmetic mean becomes a simple arithmetic mean. In this case equal importance is given to all the values.

Example 2. *Find the Arithmetic Mean from the following frequency distribution:*

Weight in kg	50	55	60	65	70	Total
No. of men	15	20	25	30	10	100

Solution: Here we shall give due importance to the frequency of each value of the variable.

Weighted Arithmetic Mean = $\frac{\Sigma fx}{N}$, where $N = \Sigma f$ = total frequency.

TABLE 4.1: CALCULATION OF WEIGHTED AM

Weight in kg x	No. of men f	fx
50	15	750
55	20	1100
60	25	1500
65	30	1950
70	10	700
Total	$100 = N$	$6000 = \Sigma fx$

$\therefore$ weighted Arithmetic Mean $(\bar{x}) = \frac{\Sigma fx}{N} = \frac{6000}{100} =$ **60 kg.**

4.4.1 Important Properties of AM

1. *The sum or total of the values is equal to the product of the number of values and their AM*

   Proof. (i) Simple AM

   If $x_1, x_2, \ldots, x_n$ be n values of a variable x, then mean $\bar{x}$ is

$$\bar{x} = \frac{\Sigma x}{n} \text{ or, } n\bar{x} = \Sigma x, \text{ i.e., } \Sigma x = n\bar{x}. \qquad (1)$$

   Proof. (ii) Weighted AM

If $x_1, x_2, \ldots, x_n$ be n values of a variable and if $f_1, f_2, \ldots, f_n$ be their respective weights, then the weighted AM $\bar{x}$ is given by

$$\bar{x} = \frac{\Sigma fx}{N} \text{ or, } N\bar{x} = \Sigma fx = f_1x_1 + f_2x_2 + \cdots + f_nx_n.$$

2. *The algebraic sum of the deviations of the values from their AM is zero.*

Proof.

Case I. *Simple AM*

If $x_1, x_2, \ldots, x_n$ are the n values of a variable x and $\bar{x}$, their AM, then $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$ are called the *deviations* of $x_1, x_2, \ldots, x_n$ respectively from $\bar{x}$.

$$\begin{aligned}\text{Algebraic sum of the deviations} &= \sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) \\ &= (x_1 + x_2 + \cdots + x_n) - (\bar{x} + \bar{x} + \cdots \text{ to } n \text{ terms}) \\ &= \Sigma x - n\bar{x} = n\bar{x} - n\bar{x} \quad [\because \Sigma x = n\bar{x} \text{ by } (1)] \\ &= 0.\end{aligned}$$

Case II. *Weighted AM*

$$\begin{aligned}\sum_{i=1}^{n} f_i(x_i - \bar{x}) &= f_1(x_1 - \bar{x}) + f_2(x_2 - \bar{x}) + \cdots + f_n(x_n - \bar{x}) \\ &= f_1x_1 - f_1\bar{x} + f_2x_2 - f_2\bar{x} + \cdots + f_nx_n - f_n\bar{x} \\ &= (f_1x_1 + f_2x_2 + \cdots + f_nx_n) - (f_1\bar{x} + f_2\bar{x} + \cdots + f_n\bar{x}) \\ &= \Sigma fx - (f_1 + f_2 + \cdots + f_n)\bar{x} \\ &= N\bar{x} - N\bar{x} \quad \left[\because \bar{x} = \frac{\Sigma fx}{N} \text{ and } f_1 + f_2 + \cdots + f_n = N\right] \\ &= 0.\end{aligned}$$

Example 3. *Verify the truth of the previous statement (No. 2 above) taking the values of a variable as* 3, 6, 12, 16, 18.

Solution:

$$\text{AM}(\bar{x}) = \frac{3+6+12+16+18}{5} = \frac{55}{5} = 11.$$

The algebraic sum of the deviations of 3, 6, 12, 16, 18 from their AM is

$$\begin{aligned}(3-11)+(6-11)+(12-11)+(16-11)+(18-11) &= -8-5+1+5+7 \\ &= 0.\end{aligned}$$

Example 4. *Verify the truth of the statement (No. 2) taking the values of a variable as* 2, 5, 9, 11 *with weights* 8, 7, 3, 2 *respectively.*

Solution: Weighted AM $(\bar{x}) = \dfrac{8 \times 2 + 7 \times 5 + 3 \times 9 + 2 \times 11}{8 + 7 + 3 + 2} = \dfrac{100}{20} = 5.$

$\therefore$ algebraic sum of the deviations from $\bar{x} = \Sigma f(x - \bar{x}) = 8(2-5) + 7(5-5) + 3(9-5) + 2(11-5)$

$$= 8 \times (-3) + 7 \times 0 + 3 \times 4 + 2 \times 6$$
$$= -24 + 0 + 12 + 12 = \mathbf{0}.$$

Note: If the deviations are taken from any arbitrary number A (not $\bar{x}$), then the sum of the deviations will not be zero, as can be easily verified.

Example 5. *If the algebraic sum of the deviations of* 12 *observations measured from* 40 *be* 60, *find the* AM. **[V.U. B.Com. 1995]**

Solution: By the question,

$$\sum_{i=1}^{12}(x_i - 40) = 60 \text{ or, } \sum_{i=1}^{12} x_i - \sum_{i=1}^{12} 40 = 60 \text{ or, } \Sigma x_i - 40 \times 12 = 60 \text{ or, } \Sigma x_i = 60 + 480 = 540.$$

Hence

$$\text{AM} = \frac{\Sigma x_i}{12} = \frac{540}{12} = \mathbf{45}.$$

3. *The sum of the squares of the deviations of the values is the minimum, when deviations are taken from the arithmetic mean.* **[B.U. B.Com.(H) 1990]**

Proof. Let $x_1, x_2, \ldots, x_n$ be n values of the variable x and let $\bar{x}$ be their AM. Suppose that A is any arbitrary number.

The deviations from A are $x_1 - A, x_2 - A, \ldots, x_n - A$. We have to show that $\sum_{i=1}^{n}(x_i - A)^2$ is minimum, when $A = \bar{x}$.

Now

$$x_i - A = (x_i - \bar{x}) + (\bar{x} - A).$$

$$\begin{aligned}
\therefore \sum_{i=1}^{n}(x_i - A)^2 &= \sum_{i=1}^{n}\{(x_i - \bar{x}) + (\bar{x} - A)\}^2 = \Sigma\left\{(x_i - \bar{x})^2 + (\bar{x} - A)^2 + 2(x_i - \bar{x})(\bar{x} - A)\right\} \\
&= \Sigma(x_i - \bar{x})^2 + \Sigma(\bar{x} - A)^2 + 2\Sigma\{(x_i - \bar{x})(\bar{x} - A)\} \\
&= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - A)^2 + 2(\bar{x} - A)\Sigma(x_i - \bar{x}) \\
&= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - A)^2 \quad [\because \Sigma(x_i - \bar{x}) = 0].
\end{aligned}$$

We see that the right-hand side is the sum of two quantities of which the first term is positive and does not depend on the value of A we choose, and the second term ≥ 0.

$$\therefore \sum_{i=1}^{n}(x_i - A)^2 \geq \sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Equality occurs when the second term is zero, i.e., when $A = \bar{x}$.

$$\therefore \sum_{i=1}^{n}(x_i - A)^2 \text{ is minimum only when } A = \bar{x}.$$

Similarly, *we can prove the result for a frequency distribution.*

4.4.2 Short-cut Method of Calculating AM for Discrete Series: Method of Assumed Mean

Simple AM

Let $x_1, x_2, \ldots, x_n$ be the n values of a variable x and $d_1, d_2, \ldots, d_n$ be the deviations of the n values from any arbitrary value A (called *assumed mean*).

Then $d_i = x_i - A$ or, $x_i = A + d_i$.

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{\Sigma(A + d_i)}{n} = \frac{nA + \Sigma d_i}{n} = \frac{nA}{n} + \frac{\Sigma d_i}{n} = A + \frac{\Sigma d_i}{n}.$$

Thus, $\bar{x} = A + \frac{\Sigma d_i}{n}$, i.e.,

$$\text{AM} = \text{Assumed mean} + \frac{\text{Sum of the deviations from } A}{\text{No. of values of the variable}}.$$

Weighted AM

$$\bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{\Sigma f_i(A + d_i)}{N} = \frac{\Sigma A f_i + \Sigma f_i d_i}{N} = A\frac{\Sigma f_i}{N} + \frac{\Sigma f_i d_i}{N} = A + \frac{\Sigma f_i d_i}{N} \quad [\because \Sigma f_i = N].$$

Thus, $\bar{x} = A + \frac{\Sigma f_i d_i}{N}$.

Example 6. *The monthly incomes of* 5 *persons are* ₹250, ₹ 360, ₹280, ₹480 *and* ₹410. *Find the Arithmetic Mean of the incomes of* 5 *persons.*

Solution: We take the assumed mean $A = 360$. Here $n = 5$, $\Sigma d = -20$.

$$\therefore \bar{x} = A + \frac{\Sigma d}{n} = 360 + \frac{-20}{5} = 360 - 4 = ₹356.$$

TABLE 4.2: CALCULATION OF DEVIATIONS

Income (₹) x	Deviation from 360($d = x - 360$)
250	−110
360	0
280	−80
480	120
410	50
Total	$-20 = \Sigma d$

Note: The students can try with another assumed mean and arrive at the same result.

Example 7. *Calculate the Arithmetic Mean of the following data:*

Value (x)	1	2	3	4	5	6	7	8	9	Total
Frequency (f)	7	11	16	17	26	31	11	1	1	121

Solution:

TABLE 4.3: CALCULATION OF AM

Value x	Frequency f	Deviations from $A(=5)[d=x-5]$	fd
1	7	−4	−28
2	11	−3	−33
3	16	−2	−32
4	17	−1	−17
5	26	0	0
6	31	1	31
7	11	2	22
8	1	3	3
9	1	4	4
Total	$121=N$	—	$-110+60=-50=\Sigma fd$

Let us take the assumed mean $A=5$. Here $N=121, \Sigma fd=-50$.

$$\therefore \text{ Arithmetic Mean} = A+\frac{\Sigma fd}{N}=5+\frac{-50}{121}=5-0.41=4.59.$$

4.4.3 Calculation of Arithmetic Mean (or simply, Mean) from a Grouped Frequency Distribution – Continuous Series

In a grouped frequency distribution, the individual values of the variable and their frequencies are not known. Here, the mid-value of a class-interval is taken to represent all the values of the variable falling within that class with a frequency equal to that of the class-interval. From the given data the mid-values of all the class-intervals are found by taking the average of the two class-limits (or class-boundaries). Then Arithmetic Mean may be calculated by any one of the methods discussed below.

Ordinary Method (or Direct Method)

This method is already discussed before. In this method, the mid-values of the class-intervals (or values) are multiplied by the corresponding class-frequencies. The sum of the products thus obtained is divided by the total frequency to get the Mean. The mean $\bar{x}$ is given by the formula

$$\bar{x}=\frac{\Sigma fx}{N}, \text{ where } x = \text{mid-value of a class and } N = \text{total frequency.}$$

Example 8. *Calculate the mean of daily wages from the following table:*

Wages (₹)	4–6	6–8	8–10	10–12	12–14
No. of workers	6	12	17	10	5

Solution:

TABLE 4.4: CALCULATION OF MEAN OF DAILY WAGES			
Class-intervals	Mid-values (₹) x	Frequency f	fx
4–6	5	6	30
6–8	7	12	84
8–10	9	17	153
10–12	11	10	110
12–14	13	5	65
Total	—	$50 = N$	$442 = \Sigma fx$

$$\therefore \text{ Mean of daily wages} = \frac{\Sigma fx}{N} = \frac{442}{50} = ₹8.84.$$

Short-cut Method (or Method of Assumed Mean)

This is similar to the method described earlier. In this method, the mid-value of one class-interval (preferably corresponding to the maximum frequency lying near the middle of the distribution) is taken as the assumed mean (or the arbitrary origin) A and the deviations from A are calculated. The mean is then given by the formula:

$$\bar{x} = A + \frac{\Sigma fd}{N}, \text{ where } d = x - A = \text{(Mid-value)}-\text{(Assumed mean)}.$$

Example 9. *Compute the Arithmetic Mean of the following frequency distribution:*

Marks	20–29	30–39	40–49	50–59	60–69	70–79
No. of students	5	11	18	22	16	8

Solution: Arithmetic Mean $= A + \dfrac{\Sigma fd}{N}$, where $d = x - A$.

TABLE 4.5: CALCULATION OF AM OF MARKS				
Class-intervals	Mid-values x	Deviations from 54.5 ($d = x - 54.5$)	Frequency f	fd
20–29	24.5	– 30	5	– 150
30–39	34.5	– 20	11	– 220
40–49	44.5	–10	18	– 180
50–59	$54.5 = A$	0	22	0
60–69	64.5	10	16	160
70–79	74.5	20	8	160
Total	—	—	$80 = N$	$-550 + 320 = -230 = \Sigma fd$

$$\therefore \text{ Arithmetic Mean} = A + \frac{\Sigma fd}{N} = 54.5 + \frac{-230}{80} = 54.5 - 2.875 = 51.625 = 51.6 \text{ (approx.)}.$$

Method of Assumed Mean Using Step Deviation

In the preceding method, the deviations of the mid-values of the class-intervals from the assumed mean were used. The deviations are often divided by a common factor i, usually the common width of the class-intervals if the classes are of equal width or by the HCF of the deviations, if the classes are not of equal width. The arithmetic mean in this case is given by

$$\textbf{Mean } (\bar{x}) = A + \frac{\Sigma f d}{N} \times i,$$

where $d = \frac{x-A}{i}$ *and* i *= the common width of the classes or HCF of the mid-values.*

Example 10. *Find the Arithmetic Mean of the weekly income from the following frequency distribution:*

Weekly income in ₹	10–15	15–20	20–25	25–30	30–35	35–40
No. of workers	200	700	900	800	600	400

Solution:

$$\text{AM } (\bar{x}) = A + \frac{\Sigma f d}{N} \times i, \text{ where } d = \frac{x-A}{i}.$$

TABLE 4.6: CALCULATION OF AM OF WEEKLY INCOME

Class-intervals	Mid-values x	$d = \frac{x-A}{i}$ $(A = 22.5, i = 5)$	Frequency f	fd
10–15	12.5	−2	200	−400
15–20	17.5	−1	700	−700
20–25	22.5 = A	0	900	0
25–30	27.5	1	800	800
30–35	32.5	2	600	1200
35–40	37.5	3	400	1200
Total	—	—	3600 = N	- 1100 + 3200 = 2100 = $\Sigma f d$

$$\therefore \text{ AM } (\bar{x}) = 22.5 + \frac{2100}{3600} \times 5 = 22.5 + 2.92 = ₹25.42.$$

4.4.4 Calculation of AM from Grouped Frequency Distribution with Open Ends

If in a grouped frequency distribution, the lower limit of the first class and the upper limit of the last class are not known, it is difficult to find the AM. When the closed classes (other than the first and last classes) are of equal widths, we take the widths of the open classes equal to the common width of closed classes and determine the AM by any of the preceding methods.

For example, consider the following data:

Marks	No. of students
Below 10	6
10–20	9
20–30	15
30–40	12
Above 40	8

Here, since the closed classes are of equal width (i.e., 10), we take the lower limit of the first class as '0' and the upper limit of the last class as 50. The first and the last class-intervals would be 0–10 and 40–50 respectively.

Let us consider another example:

Marks	No. of students
Below 5	3
5–15	6
15–30	11
30–50	8
Above 50	2

Here, since the widths of 2nd, 3rd, 4th classes are respectively 10, 15, 20 which increase by 5, we take the lower limit of the 1st class as '0' and the upper limit of the last class as '75'. The first and the last classes would be 0–5 and 50–75 respectively. The AM can be obtained by any of the previous methods.

4.4.5 Mean of Compositive Group

If a group of n_1 values has AM $\bar{x}_1$, and another group of n_2 values has AM $\bar{x}_2$, then the AM $(\bar{x})$ of the composite group (i.e., the two groups combined) of n_1+n_2 values is given by

$$\bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}.$$

Proof. Let $\bar{x}_1$ be the AM of n_1 values $x_1, x_2, \ldots, x_{n_1}$ and $\bar{x}_2$ be the AM of n_2 values $y_1, y_2, \ldots, y_{n_2}$. Then

$$\bar{x}_1=\frac{\Sigma x_i}{n_1} \text{ and } \bar{x}_2=\frac{\Sigma y_j}{n_2}$$

$$\text{or,}\quad n_1\bar{x}_1=\Sigma x_i \text{ and } n_2\bar{x}_2=\Sigma y_j.$$

Now $\bar{x}$ is the AM of (n_1+n_2) values of $x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2}$;

$$\therefore\ \bar{x}=\frac{x_1+x_2+\cdots+x_{n_1}+y_1+y_2+\cdots+y_{n_2}}{n_1+n_2}=\frac{\Sigma x_i+\Sigma y_j}{n_1+n_2}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}.$$

For r groups the AM $(\bar{x})$ is given by

$$\bar{x}=\frac{n_1\bar{x}+n_2\bar{x}_2+\cdots+n_r\bar{x}_r}{n_1+n_2+\cdots+n_r}.$$

Example 11. *The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3. Obtain the mean of the sample of size 150 obtained by combining the two samples.*

Solution: Here $n_1=50, n_2=100, \bar{x}_1=54.1, \bar{x}_2=50.3$.

$$\therefore \text{ Mean } (\bar{x})=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}=\frac{50\times 54.1+100\times 50.3}{50+100}=\frac{2705+5030}{150}=\frac{7735}{150}=\mathbf{51.57\ (approx.)}.$$

Example 12. *The AM of $7, x-2, 10, x+3$ is 9; find the value of x.* [C.U. B.Com. 2007]

Solution: AM of $7, x-2, 10, x+3$ is

$$\frac{7+(x-2)+10+(x+3)}{4}=\frac{18+2x}{4}=\frac{9+x}{2}.$$

$$\therefore\ \frac{9+x}{2}=9\ [\because \text{ AM}=9] \text{ or, } 9+x=18 \text{ or, } x=18-9=\mathbf{9}.$$

4.5 Mean: Geometric Mean (GM)

Besides AM the geometric mean of a set of values of a variable can also be taken as one of the measures of central tendency of the values.

Definition 1. *The geometric mean G of n positive values, say, $x_1, x_2, \dots, x_n$, is defined by the positive nth root of their product; thus*

$$G = \sqrt[n]{x_1 \times x_2 \times \cdots \times x_n} = (x_1 \times x_2 \times \cdots \times x_n)^{1/n}.$$

In order to calculate G we often use

$$\log G = \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n) = \frac{1}{n}\Sigma \log x.$$

(Now obtain G by taking antilog of the right-hand side.)

Illustration 1. *If x is the GM of 28 and 7, then* $x = \pm\sqrt{28 \times 7} = \pm 2 \times 7 = \pm 14$. [C.U. B.Com. 1997]

Example 13. *Find the GM of the following*: 1, 3, 9, 3. [C.U. B.Com. 2001]

Solution: GM of 1, 3, 9, 3 $= (1 \times 3 \times 9 \times 3)^{1/4} = (3^4)^{1/4} = 3$.

Example 14. *Find the GM of* 4, 6, 9 *with weights* 1, 2, 1 *respectively.* [C.U. B.Com. 1996]

Solution: Weighted GM $= (4^1 \times 6^2 \times 9^1)^{\frac{1}{(1+2+1)}} = (2^2 \times 2^2 \times 3^2 \times 3^2)^{1/4} = (2^4 \times 3^4)^{1/4} = 2 \times 3 = 6$.

Generalized Definition

If $x_1, x_2, \dots, x_n$ occur with frequencies (or weights) $f_1, f_2, \dots, f_n$ respectively, then the GM is given by

$$G = \left(x_1^{f_1} \times x_2^{f_2} \times \cdots \times x_n^{f_n}\right)^{1/N}, \text{ where } N = f_1 + f_2 + \cdots + f_n.$$

Here
$$\log G = \frac{1}{N}(f_1 \log x_1 + f_2 \log x_2 + \cdots + f_n \log x_n) = \frac{1}{N}\Sigma f \log x.$$

Hence,
$$\boldsymbol{G = \text{antilog}\left\{\frac{1}{N}\sum f \log x\right\}}.$$

Example 15. (i) *Calculate the Geometric Mean of* 3, 6, 24 *and* 48. [C.U. B.Com. 1997; V.U. B.Com.(H) 2007]

(ii) *If the GM of x*, 9, 12 *be* 6, *find the value of x.* [V.U. B.Com. 2011]

Solution: (i) GM $= (3 \times 6 \times 24 \times 48)^{1/4} = (3 \times 3 \times 2 \times 3 \times 2^3 \times 3 \times 2^4)^{1/4} = (3^4 \times 2^8)^{1/4} = 3 \times 2^2 = 12$.

(ii) By definition, $(x \times 9 \times 12)^{1/3} = 6$; or, $108x = 6^3 = 216$ or, $\boldsymbol{x = 2}$.

Example 16. *Find the GM of* 12, 18, 48, 61 *whose weights are* 5, 3, 2, 8 *respectively.*

Solution:

TABLE 4.7: CALCULATIONS FOR GM			
x	$\log x$	f	$f \log x$
12	1.0792	5	5.3960
18	1.2553	3	3.7659
48	1.6812	2	3.3624
61	1.7853	8	14.2824
Total	—	$18 = N$	$26.8067 = \Sigma f \log x$

$$\log G = \frac{1}{N}\Sigma f \log x = \frac{1}{18} \times 26.8067 = 1.4893; \; \therefore G = \text{antilog}\,(1.4893) = 30.85.$$

Uses of GM

The GM is not as widely used as the AM. But as a measure of central tendency it is sometimes more significant than the AM. It is useful in averaging rates of changes when ratio changes are more important than the absolute changes. It is used in the construction of index numbers.

4.6 Mean: Harmonic Mean (HM)

Definition 1. *The Harmonic Mean H of the n values $x_1, x_2, \ldots, x_n$ is defined by*

$$\frac{1}{H} = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n} \quad \text{or, } H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}},$$

i.e., HM is the number of values divided by the sum of the reciprocals of the values, and it cannot be defined if some of the values is zero.

Generalized Definition

If $x_1, x_2, \ldots, x_n$ have the frequencies $f_1, f_2, \ldots, f_n$ respectively, then the HM (H) is given by

$$H = \frac{N}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \cdots + \frac{f_n}{x_n}} = \frac{N}{\sum \frac{f}{x}}, \text{ where } N = \Sigma f.$$

Uses of HM

HM is of very limited use. It is useful in finding averages involving rate, time, price and ratio.

Example 17. *Find the harmonic mean of the following numbers:* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. [V.U. B.Com. 1995]

Solution:

$$\text{HM} = \frac{4}{\frac{1}{1} + \frac{1}{1/2} + \frac{1}{1/3} + \frac{1}{1/4}} = \frac{4}{1+2+3+4} = \frac{4}{10} = \frac{2}{5}.$$

Example 18. *An aeroplane flies around a square the sides of which measure* 100 km *each. The aeroplane covers at a speed of* 100 km per hour *the first side, at* 200 km per hour *the second side, at* 300 km per hour *the third side and at* 400 km per hour *the fourth side. Use the correct mean to find the average speed round the square.*

Solution: Here HM is the appropriate mean. Let the required average speed H km per hour. Then

$$H = \frac{4}{\frac{1}{100}+\frac{1}{200}+\frac{1}{300}+\frac{1}{400}} = \frac{4}{\frac{12+6+4+3}{1200}} = \frac{4\times 1200}{25} = 4\times 48 = \mathbf{192\ km/hour.}$$

Example 19. (a) *Find the* HM *of* 3, 6, 12, 24 *whose weights are* 6, 2, 4, 8 *respectively.*

(b) *Find the Arithmetic Mean of two observations if their Geometric Mean and Harmonic Mean are* 15 *and* 9 *respectively. Also find the two observation.* [V.U. B.Com.(H) 2009]

Solution: (a) $\text{HM} = \dfrac{6+2+4+8}{\frac{6}{3}+\frac{2}{6}+\frac{4}{12}+\frac{8}{24}} = \dfrac{20}{2+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \dfrac{20}{2+1} = \dfrac{20}{3} = \mathbf{6\frac{2}{3}}.$

(b) Let a and b be two observations. Then

$$\sqrt{ab} = 15 \text{ and } \frac{2}{\frac{1}{a}+\frac{1}{b}} = 9 \text{ or, } ab = 225 \tag{1}$$

$$\text{and } \frac{2ab}{a+b} = 9 \text{ or, } \frac{2\times 225}{a+b} = 9 \text{ or, } a+b = 50. \tag{2}$$

$\therefore\ (a-b)^2 = (a+b)^2 - 4ab = 2500 - 900 = 1600;\ \therefore\ a-b = \pm 40.$

If $a-b=40$, then from (2), $b+40+b=50$ or, $2b=10$ or, $b=5$ and $a=45$.

If $a-b=-40$, then from (2), $b-40+b=50$ or, $2b=90$ or, $b=45$ and $a=5$.

Hence, *the two observations are* 5 *and* 45.

4.7 Advantages and Disadvantages of AM, GM and HM

Arithmetic Mean: Advantages

- The AM is the most familiar and widely used measure of central tendency. It is simple to understand and easy to calculate.
- It is rigidly defined.
- Its calculation depends upon all the values in the series.
- It is suitable for algebraic treatment.
- It is least affected by sampling fluctuations.

Disadvantages

- It cannot be determined by inspection.
- It is very much affected by the presence of a few extremely large or small values of the variable.
- Mean cannot be calculated if a single item is missing.
- Calculation of AM from a grouped frequency distribution with open-end classes is impossible, unless some assumptions are made regarding the sizes of these classes.

Geometric Mean: Advantages

- GM is not as widely used as the AM. It is particularly suitable for averaging rates of changes.
- It is rigidly defined and it depends upon all the values in the series.
- It is suitable for algebraic treatment.
- GM is not much affected by the presence of extremely large or small values of the variable.

Disadvantages

- Unlike AM, GM is neither simple to understand nor easy to calculate.
- If any value in the series is zero, GM cannot be calculated.
- Calculation of GM is impossible unless all the values are positive.

Harmonic Mean: Advantages

- It is useful in averaging rates, ratios and prices.
- It is suitable for algebraic treatment.
- Its calculation is based on all the values in the series.

Disadvantages

- It is of very limited use and it is not easy to understand.
- HM cannot be calculated if any one of the values is zero.

Relation between AM, GM and HM

For any set of positive values of a variable, we can write

$$\text{AM} \geq \text{GM} \geq \text{HM} \text{ (\textit{equality occurring only when the values are all equal}).}$$

[C.U. B.Com. 2007]

It can be proved that

$$\frac{\text{AM}}{\text{GM}} = \frac{\text{GM}}{\text{HM}} \text{ or, } \text{AM} \times \text{HM} = (\text{GM})^2.$$

We prove these results for two positive values, say x_1 and x_2 of a variable x. We have

$$\text{AM} = \frac{x_1 + x_2}{2}, \text{ GM} = \sqrt{x_1 \cdot x_2} \text{ and HM} = \frac{2}{(1/x_1)+(1/x_2)} = \frac{2x_1x_2}{x_1+x_2}.$$

Since the square of any real quantity ≥ 0,

$$\therefore \quad \left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0 \text{ or, } x_1 + x_2 - 2\sqrt{x_1 \cdot x_2} \geq 0 \text{ or, } x_1 + x_2 \geq 2\sqrt{x_1 \cdot x_2}$$

$$\text{or, } \quad \frac{x_1 + x_2}{2} \geq \sqrt{x_1 \cdot x_2} \text{ i.e., AM} \geq \text{GM \textit{ equality occurs only when} } x_1 = x_2. \tag{1}$$

Now,

$$\text{AM} \times \text{HM} = \frac{x_1 + x_2}{2} \times \frac{2x_1x_2}{x_1 + x_2} = x_1 \cdot x_2 = \left(\sqrt{x_1 \cdot x_2}\right)^2 = (\text{GM})^2, \tag{2}$$

$$\text{i.e., } \quad \text{AM} \times \text{HM} = (\text{GM})^2. \tag{3}$$

From (2), AM × HM = GM × GM

Since by (1), AM $\geq$ GM,

$$\therefore \text{ HM} \leq \text{GM, i.e., GM} \geq \text{HM} \tag{4}$$

Combining (1) and (3) we get, AM $\geq$ GM $\geq$ HM, *equality occurring only when* $x_1 = x_2$.

[V.U. B.Com.(H) 2011]

Illustration 1.

(i) *Take three numbers* 2, 4, 8: AM = $\frac{14}{3}$;

$$GM = (2.4.8)^{1/3} = 4;\ HM = \frac{3}{1/2 + 1/4 + 1/8} = \frac{24}{7}.$$

Now see that AM × HM = $\frac{14}{3} \times \frac{24}{7} = 16 = (GM)^2$.

(ii) *If* AM *between two numbers is* 6.5 *and their* GM *is* 6, *then*

$$AM \times HM = (GM)^2 \text{ or, } 6.5 \times HM = 6^2 = 36 \text{ or, } HM = \frac{36}{6.5} = 5.54.$$

4.8 Median

Median is another measure of central tendency. If a series of values of a variable is arranged in ascending or descending order of magnitudes, then the value of the middle term or the mean of the two middle terms according as the number of values is odd or even is called the *median*. Median divides the series into two equal parts. It is a positional average and it is unaffected by the presence of an extremely large or small value. It can be calculated from a grouped frequency distribution with open-end classes.

Example 20.

(i) *Find the median of* 33, 86, 68, 32, 80, 48, 70, 64. [C.U. B.Com. 1997]

(ii) *Find the median of* 88, 72, 33, 29, 70, 86, 54, 91, 61, 57.

Solution: (i) Arranging the given numbers in ascending order of magnitudes, we get 32, 33, 48, 64, 68, 70, 80, 86.

There are 8 numbers and, therefore, there are two middle terms which are the 4th and 5th terms. Thus the two middle terms, are 64 and 68.

Hence, Median = $\frac{64+68}{2} = \frac{132}{2} = \mathbf{66}$.

(ii) Arranging the given numbers in ascending order of magnitudes, we get 29, 33, 54, 57, 61, 70, 72, 86, 88, 91.

There are 10 numbers and, therefore, there are two middle terms which are 5th and 6th terms.

∴ middle terms are 61 and 70. Hence, Median = $\frac{61+70}{2} = \frac{131}{2} = \mathbf{65.5}$.

4.8.1 Calculation of Median

Simple Series

The given values are arranged in order of magnitudes. If the number of values is odd, there is one middle term which is the median. If the number of values is even, then there are two middle terms and the AM of these two middle terms is the median.

The formula is

Median = value of the $\left(\frac{n+1}{2}\right)$th term, where n = total number of terms in the series.

Simple Frequency Distribution

In this case, we first calculate cumulative frequency (less than type) corresponding to each value of the variable. Then the value of the variable corresponding to the cumulative frequency $\frac{N+1}{2}$ is the median, where N = total frequency.

Grouped Frequency Distribution

(i) Median by Formula: The cumulative frequency (less than type) corresponding to each class-boundary is first calculated. And then Median is that value of the variable which corresponding to the cumulative frequency $\frac{N}{2}$ and the class in which the cumulative frequency $\frac{N}{2}$ lies is called the median-class, Median is given by the formula:

$$\text{Median} = l_1 + \frac{(N/2) - F}{f_m} \times c,$$

where l_1 = lower boundary of the median-class,
N = total frequency,
F = cumulative frequency below L_1 (or sum of the frequencies of all classes lower than the median-class),
f_m = frequency of the median-class,
c = width of the median class.

(ii) Median by Simple Interpolation: We first define simple interpolation. Simple interpolation is the method of computing an intermediate value of the function y for a specified value of x on the basis of given values of x and y, assuming that y change uniformly with x (i.e., assuming a linear relationship between x and y). *For example,* consider the following pairs of given values of x and y:

x	y
12	60
15 $\longrightarrow$	$\longleftarrow y_1$
20	92

Suppose, we have to compute the value of y, say y_1, when $x = 15$. If the values of y changes uniformly with those of x, then we can write,

$$\frac{15-12}{20-12} = \frac{y_1 - 60}{92-60} \quad [\because y \text{ changes uniformly with } x]$$

or, $\frac{3}{8} = \frac{y_1 - 60}{32}$ or, $\frac{3}{8} \times 32 = y_1 - 60$ or, $12 = y_1 - 60$

or, $y_1 = 12 + 60 = 72.$

Median can be determined by applying *simple interpolation* in a cumulative frequency distribution.

If the cumulative frequency $\frac{N}{2}$ lies between the cumulative frequencies F_1 and F_2, then F_1 and F_2 correspond to the lower class-boundary l_1 and upper class-boundary l_2 of the median class. In this case, Median (M) is given by

$$\frac{M - l_1}{l_2 - l_1} = \frac{(N/2) - F_1}{F_2 - F_1}.$$

(iii) Median by Graphical Method: First draw the ogive (less than or more than type) or cumulative frequency polygon taking the variable (or class-boundaries) on the horizontal axis (i.e., x-axis) and the cumulative frequency on the vertical axis (i.e., y-axis). Locate $\frac{N}{2}$ on the y-axis and from it draw a horizontal line which meets the ogive, say at P. From P draw a perpendicular on the x-axis and then read the point where it meets the x-axis from the horizontal scale. This gives the *median*.

[See Ex. 24 given below.]

Example 21. *Find the median from the following data:*

Wages (₹)	2.30	2.40	3.20	3.40	3.50	4.50	5.10	5.50
No. of persons	6	4	3	4	4	4	4	1

Solution:

TABLE 4.8: CALCULATION OF CUMULATIVE FREQUENCY

x	f	Cumulative Frequency
2.30	6	6
2.40	4	10
3.20	3	13
3.40	4	17←
3.50	4	21
4.50	4	25
5.10	4	29
5.50	1	30 = N

Here $\frac{N+1}{2} = \frac{30+1}{2} = 15.5$ and Median = the value corresponding to cumulative frequency 15.5, i.e., when the values are arranged in order of magnitudes, median is the AM of the 15th and the 16th values.

From the last column we see that 15.5 lies between 13 and 17, and 14th to 17th values are each 3.40.

$$\text{Hence, Median} = \frac{\text{15th value + 16th value}}{2} = \frac{3.40 + 3.40}{2} = 3.40.$$

Example 22. *Calculate the median from the following:*

Class-intervals	2–4	4–6	6–8	8–10
Frequency	3	4	2	1

Solution:

First method (Median by the application of formula):

TABLE 4.9: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary	Frequency	Cumulative Frequency
2–4	3	3
4–6	4	7 ←
6–8	2	9
8–10	1	10 = N

Median = the value corresponding to cumulative frequency $\frac{N}{2}(= 5)$, and 5 is greater than cumulative frequency 3, but less than the next cumulative frequency 7. Therefore, medial-class is 4–6 and hence,

$$\text{Median} = l_1 + \frac{(N/2) - F}{f_m} \times c.$$

Here $l_1 = 4$, $\frac{N}{2} = 5$, $F = 3$, $f_m = 4$, $c = 2$. Hence, Median $= 4 + \frac{5-3}{4} \times 2 = 4 + 1 = 5$.

Second method (Median by the application of simple interpolation):

Median = the value corresponding to the cumulative frequency $\frac{N}{2}$.

TABLE 4.10: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary	Cumulative Frequency (less than)
2	0
4	3
Median (M) $\longrightarrow$	$\longleftarrow \frac{N}{2} = 5$
6	7
8	9
10	$10 = N$

Since $\frac{N}{2} = 5$ lies between the cumulative frequencies 3 and 7, Median (M) must lie between 4 and 6. Now by simple interpolation,

$$\frac{M-4}{6-4} = \frac{5-3}{7-3} \text{ or, } \frac{M-4}{2} = \frac{2}{4} = \frac{1}{2} \text{ or, } M = 4 + \frac{1}{2} \times 2 = 4 + 1 = 5.$$

Hence, **median** = 5.

Example 23. *From the following distribution determine the median:*

Marks	20–29	30–39	40–49	50–59	60–69	70–79	Total
No. of students	7	16	34	55	28	20	160

[C.U. B.Com. 1996]

Solution: Median = the value corresponding to cumulative frequency (less than type) $\frac{N}{2}$.

TABLE 4.11: CALCULATIONS FOR MEDIAN

Marks (Class-intervals)	Frequency f	Cumulative Frequency (less than type)
20–29	7	7
30–39	16	23
40–49	34	57 ($= F$)
50–59	55 ($= f_m$)	112$\leftarrow$
60–69	28	140
70–79	20	$160 = N$
	$160 = N$	

$$\therefore \frac{N}{2} = \frac{160}{2} = 80.$$

Median $= l_1 + \frac{(N/2) - F}{f} \times c$.

From the last column, we see that the cumulative frequency just greater than 80 ($=\frac{N}{2}$) is 112 and the class-interval corresponding to cumulative frequency 112 is 50–59. Hence, the median class is 50–59.

Here l_1 = lower class-boundary of the median class $= 50 - 0.5 = 49.5$, $F = 57$, $f_m = 55$, $c = 10$.

Hence, Median $= 49.5 + \frac{80-57}{55} \times 10 = 49.5 + \frac{230}{55} = 49.5 + 4.18 = \mathbf{53.68}$.

Example 24. *The table below gives the frequency distribution of weights of 80 apples at random from a big consignment:*

Weight in gm	110–119	120–129	130–139	140–149	150–159	160–169	170–179	180–189
Frequency	5	7	12	20	16	10	7	3

Draw the cumulative frequency diagram and hence determine the median weight of an apple.

[C.U. B.Com. 2002 Type]

Solution: Here class-limits are given. For the cumulative frequency distribution, first we have to find the class-boundaries corresponding to the class-limits. The class-boundaries are 109.5–119.5, 119.5–129.5, etc.

TABLE 4.12: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary (Weight in gm)	Cumulative Frequency	
	Less than	More than
109.5	0	80
119.5	5	75
129.5	12	68
139.5	24	56
149.5	44	36
159.5	60	20
169.5	70	10
179.5	77	3
189.5	80	0

The cumulative frequencies (both less than and more than types) are plotted on a graph-paper (See Fig. 4.1) against the class-boundaries taking the former along the vertical axis OY and the latter along the horizontal axis OX. The two cumulative frequency diagrams (or ogives) are drawn. They meet at P.

Locate $\frac{N}{2} = 40$ on the Y-axis and from it draw a horizontal line which meets the ogive at P. From P draw a perpendicular on OX. If M be the foot of this perpendicular, then Median = OM = 139.5 + 8 = **147.5 gm.**

Note: Quartiles can also be determined from the cumulative frequency polygon (or ogive) by using the following definitions:

$$Q_1 = \text{the value corresponding to cumulative frequency } \frac{N}{4};\ \frac{N}{4} = 20$$

$$\text{and}\quad Q_3 = \text{the value corresponding to cumulative frequency } \frac{3N}{4};\ \frac{3N}{4} = 60.$$

From the ogive (Fig. 4.1), $Q_1 = 136.5$, $Q_3 = 159.5$. [For more details, see Section **4.9.1**.]

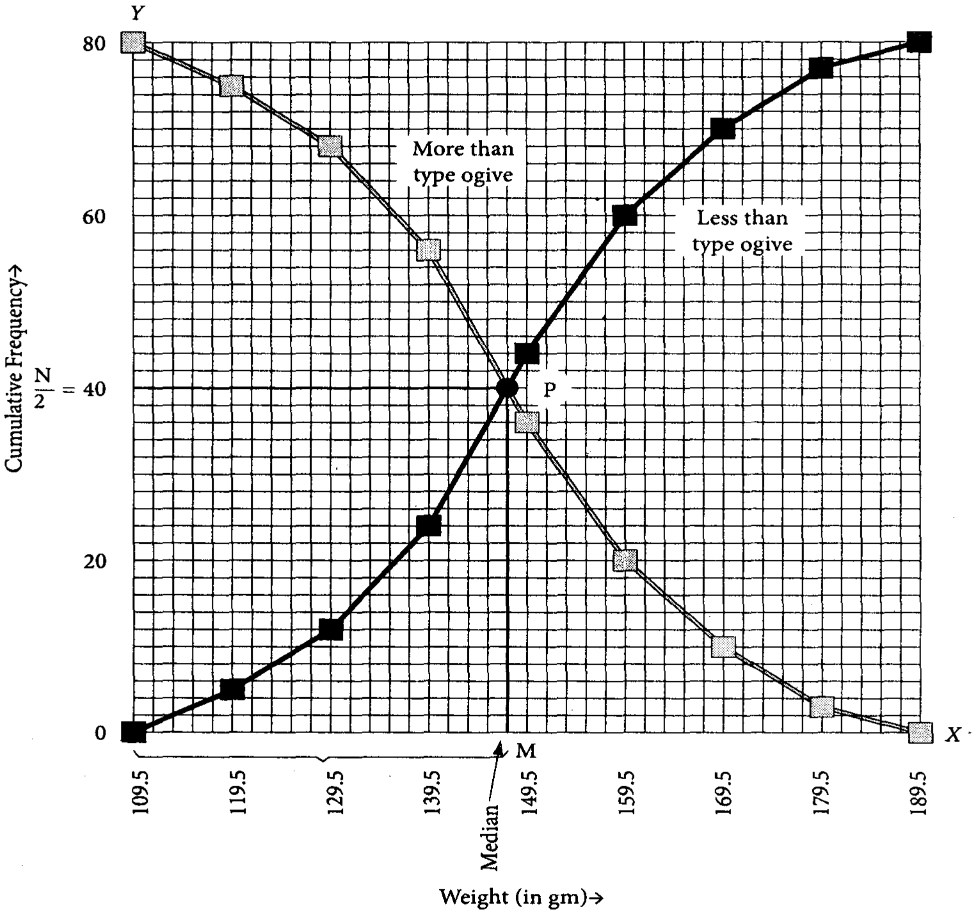


Fig. 4.1: Cumulative Frequency Diagram.

4.8.2 Advantages and Disadvantages of Median

Advantages

- It is simple to understand and easy to calculate.
- It is rigidly defined.
- It is unaffected by the extreme values.

Disadvantages

- It is not based on all the values.
- It is much affected by sampling fluctuations in comparison to mean.
- It is not suitable for algebraic treatment.
- Its calculation depends on the arrangement of the data in order of magnitudes.

4.9 Quartiles, Deciles and Percentiles

If a series of values of a variable is arranged in ascending order of magnitude, the *middlemost value* which divides the series into two equal parts is the *median*. By extending the idea, we can find *three values* Q_1, Q_2

and Q_3 which would divide the series into four equal parts. These three values Q_1, Q_2 and Q_3 are called the *first* (or *lower*) *Quartile, second Quartile* and *third* (or *upper*) *Quartile* respectively. Obviously, the *second quartile* Q_2 *is the median.*

Similarly, the *nine values* which divide the series into *ten equal parts* are called *Deciles* which are denoted by $D_1, D_2, \ldots, D_9$ and the 99 *values* dividing the data into 100 equal parts are called the *Percentiles* denoted by $P_1, P_2, \ldots, P_{99}$ respectively. Quartiles, Deciles and Percentiles are also called *partition values.*

4.9.1 Calculation of Quartiles, Deciles and Percentiles

The calculation of Quartiles, Deciles and Percentiles is similar to that of Median.

For a simple series (or a simple frequency distribution) the data are first arranged in ascending order of magnitude. Then for N values of the series (or frequency distribution),

$$Q_1 = \text{the value of the } \frac{N+1}{4}\text{th item,}$$

$$Q_3 = \text{the value of the } \frac{3(N+1)}{4}\text{th item,}$$

$$D_7 = 7\text{th decile} = \text{the value of the } \frac{7(N+1)}{10}\text{th item,}$$

$$D_9 = 9\text{th decile} = \text{the value of the } \frac{9(N+1)}{10}\text{th item,}$$

$$P_{58} = 58\text{th percentile} = \text{the value of the } \frac{58(N+1)}{100}\text{th item.}$$

For a grouped frequency distribution, cumulative frequencies (less than type) are first calculated. Then

$$Q_1 = \text{the value corresponding to cumulative frequency } \frac{N}{4}, \text{ where } N = \Sigma f = \text{total frequency}$$

$$Q_3 = \text{the value corresponding to cumulative frequency } \frac{3N}{4},$$

$$D_6 = 6\text{th decile} = \text{the value corresponding to cumulative frequency } \frac{6N}{10},$$

$$P_{35} = 35\text{th percentile} = \text{the value corresponding to cumulative frequency } \frac{35N}{100}.$$

$$Q_1 = l_1 + \frac{\frac{N}{4} - F_1}{f_1} \times c_1,$$

where l_1 = lower boundary of the Q_1-class (i.e., the class in which cumulative frequency $\frac{N}{4}$ falls),

F_1 = cumulative frequency below l_1,

f_1 = frequency of the Q_1-class, c_1 = width of Q_1-class.

Similarly,

$$Q_3 = l_3 + \frac{\frac{3N}{4} - F_3}{f_3} \times c_3, D_6 = l_6 + \frac{\frac{6N}{10} - F_6}{f_6} \times c_6, P_{35} = l_{35} + \frac{\frac{35N}{100} - F_{35}}{f_{35}} \times c_{35}.$$

For a symmetrical distribution, $Q_2 - Q_1 = Q_3 - Q_2$ or, $Q_1 + Q_3 = 2Q_2$.

Illustration 1. *If for a symmetrical distribution,* $Q_1 = 24$ *and* $Q_3 = 42$, *then*

$$2Q_2 = 24 + 42 = 66 \text{ or, } \mathbf{Q_2 = 33}.$$

Uses of Quartiles

Quartiles are used for measuring central tendency, dispersion and skewness. Second Quartile Q_2 is the median which is a measure of central tendency. Quartiles Q_1 and Q_3 are used to define Quartile Deviation which is a measure of dispersion. Skewness based on Quartiles Q_1, Q_2 and Q_3 is given by

$$\text{Skewness} = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{(Q_3 - Q_2) + (Q_2 - Q_1)} = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}.$$

Example 25. (i) *Find the quartiles of the following numbers*: 29, 12, 26, 19, 24, 36, 21, 33, 35. *Find also the 3rd and 7th deciles.*

(ii) *Find the 23rd and 57th percentiles of the following values:* 1, 3, 5, 7, ..., 197.

Solution: (i) Arranging the given numbers in ascending order of magnitude, we get, 12, 19, 21, 24, 26, 29, 33, 35, 36.

Here $n = 9$,

$$\frac{n+1}{4} = \frac{9+1}{4}, \frac{n+1}{2} = \frac{9+1}{2}, \frac{3(n+1)}{4} = \frac{3(9+1)}{4}.$$

$\therefore$ 1st Quartile (Q_1) = value of the $\frac{9+1}{4}$th, i.e., $2\frac{1}{2}$th item

= value of the 2nd item + $\frac{1}{2}$ (3rd item − 2nd item)

$= 19 + \frac{1}{2} \times 2 = \mathbf{20}$;

2nd Quartile (Q_2) = value of the $\frac{9+1}{2}$th, i.e., 5th item = **26**;

and 3rd Quartile (Q_3) = value of the $\frac{3(9+1)}{4}$th, i.e., $7\frac{1}{2}$th item

= value of the 7th item + $\frac{1}{2}$ (8th item − 7th item)

$= 33 + \frac{1}{2} \times 2 = \mathbf{34}$.

Deciles. 3rd decile $= D_3$ = the value of the $\frac{3(N+1)}{10}$th item = $\frac{3(9+1)}{10}$th item = 3rd term = **21** and 7th decile $= D_7$ = the value of the $\frac{7(N+1)}{10}$th item = 7th term = **33**.

(ii) The given values are in ascending order of magnitude with common difference $d = 2$, i.e., the values are in A.P. with $a = 1$ and $d = 2$. If N be the number of values, then

$$1 + (N-1) \times 2 = 197 \text{ or, } (N-1) \times 2 = 196 \text{ or, } N - 1 = 98 \text{ or, } N = 98 + 1 = 99.$$

$\therefore$ there are 99 values and $\frac{N+1}{100} = \frac{99+1}{100} = 1$.

$\therefore$ 23rd percentile = $\frac{23(N+1)}{100}$th value = $\left(\frac{23 \times 100}{100}\right)$th, i.e., 23rd value

$= 1 + (23 - 1) \times 2 = 1 + 22 \times 2 = \mathbf{45}$

and 57th percentile = $\frac{57(N+1)}{100}$th value = 57th value $= 1 + (57 - 1) \times 2 = 1 + 56 \times 2 = \mathbf{113}$.

Remarks. For computing *Quartiles, Deciles and Percentiles of a simple series of values*, see worked-out Ex. 25. Calculations of Deciles and Percentiles for a simple frequency distribution (ungrouped) and for a grouped frequency distribution are similar to that of quartiles as shown in Exs 26 and 27. Only proper formulae are to be used in each case.

Example 26. *Obtain the quartiles of the following distribution:*

Ages (in years)	50	52	54	58	60	62	64	66	68	70
Frequency	4	12	18	23	30	26	22	16	5	4

Solution: For an ungrouped frequency distribution, if $N = \Sigma f$ = total frequency, then

Q_1 = the value corresponding to cumulative frequency $\frac{N+1}{4}$,

Q_2 = the value corresponding to cumulative frequency $\frac{2(N+1)}{4}$, i.e., $\frac{N+1}{2}$,

Q_3 = the value corresponding to cumulative frequency $\frac{3(N+1)}{4}$.

				Q_1 ↓	Q_2 ↓		Q_3 ↓			
Age (years)	50	52	54	58	60	62	64	66	68	70
Cumulative Frequency (up to)	4	16	34	57	87	113	135	151	156	160 = N

Q_1 = the value corresponding to c.f. $\frac{161}{4}$, i.e., 40.25 = **58.**

Q_2 = the value corresponding to c.f. $\frac{161}{2}$, i.e., 80.5 = **60.**

Q_3 = the value corresponding to c.f. $\frac{3(161)}{4}$, i.e., 120.75 = **64.**

Example 27. *Find the Quartiles from the following distribution:*

Age (years)	No. of employees	Age (years)	No. of employees
Below 20	13	35–40	112
20–25	29	40–45	94
25–30	46	45–55	45
30–35	60	55 and above	21

Solution:

TABLE 4.13: CALCULATION OF CUMULATIVE FREQUENCY

Age (years)	Cumulative Frequency (less than)
20	13
25	42
30	88
$Q_1 \longrightarrow$	$\longleftarrow$ 105 = $N/4$
35	148
$Q_2 \longrightarrow$	$\longleftarrow$ 210 = $N/2$
40	260
$Q_2 \longrightarrow$	$\longleftarrow$ 315 = $3N/4$
45	354
55	399
55 and above	420 = N

Here $\frac{N}{4} = 105$, $\frac{N}{2} = 210$ and $\frac{3N}{4} = 315$.
We see that Q_1-class is 30–35, Q_2-class is 35–40 and Q_3-class is 40–45.

$$Q_1 = l_1 + \frac{\frac{N}{4} - F_1}{f_1} \times c_1 = 30 + \frac{105-88}{60} \times 5 = 30 + \frac{17}{12} = 31.42 = 31.4,$$

$$Q_2 = l_2 + \frac{\frac{N}{2} - F_2}{f_2} \times c_2 = 35 + \frac{210-148}{112} \times 5 = 35 + \frac{62}{112} \times 5 = 37.8,$$

$$Q_3 = l_3 + \frac{\frac{3N}{4} - F_3}{f_3} \times c_3 = 40 + \frac{315-260}{94} \times 5 = 40 + \frac{55}{94} \times 5 = 42.9.$$

Note: (i) Skewness $= \frac{(Q_3-Q_2)-(Q_2-Q_1)}{(Q_3-Q_2)+(Q_2-Q_1)} = \frac{(42.9-37.8)-(37.8-31.4)}{(42.9-37.8)+(37.8-31.4)} = \frac{5.1-6.4}{5.1+6.4} = \frac{-1.3}{11.5} \doteq -0.11.$

(ii) For computation of *Deciles* and *Percentiles* from grouped frequency distribution, see Ex. 16 in **Section 4.11**.

4.10 Mode

Definition 1. *Mode is a measure of central tendency of a variable. Mode of a series of values of a variable is that value which occurs with the maximum frequency, i.e., it is the most common value.*

Mode is the most typical value which sometimes represents the true characteristic of the distribution. It is often used in business. In many situations Mode is more suitable than Mean or Median as a measure of central tendency. *For example,* when we speak of *most common wage,* we mean modal wage (i.e., the wage the largest number of workers receive). By *most common size of shoe,* we mean the modal size which is most usually purchased by the customers.

4.10.1 Calculation of Mode

- In a simple series of values of a variable, Mode is that value which occurs the maximum number of times. *For example,* the series of values 2, 2, 3, 4, 4, 4, 5, 5, 6 has mode 4, because 4 occurs 3 times and other values do not occur so often.
- In the case of simple frequency distribution, Mode can be found by inspection only. It is that value which occurs with maximum frequency.
- In the case of a grouped frequency distribution, the exact values of the variable are not known and as such it is very difficult to locate mode accurately. In such cases, an approximate value of the mode may be found either by graphical method or by using the following *empirical relation* between Mean, Median and Mode:

$$\textbf{Mean} - \textbf{Mode} = \textbf{3 (Mean} - \textbf{Median)}.$$

When all the classes are of equal width, Mode is usually calculated by the formula:

$$\text{Mode} = L + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i,$$

where L = lower boundary of the modal class (modal class is the class with the maximum frequency),
f_m = frequency of the modal class or maximum frequency,
f_1 = frequency of the class just preceding the modal class,
f_2 = frequency of the class just succeeding the modal class,
i = common width of the classes.

In graphical method, the histogram for the given data is first drawn. In the histogram, we join the top right angular point of the highest rectangle to the top right angular point of the preceding rectangle and then the top left angular point of the highest rectangle to the top left angular point of the succeeding rectangle by straight lines. From the point of intersection of these two lines a perpendicular is drawn on the horizontal axis OX. The foot of this perpendicular on OX is read from the horizontal scale. This reading gives the mode of the given frequency distribution (See Ex. 31).

When all the classes are not of equal widths, we cannot apply the formula or use histogram to compute mode. In such cases, an approximate value of Mode may be determined by using the empirical relation if mean and median are known.

Example 28. *Find the Mode of the following numbers:* 7, 4, 3, 5, 6, 3, 3, 2, 4, 3, 4, 3, 3, 4, 4, 3, 2, 2, 4, 3, 5, 4, 3, 4, 3, 4, 3, 1, 2, 3.

Solution: Let us first find the frequencies of the numbers 1, 2, 3, 4, 5, 6, 7.

Number	1	2	3	4	5	6	7
Frequency	1	4	12	9	2	1	1

From the above frequency distribution, we see that the value 3 occurs with the maximum frequency 12.

Hence, **Mode = 3.**

Example 29. (a) *In a moderately asymmetric distribution the mode and mean are respectively* ₹12.30 *and* ₹18.48. *Find the median.* [C.U. B.Com. 2001]

(b) *For a distribution, Arithmetic mean* = ₹22, *Median* = ₹20, *then find the value of Mode.* [C.U. B.Com. 2008]

Solution: (a) In a moderately asymmetric distribution, we have

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median}) \text{ or, } 18.48 - 12.30 = 3(18.48 - \text{Median})$$

$$\text{or, } 6.18 = 55.44 - 3 \times \text{Median} \text{ or, } 3 \times \text{Median} = 55.44 - 6.18 = 49.26$$

$$\text{or, } \text{Median} = \frac{49.26}{3} = ₹\mathbf{16.42}.$$

(b) Mean − Mode = 3 (Mean − Median) or, 22 − Mode = 3(22 − 20) = 6 or, **Mode** = 22 − 6 = **₹16.**

Example 30. *Find the Mode from the following frequency distribution:*

Output in units	**No. of workers**
300 to 309	9
310 to 319	20
320 to 329	24
330 to 339	38
340 to 349	48
350 to 359	27
360 to 369	17
370 to 379	6

Solution: Here the class-interval 340–349 has the maximum frequency 48. Therefore, modal class is 340–349 and the corresponding class-boundaries are 339.5–349.5.

Now,
$$\text{Mode} = L + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i.$$

Here $L = 339.5$, $f_m = 48$, $f_1 = 38$, $f_2 = 27$, $i = 10$.

$$\therefore \text{ Mode} = 339.5 + \left(\frac{48-38}{2\times 48 - 38 - 27}\right) \times 10 = 339.5 + \frac{10}{96-65} \times 10 = 339.5 + \frac{100}{31} = 339.5 + 3.23$$
$$= 342.73.$$

4.10.2 Advantages and Disadvantages of Mode

Advantages

- The mode is easily understood by the common man.
- It is not affected by the extreme values of the variable.
- It is determined by inspection from a simple frequency distribution.
- It can be calculated from a grouped frequency distribution with open-end classes, provided the closed classes are of equal width.

Disadvantages

- Mode is not well defined and is not always possible to find a well defined mode.
- It is not suitable for algebraic treatment.
- It is not based on all the values of the variable.
- It is affected by sampling fluctuations.

Example 31. *Below is given the frequency distribution of weights of a group of* 60 *students in a class in a college*:

Weight in kg	30–34	35–39	40–44	45–49	50–54	55–59	60–64
No. of students	3	5	12	18	14	6	2

Draw histogram for this distribution and find modal value.

Solution: Here class-limits are given; the corresponding class-boundaries of the given classes are 29.5–34.5, 34.5–39.5, ..., 59.5–64.5.

The histogram for the given data has been drawn in the following figure.

Since the maximum frequency is 18, the modal class is 45–49. Modal value is indicated in the figure by an arrow.

From the figure,

Mode = 44.5 + 3 = 47.5 kg,

i.e., Modal weight is **47.5 kg.**

Note: We can determine modal value from the histogram approximately. Students may calculate mode first by using the formula and then determine the mode from the histogram by checking its approximate position.

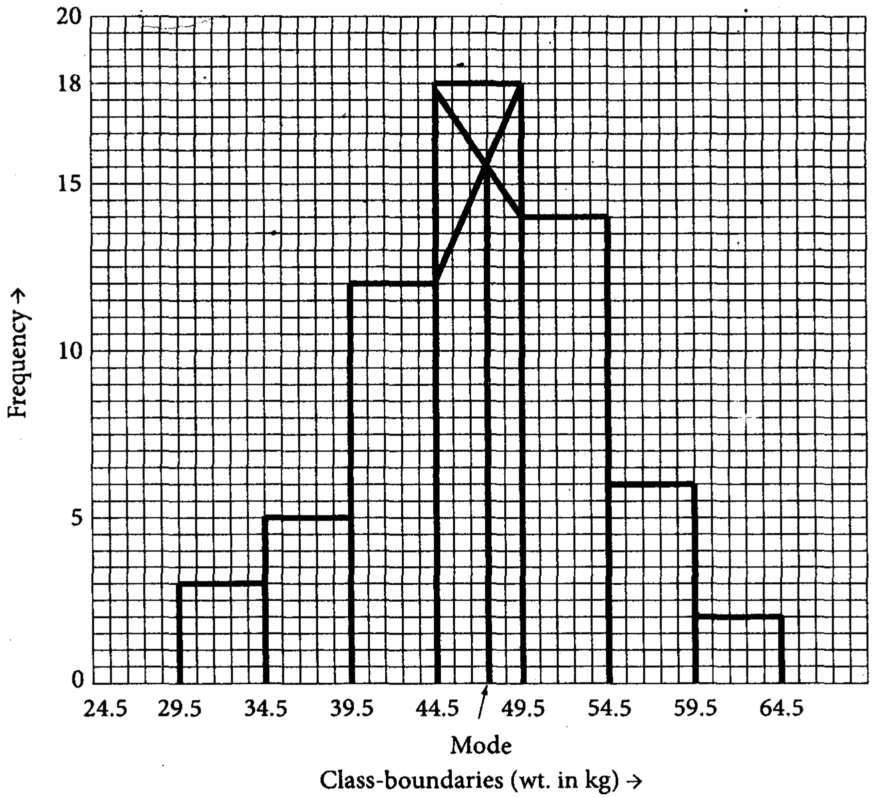


Fig. 4.2: Histogram for the given data.

4.11 Miscellaneous Examples

Example 32. *Find the arithmetic mean of the numbers* $1, 3, 5, \ldots, (2n-1)$.

Solution: $1, 3, 5, \ldots, (2n-1)$ are in A.P. with common difference 2.

$$\therefore\ 1+3+5+\cdots+(2n-1) = \frac{n}{2}\{2\cdot 1+(n-1)\cdot 2\} = \frac{n}{2}\{2+2n-2\} = \frac{n}{2}\times 2n = n^2.$$

$$\therefore\ \text{required AM} = \frac{1+3+5+\cdots+(2n-1)}{n} = \frac{n^2}{n} = \boldsymbol{n}.$$

Example 33. *Find the geometric mean of the numbers* $1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{n-1}}$.

Solution: Here the no. of numbers is n.

$$\text{Required GM} = \left(1\cdot\frac{1}{2}\cdot\frac{1}{2^2}\cdots\frac{1}{2^{n-1}}\right)^{1/n} = \left(\frac{1}{2\cdot 2^2\cdot 2^3\cdots 2^{n-1}}\right)^{1/n} = \frac{1}{\{2^{1+2+3+\cdots+(n-1)}\}^{1/n}}$$

$$= \frac{1}{\left\{2^{\frac{(n-1)(n-1+1)}{2}}\right\}^{1/n}} = \frac{1}{2^{\frac{(n-1)n}{2}\times\frac{1}{n}}} = \frac{\mathbf{1}}{\mathbf{2^{\frac{n-1}{2}}}}.$$

Example 34. *The average weight of the following distribution is* 58.5 kg:

Weight in kg	50	55	60	$x+12.5$	70	Total
No. of men	1	4	2	2	1	10

Find the value of x.

Solution:

TABLE 4.14: CALCULATION OF AVERAGE WEIGHT

Weight X	Frequency f	fX
50	1	50
55	4	220
60	2	120
$x+12.5$	2	$2x+25$
70	1	70
Total	$10=N$	$2x+485=\Sigma fX$

$$\therefore \text{ average weight} = \frac{\Sigma fX}{N} = \frac{2x+485}{10};$$

$$\therefore \frac{2x+485}{10} = 58.5 \text{ or, } 2x+485=585 \text{ or, } 2x=100.$$

Hence, $x = 50$.

Example 35. *The following are the monthly salaries of* 20 *employees in a firm*: (in ₹) 130, 62, 145, 118, 125, 76, 151, 142, 110, 98, 65, 116, 100, 103, 71, 85, 80, 122, 132, 95. *The firm gives bonuses of* (in ₹) 10, 15, 20, 25 *and* 30 *respectively in salary subgroups of* ₹60–80, ₹81–100, ₹101–120, ₹ 121–140, ₹141–160. *Find the average bonus paid per employee.*

Solution:

TABLE 4.15: CALCULATION OF AVERAGE BONUS

Class-intervals	Tally Marks	Frequency f	Bonus x	fx
60–80	~~////~~	5	10	50
81–100	////	4	15	60
101–120	////	4	20	80
121–140	////	4	25	100
141–160	///	3	30	90
Total	—	$20=N$	—	$380=\Sigma fx$

$$\therefore \text{ average bonus per employee} = \frac{\Sigma fx}{N} = \frac{380}{20} = ₹19.$$

Example 36. *The expenditure of* 1000 *families is given as under*:

Expenditure (in ₹)	40–59	60–79	80–99	100–199	120–139
No. of families	50	?	500	?	50

The median and mean for the distribution are both ₹87.50. Calculate the missing frequencies.

[N.B.U. B.Com. 1995; C.U. B.Com. 2003]

Solution: Let the missing frequencies be f_1 and f_2 respectively.

TABLE 4.16: CALCULATION OF DEVIATIONS

Class-intervals	Mid-value x	Frequency f	$d=(x-89.5)/20$	fd
40–59	49.5	50	−2	−100
60–79	69.5	f_1	−1	$-f_1$
80–99	$89.5=A$	500	0	0
100–119	109.5	f_2	1	f_2
120–139	129.5	50	2	100
Total	...	$600+f_1+f_2=N=1000$	...	$f_2-f_1=\Sigma fd$

$$\text{Mean}=A+\frac{\Sigma fd}{N}\times i \quad \text{or,} \quad 87.50=89.5+\frac{f_2-f_1}{1000}\times 20 \text{ or, } -2=\frac{f_2-f_1}{50}$$

$$\text{or,} \quad -100=f_2-f_1 \text{ or, } f_1-f_2=100. \tag{1}$$

Also
$$600+f_1+f_2=1000 \text{ or, } f_1+f_2=400. \tag{2}$$

Solving (1) and (2), we get $f_1=250$ and $f_2=150$.

Hence, the required missing frequencies are respectively 250 and 150.

Note: The problem can also be solved by using the formula for Median.

Example 37. *The Arithmetic Mean of the following frequency distribution is* 46.5. *Find the missing frequency of the distribution:*

Marks	20–30	30–40	40–50	50–60	60–70
No. of students	10	20	?	25	15

Solution: Let the missing frequency be y.

TABLE 4.17: CALCULATIONS FOR ARITHMETIC MEAN

Class-interval	Mid-value x	Frequency f	$d=x-A$ $(A=45)$	fd
20–30	25	10	−20	−200
30–40	35	20	−10	−200
40–50	45	y	0	0
50–60	55	25	10	250
60–70	65	15	20	300
Total	—	$70+y=N$		$550-400=150=\Sigma fd$

$$\text{Arithmetic Mean} = A + \frac{\Sigma fd}{N} \quad \text{or,} \quad 46.5 = 45 + \frac{150}{70+y} \quad [\text{since AM} = 46.5]$$

$$\text{or,} \quad 46.5 - 45 = \frac{150}{70+y}$$

$$\text{or,} \quad 1.5 = \frac{150}{70+y}$$

$$\text{or,} \quad 105 + 1.5y = 150$$

$$\text{or,} \quad 1.5y = 150 - 105 = 45.$$

$$\text{or,} \quad y = \frac{45}{1.5} = \frac{45 \times 10}{15} = 30.$$

Hence, the required missing frequency is **30**.

Example 38. *The Mode of the following frequency distribution of heights of students of a class is* 153 cm. *Find missing frequency of the distribution:*

Heights in cm	140–145	145–150	150–155	155–160	160–165
No. of students	10	20	35	?	10

Solution: Let the missing frequency be y. Given, Mode = 153 cm.

$\therefore$ modal class is 150–155 and maximum frequency (f_m) = 35.

Now,

$$\text{Mode} = L + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i.$$

Here $f_m = 35$, $L = 150$, $f_1 = 20$, $f_2 = y$ and $i = 5$.

$$\therefore \quad 153 = 150 + \left(\frac{35-20}{2 \times 35 - 20 - y}\right) \times 5$$

$$\text{or,} \quad 153 - 150 = \frac{15 \times 5}{50 - y} \text{ or, } 3 = \frac{75}{50-y}$$

$$\text{or,} \quad 150 - 3y = 75 \text{ or, } 150 - 75 = 3y \text{ or, } y = \frac{75}{3} = 25.$$

Hence, the required missing frequency = **25**.

Example 39. *The Median of the following frequency distribution is* ₹74. *Find the missing frequency of the distribution:*

Daily wages (₹)	40–50	50–60	60–70	70–80	80–90	90–100
No. of workers	5	15	20	?	20	15

Solution: Given, Median = ₹74; $\therefore$ median class is 70–80.

Let the missing frequency be y.

TABLE 4.18: CUMULATIVE FREQUENCY DISTRIBUTION (*Less than* type)		
Class-interval	Frequency	Cumulative Frequency *Less than*
40–50	5	5
50–60	15	20
60–70	20	$40 = F$
→ 70–80	y	$40 + y \leftarrow \frac{N}{2}$
80–90	20	$60 + y$
90–100	15	$75 + y = N$

Now,

$$\text{Median} = l_1 + \frac{\frac{N}{2} - F}{f_1} \times c.$$

Here $l_1 = 70$, $F = 40$, $f_1 = y$, $c = 10$.

$$\text{or,} \quad 74 = 70 + \frac{\frac{75+y}{2} - 40}{y} \times 10$$

$$\text{or,} \quad 74 - 70 = \frac{(75 + y) \times 5 - 400}{y}$$

$$\text{or,} \quad 4y = 375 + 5y - 400 = 5y - 25$$

$$\text{or,} \quad 25 = 5y - 4y = y.$$

Hence, the required missing frequency is 25.

Example 40. *Calculate the arithmetic mean and the median of the frequency distribution given below. Hence calculate the mode using the empirical relation between the three:*

Class-limits	130–134	135–139	140–144	145–149	150–154	155–159	160–164
Frequency	5	15	28	24	17	10	1

Solution:

TABLE 4.19: CALCULATION OF AM				
Class-intervals	Mid-values x	$d = \frac{x-A}{i}$ $(i = 5)$	Frequency f	fd
130–134	132	−3	5	−15
135–139	137	−2	15	−30
140–144	142	−1	28	−28
145–149	$147 = A$	0	24	0
150–154	152	1	17	17
155–159	157	2	10	20
160–164	162	3	1	3
Total	...	...	$100 = N$	$-33 = \Sigma fd$

$$\therefore \text{ Arithmetic Mean} = A + \frac{\Sigma fd}{N} \times i = 147 + \frac{-33}{100} \times 5 = 147 - 1.65 = 145.35.$$

TABLE 4.20: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary	Cumulative Frequency (less than)
129.5	0
134.5	5
139.5	20
144.5	48
Median$\longrightarrow$	$\longleftarrow 50 = \frac{N}{2}$
149.5	72
154.5	89
159.5	99
164.5	$100 = N$

Here $\frac{N}{2} = \frac{100}{2} = 50$. Median (M) = the value corresponding to cumulative frequency 50.

$\therefore$ median class is 145–149 and Median $= l_1 + \frac{\frac{N}{2} - F}{f_m} \times c$.

Here $l_1 = 144.5$, $\frac{N}{2} = 50$, $F = 48$, $f_m = 24$, $c = 5$.

$$\therefore \text{ Median} = 144.5 + \frac{50-48}{24} \times 5 = 144.5 + \frac{5}{12} = 144.5 + 0.42 = \mathbf{144.92}.$$

The empirical relation between Mean, Median and Mode is

Mean − Mode = 3 (Mean − Median) or, 145.35 − Mode = 3 (145.35 − 144.92) = 3 × 0.43 or, 145.35 − 1.29 = Mode, i.e., Mode = **144.06.**

Example 41. *The AM of the following distribution is* 28.8. *Find the missing frequency:*

Marks	0–10	10–20	20–30	30–40	40–50	50–60
No. of students	4	6	20	?	7	3

[C.U. B.Com. 1996]

Solution: Let z be the missing frequency. Then we have

TABLE 4.21: CALCULATIONS FOR ARITHMETIC MEAN

Class-interval	Mid-value x	Frequency f	$d = x - A$ $= x - 35$	fd
0–10	5	4	−30	−120
10–20	15	6	−20	−120
20–30	25	20	−10	−200
30–40	35	z	0	0
40–50	45	7	10	70
50–60	55	3	20	60
Total	—	$40 + z = N$		$-440 + 130 = -310 = \Sigma fd$

$$\text{Mean} = A + \frac{\Sigma fd}{N} = 35 - \frac{310}{40+z}. \text{ Also Mean} = 28.8.$$

$$\therefore\ 28.8 = 35 - \frac{310}{40+z} \quad \text{or,} \quad \frac{310}{40+z} = 35 - 28.8 = 6.2$$

$$\text{or,} \quad 310 = 248 + 6.2z$$

$$\text{or,} \quad 6.2z = 62$$

$$\text{or,} \quad z = \frac{62}{6.2} = 10.$$

Hence, the missing frequency is **10**.

Example 42. *The Median and Mode of the following frequency distribution are known to be 27 and 26 respectively. Find the values of a and b:*

Values	0–10	10–20	20–30	30–40	40–50
Frequency	3	a	20	12	b

[C.U. B.Com. 2001]

Solution: Median = 27 and Mode = 26; Median and Modal classes are the same which is 20–30.

$\therefore\ L = 20,\ f_m = 20,\ f_1 = a$ and $f_2 = 12$. Also $i = 10$.

Now,

$$\text{Mode} = L + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i. \text{ Also Mode} = 26.$$

$$\text{or,} \quad 26 = 20 + \left(\frac{20-a}{2\times 20 - a - 12}\right) \times 10 \text{ or, } 6 = \frac{(20-a)10}{28-a}$$

$$\text{or,} \quad 168 - 6a = 200 - 10a \text{ or, } 4a = 200 - 168 = 32 \text{ or, } a = 8.$$

Again,

$$\text{Median} = l_1 + \frac{\frac{N}{2} - F}{f_m} \times c.$$

Here $l_1 = 20$, $N = 3 + a + 20 + 12 + b = 35 + a + b$, $F = 3 + a$, $f_m = 20$, $c = 10$ and Median = 27.

$$\therefore \quad 27 = 20 + \frac{\frac{35+a+b}{2} - (3+a)}{20} \times 10 = 20 + \frac{35 + b - a - 6}{4}$$

$$\text{or,} \quad 7 \times 4 = 29 + b - a \text{ or, } a - b = 1$$

$$\text{or,} \quad 8 - b = 1 \text{ or, } b = 8 - 1 = 7.$$

Hence, $\boldsymbol{a = 8}$ and $\boldsymbol{b = 7}$.

Example 43. *In the following frequency distribution, two class-frequencies are missing:*

Intelligence Quotient	55–64	65–74	75–84	85–94	95–104	105–114	115–124	125–134	135–144
No. of Students	2	19	78	?	301	?	92	14	4

Is it however known that the total frequency is 900 *and the Median* 100.048. *Find the two missing frequencies.*

Solution: Let the two missing frequencies be f_1 and f_2.

Median = the value corresponding to cumulative frequency $N/2$.

The class-boundaries of the given classes are 54.5–64.5, 64.5–74.5, ..., 134.5–144.5.

Here Median = 100.048 and $N/2 = 900/2 = 450$.

TABLE 4.22: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary	Cumulative Frequency (less than)
54.5	0
64.5	2
74.5	21
84.5	99
94.5	$99 + f_1$
100.048 →	$\leftarrow 450 = \frac{N}{2}$
104.5	$400 + f_1$
114.5	$400 + f_1 + f_2$
124.5	$492 + f_1 + f_2$
134.5	$506 + f_1 + f_2$
144.5	$510 + f_1 + f_2 = N = 900$

By simple interpolation, we get

$$\frac{100.048 - 94.5}{104.5 - 94.5} = \frac{450 - (99 + f_1)}{(400 + f_1) - (99 + f_1)}$$

$$\text{or,} \quad \frac{5.548}{10} = \frac{351 - f_1}{301} \quad \text{or,} \quad \frac{5.548 \times 301}{10} = 351 - f_1$$

$$\text{or,} \quad 167 = 351 - f_1, \quad \left[\because \frac{5.548 \times 301}{10} = \frac{1669.948}{10} = 166.9948 = 167\right]$$

$$\text{or,} \quad f_1 = 351 - 167 = 184.$$

Again, $510 + f_1 + f_2 = 900$ or, $f_1 + f_2 = 390$ or, $184 + f_2 = 390$ or, $f_2 = 390 - 184 = 206$.

Hence, the two required missing frequencies are **184** and **206**.

Otherwise:

$$\text{Median} = l_1 + \frac{\frac{N}{2} - F}{f_m} \times c \quad \text{or,} \quad 100.048 = 94.5 + \frac{450 - (99 + f_1)}{301} \times 10$$

$$\text{or,} \quad 5.548 = \frac{(351 - f_1)}{301} \times 10$$

$$\text{or,} \quad \frac{5.54 \times 301}{10} = 351 - f_1, \text{ etc.}$$

Example 44. *Calculate the median and mode of the following:*

Annual Sales (₹'000)	Frequency	Annual Sales (₹'000)	Frequency
less than 10	4	less than 40	55
less than 20	20	less than 50	62
less than 30	35	less than 60	67

Is it possible to calculate the arithmetic mean? If possible, calculate it. [Utkal U. B.Com. 2000]

Solution: Here cumulative frequency distribution (less than type) is given.

Median = the value corresponding to cumulative frequency $N/2$.

TABLE 4.23: CALCULATION OF CUMULATIVE FREQUENCY

Value (Annual Sales) in ₹'000	Cumulative Frequency (less than)
10	4
20	20
Median $(M) \rightarrow$	$\leftarrow 33.5 = \frac{N}{2}$
30	35
40	55
50	62
60	$67 = N$

$N = 67; \therefore \frac{N}{2} = 33.5.$

By Simple Interpolation, we get

$$\frac{M-20}{30-20} = \frac{33.5-20}{35-20} \quad \text{or,} \quad \frac{M-20}{10} = \frac{13.5}{15}$$

$$\text{or,} \quad M - 20 = \frac{13.5}{15} \times 10$$

$$\text{or,} \quad M - 20 = \frac{135}{15} = 9$$

$$\text{or,} \quad M = 9 + 20 = 29.$$

Hence, Median = **₹29,000.**

Using formula,

$$\text{Median} = l_1 + \frac{\frac{N}{2} - F}{f_m} \times c = 20 + \frac{33.5-20}{15} \times 10 = 20 + \frac{135}{15} = \mathbf{29}.$$

To calculate Mode, we construct grouped frequency distribution.

TABLE 4.24: ORDINARY FREQUENCY DISTRIBUTION

Class-intervals	Frequency
0–10	4
10–20	16
20–30	15
30–40	20
40–50	7
50–60	5
Total	67

Maximum frequency is 20 and, therefore, the modal class is 30–40.

$$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i = 30 + \left(\frac{20-15}{2 \times 20 - 15 - 7}\right) \times 10$$
$$= 30 + \frac{50}{18} = 30 + 2.778 = 32.778 = ₹32,778.$$

Yes. It is possible to calculate the arithmetic mean from the ordinary frequency distribution shown in Table 4.25.

TABLE 4.25: CALCULATIONS FOR AM

Class-interval	Mid-value x	Frequency f	$d = \frac{x-A}{i} = \frac{x-25}{10}$ $(A=25)$	fd
0–10	5	4	−2	−8
10–20	15	16	−1	−16
20–30	$25 = A$	15	0	0
30–40	35	20	1	20
40–50	45	7	2	14
50–60	55	5	3	15
Total		$67 = N$		$-24 + 49 = 25 = \Sigma fd$

$$\text{AM}\ (\bar{x}) = A + \frac{\Sigma fd}{N} \times i = 25 + \frac{25}{67} \times 10 = 25 + \frac{250}{67} = 25 + 3.731 = 28.731 = ₹28,731.$$

Example 45. *The table below gives the number* (F) *of candidates obtaining marks x or higher in a certain examination*:

x	10	20	30	40	50	60	70	80	90	100
F	140	133	118	100	75	45	25	9	2	0

Calculate the mean and the median marks obtained by the candidates.

Solution: Here cumulative frequency distribution (more than type) is given. We first construct ordinary (grouped) frequency distribution and then calculate mean.

TABLE 4.26: CALCULATIONS FOR MEAN

Class-intervals	Mid-value x	Frequency f	$d=\frac{x-A}{i}$ $(i=10)$	fd
10–20	15	7	−4	−28
20–30	25	15	−3	−45
30–40	35	18	−2	−36
40–50	45	25	−1	−25
50–60	$55=A$	30	0	0
60–70	65	20	1	20
70–80	75	16	2	32
80–90	85	7	3	21
90–100	95	2	4	8
Total	...	$140=N$	...	$-134+81=-53=\Sigma fd$

$$\text{Mean}=A+\frac{\Sigma fd}{N}\times i=55+\left(\frac{-53}{140}\right)\times 10=55-\frac{53}{14}=55-3.79=51.21\ \textbf{(approx.)}.$$

Example 46. *Find the missing frequencies in the following distribution when the mean is* 11.09:

Class-limits	9.3–9.7	9.8–10.2	10.3–10.7	10.8–11.2	11.3–11.7	11.8–12.2	12.3–12.7	12.8–13.2	Total
Frequency	2	5	f_3	f_4	14	6	3	1	60

Solution: Mean $(\bar{x})=11.09$, $N=\Sigma f=60$.

TABLE 4.27: CALCULATIONS FOR MEAN

Class-limits	Mid-value x	Frequency f	$d=\frac{x-11}{i}$ $i=0.5$	fd
9.3–9.7	9.5	2	−3	−6
9.8–10.2	10.0	5	−2	−10
10.3–10.7	10.5	f_3	−1	$-f_3$
10.8–11.2	$11.0=A$	f_4	0	0
11.3–11.7	11.5	14	1	14
11.8–12.2	12.0	6	2	12
12.3–12.7	12.5	3	3	9
12.8–13.2	13.0	1	4	4
Total		$31+f_3+f_4=N=60$	—	$23-f_3=\Sigma fd$

Mean $(\bar{x})=A+\frac{\Sigma fd}{N}\times i$, where $d=\frac{x-A}{i}$. Here $A=11$, $i=0.5$.
or, $11.09=11.0+\frac{23-f_3}{60}\times 0.5$, or, $11.09-11.0=\frac{23-f_3}{60}\times\frac{1}{2}$
or, $0.09\times 120=23-f_3$ or, $f_3=23-10.8=12.2$ [$\because$ $0.09\times 120=10.8$].
But the value of f_3 cannot be a fraction; $\therefore$ $f_3=12$.
Again, $31+f_3+f_4=60$ or, $31+12+f_4=60$ or, $f_4=60-43=17$.
Hence, the required missing frequencies are $\boldsymbol{f_3=12}$ **and** $\boldsymbol{f_4=17}$.

Example 47. *Find the third and the 8th deciles and also the 35th and 75th percentiles from the following frequency distribution:*

Class-interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	10	15	25	40	50	30	20	10

Solution:

TABLE 4.28: CALCULATIONS OF CUMULATIVE FREQUENCY

Class-boundary	Cumulative Frequency ("less than")
0	0
10	10
20	25
30	50
$D_3 \longrightarrow$	$\longleftarrow 60 = \frac{3N}{10}$
$P_{35} \longrightarrow$	$\longleftarrow 70 = 35N/100$
40	90
50	140
$P_{75} \longrightarrow$	$\longleftarrow 150 = 75N/100$
$D_8 \longrightarrow$	$\longleftarrow 160 = \frac{8N}{10}$
60	170
70	190
80	$200 = N$

$N = 200$

For D_3, $\frac{3N}{10} = 60$

For D_8, $\frac{8N}{10} = 160$

For P_{35}, $\frac{35N}{100} = 70$

For P_{75}, $\frac{75N}{100} = 150$

$$D_3 = \text{3rd Decile} = l_3 + \frac{\frac{3N}{10} - F_3}{f_3} \times c_3 = 30 + \frac{60-50}{40} \times 10 = 30 + 2.5 = 32.5.$$

[$\because$ D_3-class is 30–40]

$$D_8 = \text{8th Decile} = l_8 + \frac{\frac{8N}{10} - F_8}{f_8} \times c_8 = 50 + \frac{160-140}{30} \times 10 = 50 + 6.67 = 56.67.$$

[$\because$ D_8-class is 50–60]

$$P_{35} = \text{35th Percentile} = l_{35} + \frac{\frac{35N}{100} - F_{35}}{f_{35}} \times c_{35} = 30 + \frac{70-50}{40} \times 10 = 35$$

$$\text{and } P_{75} = \text{75th Percentile} = l_{75} + \frac{\frac{75N}{100} - F_{75}}{f_{75}} \times c_{75} = 50 + \frac{150-140}{30} \times 10 = 50 + 3.33 = 53.33.$$

EXERCISES ON CHAPTER 4(I)

Theory

1. Explain what is meant by Central Tendency of data. What are the common measures of central tendency?

2. What do you understand by 'Average'? Define at least 4 types of such averages, and then state their advantages and disadvantages. Give example of each type.

3. (a) Define Arithmetic Mean, Geometric Mean and Harmonic Mean, and compare their relative advantages and disadvantages.
   (b) Define Median of a distribution. Give one example.
   (c) What are the requisites of a good average? [Utkal U. B.Com. 2000]

4. (a) Write short notes on Mean, Median and Mode.
   (b) Point out the merits and demerits of the mean, the median and the mode as measures of central tendency as well as the situations where to use each.
   (c) What are quartiles of a distribution? How do you use them in measuring dispersion?
   (d) What do you understand by Weighted Mean?

5. Define and discuss the 'quartiles' of a distribution. Explain their uses.

6. (a) Explain the terms: Simple Mean and Weighted Mean.
   (b) Distinguish between simple and weighted averages, and state the circumstances under which the latter should be employed.

7. Give a critical review of the different measures of central tendency with examples.

8. (a) Prove that the Arithmetic Mean of two unequal positive numbers x_1, x_2 is greater than their Geometric Mean.

   [**Hints:** Since square of any real number > 0, if the number $\neq 0$;
   $\therefore$ $(\sqrt{x_1}-\sqrt{x_2})^2 > 0$, where $x_1 \neq x_2$ or, $x_1 + x_2 - 2\sqrt{x_1x_2} > 0$ or, $x_1 + x_2 > 2\sqrt{x_1x_2}$ or, $\frac{x_1+x_2}{2} > \sqrt{x_1x_2}$, i.e., AM > GM]

   (b) For any two positive numbers, prove that AM $\geq$ GM $\geq$ HM. [V.U. B.Com.(H) 2011]

9. Explain mean, median and mode as measures of central tendency. Give a comparative study of their relative advantages and disadvantages as measures of central tendency. [C.U. B.Com.(H) 1991]

Problems (A)

1. (a) The weekly wages of 5 labourers are (in ₹) 40, 60, 36, 45, 25. Calculate their AM.
   (b) Find the arithmetic mean of 14, 16, 19, 25, 21.
   (c) If the mean of 7, $x-3$, 10, $x+3$ and $x-5$ is 15, find x. [C.U. B.Com. 2002]

2. (a) Find the Geometric Mean and the Harmonic Mean of the numbers 1, 9, 81.

(b) Find the Geometric Mean of:

(i) 1, 3, 9.

(ii) 2, 9, 12.

(iii) 3, 8, 9.

(iv) 6, 24, 12.

(v) 2, 8, 9 with weights 1, 1, 2.

(c) If the GM of a, 4, 8 be 6, find a. [C.U. B.Com. 2006]

3. Calculate the Arithmetic Mean, Geometric Mean and Harmonic Mean of four numbers:

(a) 3, 6, 24 and 48. [C.U. B.Com. 1997]

(b) 4, 6, 12 and 72.

(c) 24, 72, 108 and 144.

4. Find the Harmonic Mean of the following numbers:

(a) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$.

(b) $1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}$.

5. (a) Find the Median of the following numbers:

(i) 7, 2, 5, 9, 6.

(ii) 8, 3, 11, 7, 12, 6, 9.

(iii) 33, 86, 68, 32, 80, 48, 70.

(iv) 25, 15, 23, 40, 27, 25, 23, 25 and 20. [V.U. B.Com.(H) 2008]

(b) Find the Mode of the numbers:

(i) 5, 3, 27, 5, 9, 3, 8, 5.

(ii) 4, 3, 2, 5, 3, 4, 5, 3, 7, 3, 2, 6.

(iii) 3, 2, 5, 4, 4, 2, 4, 3, 3, 4, 4, 5, 4, 2, 4, 4, 2, 4, 5, 4, 4. [C.U. B.Com. 2000]

(c) Find the Mode and Median of the following numbers:

(i) 25, 1275, 748, 162, 967, 162;

(ii) 108, 94, 107, 84, 108, 79, 115, 119, 98, 102, 122. [V.U. B.Com.(H) 2007]

6. Find the Median of the following numbers:

(a) 3, 9, 7, 4, 8, 6. [C.U. B.Com. 1995]

(b) 94, 33, 86, 68, 32, 80, 48, 70. [C.U. B.Com. 2007]

(c) 79, 82, 36, 38, 51, 72, 68, 70, 64, 63.

7. (a) Find the Mean and Mode of the numbers: 4, 3, 2, 5, 3, 4, 5, 1, 7, 3, 2, 1.

(b) Find the Mean and the Mode of the set of numbers: 7, 4, 10, 15, 7, 3, 5, 2, 9, 12.

(c) Find the Median and Mode of the numbers: 3, 2, 5, 4, 4, 2, 4, 3, 3, 4, 4, 5, 4, 2, 4, 4, 2, 4, 5, 4, 4. [C.U. B.Com. 2000]

(d) Find the Median and Mode of the numbers: 4, 10, 7, 15, 7, 3, 5, 3, 7. [C.U. B.Com. 1990]

(e) Find the Mean, Median and Mode of the following numbers: 7, 4, 3, 5, 6, 3, 3, 2, 4, 3, 4, 3, 3, 4, 4, 3, 2, 2, 4, 3, 5, 4, 3, 4, 3, 4, 3, 1, 2, 3.

8. Find the AM of the following distribution:

Weights (in pounds)	100	110	120	130	140
No. of men	15	20	25	30	10

9. Calculate the Arithmetic Mean and Mode from the following data:

Value	1	2	3	4	5	6	7	8	9	Total
Frequency	7	11	16	17	26	31	11	1	1	121

10. Find the mean of weekly wages from the following frequency distribution:

Wages (in ₹)	30–40	40–50	50–60	60–70	70–80	80–90
No. of workers	10	20	40	16	8	6

11. (a) Find the arithmetic mean of the weekly income from the following frequency distribution:

Weekly income (in ₹)	20–25	25–30	30–35	35–40	40–45	45–50
No. of workers	200	700	900	800	600	400

(b) Calculate the mean from the following table:

Monthly wages (in ₹)	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90
No. of workers	15	44	104	225	310	355	108	72	17

[B.U. B.Com. 1998]

12. Find the mode of the following distributions:

(a)

Marks	5	10	15	20	25	30	35
No. of students	2	6	10	15	12	8	4

(b)

Marks	0–10	10–20	20–30	30–40	40–50
No. of students	4	12	18	14	8

13. (a) Find the median from the following frequency distribution:

Marks	0–10	10–20	20–30	30–40	40–50
No. of students	10	20	35	25	10

(b) Find the median of the following distribution:

Weight in lb	160–162	163–165	166–168	169–171	172–174
No. of men	15	54	126	81	24

[C.U. B.Com. 2007]

14. (a) A sample of size 50 has mean 54.4 and another sample of size 100 has mean 50.3. If the two samples are pooled together, find the mean of the combined sample. **[B.U. B.Com. 1990]**

(b) The mean annual salary of all employees in a company is ₹25,000. The mean salaries of male and female employees are ₹27,000 and ₹ 17,000 respectively. Find the percentage of males and females employed in the company. [CA Foun. Nov. 1995]

15. (a) If the Mean and Mode of a certain set of numbers be 60.4 and 50.2 respectively, find approximately the value of the Median.

(b) For a moderately asymmetric distribution, Median = 27, Mean =26. Find the mode.

[C.U. B.Com. 2000]

(Use empirical relation between Mean, Median and Mode.)

16. (a) The table below gives the marks obtained in Statistics by 60 students:

Class-interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	5	8	11	15	13	6	2

With the help of the cumulative frequency diagram or otherwise, determine the median mark of a student. [C.U. B.Com. 2004]

(b) Obtain the quartiles of the following distribution:

Ages (in years)	50	52	54	58	60	62	64	66	68	70
Frequency	4	12	18	23	30	26	22	16	5	4

[C.U. B.Com. 2002]

17. Below is given the frequency distribution of weights of a group of 60 students in a class in a school:

Weight (in kg)	30–34	35–39	40–44	45–49	50–54	55–59	60–64
No. of students	3	5	12	18	14	6	2

Draw the cumulative frequency diagrams and hence determine the median weight. Find also the quartiles of the distribution.

18. The table below gives the frequency distribution of weights of 85 apples:

Weight in gm	110-119	120-129	130-139	140-149	150-159	160-169	170-179	180-189
Frequency	5	8	12	18	22	9	7	4

Determine the median weight of an apple.

19. (a) The frequency distribution of wages of 100 workers is as follows:

Wages (₹)	250–259	260–269	270–279	280–289	290–299	300–309
No. of workers	8	16	30	34	10	2

Calculate the value of the median from the above data.

(b) Find the mode of the following distribution:

Weight (in gm)	410-419	420-429	430-439	440-449	450-459	460-469	470-479
Frequency	14	20	39	54	45	18	10

20. (a) Marks obtained by 22 students are given below:

Marks obtained	0–10	10–20	20–30	30–40
No. of students	2	4	9	7

Find the mode of the above distribution.

(b) Find the mode of the following data:

Monthly wages (₹)	125–175	175–225	225–275	275–325	325–375	375–425	425–475
No. of workers	8	10	25	35	12	10	4

(c) Find the mode of the following frequency distribution:

Age (in years)	16-20	21-25	26-30	31-35	36-40	41-45	46-50
Population (in thousand)	4	6	11	5	7	8	3

[C.U. B.Com. 2007]

21. (a) The AM of the following frequency distribution is 36.3. Find the missing frequency:

Marks	10–20	20–30	30–40	40–50	50–60
No. of students	12	18	?	25	15

(b) The arithmetic mean calculated from the following frequency distribution is known to be 67.45 inches. Find the value of f_3.

Height (inches)	60–62	63–65	66–68	69–71	72–74
Frequency	15	54	f_3	81	24

[C.U. B.Com.(H) 2008]

22. The median of the following distribution is 33. Find the missing frequency.

Marks	0–10	10–20	20–30	30–40	40–50	50–60
No. of students	8	12	21	?	20	9

23. The mode of the following frequency distribution is ₹66. Find the missing frequency.

Daily wages (₹)	30–40	40–50	50–60	60–70	70–80	80–90
No. of workers	8	16	22	28	?	12

[C.U. B.Com. 2005]

24. (a) Construct the grouped frequency distribution from the following data and hence find its Arithmetic Mean:

Marks	No. of students
Less than 10	175
Less than 20	360
Less than 30	680
Less than 40	790
Less than 50	900
Less than 60	1000

[C.U. B.Com. 1995; V.U. B.Com. 1998]

(b) Construct a grouped frequency distribution from the following data and hence find the median and mode:

Marks obtained	No. of students
Below 10	175
Below 20	360
Below 30	680
Below 40	790
Below 50	900
Below 60	1000

[C.U. B.Com. 2008]

25. Construct the grouped frequency distribution from the following data and hence find its AM and Median:

Value	Below 10	Below 20	Below 30	Below 40	Below 50	Below 60	Below 70	Below 80
Frequency	4	16	40	76	96	112	120	125

[N.B.U. B.Com. 1994]

26. The AM and GM of two numbers are 25 and 15 respectively, find the two numbers.

[C.U. B.Com. 2008]

[Hints: Let the two numbers be x and y. Then $\frac{x+y}{2} = 25$ and $\sqrt{xy} = 15$ or, $x + y = 50$ and $xy = 225$.

$\therefore (x-y)^2 = (x+y)^2 - 4xy = (50)^2 - 4 \times 225 = 2500 - 900 = 1600$; $\therefore x - y = \pm 40$.

Solving $x + y = 50$ and $x - y = 40$, $x = 45$, $y = 5$. If $x + y = 50$, $x - y = -40$, then $x = 5$, $y = 45$. Hence, *the two numbers are* 5 and 45.]

27. An incomplete frequency distribution is given below:

Variable	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	12	30	f_1	65	f_2	25	18

If Median Value is 46 and total frequency is 229, find the two missing frequencies. [V.U. B.Com.(H) 2007]

[Hints: See worked-out Ex. 12 in Section 4.11. Here $N = 229$, Median = 46, $\frac{N}{2} = 114.5$, Median class is 40–50, $F = 42 + f_1$, $f_m = 65$, $l_1 = 40$, $c = 10$, $f_1 + f_2 = 229 - 150 = 79$. Now apply formulae for Median and fixed the relation $10f_1 = 335$ which gives $f_1 = 34$.]

Problems (B)

1. (a) The monthly incomes of 5 labourers are ₹150, ₹140, ₹165, ₹170 and ₹180. Calculate the Arithmetic Mean and the Geometric Mean.

   (b) The monthly incomes of 6 labourers are (in ₹) 70, 42, 85, 75, 68, 55. Calculate the AM and GM.

   (c) For a moderately asymmetrical distribution, median = 27, mean = 26. Find mode.

   [**Hints:** Use the formula: Mean − Mode = 3 (Mean − Median). **[C.U. B.Com. 2000]**]

   (d) Calculate the GM and HM of three numbers 1, 9 and 81.

2. Find the Geometric Mean of the following numbers, correct to two decimal places:

   (a) 90, 25, $\frac{125}{3}$, 81.

   (b) 126, 184, 267, 375, 458. **[N.B.U. B.Com. 1997]**

3. The price of a certain security in Bombay Stock Exchange on a certain day are given below. Find the median price:

$$100\tfrac{5}{16},\quad 100\tfrac{3}{8},\quad 100\tfrac{1}{4},\quad 100\tfrac{5}{16},\quad 100\tfrac{3}{16},\quad 100\tfrac{1}{4},\quad 100\tfrac{3}{8},\quad 100\tfrac{9}{32},$$
$$100\tfrac{11}{32},\quad 100\tfrac{1}{16},\quad 100\tfrac{1}{8},\quad 100,\quad 99\tfrac{7}{8},\quad 99\tfrac{9}{32},\quad 99\tfrac{11}{32},\quad 99\tfrac{3}{8},\quad 99\tfrac{1}{4}.$$

4. (a) The average weight of the following frequency distribution is 117 lb:

Weight in lb	100	110	120	$x+25$	140	Total
No. of persons	1	4	2	2	1	10

   Find the value of x.

   (b) The mean of 20 observations is 85; but it was later found that two of the observations were wrongly read as 75 and 70 instead of 57 and 60. Find the actual mean.

   Solution: By definition, $\frac{\Sigma x}{20}$ = mean = 85 or, $\Sigma x = 85 \times 20 = 1700$.

   Correct value of $\Sigma x = 1700 - (75+70) + (57+60) = 1672$.

   $\therefore$ actual mean $= \frac{\text{Correct value of } \Sigma x}{20} = \frac{1672}{20} = \mathbf{83.6}$.

   (c) The AM of 25 observations is 44; later on it was reported that two of the observations 34 and 46 were copied as 28 and 42. Find the actual AM.

5. (a) Form a frequency distribution with 8 classes from the following data and work out the mean, the median and the mode from it.

   Data: In a workshop employing 30 persons the daily wages paid are as follows:

(in ₹)	2.30,	3.50,	2.30,	2.40,	3.20,	5.10,	4.50,	3.50,	2.30,	3.40
	2.30,	4.50,	5.10,	4.50,	5.50,	2.40,	3.50,	3.20,	2.30,	3.40
	4.50,	2.30,	3.50,	3.20,	2.40,	5.10,	3.40,	2.40,	5.10,	3.40

[V.U. B.Com. 1997]

(b) The heights of students of a college are given below. Find the median height:

Height in cm	160–163	164–167	168–171	172–175	176–179	180–183	184–187
No. of students	22	80	98	148	104	43	5

[C.U. B.Com. 1994]

6. Find the Mean and the Mode from the following frequency distributions:

(a)

Marks obtained	No. of candidates
0–9	6
10–19	29
20–29	87
30–39	181
40–49	247
50–59	263
60–69	133
70–79	43
80–89	9
90–99	2
Total	1000

(b)

Output in units	No. of workers
300 to 309	9
310 to 319	20
320 to 329	24
330 to 339	38
340 to 349	48
350 to 359	27
360 to 369	17
370 to 379	6

7. (a) Find the Mode from the following distribution:

Marks	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45
No. of candidates	7	10	16	32	24	18	10	5	1

(b) Find the mode of the following frequency distribution. The monthly incomes of 300 workers of a factory are as follows:

Monthly incomes (₹)	1000–1100	1100–1200	1200–1300	1300–1400	1400–1500	1500–1600	1600–1700	1700–1800
No. of workers	16	24	59	100	41	31	19	10

[C.U. B.Com. 1997]

8. (a) Calculate the AM and the Median of the frequency distribution given below. Hence calculate the mode using the empirical relation between the three:

Class-limits	130–134	135–139	140–144	145–149	150–154	155–159	160–164
Frequency	5	15	28	24	17	10	1

[B.U. B.Com. 1990]

(b) Find the Mean and Median of height from the following table:

Height (x cm)	158–161	162–165	166–169	170–173	174–177	178–181	Total
No. of men (f)	11	23	31	18	12	5	100

[V.U. B.Com. 1995]

9. In a small town, a survey was conducted in respect of profit made by retail shops. The following results were obtained:

Profit or Loss in '000 ₹	No. of shops	Profit or Loss in '000 ₹	No. of shops
−4 to −3	4	1 to 2	56
−3 to −2	10	2 to 3	40
−2 to −1	22	3 to 4	24
−1 to 0	28	4 to 5	18
0 to 1	38	5 to 6	10

Calculate: (a) the average profit made by a retail shop; (b) total profit by all shops.

10. (a) Find out the missing frequencies of the following data. AM is 67.45 inches:

Height (inches)	60–62	63–65	66–68	69–71	72–74	Total
No. of students	5	18	f_3	f_4	8	100

[C.U. B.Com. 2007]

(b) An incomplete frequency distribution is given below:

Variable	10–20	20–30	30–40	40–50	50–60	60–70	70–80	Total
Frequency	15	12	?	56	?	32	18	200

You are given that the median value is 48.93. Using the median formula fill in the missing frequencies. [V.U. B.Com. 1996]

(c) Find the missing frequencies in the following frequency distribution when the mean is 11.09:

Class-limits	9.3-9.7	9.8-10.2	10.3-10.7	10.8-11.2	11.3-11.7	11.8-12.2	12.3-12.7	12.8-13.2	Total
Frequency	2	5	f_3	f_4	14	6	3	1	60

[C.U. B.Com. 2002]

11. (a) Form an ordinary frequency table from the following cumulative distribution of marks obtained by 22 students and calculate —

(i) Arithmetic Mean,
(ii) Median,
(iii) Mode.

Marks	No. of students
Below 10	3
Below 20	8
Below 30	17
Below 40	20
Below 50	22

[C.U. B.Com.(H) 1990; V.U. B.Com.(H) 2009]

(b) Calculate the values of median from the following frequency distribution:

Marks	No. of students
less than 10	160
less than 20	210
less than 30	350
less than 40	480
less than 50	730
less than 60	1000

[C.U. B.Com. 2008 Type]

12. (a) Calculate the quartiles of the following data:

Class-limits	Frequency	Class-limits	Frequency
10–19	5	50–59	25
20–29	9	60–69	15
30–39	14	70–79	8
40–49	20	80–89	4

Find also skewness based on quartiles.

(b) Calculate the quartiles from the following data:

x	4–8	8–12	12–16	16–20	20–24	24–28
f	5	8	18	25	14	10

[C.U. B.Com. 2003]

13. You are given the following incomplete frequency distribution. It is known that the total frequency is 1000 and that the median is 413.11. Estimate by calculation the missing frequencies and find the value of the mode.

Value	Frequency	Value	Frequency
300–325	5	400–425	326
325–350	17	425–450	?
350–375	80	450–475	88
375–400	?	475–500	9

14. (a) Find the Median and Mode from the following table:

Age	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–60
No. of men	5	70	100	180	150	120	70	60

[C.U. B.Com. 1991]

(b) Determine the Median and Modal Price of the following distribution:

Price of the Commodity (₹)	·310–319	320–329	330–339	340–349	350–359	360–369	370–379	380–389	Total
Frequency	5	8	12	28	32	9	7	4	105

[C.U. B.Com. 2000]

15. You are given below a certain statistical distribution:

Value	Frequency
Less than 100	40
100–200	89
200–300	148
300–400	64
400 and above	39
Total	380

Calculate the most suitable average giving reasons for your choice.

16. The table below gives the number (F) of candidates obtaining marks x or higher in a certain examination (all marks are given in whole numbers):

x	10	20	30	40	50	60	70	80	90	100
F	140	133	118	100	75	45	25	9	2	0

Calculate the mean and the median marks obtained by the candidates.

17. The following table shows the marks in Statistics, secured by 60 students:

Class-interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	5	8	11	15	13	6	2

With the help of a cumulative frequency diagram determine the median mark of the students.

[C.U. B.Com. 2002]

[**Hints:** See worked-out Ex. 24 in **Section 4.8.1.**]

18. Form a frequency distribution with 8 classes of equal class-intervals from the following data and work out the mean, median and mode from it:

Data: In a workshop employing 30 persons the hourly wages paid are as follows:

₹ 2.30, 3.50, 2.30, 2.40, 3.20, 5.10, 4.50, 5.30, 2.30, 3.40,
2.30, 4.50, 5.10, 4.50, 5.50, 2.40, 3.50, 3.20, 2.30, 3.40,
4.50, 2.30, 3.50, 3.20, 2.40, 5.10, 3.40, 2.40, 5.10, 3.40.

[V.U. B.Com. 1997]

ANSWERS

A

1. (a) 41.2;
   (b) 19;
   (c) 21.
2. (a) GM = 9, HM = $2\frac{61}{91}$;
   (b) (i) 3;
   (ii) 6;
   (iii) 6;
   (iv) 12;
   (v) 6.
   (c) $a = \frac{27}{4}$.
3. (a) AM = 20.25, GM = 12, HM = 7.1;
   (b) AM = 23.5, GM = 12, HM = 7.8;
   (c) AM = 87, GM = 72, HM = 55.74.
4. (a) $\frac{2}{n+1}$;
   (b) $\frac{1}{n}$.
5. (a) (i) 6;
   (ii) 8;
   (iii) 68;
   (iv) 25.
   (b) (i) 5;
   (ii) 3;
   (iii) 4;
   (c) (i) Mode = 162, Median = 455;
   (ii) Mode = 108, Median = 107.
6. (a) 6.5;
   (b) 69;
   (c) 66.
7. (a) Mean = 3.33, Mode = 3,
   (b) 7.4 and 7;
   (c) 3.62, 4;
   (d) 7 and 7;
   (e) Mean = 3.47, Median = 3, Mode = 3.
8. 120 lb;
9. AM = 4.59, Mode = 6.
10. ₹56.
11. (a) ₹35.41;
    (b) ₹46.84.
12. (a) 20;
    (b) 26.
13. (a) 25.71;
    (b) 167.43 lb.
14. (a) 51.7;
    (b) 80% and 20%.
15. (a) 57;
    (b) 29.
16. (a) 34;
    (b) 58, 60, 64.
17. Median wt. = 47.3 kg,
    Q_1 = 42.4 kg,
    Q_3 = 52.0 kg.
18. 149.22 gm.
19. (a) ₹278.19;
    (b) 445.75 gm.
20. (a) 27.14;
    (b) ₹290.15;
    (c) 27.77 years.
21. (a) 30; (b) $f_3 = 126$.
22. 30.
23. 24.
24. (a) 25.45.
    (b) 24.375; 23.91.
25. 37.88; 36.25.
26. 5 and 45.
27. $f_1 = 34$ and $f_2 = 45$.

B

1. (a) AM = ₹161, GM = ₹160.30;
   (b) AM = ₹65.83, GM = ₹64.21;
   (c) 29;
   (d) 9 and $\frac{243}{11}$.
2. (a) 69.08;
   (b) 251.5.
3. $100\frac{3}{16}$.
4. (a) $x = 100$;
   (c) 44.4.
5. (a) Mean = ₹3.48, Median = ₹3.40, Mode = ₹2.30;
   (b) 172.85 cm.

6. (a) Mean = 46.88, Mode = 50.60;
   (b) Mean = 339.05, Mode = 342.73.
7. (a) 18.83;
   (b) ₹1,341.
8. (a) AM = 145.35, Median = 144.92, Mode = 144.06;
   (b) 167.98 cm, 167.56 cm.
9. Mean = ₹1348, Total Profit = ₹3,37,000.
10. (a) $f_3 = 42$, $f_4 = 27$;
    (b) 23 and 44;
    (c) 12, 17.
11. (a) AM = 23.18, Median = 23.33, Mode = 24;
    (b) 40.8.
12. (a) $Q_1 = 37.36$, $Q_2 = 50.3$, $Q_3 = 60.83$, Skewness = -0.1027;
    (b) 13.56, 17.44, 21.14.
13. 227 and 248, Mode = 413.98.
14. (a) Median = 40.75, Mode = 38.64;
    (b) ₹349.32, ₹350.98.
15. Median = 241.22.
16. Mean = 50.715 or 51.214, Median = 51.166 or 51.67.
17. 34.
18. ₹3.58, ₹3.40 and 2.30.

EXERCISES ON CHAPTER 4(II)

(Miscellaneous Problems)

1. (a) Find the quartiles of the values: 35, 30, 48, 40, 25, 36, 45.
   (b) Find the deciles of the following values: 50, 80, 60, 30, 40, 10, 45, 20, 25, 75, 55, 65, 35, 52, 44, 38, 63, 34, 46.
   How many deciles lie below the first quartile?
   (c) The numbers 3.2, 5.8, 7.9 and 4.5 have frequencies x, $x+2$, $x-3$ and $x+6$ respectively. If the arithmetic mean is 4876, find the value of x.

2. (a) The mean wage of 100 workers working in a factory running two shifts of 60 and 40 workers respectively is ₹38. The mean wage of 60 labourers working in the morning shift is ₹40. Find the mean wage of labourers working in the evening shift.
   (b) The algebraic sum of the deviations of 25 observations measured from 45 is -55; find the AM of the observations. **[C.U. B.Com. 1991]**

   [**Hints:** $\sum_{i=1}^{25}(x_i - 45) = -55$ or, $\sum_{i=1}^{25} x_i - \sum_{i=1}^{25} 45 = -55$ or, $\sum_{i=1}^{25} x_i - 25 \times 45 = -55$
   or, $\sum_{i=1}^{25} x_i = 1125 - 55 = 1070$. $\therefore$ mean $= \frac{\Sigma x_i}{n} = \frac{1070}{25} = 42.8$.]

   (c) For some symmetrical distribution, $Q_1 = 24$ and $Q_3 = 42$; find Median. **[C.U. B.Com. 1998]**

3. Find the missing frequency from the following frequency distribution if Mean is 38:

Marks	10	20	30	40	50	60	70
No. of students	8	11	20	25	?	10	3

4. (a) The mean of 20 observations is 16.5. If by mistake one observation was copied 12 instead of 21, find the correct value of mean.

(b) The mean marks of 100 students was found to be 40. Later on it was discovered that a mark 53 was misread as 83. Find the correct mean mark. [C.U. B.Com. 1990]

[Hints: Mean = $\frac{\Sigma x}{n}$ or, $40 = \frac{\Sigma x}{100}$ or, $\Sigma x = 4000$. Correct value of $\Sigma x = 4000 - 83 + 53 = 3970$.
$\therefore$ correct Mean = $\frac{\text{Correct}\Sigma x}{100} = \frac{3970}{100} = 39.7$.]

5. The mean weekly salary paid to 77 employees in a company was ₹78. The mean salary of 32 of them was ₹75 and of other 25 was ₹82. What was the mean salary of the remaining?

6. (a) The mean annual salary paid to all employees of a company was ₹5000. The mean annual salaries paid to male and female employees were ₹ 5200 and ₹4200 respectively. Determine the percentage of males and females employed by the company.

(b) The mean monthly salary paid to all employees in a certain company was ₹600. The mean monthly salaries paid to male and female employees were ₹620 and ₹520 respectively. Obtain the percentage of male and female employees in the company.

7. An aeroplane flies along the four sides of a square at speeds of 150, 300, 450 and 600 km per hour respectively. Using an appropriate mean, find the average speed of the plane around the square.

8. Draw an ogive for the following distribution. Locate the median from the graph and verify the result by direct application of the formula. How many workers earned monthly wages between ₹60 and ₹72?

Monthly Wages (₹)	50–55	55–60	60–65	65–70	70–75	75–80	80–100
No. of workers	6	10	22	30	16	12	15

9. (a) Find the median and mode of the following frequency distribution:

No. of days absent (less than)	5	10	15	20	25	30	35	40	45
No. of students	29	224	465	582	634	644	650	653	655

(b) The frequency distributions of marks obtained by 1000 students in an examination are given below. Calculate the Arithmetic Mean:

Marks obtained	No. of students	Marks obtained	No. of students
0–9	25	50–59	225
10–19	37	60–69	46
20–29	81	70–79	22
30–39	290	80–89	17
40–49	253	90–99	4

10. The following table gives the distribution of 100 families according to expenditure. If Mode of the distribution is 24, find the missing frequencies x and y :

Expenditure	0–10	10–20	20–30	30–40	40–50
No. of families	14	x	27	y	15

11. Following is the distribution of marks in Law obtained by 50 students:

Marks more than	0	10	20	30	40	50
No. of students	50	46	40	20	10	3

Calculate the median marks. If 60% of the students pass this test, find the minimum marks obtained by a pass candidate.

12. The AM, the mode and the median of a group of 75 observations were calculated to be 27, 34 and 29 respectively. It was later discovered that one observation was wrongly read as 43 instead of 53. Examine to what extent the calculated values of the three averages will be affected by the discovery of the error.

13. The median and the mode of the following daily wage distribution of 230 workers are known to be ₹33.5 and ₹34 respectively. Three frequency values from the table are, however, missing. Find these missing values?

Wages in ₹	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of workers	4	16	?	?	?	6	4

14. (a) In a distribution, the difference of two quartiles is 2.03; their sum is 72.67 and median 36.18, find the coefficient of skewness.

(b) For an income distribution of a group of men, 20% of men have income below ₹35, 35% below ₹75, 60% below ₹175 and 80% below ₹250. The first and third quartiles are ₹ 55 and ₹ 200. Put the above information in a cumulative frequency distribution and find the median.

[Hints: Let the total number of men in the group be 100. Then

Income (₹) (less than)	35	55 (= Q_1)	75	175	200 (= Q_3)	250	250 and above
Percentage of men (c.f.)	20	25	35	60	75	80	100

Use simple interpolation to find median.]

15. Show that the weighted arithmetic mean of first n natural numbers whose weights are equal to the corresponding numbers is equal to $\frac{2n+1}{3}$.

16. The following table gives the weekly wages in ₹in a certain commercial organization:

Weekly Wages	30–	32–	34–	36–	38–	40–	42–	44–	46–	48–50
Frequency	3	8	24	31	50	61	38	21	12	2

Find (a) the median and the first quartile; (b) the number of wage-earners receiving between ₹ 37 and ₹ 47 per week.

17. Using suitable formulae calculate the mean and the median from the following data:

Mid-value	115	125	135	145	155	165	175	185	195	Total
Frequency	6	25	48	72	116	60	38	22	3	390

18. (a) Draw an ogive to illustrate the following data and from it determine: (a) median and quartile wages, and (b) the eighth decile wages:

Weekly Wages (in ₹) x	30–	32–	34–	36–	38–	40–	42–	44–	46–	48–50	Total
Number of Wage-earners	2	9	25	30	49	62	39	20	11	3	250

(b) The table below gives the marks obtained by 590 students in a certain examination. Find the number of students obtained less than 50 marks in that examination:

Marks	Less than 20	20–40	40–60	60–80	80–100
No. of students	250	120	100	70	50

[C.U. B.Com. 2004]

19. Calculate the median and mode of the following:

Annual Sales (₹ '000)	Frequency	Annual Sales (₹ '000)	Frequency
Less than 10	4	Less than 40	55
Less than 20	20	Less than 50	62
Less than 30	35	Less than 60	67

Is it possible to calculate the arithmetic mean? If possible, calculate it.

20. By using the quartiles find a measure of skewness for the following distribution:

Annual Sales (₹ '000)	Number of Firms
Less than 20	30
Less than 30	225
Less than 40	465
Less than 50	580
Less than 60	634
Less than 70	644
Less than 80	650
Less than 90	665
Less than 100	680

21. Construct the grouped frequency distribution from the following data and hence find its Arithmetic Mean:

Marks	Less than 10	Less than 20	Less than 30	Less than 40	Less than 50	Less than 60
No. of students	175	360	680	790	900	1000

[C.U. B.Com. 1995; V.U. B.Com.(H) 2007]

ANSWERS

1. (a) 30, 36 and 45;
   (b) 20, 30, 35, 40, 45, 50, 55, 63, 75; two deciles 20 and 30 lie below the first quartile;
   (c) 5.
2. (a) ₹ 35;
   (b) ₹ 42.8;
   (c) ₹ 33.
3. 18.
4. (a) 16.95;
   (b) 39.7.
5. ₹ 77.80.
6. (a) 80% and 20%;
   (b) 80% and 20%.
7. 288 km per hour.
8. 67.9, 58.
9. (a) 12.14 days, 11.35 days;
   (b) 42.58.
10. 23 and 21.
11. 27.5; 25.
12. Correct AM = 27.13; Median and Mode will remain unchanged.
13. 60, 100 and 40.
14. (a) 0.15;
    (b) ₹ 135.
16. (a) ₹ 40.30, ₹ 37.77;
    (b) 192.
17. 153.64; 153.8.1700*50
18. (a) ₹ 40.30, ₹ 37.80,
    ₹ 42.50, ₹ 43.20;
    (b) 420.
19. ₹ 29,000, ₹ 32,778.
20. 0.09.
21. 23.91.

Chapter 5

Measures of Dispersion

5.1 Introduction

We have seen how statistical data in the form of a frequency distribution can be represented by a single typical value, known as *statistical average* (or *simply average*). An average may give a good idea of the type of data, but it alone cannot reveal all the characteristics of the data. It cannot tell us in what manner all the values of the variable are scattered (or dispersed) about the average. Two series having the same number of values may have the same mean, but still the values in one may be widely dispersed and the values in the other may be close to one another. Thus it is necessary to know the scatter or dispersion of the values from their mean. In this chapter we shall study the different measures of dispersion of the values of a variable.

5.2 Dispersion

The variation or scattering or deviation of the different values of a variable from their average is known as *dispersion*. Dispersion indicates the extent to which the values vary among themselves.

Measures of Dispersion

There are seven measures of dispersion of which four are absolute measures and three are relative measures.

Absolute Measures

1. Range, 2. Quartile Deviation, 3. Mean Deviation, 4. Standard Deviation.

Relative Measures

1. Coefficient of Variation, 2. Coefficient of Quartile Deviation, 3. Coefficient of Mean Deviation.

5.3 Range

Definition 1. *Range is the simplest absolute mean of dispersion. It is the difference between the largest and the smallest values of a variable.*

It is simple to understand and easy to calculate, but it is not generally used for practical purposes due to its dependence entirely on the two extreme values (i.e., the largest and the smallest values). Range of the values (in ₹) 8, 5, 10, 7, 12, 6 is $12 - 5 = 7$ (₹), since the largest value = 12 and the smallest value = 5.

In a grouped frequency distribution, the range is measured by the difference between the upper boundary of the highest class and the lower boundary of the lowest class. It is also measured by the difference between the mid-values of the highest and the lowest classes.

$$\text{Coefficient of Range} = \frac{\text{Range}}{\text{Sum of the highest and the lowest values}}.$$

Advantages and Disadvantages

Range is simple to understand and easy to calculate. But it has many disadvantages. It is very much affected by the presence of an extremely high or low value. It is not based on all the values of the variable. It cannot be calculated from grouped frequency distribution with open-end classes.

5.4 Quartile Deviation or Semi-interquartile Range

Definition 1. *Quartile Deviation* (Q) *is an absolute measure of dispersion and is defined by the formula* $Q = \frac{(Q_3 - Q_1)}{2}$, *where* Q_1 *and* Q_3 *are the first (or lower) and the third (or upper) quartiles respectively.*

As it is based only on Q_1 and Q_3, it does not take into account the variability of all the values and hence it is not very much used for practical purposes.

Example 1. *Find the Quartile Deviation of the daily wages (in ₹) of 7 persons given:* 12, 7, 15, 10, 19, 17, 25.

Solution: Arranging the given values in ascending order of magnitudes, we get 7, 10, 12, 15, 17, 19, 25.

Here $n = 7$,

$$\frac{n+1}{4} = \frac{7+1}{4} = 2, \quad \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = 6.$$

$\therefore Q_1 = \frac{n+1}{4}$th value = 2nd value = 10 and $Q_3 = \frac{3(n+1)}{4}$th value = 6th value = 19.

$$\text{Hence, Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{19 - 10}{2} = ₹\ 4.50.$$

Advantages and Disadvantages

Quartile Deviation is easy to calculate and its calculation depends on the first and the third quartiles. It is not based on all the values of the variable and so it does not take into account the variation of each observation about the average (central value). It can be calculated from a grouped frequency distribution with open-end classes.

5.5 Mean Deviation (or Average Deviation or Mean Absolute Deviation)

Definition 1. *Mean Deviation of a series of values of a variable is the arithmetic mean of all the absolute deviations (i.e., difference without regard to sign) from any one of its averages (Mean, Median or Mode, but usually Mean or Median). It is an absolute measure of dispersion.*

Mean Deviation of a set of n values $x_1, x_2, \ldots, x_n$ about their AM is defined by

$$\textit{Mean Deviation about mean} = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} = \frac{\sum_{i=1}^{n} |d_i|}{n} = \frac{\Sigma|d|}{n}, \quad \text{i.e.,} \quad \textbf{Mean Deviation} = \frac{\Sigma|d|}{n},$$

where $\bar{x}$ = AM and $|d_i| = |x_i - \bar{x}|$ = absolute deviation of x_i from $\bar{x}$.

$$\textit{Mean Deviation about median } M = \frac{\sum_{i=1}^{n}|x_i - M|}{n} = \frac{\sum_{i=1}^{n}|d_i|}{n}, \text{ where } |d_i| = |x_i - M|.$$

For a frequency distribution,

$$\textbf{Mean Deviation about mean } \bar{x} = \frac{\Sigma f\,|x-\bar{x}|}{\Sigma f} = \frac{\Sigma f\,|d|}{N},$$

where x = value or mid-value according as the data is ungrouped or grouped and $\bar{x}$ = Mean and

$$\textbf{Mean Deviation about Median } M = \frac{\Sigma|x-M|}{\Sigma f} = \frac{\Sigma|d|}{N},$$

where $|d| = |x - M|$.

Note: The expression $|d|$ is read as mod. d and gives only numerical or absolute value of d without regard to sign. Thus $|-3| = 3$, $|+4| = 4$, $|-0.56| = 0.56$.

The reason for taking only the absolute and not the algebraic values of the deviations is that the algebraic sum of the deviations of the values from their mean is zero.

Advantages and Disadvantages

Mean Deviation is based on all the values of the variable and sometimes gives fairly good result as a measure of dispersion. However, the practice of neglecting signs and taking absolute deviations for the calculation of Mean Deviation seems rather unjustified and this makes algebraic treatment difficult.

Coefficient of Mean Dispersion

The coefficient of Mean Dispersion is defined by the formula:

$$\textbf{Coefficient of Mean Dispersion} = \frac{\textbf{Mean Deviation from Mean}}{\textbf{Mean}} \text{ or, } \frac{\textbf{Mean Deviation from Median}}{\textbf{Median}}.$$

Example 2. *Calculate the Mean Deviation of the following values about the median in respect of the following data:* 8, 15, 53, 49, 19, 62, 7, 15, 95, 77. [C.U. B.Com. 2008 Type]

Solution: Arranging the given values in ascending order of magnitude, we get 7, 8, 15, 15, 19, 49, 53, 62, 77, 95.

$$\text{Median } (M) = \frac{n+1}{2}\text{th value} = \frac{10+1}{2}\text{th value} = 5.5\text{th value}$$

$$= \text{Mean of the 5th and 6th values} = \frac{19+49}{2} = \frac{68}{2} = 34.$$

Absolute deviations of the values from the median 34 are respectively 27, 26, 19, 19 15, 15, 19, 28, 43, 61.

$$\therefore \text{ Mean Deviation about the Median} = \frac{\Sigma|d|}{n} = \frac{27+26+19+19+15+15+19+28+43+61}{10}$$

$$= \frac{272}{10} = 27.2.$$

Example 3. *Find the Mean Deviation about the Arithmetic Mean of the numbers* 31, 35, 29, 63, 55, 72, 37. [C.U. B.Com. 2003]

Solution: $$\text{Arithmetic Mean } (\bar{x}) = \frac{31+35+29+63+55+72+37}{7} = \frac{322}{7} = 46.$$

TABLE 5.1: CALCULATION OF ABSOLUTE DEVIATIONS

Value x	Deviation from Mean $(d = x - \bar{x} = x - 46)$	Absolute Deviation $\lvert d \rvert$
31	−15	15
35	−11	11
29	−17	17
63	17	17
55	9	9
72	26	26
37	−9	9
Total	—	$104 = \Sigma \lvert d \rvert$

$\therefore$ the required Mean Deviation about the Mean $= \dfrac{\Sigma \lvert d \rvert}{n} = \dfrac{104}{7} = \mathbf{14.86.}$

Example 4. *Find the mean deviation of the following series:*

x	10	11	12	13	14	Total
Frequency	3	12	18	12	3	48

Solution:

TABLE 5.2: CALCULATIONS FOR MEAN DEVIATION

x	f	fx	$\lvert x - \bar{x} \rvert$	$f \lvert x - \bar{x} \rvert$
10	3	30	2	6
11	12	132	1	12
12	18	216	0	0
13	12	156	1	12
14	3	42	2	6
Total	$48 = N$	$576 = \Sigma fx$		$36 = \Sigma f \lvert x - \bar{x} \rvert$

$$\bar{x} = \frac{\Sigma fx}{N} = \frac{576}{48} = 12.$$

$\therefore$ Mean Deviation about the Mean $= \dfrac{\Sigma f \lvert x - \bar{x} \rvert}{N} = \dfrac{36}{48} = \mathbf{0.75.}$

TABLE 5.3: CALCULATION OF CUMULATIVE FREQUENCY

x	f	Cumulative Frequency (less than)
10	3	3
11	12	15
12	18	33
13	12	45
14	3	$48 = N$

$$\text{Median} = \left(\frac{N+1}{2}\right)\text{th value} = \left(\frac{49}{2}\right)\text{th, i.e., 24.5th value} = \frac{1}{2}\text{ (24th value + 25th value).}$$

From the 3rd column, we see that 16th to 33rd values are each 12.

$\therefore$ 24th and 25th values are each 12. $\therefore$ Median $= \dfrac{12+12}{2} = 12.$

Hence, Mean Deviation about the Median is also 0.75.

Example 5. *Calculate mean deviation from the median from the following data:*

Class-intervals	2–4	4–6	6–8	8–10
Frequency	3	4	2	1

Solution:

TABLE 5.4: CALCULATION OF CUMULATIVE FREQUENCY

Class-boundary	Cumulative Frequency (less than)
2	0
4	3
Median⟶	⟵ $5 = N/2$
6	7
8	9
10	$10 = N$

Median = the value corresponding to Cumulative Frequency $\frac{N}{2}$, i.e., 5, and 5 lies between 3 and 7. Therefore, Median class is 4–6;

$$\therefore \text{ Median } (M) = l_1 + \frac{(N/2)-F}{f_m} \times c = 4 + \frac{5-3}{4} \times 2 = 4 + 1 = 5.$$

TABLE 5.5: CALCULATION OF MEAN DEVIATION ABOUT M

Class-intervals	Mid-value x	$d = x - 5$	$\lvert d\rvert$	f	$f\lvert d\rvert$
2–4	3	−2	2	3	6
4–6	5	0	0	4	0
6–8	7	2	2	2	4
8–10	9	4	4	1	4
Total				$10 = N$	$14 = \Sigma f\lvert d\rvert$

Hence, the required *Mean Deviation from the median* $= \dfrac{\Sigma f|d|}{N} = \dfrac{14}{10} = 1.4.$

5.6 Standard Deviation

Definition 1. *It is the most important absolute measure of dispersion. Standard Deviation (SD) of a set of values of a variable is defined as the positive square root of the arithmetic mean of the squares of all the deviations of the values from their arithmetic mean. In short, it may be defined as the Square Root of the Mean of the squares of deviations from Mean.*

SD is usually denoted by the Greek small letter σ (pronounced Sigma).

If $x_1, x_2, \ldots, x_n$ be a series of values of a variable and $\bar{x}$ their AM, then SD (σ) is defined by

$$\sigma = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}.$$

The square of Standard Deviation is known as **Variance,**

$$\text{i.e., Variance } (\sigma^2) = (\text{SD})^2 = \frac{\Sigma(x-\bar{x})^2}{n}.$$

SD *is often defined as the positive square root of Variance.*

Example 6. (a) *Find the standard deviation of the following numbers:* 1, 2, 3, 4, 5, 6, 7, 8, 9. *What is the variance in this case?*

(b) *Prove that the standard deviation calculated from two variables* x_1 *and* x_2 *of a variable* x *is equal to half their difference.* [V.U. B.Com.(H) 2010]

Solution: (a) AM $(\bar{x}) = \dfrac{1+2+3+4+5+6+7+8+9}{9} = \dfrac{45}{9} = 5.$

The deviations of the numbers from the AM 5 are respectively, $-4, -3, -2, -1, 0, 1, 2, 3, 4$.

The squares of the deviations from AM are 16, 9, 4, 1, 0, 1, 4, 9, 16.

$$\therefore \text{ SD} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{16+9+4+1+0+1+4+9+16}{9}} = \sqrt{\frac{60}{9}} = \frac{2}{3}\sqrt{15} = 2.58.$$

$$\therefore \text{ Variance} = (\text{SD})^2 = \left(\sqrt{\frac{60}{9}}\right)^2 = \frac{60}{9} = \frac{20}{3} = 6\frac{2}{3} \text{ or, } 6.67.$$

Otherwise. By formula (C) of **Section 4.6.1**, we have SD $= \sqrt{\dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2}$.

Now,

$$\Sigma x = 1+2+\cdots+9 = \frac{9\cdot(9+1)}{2} = 45$$

$$\text{and } \Sigma x^2 = 1^2+2^2+\cdots+9^2 = \frac{9(9+1)(2\cdot 9+1)}{6} = \frac{9\cdot 10\cdot 19}{6} = 285.$$

$$\therefore \text{ SD} = \sqrt{\frac{285}{9} - \left(\frac{45}{9}\right)^2} = \sqrt{\frac{285-225}{9}} = \sqrt{\frac{60}{9}} = 2.58.$$

(b) We have

$$\bar{x} = \frac{x_1+x_2}{2},\; x_1-\bar{x} = x_1 - \frac{x_1+x_2}{2} = \frac{x_1-x_2}{2} \text{ and } x_2-\bar{x} = x_2 - \frac{x_1+x_2}{2} = \frac{x_2-x_1}{2}.$$

Now,

$$(\text{SD of } x_1 \text{ and } x_2)^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2}{2} = \frac{1}{2}\left[\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{x_2-x_1}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{(x_1-x_2)^2+(x_2-x_1)^2}{4}\right] = \frac{2(x_1-x_2)^2}{8} = \frac{(x_1-x_2)^2}{4}.$$

Hence, SD *of* x_1 *and* $x_2 = \frac{1}{2}|x_1-x_2| =$ *half of the difference of* x_1 *and* x_2.

Example 7.

(i) *Calculate the variance of* 1, 5, 6. *What is the* SD *in this case?* [C.U. B.Com. 2000]

(ii) *Obtain the standard deviation of* 4, 8, 10, 12, 16. [C.U. B.Com. 2004; V.U. B.Com.(H) 2010]

Solution: (i) Mean $(\bar{x}) = \dfrac{1+5+6}{3} = 4$ and

$$\Sigma(x-\bar{x})^2 = (1-4)^2 + (5-4)^2 + (6-4)^2 = 9+1+4 = 14.$$

Hence, $$\text{Variance} = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{14}{3} = 4.67 \text{ and SD} = \sqrt{4.67} = 2.16.$$

(ii) Mean $(\bar{x}) = \dfrac{4+8+10+12+16}{5} = 10$

and Standard Deviation $= \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}}$; $n = 5$.

Now $$\Sigma(x-\bar{x})^2 = (4-10)^2 + (8-10)^2 + (10-10)^2 + (12-10)^2 + (16-10)^2$$
$$= 36+4+0+4+36 = 80.$$

$$\text{Hence, standard deviation} = \sqrt{\frac{80}{5}} = \sqrt{16} = 4.$$

5.6.1 Calculation of SD (σ)

The algebraic formulae for Standard Deviation are:

- For a *simple series,*

$$\textbf{SD } (\boldsymbol{\sigma}) = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma d^2}{n}}, \textit{ where } d = x - \bar{x}. \qquad \text{(A)}$$

- For a *frequency distribution,*

$$\textbf{SD } (\boldsymbol{\sigma}) = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma f d^2}{N}}, \textit{ where } d = x - \bar{x} \textit{ and } \Sigma f = N. \qquad \text{(B)}$$

By slight algebraic manipulation, the above formulae can be expressed in the following equivalent forms:

- For a *simple series,*

$$\textbf{SD } (\boldsymbol{\sigma}) = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}. \qquad \text{(C)}$$

- For a *frequency distribution,*

$$\textbf{SD } (\boldsymbol{\sigma}) = \sqrt{\frac{\Sigma f x^2}{N} - \left(\frac{\Sigma f x}{N}\right)^2}. \qquad \text{(D)}$$

Proof. (i) For a simple series,

$$\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{1}{n}\Sigma\left(x^2 - 2\bar{x}x + \bar{x}^2\right) = \frac{1}{n}\left(\Sigma x^2 - 2\bar{x}\Sigma x + n\bar{x}^2\right)$$
$$= \frac{1}{n}\left(\Sigma x^2 - 2\bar{x}\cdot n\bar{x} + n\bar{x}^2\right) = \frac{1}{n}\left(\Sigma x^2 - n\bar{x}^2\right) \left[\because \bar{x} = \frac{\Sigma x}{n} \text{ or, } n\bar{x} = \Sigma x\right]$$
$$= \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2.$$

$$\therefore \sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}.$$

(ii) For a frequency distribution,

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{N} = \frac{1}{N}\Sigma f\left(x^2 - 2\bar{x}\cdot x + \bar{x}^2\right) = \frac{1}{N}\left(\Sigma f x^2 - 2\bar{x}\Sigma f x + \bar{x}^2 \Sigma f\right)$$

$$= \frac{1}{N}\left(\Sigma f x^2 - 2\bar{x}\cdot N\bar{x} + \bar{x}^2 \cdot N\right) \left[\because \bar{x} = \frac{\Sigma f x}{N} \text{ and } \Sigma f = N\right]$$

$$= \frac{1}{N}\left(\Sigma f x^2 - N\bar{x}^2\right) = \frac{\Sigma f x^2}{N} - \bar{x}^2 = \frac{\Sigma f x^2}{N} - \left(\frac{\Sigma f x}{N}\right)^2.$$

$$\therefore \sigma = \sqrt{\frac{\Sigma f x^2}{N} - \left(\frac{\Sigma f x}{N}\right)^2}.$$

Theorem 1. *Prove that the standard deviation is independent of the choice of origin.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a set of values of a variable and let the origin of x be shifted to an arbitrary constant A. Let $d_1, d_2, \ldots, d_n$ be the deviations of the values from A, i.e., $d = x - A$. Then by definition,

$$\sigma_x^2 = \frac{\Sigma(x-\bar{x})^2}{n} \text{ and } \sigma_d^2 = \frac{\Sigma\left(d-\bar{d}\right)^2}{n}.$$

Now,

$$x - \bar{x} = (x-A) - (\bar{x}-A) = d - \left(\frac{\Sigma x}{n} - A\right) = d - \left(\frac{\Sigma x - nA}{n}\right) = d - \frac{\Sigma(x-A)}{n} \quad [\because \Sigma A = nA]$$

$$= d - \frac{\Sigma d}{n} = d - \bar{d}; \therefore (x-\bar{x})^2 = \left(d-\bar{d}\right)^2.$$

$$\therefore \sigma_x^2 = \sigma_d^2 \text{ or, } \sigma_x = \sigma_d.$$

Similarly, we can prove the result for a frequency distribution. Hence, *the theorem follows.*

Theorem 2. *Prove that the standard deviation is independent of any change of origin, but is dependent on the change of scale.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a set of n values of a variable x. Let us change the origin of x to A (arbitrary) and the scale to i and let $d = \frac{x-A}{i}$. Thus, $x - A = id$.

By definition,

$$\sigma_x^2 = \frac{\Sigma(x-\bar{x})^2}{n} \text{ and } \sigma_d^2 = \frac{\Sigma\left(d-\bar{d}\right)^2}{n}.$$

Now,

$$x - \bar{x} = (x-A) - (\bar{x}-A) = id - \left(\frac{\Sigma x}{n} - A\right) = id - \frac{(\Sigma x - nA)}{n} = id - \frac{\Sigma(x-A)}{n} \quad [\because \Sigma A = nA]$$

$$= id - \frac{\Sigma id}{n} = id - i\frac{\Sigma d}{n} = id - i\bar{d} = i\left(d-\bar{d}\right).$$

$$\therefore \quad \sigma_x^2 = \frac{\Sigma\{i(d-\bar{d})\}^2}{n} = i^2\frac{\Sigma(d-\bar{d})^2}{n} = i^2\sigma_d^2$$

$$\text{or,} \quad \sigma_x = |i|\sigma_d \quad [\because \sigma_x \text{ and } \sigma_d \text{ are positive.}] \tag{1}$$

We see that the new origin A is absent, but the new scale i is present in (1).

Hence, SD is independent of the change of origin, but is dependent on the change of scale.

Similarly, we can prove the theorem for a frequency distribution.

Determination of SD (σ) when the deviations are taken from an Arbitrary Origin

We know from Theorem 1 that $\sigma_x = \sigma_d$, where $d = x - A$ = deviation of the values from A.

$$\therefore \sigma_x = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \text{ for a simple series} \tag{E}$$

$$\text{and } \sigma_x = \sqrt{\frac{\Sigma f d^2}{N} - \left(\frac{\Sigma f d}{N}\right)^2} \text{ for a frequency distribution.} \tag{F}$$

In a grouped frequency distribution if step deviations are considered, the standard deviation is given by

$$\sigma_x = \sqrt{\frac{\Sigma f d^2}{N} - \left(\frac{\Sigma f d}{N}\right)^2} \times i, \text{ where } d = \frac{x-A}{i} = \text{step deviation.} \tag{G}$$

Here i = HCF of the deviations. If the classes are of equal width, then i will be equal to this common width.

Advantages and Disadvantages

Standard Deviation is the most important and widely used among the measures of dispersion and it possesses almost all the requisites of a good measure of dispersion. It is rigidly defined and based on all the values of the variable.

It is suitable for algebraic treatment. SD is less affected by sampling fluctuations than any other absolute measure of dispersion.

SD is difficult to understand. The process of squaring the deviations from their AM and then taking the square root of the AM of these squared deviations is a complicated affair.

The calculation of SD can be made easier by changing the origin and the scale conveniently.

It cannot be used to compare the variability of two or more distributions given in different units.

Standard Deviation of Composite Set

If a set (or sample) of n_1 observations has mean $\bar{x}_1$ and SD σ_1, and another set (or sample) of n_2 observations has mean $\bar{x}_2$ and SD σ_2, then the mean $\bar{x}$ and SD (σ) of the composite set of (n_1+n_2) observations are given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} \text{ and } \sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1+n_2},$$

where $d_1 = \bar{x}_1 - \bar{x}$ *and* $d_2 = \bar{x}_2 - \bar{x}$.

If there are three or more sets of observations, then Mean and SD of the composite set are given by $N\bar{x} = \Sigma n_i x_i$ and $N\sigma^2 = \Sigma n_i \sigma_i^2 + \Sigma n_i d_i^2$, where $d_i = \bar{x}_i - \bar{x}$ and $N = n_1 + n_2 + \cdots = \Sigma n_i$.

Example 8.

(i) *If in a distribution,* $n=10$, $\Sigma x=20$, $\Sigma x^2=200$, *then find the value of SD.* [C.U. B.Com. 2006]

(ii) *Two variables x and y are related by* $y=10-3x$. *If the SD of x is* 4, *what will be the SD of y?* [C.U. B.Com. 2008]

Solution: (i) $\text{SD} = \sqrt{\dfrac{\Sigma x^2}{n}-\left(\dfrac{\Sigma x}{n}\right)^2} = \sqrt{\dfrac{200}{10}-\left(\dfrac{20}{10}\right)^2} = \sqrt{20-2^2} = \sqrt{16} = 4.$

(ii) $y=10-3x$;

$$\therefore\ \Sigma y = \Sigma(10-3x) = \Sigma 10 - 3\Sigma x = 10n - 3\Sigma x,$$

where the variable x has n values.

$$\therefore\ \bar{y} = \frac{\Sigma y}{n} = \frac{10n}{n} - \frac{3\Sigma x}{n} = 10 - 3\bar{x};\ \therefore\ y-\bar{y} = -3(x-\bar{x}).$$

$$\therefore\ \text{SD of } y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n}} = \sqrt{\frac{\Sigma\{-3(x-\bar{x})\}^2}{n}} = \sqrt{\frac{9\Sigma(x-\bar{x})^2}{n}}$$
$$= 3\ (\text{SD of } x) = 3\times 4 = \mathbf{12}\ [\because\ \text{SD of } x = 4].$$

Example 9. *Find the Standard Deviation for the following data:* 49, 63, 46, 59, 65, 52, 60, 54.

Solution:

First Method: Applying formula (A)

TABLE 5.6: CALCULATION OF SD

Value x	**Deviation from Mean 56** $(d=x-\bar{x})$	d^2
49	−7	49
63	7	49
46	−10	100
59	3	9
65	9	81
52	−4	16
60	4	16
54	−2	4
$448=\Sigma x$	—	$324=\Sigma d^2$

$$\text{Mean }(\bar{x}) = \frac{\Sigma x}{n} = \frac{448}{8} = 56.$$

$$\text{Standard Deviation} = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{324}{8}} = \sqrt{40.5} = \mathbf{6.36}.$$

Second Method: Applying formula (E)

TABLE 5.7: CALCULATION OF SD

Value x	Deviation from Arbitrary Origin $A(d = x - A = x - 54)$	d^2
49	−5	25
63	9	81
46	−8	64
59	5	25
65	11	121
52	−2	4
60	6	36
54	0	0
Total	$16 = \Sigma d$	$356 = \Sigma d^2$

Here $n = 8$,

$$\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} = \sqrt{\frac{356}{8} - \left(\frac{16}{8}\right)^2} = \sqrt{44.5 - 4} = \sqrt{40.5} = \mathbf{6.36.}$$

Example 10. *Find out Standard Deviation from the following table giving the age distribution of* 540 *members of a Parliament:*

Age in Years	30	40	50	60	70
No. of Members	64	132	153	140	51

[C.U. B.Com. 2000]

Solution:

TABLE 5.8: CALCULATION OF SD

Age in Years x	Deviations from $A(=50)$ i.e., $d = x - 50$	No. of Members f	fd	fd^2
30	−20	64	−1280	25600
40	−10	132	−1320	13200
$50 = A$	0	153	0	0
60	10	140	1400	14000
70	20	51	1020	20400
Total	...	$540 = N$	$-2600 + 2420 = -180 = \Sigma fd$	$73200 = \Sigma fd^2$

$$\therefore \text{ Standard Deviation} = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} = \sqrt{\frac{73200}{540} - \left(\frac{-180}{540}\right)^2} = \sqrt{135.5555 - 0.1111}$$

$$= \sqrt{135.4444} = \mathbf{11.64 \text{ years (approx.).}}$$

Example 11. *Find the SD (Standard Deviation) from the following frequency distribution:*

Heights in inches	59–61	61–63	63–65	65–67	67–69	Total
No. of Students	4	30	45	15	6	100

[C.U. B.Com. 2004]

Solution:

TABLE 5.9: CALCULATION OF SD

Heights in inches	Mid-value x	Frequency f	Deviation from $A(=64)d=x-64$	fd	fd^2
59–61	60	4	−4	−16	64
61–63	62	30	−2	−60	120
63–65	64	45	0	0	0
65–67	66	15	2	30	60
67–69	68	6	4	24	96
Total		$100=N$		$-22=\Sigma fd$	$340=\Sigma fd^2$

$$\therefore \text{ the required SD} = \sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd}{N}\right)^2} = \sqrt{\frac{340}{100}-\left(\frac{-22}{100}\right)^2} = \sqrt{3.40-0.0484} = \sqrt{3.3516}$$

$= \mathbf{1.83}$ **inches (approx.).**

Example 12. *The mean of* 5 *observations is* 4.4 *and the variance is* 8.24. *If three of the five observations are* 1, 2 *and* 6, *find the other two.* [C.U. B.Com. 2004]

Solution: Let the other two observations be x_1 and x_2; Mean = 4.4 and variance = 8.24.

$$\therefore \frac{\Sigma x}{n}=4.4 \quad \text{or,} \quad \frac{\Sigma x}{5}=4.4$$

$$\text{or,} \quad 1+2+6+x_1+x_2=4.4\times 5=22$$

$$\text{or,} \quad x_1+x_2=13. \tag{1}$$

Again,

$$\frac{\Sigma x^2}{5}-\left(\frac{\Sigma x}{5}\right)^2=8.24, \quad [\because \text{ Variance}=8.24]$$

$$\text{or,} \quad \frac{\Sigma x^2}{5}-(4.4)^2=8.24 \text{ or, } \Sigma x^2=(8.24+19.36)\times 5=27.6\times 5=138$$

$$\text{or,} \quad 1^2+2^2+6^2+x_1^2+x_2^2=138 \text{ or, } x_1^2+x_2^2=138-41=97$$

$$\text{or,} \quad (x_1+x_2)^2-2x_1x_2=97 \text{ or, } (13)^2-97=2x_1x_2$$

$$\text{or,} \quad 2x_1x_2=169-97=72.$$

$$\therefore (x_1-x_2)^2=x_1^2+x_2^2-2x_1x_2=97-72=25 \text{ or, } x_1-x_2=\pm 5. \tag{2}$$

$\therefore x_1-x_2=5$, taking positive sign from (2).

Solving (1) and (2), $x_1=9$ and $x_2=4$.

Hence, *the other two observations are* **9** *and* **4.**

Example 13. *The means of two samples of sizes* 50 *and* 100 *respectively are* 54.4 *and* 50.3 *and the standard deviations are* 8 *and* 7. *Obtain the mean and standard deviation of the sample size* 150 *obtained by combining the two samples.* [*Give answers correct to one decimal place.*]

Solution:

TABLE 5.10: MEAN AND SD OF COMBINED SET

	Set 1	Set 2	Combined Set
No. of Observations	$n_1 = 50$	$n_2 = 100$	$N = n_1 + n_2 = 150$
Mean	$\bar{x}_1 = 54.4$	$\bar{x}_2 = 50.3$	$\bar{x} = ?$
SD	$\sigma_1 = 8$	$\sigma_2 = 7$	$\sigma = ?$

Mean ($\bar{x}$) and SD (σ) of combined set are given by

$$\bar{x} = \frac{n_1\bar{x} + n_2\bar{x}_2}{n_1 + n_2} = \frac{50 \times 54.4 + 100 \times 50.3}{50 + 100} = \frac{2720 + 5030}{150} = \frac{7750}{150} = 51.67 = 51.7$$

$$\text{and } \sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2},$$

where $d_1 = \bar{x}_1 - \bar{x} = 54.4 - 51.7 = 2.7$,
$d_2 = \bar{x}_2 - \bar{x} = 50.3 - 51.7 = -1.4$;

$$\therefore \sigma^2 = \frac{50 \times 64 + 100 \times 49 + 50 \times (2.7)^2 + 100 \times (-1.4)^2}{150} = \frac{3200 + 4900 + 364.50 + 196}{150}$$

$$= \frac{8660.5}{150} = 57.737.$$

$\therefore \sigma = \sqrt{57.737} = 7.59 = 7.6.$

Example 14. *A student obtained the mean and SD of* 100 *observations as* 40.1 *and* 5.0 *respectively. It was later found that he copied* 50 *wrongly instead of the correct value* 40. *Find the correct mean and correct* SD.

[C.U. B.Com. 2000]

Solution: By the question, $\frac{\Sigma x}{100} = 40.1$ or, $\Sigma x = 40.1 \times 100 = 4010.$

$\therefore$ correct Σx for 100 observations = Incorrect $\Sigma x - 50 + 40 = 4010 - 10 = 4000.$

$$\therefore \text{ correct mean} = \frac{\text{Correct } \Sigma x}{100} = \frac{4000}{100} = \mathbf{40}.$$

Again, incorrect SD = 5 or, $\frac{\Sigma x^2}{100} - \left(\frac{\Sigma x}{100}\right)^2 = (5)^2 = 25$

or, $\frac{\Sigma x^2}{100} = 25 + (40.1)^2$

or, $\frac{\Sigma x^2}{100} = 25 + 1608.01 = 1633.01$

or, $\Sigma x^2 = 1633.01 \times 100 = 163301.$

$\therefore$ correct Σx^2 for 100 observations $= 1633.01 - (50)^2 + (40)^2 = 163301 - 900 = 162401$.

$$\therefore \text{ correct SD} = \sqrt{\frac{\text{Correct } \Sigma x^2}{100} - \left(\frac{\text{Correct } \Sigma x}{100}\right)^2} = \sqrt{\frac{162401}{100} - (40)^2}$$

$$= \sqrt{1624.01 - 1600} = \sqrt{24.01} = 4.9.$$

5.7 Relative Measures of Dispersion

The four measures of dispersion discussed earlier, viz., Range, Quartile Deviation, Mean Deviation and Standard Deviation are all absolute measures of dispersion and all of them are expressed in the same units in which the original data are given. Absolute measures are used for measuring dispersion of any distribution and they can also be employed for comparing the dispersion of two or more distributions only when the distributions are given in the same units.

Absolute measures expressed as percentage of a measure of a central tendency (Mean or Median) gives relative measures of dispersion. Relative measures are independent of the units of measurement and hence they are used for the comparison of dispersion (or variability) of two or more distributions given in different units. The three relative measures of dispersion as stated earlier are discussed below.

Coefficient of Variation

Definition 1. *Coefficient of variation is the most important relative measure of dispersion and is defined by the formula:*

$$\textbf{Coefficient of Variation} = \frac{\textbf{Standard Deviation}}{\textbf{Mean}} \times 100\%.$$

Coefficient of variation is thus the ratio of the Standard Deviation to the Mean, expressed as a percentage. According to Karl Pearson, Coefficient of variation is the percentage variation in the mean.

Coefficient of Quartile Deviation

Definition 2. *Coefficient of Quartile Deviation is a relative measure of dispersion and is defined by*

$$\textbf{Coefficient of Quartile Deviation} = \frac{\textbf{Quartile Deviation}}{\textbf{Median}} \times 100\%.$$

Coefficient of Mean Deviation

Definition 3. *It is a relative measure of dispersion. Coefficient of Mean Deviation is defined by*

$$\textbf{Coefficient of Mean Deviation} = \frac{\textbf{Mean Deviation}}{\textbf{Mean or Median}} \times 100\%.$$

Example 15.

(i) *Find the coefficient of variation when standard deviation* = 5 *and mean* = 15.

(ii) *Find the coefficient of quartile deviation when* $Q_1 = 6$, $Q_2 = 12$ *and* $Q_3 = 21$.

(iii) *Find the coefficient of mean deviation when mean* = 16 *and mean deviation about mean is* 4.

Solution: (i) Coefficient of variation $= \dfrac{\text{Standard deviation}}{\text{Mean}} \times 100\% = \dfrac{5}{15} \times 100\% = \mathbf{33\tfrac{1}{3}\%}$.

(ii) Quartile Deviation $= \dfrac{Q_3 - Q_1}{2} = \dfrac{21 - 6}{2} = \dfrac{15}{2} = 7.5$ and Median $= Q_2 = 12$.

$\therefore$ coefficient of Quartile Deviation $= \frac{7.5}{12} \times 100\% = 62.5\%$.

(iii) Coefficient of Mean Deviation about mean $= \frac{\text{Mean Deviation}}{\text{Mean}} \times 100\% = \frac{4}{16} \times 100\% = 25\%$.

Example 16. (i) *Find the mean if coefficient of variation* = 5% *and variance* = 4. [C.U. B.Com. 1997]

(ii) *Find CV if the sum of squares of the deviations of* 10 *observations taken from the mean* 50 *is* 250. [CA Foun. May 1999]

Solution: (i) Variance = 4; $\therefore$ SD $= \sqrt{\text{Variance}} = \sqrt{4} = 2$, CV = 5%, Mean = ?

Now,
$$\text{CV} = \frac{\text{SD}}{\text{Mean}} \times 100\% \text{ or, } 5\% = \frac{2}{\text{Mean}} \times 100\% \text{ or, Mean} = \frac{200}{5} = 40.$$

(ii) Here $n = 10$, Mean = 50, $\Sigma(x-50)^2 = 250$, CV = ?

$$\text{SD} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma(x-50)^2}{10}} = \sqrt{\frac{250}{10}} = \sqrt{25} = 5.$$

Hence
$$\text{CV} = \frac{\text{SD}}{\text{Mean}} \times 100\% = \frac{5}{50} \times 100\% = 10\%.$$

Example 17. *Find the standard deviation and coefficient of variation from the following table giving the marks of* 150 *students*:

Marks	No. of Students	Marks	No. of Students
1–10	5	51–60	22
11–20	12	61–70	15
21–30	20	71–80	6
31–40	25	81–90	4
41–50	40	91–100	1

Solution:

TABLE 5.11: CALCULATIONS FOR MEAN AND SD

Class-intervals	Mid-values x	Frequency f	$d = \frac{x-A}{i}$ (here $i = 10$)	fd	fd^2
1–10	5.5	5	−5	−25	125
11–20	15.5	12	−4	−48	192
21–30	25.5	20	−3	−60	180
31–40	35.5	25	−2	−50	100
41–50	45.5	40	−1	−40	40
51–60	$55.5 = A$	22	0	0	0
61–70	65.5	15	1	15	15
71–80	75.5	6	2	12	24
81–90	85.5	4	3	12	36
91–100	95.5	1	4	4	16
Total	—	$150 = N$	—	$-180 = \Sigma fd$	$728 = \Sigma fd^2$

$$\therefore\ \text{SD} = \sqrt{\frac{\Sigma f d^2}{N} - \left(\frac{\Sigma f d}{N}\right)^2} \times i = \sqrt{\frac{728}{150} - \left(\frac{-180}{150}\right)^2} \times 10 = \sqrt{\frac{109200 - 32400}{(150)^2}} \times 10$$

$$= \frac{10}{150}\sqrt{76800} = \frac{10}{15} \times 16\sqrt{3} = \frac{32 \times 1.7325}{3} = 32 \times 0.5775 = 18.48.$$

$$\bar{x} = A + \frac{\Sigma f d}{N} \times i = 55.5 + \left(\frac{-180}{150}\right) \times 10 = 55.5 - 12 = 43.5.$$

$$\therefore\ \text{Coefficient of Variation} = \frac{\text{SD}}{\text{Mean}} \times 100\% = \frac{18.48}{43.5} \times 100\% = \frac{18480}{435}\% = \frac{3696}{87}\%$$

$$= 42.48\% = 42.5\%.$$

Example 18. (i) *Find the Mean Deviation about the Median in respect of the following numbers*: 46, 79, 26, 85, 39, 65, 99, 29, 56, 72. *Find also the coefficient of Mean Deviation.* [C.U. B.Com. 2005 Type]

(ii) *Find the Mean Deviation about Median of the following numbers*:

50, 46, 79, 26, 85, 39, 65, 29, 59, 73, 85, 64, 72, 65. [C.U. B.Com. 2001]

Solution: (i) Arranging the given numbers in ascending order of magnitude, we get 26, 29, 39, 46, 56, 65, 72, 79, 85, 99.

$$\text{Median} = \left(\frac{n+1}{2}\right)\text{th value} = \left(\frac{10+1}{2}\right)\text{th value} = 5.5\text{th value}$$

$$= \frac{\text{5th value + 6th value}}{2} = \frac{56+65}{2} = \frac{121}{2} = 60.5.$$

Absolute deviations of the values from the median 60.5 are respectively 34.5, 31.5, 21.5, 14.5, 4.5, 4.5, 11.5, 18.5, 24.5, 38.5.

$$\therefore\ \text{Mean Deviation about the Median} = \frac{34.5+31.5+21.5+14.5+4.5+4.5+11.5+18.5+24.5+38.5}{10}$$

$$= \frac{204}{10} = 20.4.$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Median}} \times 100\% = \frac{20.4}{60.5} \times 100\% = \frac{204}{605} \times 100 = 33.72\%.$$

(ii) Arranging the given numbers in ascending order of magnitude, we get

26, 29, 39, 46, 50, 59, 64, 65, 65, 72, 73, 79, 85, 85. Here $n = 14$.

$$\text{Median } (M) = \text{the value of the } \left(\frac{n+1}{2}\right)\text{th term} = \text{the value of the (7.5)th term}$$

$$= \frac{\text{7th term + 8th term}}{2} = \frac{64+65}{2} = 64.5.$$

$$\text{Mean Deviation about Median} = \frac{\Sigma|x-M|}{n} = \frac{|26-64.5|+|29-64.5|+|39-64.5|+\cdots+|85-64.5|}{14}$$

$$= \frac{38.5+35.5+25.5+18.5+14.5+5.5+0.5+0.5+0.5+7.5+8.5+14.5+20.5+20.5}{14}$$

$$= \frac{211}{14} = 15.07.$$

Note: Coefficient of Mean Deviation = $\frac{15.07}{64.5} \times 100\% = 23.36\%$.

Example 19. *In two factories A and B, engaged in the same industry in the area, the average weekly wages (in ₹) and the SD are as follows:*

Factory	Average	SD	No. of Employees
A	34.5	5.0	476
B	28.5	4.5	524

(i) *Which factory A or B gives a pay out of larger amount on weekly wages?*

(ii) *Which factory A or B has greater variability in individual wages?*

Solution: (i) Total weekly wages paid by factory A = ₹ 34.5 × 476 = ₹ 16,422, and total weekly wages paid by factory B = ₹ 28.5 × 524 = ₹ 14,934.

∴ **factory *A* pays larger amount as weekly wages.**

(ii) Coefficient of variation for Factory A

$$= \frac{\text{SD}}{\text{Mean}} \times 100\% = \frac{5.0}{34.5} \times 100\% = \frac{5000}{345} = 14.49\%$$

and Coefficient of variation for factory B

$$= \frac{\text{SD}}{\text{Mean}} \times 100\% = \frac{4.5}{28.5} \times 100\% = \frac{4500}{285} = 15.8\%.$$

∴ **factory *B* has greater variability in individual wages.**

Example 20. *For a distribution of* 280 *observations mean and standard deviation were found to be* 54 *and* 3 *respectively. On checking it was discovered that two observations which should correctly read as* 62 *and* 82 *had been wrongly recorded as* 64 *and* 80 *respectively. Calculate the correct values of mean and standard deviation.*

[V.U. B.Com.(H) 2008]

Solution: Given, $N = 280, \bar{x} = \text{mean} = 54, \sigma = \text{SD} = 3$. Also $\bar{x} = \frac{\Sigma x}{N}$.

$\therefore \frac{\Sigma x}{280} = 54$ or, $\Sigma x = 54 \times 280 = 15120$, which is incorrect Σx.

$\therefore$ correct $\Sigma x = 15120 - (64 + 80) + (62 + 82) = 15120$.

Hence *correct value of mean* $= \frac{\text{correct } \Sigma x}{280} = \frac{15120}{280} = 54$.

Again, incorrect SD = 3, or, $\frac{\Sigma x^2}{280} - \left(\frac{\Sigma x}{280}\right)^2 = 3^2$ or, $\frac{\Sigma x^2}{280} = 9 + (54)^2 = 2925$.

$\therefore$ incorrect $\Sigma x^2 = 2925 \times 280 = 819000$.

$$\therefore \text{ correct } \Sigma x^2 = 81900 - \left\{(64)^2 + (80)^2\right\} + \left\{(62)^2 + (82)^2\right\} = 819000 - (4096 + 6400) + (3844 + 6724)$$
$$= 819000 - 10496 + 10568 = 819072.$$

$$\text{Hence correct value of SD} = \sqrt{\frac{\text{correct } \Sigma x^2}{280} - \left(\frac{\text{correct } \Sigma x}{280}\right)^2} = \sqrt{\frac{819072}{280} - \left(\frac{15120}{280}\right)^2}$$
$$= \sqrt{2925.26 - 2916} = \sqrt{9.26} = 3.04.$$

Example 21. *A factory produces two types of electric lamps A and B. In an experiment relating to their life, the following results were obtained:*

Length of Life (in hours)	No. of Lamps A	No. of Lamps B
500–700	5	4
700–900	11	30
900–1100	26	12
1100–1300	10	8
1300–1500	8	6
Total	60	60

Compare the variability of the life of the two varieties using Coefficient of Variation.

Solution:

TABLE 5.12: CALCULATION FOR MEAN AND SD

Mid-value (x)	$d=\frac{x-1000}{200}$	Type A			Type B		
		f_1	f_1d	f_1d^2	f_2	f_2d	f_2d^2
600	−2	5	−10	20	4	−8	16
800	−1	11	−11	11	30	−30	30
1000	0	26	0	0	12	0	0
1200	1	10	10	10	8	8	8
1400	2	8	16	32	6	12	24
Total	—	$60=N_1$	$5=\Sigma f_1d$	$73=\Sigma f_1d^2$	$60=N_2$	$-18=\Sigma f_2d$	$78=\Sigma f_2d^2$

Electric Lamp A

$$\text{Mean}=A_1+\frac{\Sigma f_1d}{N_1}\times i=1000+\frac{5}{60}\times 200=1016.67 \text{ hours.}$$

$$\text{SD}=\sqrt{\frac{\Sigma f_1d^2}{N_1}-\left(\frac{\Sigma f_1d}{N_1}\right)^2}\times i=\sqrt{\frac{73}{60}-\left(\frac{5}{60}\right)^2}\times 200=\sqrt{1.2167-0.0069}\times 200$$

$$=1.099\times 200=219.8 \text{ hours.}$$

$$\therefore \text{ Coefficient of Variation}=\frac{\text{SD}}{\text{Mean}}\times 100\%=\frac{219.8}{1016.67}\times 100\%=\mathbf{21.62\%}.$$

Electric Lamp B

$$\text{Mean}=A_2+\frac{\Sigma f_2d}{N_2}\times i=1000+\left(\frac{-18}{60}\right)\times 200=940 \text{ hours.}$$

$$\text{SD}=\sqrt{\frac{\Sigma f_2d^2}{N_2}-\left(\frac{\Sigma f_2d}{N_2}\right)^2}\times i=\sqrt{\frac{78}{60}-\left(\frac{-18}{60}\right)^2}\times 200=\sqrt{1.3-0.09}\times 200=1.1\times 200$$

$$=220 \text{ hours.}$$

$$\therefore \text{ Coefficient of Variation}=\frac{\text{SD}}{\text{Mean}}\times 100\%=\frac{220}{940}\times 100\%=\mathbf{23.40\%}.$$

Hence, *the life of type B electric lamps is more variable than that of type A electric lamps.*

EXERCISES ON CHAPTER 5(I)

Theory

1. What do you understand by Dispersion? Discuss its importance in Statistics and state also how to measure it.

2. Name any three measures of dispersion that you are acquainted with and describe one of those with a suitable illustration.

3. (a) Explain how Dispersion is measured.
   (b) Explain and illustrate Measures of Dispersion.
   (c) Define Range and Quartile Deviation.

4. Why can't we use $\frac{\Sigma x}{n}$ as a measure of dispersion? How does Mean Deviation overcome this problem? How does SD overcome this problem?

5. Describe the absolute measures of dispersion. Discuss their relative advantages and disadvantages.

6. (a) What do you mean by Standard Deviation? [C.U. B.Com. 1996]
   (b) Define Mean Deviation. [C.U. B.Com. 1987; B.U. B.Com. 1990]
   (c) Define Standard Deviation. [V.U. B.Com.(H) 2007]

7. Explain the advantages and disadvantages of the different measures of dispersion.

8. Distinguish between absolute and relative measures of dispersion.

9. What are relative measures of variability of observations? Discuss their various uses.

10. Define Coefficient of Variation. What are the special uses of this measure?

11. Define the terms Coefficient of Variation and Coefficient of Mean Deviation.

Problems (A)

1. Find the Range of the daily wages of 8 persons in (a) and 10 persons in (b) given below:
   (a) ₹ 9, ₹ 7, ₹ 25, ₹ 18, ₹ 38, ₹ 12, ₹ 30, ₹ 35;
   (b) ₹ 24, ₹ 18, ₹ 25, ₹ 16, ₹ 20, ₹ 28, ₹ 22, ₹ 17, ₹ 21, ₹ 27.

2. (a) Find the Quartile Deviation of the monthly income of 7 men given below:
   ₹ 350, ₹ 840, ₹ 650, ₹ 710, ₹ 980, ₹ 575, ₹ 290.
   (b) Find the *Quartile Deviation* of the following data:
   12, 10, 17, 14, 19, 21, 27, 30, 32, 28, 34. [C.U. B.Com. 2006]

3. Find the Mean Deviation about the Median of the following:
   (a) (i) 13, 84, 68, 24, 96, 139, 84, 27;
   (ii) 8, 15, 53, 49, 19, 62, 7, 15, 95, 77.
   (b) 46, 79, 26, 85, 39, 59, 73. [C.U. B.Com. 2005]

(c) Find the Mean Deviation about the AM of each of the following distributions:

(i) 27, 33, 49, 61, 76, 104, 126;

(ii) 29, 35, 51, 63, 78, 106, 128;

(iii) 31, 35, 29, 63, 55, 72, 37. [B.U. B.Com. 1990]

(d) Find the Mean Deviation about the AM and the Median in respect of the following numbers: 50, 10, 94, 206, 80. [V.U. B.Com.(H) 2009]

(e) Find the Mean Deviation about median of the following numbers:
46, 79, 26, 85, 39, 65, 99, 29, 56. [C.U. B.Com. 2008]

4. (a) For a set of Ungrouped Values the following sums are found: $N = 15$, $\Sigma x = 480$, $\Sigma x^2 = 15{,}735$. Find the Mean and the Standard Deviation.

(b) Calculate the Variance of 1, 5, 6. [C.U. B.Com. 1990]

(c) The Standard Deviation of a set of 30 items is 9.5. Find the Standard Deviation if every item is decreased by 5. [CA Foun. Nov. 1998]

(d) The Standard Deviation of a set of 50 items is 8. Find the standard deviation, if each item is multiplied by 2. [CA Foun. May, 1996]

5. (a) Calculate the Standard Deviation from the following series: 20, 85, 120, 60, 40.

(b) Find the Standard Deviation for the following distribution:

(i) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

(ii) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11;

(iii) 4, 6, 9, 12, 14, 16, 18, 20, 22.

(c) Prove that the variance of the first n positive integers is $\frac{(n^2-1)}{12}$.

6. (a) Calculate the Mean Deviation of the following distribution:

Value	5	7	9	11	13	Total
Frequency	4	10	22	10	4	50

(b) Calculate the Mean Deviation about the Arithmetic Mean of the following distribution:

x	5	15	25	35	45	55	65
f	4	6	10	20	10	6	4

[C.U. B.Com. 1997]

(c) Calculate the Mean Deviation about AM of the following series:

Marks (x)	5	10	15	20	Total
No. of Students (f)	8	2	4	6	20

7. Find out the Standard Deviation from the following table giving the weights of 200 persons:

Weight in kg	50	55	60	65	70	Total
No. of Mean	30	40	65	50	15	200

8. (a) Find the Mean Deviation about the median from the following distribution:

Class-intervals	2–6	6–10	10–14	14–18
Frequency	6	8	4	2

(b) Find the mean deviation about mean of the following distribution:

Class-intervals	2–6	6–10	10–14	14–18
Frequency	6	4	8	2

[C.U. B.Com. 2007]

9. Find the Standard Deviation from the following frequency distribution:

(a)

Daily Wages (₹)	20–24	25–29	30–34	35–39
No. of Workers	16	28	14	12

[C.U. B.Com. 1990]

(b)

Weight (lb)	120–124	125–129	130–134	135–139	140–144	145–149
No. of Boys	12	25	28	15	12	8

[B.U. B.Com. 1990]

(c)

Height (cm)	160–163	164–167	168–171	172–175	176–179	180–183	184–187
No. of Students	22	80	98	148	14	43	5

[C.U. B.Com. 1994]

10. A sample of size 15 has mean 3.5 and standard deviation 3.0. Another sample of size 22 has mean 4.7 and standard deviation 4.0. If the two samples are pooled together, find the mean and the standard deviation of the combined sample.

11. Find the coefficient of mean deviation of the series:

(a) 10, 20, 40, 60, 70, 100;

(b) 487, 508, 620, 382, 408, 266, 186, 218.

12. (a) Find the coefficient of variation of the following values:

(i) 5, 10, 30, 40, 65;

(ii) 40, 30, 80, 60, 50, 90, 70.

(b) Find Mean if CV = 59% and Variance = 4. [C.U. B.Com. 2001]

(c) CV = 60% and Variance = 36, find Mean. [C.U. B.Com. 2002]

[Hints: $CV = \frac{SD}{Mean} \times 100\%$ or, $60\% = \frac{\sqrt{Variance}}{Mean} \times 100\% = \frac{\sqrt{36}}{Mean} \times 100\% = \frac{600}{Mean}\%$ or, $\frac{600}{Mean} = 60$ or, $Mean = \frac{600}{60} = 10$.]

Problems (B)

1. (a) From the following array, find out the Mean Deviation: 7, 9, 16, 24, 26, 31 and 39.

(b) Find the mean deviation about AM of the first ten natural numbers.

2. (a) The Standard Deviation of $1, 2, 3, \ldots, n$ is $\sqrt{14}$; find n.

(b) If CV = 40% and variance = 16, find mean. [C.U. B.Com. 2002]

3. Find the Standard Deviation of:

(a) (i) 4, 5, 6, 6, 7, 8;

(ii) 1, 5, 3, 8, 2; [C.U. B.Com. 2007]

(b) 3, 5, 11, 7, 8, 10, 9, 12, 14, 11;

(c) 240.12, 240.13, 240.15, 240.12, 240.17, 240.15, 240.17, 240.16, 240.22, 240.21.

4. Find out the range of the following data:

Height (inches)	60–62	63–65	66–68	69–71	72–74
No. of Students	8	27	42	18	5

5. (a) Find the Mean Deviation about the AM from the following data:

Daily Wages (₹)	8–11	12–15	16–19	20–23	24–27
No. of Workers	5	11	20	10	4

(b) Calculate the mean deviation from the mean for the following data:

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of Students	6	5	8	15	7	6	3

[CA Foun. May 1999]

6. Find the Standard Deviation from the following table:

(a)

x	10	20	30	40	50	60	Total
f	9	18	25	27	14	7	100

[C.U. B.Com. 1991]

(b)

x	62	64	66	68
f	8	13	17	12

[V.U. B.Com.(H) 2007]

[Hints:

x	f	$d = x - 65$	fd	fd^2
62	8	−3	−24	72
64	13	−1	−13	13
66	17	1	17	17
68	12	3	36	108
Total	50 $= N$	—	16 $= \Sigma fd$	210 $= \Sigma fd^2$

$$SD = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} = \sqrt{\frac{210}{50} - \left(\frac{16}{50}\right)^2}$$

$$= \sqrt{4.2 - (0.32)^2} = \sqrt{4.2 - 0.1024} = \sqrt{4.0976}$$

$= 2.02$]

7. (a) Find the Standard Deviation of the following frequency distribution of the daily wages of 500 workers in a factory:

Daily Wages	25	36	45	55	65
No. of Workers	60	130	150	130	30

(b) Find the Standard Deviation from the following table giving the age distribution of 570 members of a Parliament:

Age (Years)	30	40	50	60	70
No. of Members	64	142	163	140	·61

[C.U. B.Com. 2000]

8. (a) Find the SD of the following frequency distribution:

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of Students	3	12	25	30	15	10	5

[C.U. B.Com. 2007]

(b) Find the Standard Deviation from the following figures:

Marks	No. of Persons	Marks	No. of Persons
0–10	5	40–50	30
10–20	10	50–60	20
20–30	20	60–70	10
30–40	40	70–80	4

(c) Find the AM and SD from the following frequency distribution:

Weekly Wages (₹)	141–150	151–160	161–170	171–180	181–190	191–200	201–210
No. of Workers	5	8	15	25	20	17	10

[N.B.U. B.Com. 1996; Utkal U. B.Com. 2000 Type]

(d) Find the variance of the following frequency distribution:

Marks obtained	20–30	30–40	40–50	50–60	60–70
No. of Students	2	35	46	12	5

[C.U. B.Com. 2008]

9. (a) Find the Standard Deviation from the following frequency distribution:

Heights in inches	No. of Students
Over 60 but not more than 62	35
Over 62 but not more than 64	27
Over 64 but not more than 66	20
Over 66 but not more than 68	13
Over 68 but not more than 70	5
	100

(b) Find the Standard Deviation of the following distribution:

Weight (kg)	45–50	50–55	55–60	60–65	65–70
No. of Persons	10	16	32	28	14

[C.U. B.Com. 1996]

(c) Frequency distribution of marks of 100 students are given below:

Marks	20–29	30–39	40–49	50–59	60–69	70–79	80–89
No. of Students	5	15	18	26	16	14	6

Find the Standard Deviation. **[C.U. B.Com. 2003; N.B.U. B.Com. 1994]**

10. Calculate the Variance from the data:

Age (years)	10–19	20–29	30–39	40–49	50–59	60–69
Frequencies	3	61	50	32	20	4

11. Find the Standard Deviation of the following distribution:

Turnover in (₹'000) p.a.	50–100	100–150	150–200	200–250	250–300	300–350	350–400
No. of Firms	5	8	9	12	18	23	17

12. (a) The following table gives the frequency distribution of marks obtained by 150 students in a certain examination:

Marks	No. of Students	Marks	No. of Students
0–10	7	40–50	30
10–20	10	50–60	28
20–30	20	60–70	10
30–40	40	70–80	5

From the above distribution calculate Mean and Standard Deviation and also Coefficient of Variation. **[Agra U. B.Com.]**

(b) Find the Standard Deviation of the following distribution:

Weight (pounds)	120–124	125–129	130–134	135–139	140–144	145–149
No. of Students	12	25	28	15	12	8

[C.U. B.Com. 1997]

13. (a) Find the Coefficient of Variation of the following data:

Marks	0–10	10–20	20–30	30–40	40–50
No. of Students	4	10	16	12	8

(b) Find the Coefficient of Variation for the following distribution:

Weight in gm	Frequency	Weight in gm	Frequency
110–119	5	150–159	16
120–129	7	160–169	10
130–139	12	170–179	7
140–149	20	180–189	3

14. From the following data determine in which firm A or B, there is greater variability in individual wages:

	Firm A	Firm B
Average Monthly Wages	₹ 52.50	₹ 47.50
Variance of Distribution of Wages	₹ 100.00	₹ 121.00

15. An analysis of the monthly wages paid to workers in two firms A and B, belonging to the same industry, gives the following results:

	Firm A	Firm B
No. of Wage Earners	**500**	**650**
Average Monthly Wages	₹ 50	₹ 45
Standard Deviation of the Distribution of Wages	₹ $(\sqrt{90})$	₹ $(\sqrt{120})$

Answer to the following questions with proper justifications:

(a) Which firm, A or B, gives a pay out of larger amount as monthly wages?

(b) In which firm, A or B, is greater variability in individual wages?

(c) What are the measures of: (i) average monthly wages and (ii) standard deviation in the distribution of individual wages of all workers in the two firms taken together?

16. The scores of two batsmen, A and B, in ten innings during a certain season, are as under:

A	32	28	47	63	71	39	10	60	96	14
B	19	31	48	53	67	90	10	62	40	80

Find which of the batsmen is more consistent in scoring. [C.U. B.Com. 2006]

17. The mean and SD of income of 50 men are ₹ 3200 and ₹ 525 respectively. The same for 40 women are ₹ 2850 and ₹ 460 respectively. Find the SD of income for the combined group. [C.U. B.Com. 2001]

[**Hints:** Mean ($\bar{x}$) of the combined group is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} = \frac{50\times 3200 + 40\times 2850}{50+40} = \frac{274000}{90} = 3044.40.$$

$d_1 = \bar{x}_1 - \bar{x} = 3200 - 3044.40 = 155.60$ and $d_2 = \bar{x}_2 - \bar{x} = 2850 - 3044.40 = -194.4$.

Now, $\sigma^2 = \dfrac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1+n_2} = \dfrac{50\left\{(525)^2 + (155.6)^2\right\} + 40\left\{(460)^2 + (-194.4)^2\right\}}{50+40}$, etc.]

ANSWERS

A

1. (a) ₹ 31; (b) 8. (c) (i) 29.14;
   (b) ₹ 12.
3. (a) (i) 33.88 (approx.); (ii) $29\frac{1}{7}$;
   (ii) 27.2; (iii) 14.86;
2. (a) ₹ 245; (b) 18; (d) 49.6; 48.

(e) 20.89.

4. (a) Mean = 32; SD = 5;
   (b) $\frac{14}{3}$;
   (c) 9.5;
   (d) 16.

5. (a) 34.9;
   (b) (i) 2.87;
   (ii) 3.16 (approx.);
   (iii) 5.87.

6. (a) 1.44;
   (b) 11.333;
   (c) 6.

7. 5.79 (approx.).

8. (a) 2.8;
   (b) 3.6.

9. (a) ₹ 5.04;
   (b) 7.18 lb;
   (c) 5.38 cm.

10. Mean = 4.21; SD = 3.68.

11. (a) 53.34%;
    (b) 30.73%.

12. (a) (i) 72.26%;
    (ii) 33.33%;
    (b) 3.39;
    (c) 10.

B

1. (a) 9.47 about mean and 9.14 about median;
   (b) 2.5.

2. (a) $n = 13$;
   (b) 10.

3. (a) (i) 1.29,
   (ii) 2.5;
   (b) 3.16;
   (c) 0.0325.

4. 15 inches.

5. (a) ₹ 3.21;
   (b) 13.184.

6. (a) 13.56; (b) 1.97.

7. (a) 11.07 units;
   (b) 11.712 years.

8. (a) 14.12;
   (b) 15.69 marks;
   (c) ₹ 179.30, ₹ 15.99;
   (d) 72.11.

9. (a) 2.41 inches;
   (b) 5.83 kg;
   (c) 15.75.

10. 129 square years.

11. ₹ 88.48 thousand.

12. (a) AM = 40; SD = 16.3; CV = 40.7%;
    (b) 7.18 points.

13. (a) 43.19%;
    (b) 11.75%.

14. *B* has greater variability in individual wages.

15. (a) *B* pays larger amount as monthly wages;
    (b) Greater variability in individual wages in firm *B*;
    (c) Average = ₹ 47.29 and SD = 10.604.

16. *B* is more consistent in his scoring than *A*. [CV of *A* = 55.42%, CV of *B* = 48.86%.]

17. ₹ 526.70.

EXERCISES ON CHAPTER 5(II)

Miscellaneous Problems

1. Find the range of the following distributions:

(a)

Value	5	10	25	35	50
Frequency	3	8	14	10	5

(b)

Marks	1–10	11–20	21–30	31–40	41–50
No. of Students	8	12	20	10	6

2. Find the Quartile Deviation of the following distributions:

(a)

Heights (inches)	60	62	64	66	68	70	72
No. of Students	4	10	18	26	20	12	5

(b)

Weights (kg)	40–45	45–50	50–55	55–60	60–65	65–70
No. of Men	10	22	28	20	12	8

3. (a) Find out the Arithmetic Mean and Standard Deviation from the following data:

Variable	Frequency	Variable	Frequency
5–10	2	20–25	54
10–15	9	25–30	11
15–20	29	30–35	6

(b) Calculate the SD (σ) of the following frequency distribution:
Wages in a sample of factory workers in Kolkata, 1980.

Weekly Wages (₹)	30–34.99	35–39.99	40–44.99	45–49.99	50–54.99	55–59.99	60–64.99	65–69.99
No. of Employees	3	9	15	27	18	12	9	7

(c) Calculate the Mean Deviation about the Arithmetic Mean of the following distribution:

x	10	11	12	13	14
f	1	2	4	2	1

[C.U. B.Com. 1995]

(d) Define SD. Find the Standard Deviation from the following frequency distribution:

Earned Profit (₹ '000)	50–100	100–150	150–200	200–250	250–300	300–350	350–400
No. of Company	3	8	9	12	18	23	17

[V.U. B.Com. 1997]

4. (a) Two samples of sizes 40 and 50 respectively have the same mean 53, but different standard deviations 19 and 8 respectively. Find the standard deviation of the combined sample of size 90.

(b) The mean and variance of a group of 250 items was 15.6 and 13.44 respectively. If the mean and standard deviation of 100 of these items are 15 and 3 respectively, find the mean and standard deviation of the remaining 150 items. [C.U. B.Com. 2004]

[**Hints:** $N = 250$, $\bar{x} = 15.6$, $\sigma^2 = 13.44$; $n_1 = 100$, $\bar{x}_1 = 15$, $\sigma_1 = 3$, $\bar{x}_2 = ?$, $\sigma_2 = ?$, $n_2 = 250 - 100 = 150$; $(N = n_1 + n_2)$.
Now $\bar{x} = \frac{n_1 x_1 + n_2 x_2}{n_1 + n_2} \Rightarrow 15.6 = \frac{100 \times 15 + 150 \times \bar{x}_2}{100 + 150} \Rightarrow 150\bar{x}_2 = 15.6 \times 250 - 1500 = 2400 \Rightarrow \bar{x}_2 = 16$.
$d_1 = \bar{x}_1 - \bar{x} = 15 - 15.6 = -0.6$, $d_2 = \bar{x}_2 - \bar{x} = 16 - 15.6 = 0.4$.
Now find σ_2 from $\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$.]

5. (a) Calculate the appropriate measure of dispersion from the following data:

Wages in ₹ per Week	No. of Wage Earners
Less than 35	14
35–37	62
38–40	99
41–43	18
Over 43	7

(b) Find the Mean Deviation from the Mean for the following data:

Class-interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	8	12	10	8	3	2	7

[CA Foun. Nov. 1996]

6. (a) The mean and the standard deviation of a sample of size 10 were found to be 9.5 and 2.5 respectively. Later on an additional observation became available. This was 15.0 and was included in the original sample. Find the mean and the standard deviation of the 11 observations.

[Hints: For the additional observation, $\bar{x}_2 = \frac{15}{1} = 15$ and $\sigma_2 = \sqrt{\frac{(15-15)^2}{1}} = 0, n_2 = 1.$]

(b) For a group containing 100 observations, the arithmetic mean and standard deviation are 8 and $\sqrt{10.5}$, respectively. For 50 observations selected from these 100 observations the mean and the standard deviations are 10 and 2 respectively. Find the arithmetic mean and the standard deviation of the other half.

(c) The first of two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{13.44}$, find the standard deviation of the second group.

7. (a) A student obtained the mean and SD of 100 observations as 40 and 5.1 respectively. It was later found that he had wrongly copied one observation as 50, the correct figure being 40. Calculate the correct mean and correct SD

(b) The mean and SD of 20 items is found to be 10 and 2 respectively. At the time of checking it was found that one item 8 was incorrect. Calculate the mean and SD if (i) the wrong item is omitted, and (ii) it is replaced by 12.

(c) The mean and the standard deviation of a characteristic of 100 items were found to be 60 and 10 respectively. At the time of calculations, two items were wrongly taken as 5 and 45 instead of 30 and 20. Calculate the corrected mean and corrected standard deviation.

[CA Foun. June 1993]

8. In a distribution of 200 boys, where 0–5, 5–10, etc. are groups, mean and standard deviations are 40 and 15 respectively. On checking it was found that the marks obtained by one student was wrongly entered as 53 in place of 43. Find the correct mean and standard deviation.

9. The following table gives the heights of students in a class. Find out the Quartile Deviation:

Heights in inches	50–53	53–56	56–59	59–62	62–65	65–68
No. of Students	2	7	24	7	13	3

10. Prices of a particular commodity in five years in two cities are given below:

Prices in City A	Prices in City B
20	10
22	20
19	18
23	12
26	15

Find from the above data the city which had more stable prices.

11. (a) From the following data calculate Mean and Standard Deviation:

Age Group	No. of Employees
Below 20	20
20–25	26
25–30	44
30–35	60
35–40	101
40–45	109
45–50	84
50–55	56
55 and above	10

What inference will you draw from the above?

(b) Find the SD of the following distribution:

Weights (kg)	50–52	52–54	54–56	56–58	58–60
No. of Students	17	35	28	15	5

[Hints: See worked-out Ex. 11. [C.U. B.Com. 2005]]

12. Calculate Quartile Deviation and Standard Deviation from the following data:

Expenditure on Food	No. of Families of Factory Employees
55.5–57.5	2
57.5–59.5	4
59.5–61.5	9
61.5–63.5	30
63.5–65.5	23
65.5–67.5	20
67.5–69.5	9
69.5–71.5	2
71.5–73.5	1

13. Find the Coefficient of Variation of the marks of Business Mathematics and Statistics obtained by the students of a college:

Marks Obtained	20–30	30–40	40–50	50–60	60–70
No. of Students	2	35	46	12	5

[C.U. B.Com. 1994]

14. From the data given below, state which series is more consistent:

Variable	Series A	Series B
10–20	10	18
20–30	18	22
30–40	32	40
40–50	40	32
50–60	22	18
60–70	18	10

15. A purchasing agent obtained samples of incandescent lamps from two suppliers. He had the samples tested in his own laboratory for length of life with the following results:

	Samples from	
Length of Life (in hours)	**Co. A**	**Co. B**
700 and under 900	10	3
900 and under 1100	16	42
1100 and under 1300	26	12
1300 and under 1500	8	3

Which company's lamps are more uniform?

16. The number of workers employed, the mean wage (in ₹) p.m. and the standard deviation (in ₹) in each section of a factory are given below. Calculate the mean wage and standard deviation of all workers taken together:

Section	No. of Workers Employed	Mean Wage (in ₹)	Standard Deviation (in ₹)
A	50	113	6
B	60	120	7
C	90	115	8

17. Find the missing information from the following:

	Group I	Group II	Group III	Combined
No.	50	?	90	200
Standard Deviation	6	7	?	7.746
Mean	113	?	115	116

[Hints: $50 + n_2 + 90 = 200$ or, $n_2 = 60$; $116 = \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times \bar{x}_2 + 90 \times 115}{200}$, etc.]

ANSWERS

1. (a) 45;
   (b) 50.
2. (a) 2 inches;
   (b) 5.17 kg.
3. (a) 21.15, 4.98;
   (b) ₹ 8.76;
   (c) 0.8;
   (d) ₹ 88.48 thousand.
4. (a) 14;
   (b) Mean = 16, SD = 4.
5. (a) QD = ₹ 1.50 and Coefficient of QD = 3.92;
   (b) 16.
6. (a) 10.0 and 2.86;
   (b) 6 and 3;
   (c) 4.
7. (a) 39.9, 5;
   (b) (i) 10.1053, 1.997;
   (ii) 10.2, 1.99;
   (c) 60; 9.62.
8. 39.95 and 14.97.
9. 2.92 inches.
10. City A has more stable prices.
11. (a) 39.25 and 9.60;
    (b) 2.18 kg.
12. 2.015, 2.97.
13. 19.6%.
14. Series B is more consistent.
15. Lamps of Company B are more uniform.
16. ₹ 116 and ₹ 7.75.
17. $n_2 = 60$, $\bar{x}_2 = 120$, $\sigma_3 = 8$.

Chapter 6

Moments, Measures of Skewness and Kurtosis

"A quantity of data which by its mere bulk may be incapable of entering the mind is to be replaced by relatively few quantities which shall adequately represent the whole, or which shall contain as much as possible, ideally the whole of the relevant information contained in the original data."

— R. A. Fisher

6.1 Introduction

We have already seen in the Chapters 4 and 5 how Measures of Central Tendency and Measures of Dispersion reveal the characteristics of frequency distributions. Frequency distributions differ in three ways: (1) average value, (2) variability or dispersion and (3) shape. In this chapter we shall study the relationship between shapes of frequency distribution and averages.

6.2 Moments

Definition 1. *If $x_1, x_2, \ldots, x_n$ are the n values of a variable x and A is any arbitrary constant, then the rth moment (m'_r) about A is defined by*

$$m'_r = \frac{\Sigma(x-A)^r}{n}, \text{ where } r = 1,2,3,\ldots.$$

For a frequency distribution, the rth moment (m'_r) about A is defined by

$$m'_r = \frac{\Sigma f(x-A)^r}{N}, \text{ where } N = \Sigma f \text{ and } r = 1,2,3,\ldots.$$

In particular,
(i) If $A = 0$, then the rth moment about zero is $\frac{\Sigma x^r}{n}$, which is also called the *rth raw moment.*

$$\therefore \quad \text{1st moment about zero} = \frac{\Sigma x}{n} = \text{mean } (\bar{x});$$

$$\text{2nd moment about zero} = \frac{\Sigma x^2}{n}, \text{ and so on.}$$

[1]In some books, μ'_r is taken instead of m'_r and μ_r instead of m_r.

For a frequency distribution, the rth moment about zero (or rth raw moment) is $\frac{\Sigma f x^r}{N}$, where $N=\Sigma f$ and x = value or mid-value of a class-interval.

$$\therefore \quad \text{1st moment about zero} = \frac{\Sigma f x}{N} = \text{mean } (\bar{x}),$$

$$\text{2nd moment about zero} = \frac{\Sigma f x^2}{N}, \text{ and so on.}$$

(ii) If $A=\bar{x}$, then the rth central moment (m_r) or the rth moment about the mean is defined by

$$m_r = \frac{\Sigma(x-\bar{x})^r}{n}, \text{ where } r=1,2,3,\ldots.$$

$$\therefore \quad m_1 = \text{1st moment about the mean} = \frac{\Sigma(x-\bar{x})}{n} = 0, \ [\because \Sigma(x-\bar{x})=0.]$$

$$m_2 = \text{2nd moment about the mean} = \frac{\Sigma(x-\bar{x})^2}{n} = (\text{SD})^2 = \text{variance } \sigma^2, \text{ and so on.}$$

For a frequency distribution, the rth central moment or the rth moment about the mean is defined by

$$m_r = \frac{\Sigma f(x-\bar{x})^r}{N}, \text{ where } N=\Sigma f, \text{ and } r=1,2,3,\ldots.$$

$$\therefore \quad m_1 = \text{1st moment about the mean} = \frac{\Sigma f(x-\bar{x})}{N} = 0,$$

$$m_2 = \text{2nd moment about the mean} = \frac{\Sigma f(x-\bar{x})^2}{N} = (\text{SD})^2 = \text{variance } \sigma^2, \text{ and so on.}$$

From the above definitions, we see that
- 1st moment about zero = mean ($\bar{x}$),
- 1st moment about the mean is always zero,
- 2nd moment about the mean = (SD)^2 = variance.

6.2.1 Relation between the moments about the mean m_r and the moments about any arbitrary point m'_r

We have

$$m_r = \frac{\Sigma f(x-\bar{x})^r}{N} \text{ and } m'_r = \frac{\Sigma f(x-A)^r}{N}.$$

If $d=x-A$, then

$$\bar{x} = \frac{\Sigma f x}{N} = \frac{\Sigma f(A+d)}{N} = \frac{A\Sigma f + \Sigma f d}{N} = A + \frac{\Sigma f d}{N};$$

$$\therefore \ \bar{x} - A = \frac{\Sigma f(x-A)}{N} = m'_1 \text{ or, } \bar{x} = A + m'_1.$$

We can write,

$$x - \bar{x} = (x-A) - (\bar{x}-A) = d - m'_1;$$

$$\therefore \ (x-\bar{x})^r = \left(d - m'_1\right)^r = d^r - {}^rC_1 d^{r-1} \cdot m'_1 + {}^rC_2 d^{r-2} m_1'^2 - \cdots + (-1)^r m_1'^r.$$

[using Binomial Theorem]

Summing over all the n values,

$$\Sigma f(x-\bar{x})^r=\Sigma f d^r-{}^rC_1 m_1'\Sigma f d^{r-1}+{}^rC_2 m_1'^2\Sigma f d^{r-2}-\cdots+(-1)^r m_1'^r\Sigma f.$$

Dividing both sides by N, we get

$$\frac{1}{N}\Sigma f(x-\bar{x})^r=\frac{1}{N}\Sigma f d^r-{}^rC_1 m_1'\cdot\frac{1}{N}\Sigma f d^{r-1}+{}^rC_2 m_1'^2\cdot\frac{1}{N}\Sigma f d^{r-2}+\cdots+(-1)^r m_1'^r,$$

$$[\because \Sigma f=N]$$

$$\text{or,}\quad m_r=m_r'-{}^rC_1 m_1'\cdot m_{r-1}'+{}^rC_2 m_1'^2 m_{r-2}'-\cdots+(-1)^r\cdot m_1'^r.$$

In particular, putting $r=1,2,3,4$, we get

$$\begin{aligned}
m_1&=m_1'-m_1'=0,\\
m_2&=m_2'-2m_1'^2+m_1'^2=m_2'-m_1'^2,\\
m_3&=m_3'-3m_1'm_2'+3m_1'^2m_1'-m_1'^3=m_3'-3m_2'm_1'+2m_1'^3\\
m_4&=m_4'-4m_1'm_3'+6m_1'^2m_2'-4m_1'^3m_1'+m_1'^4=m_4'-4m_3'm_1'+6m_2'm_1'^2-3m_1'^4.
\end{aligned}$$

Similarly, taking $x-A=(x-\bar{x})+(\bar{x}-A)=(x-\bar{x})+m_1'$ and proceeding as above, we can prove that

$$\begin{aligned}
m_1'&=m_1+m_1',\ \text{i.e., } m_1'=m_1'\ [\because\ m_1=0],\\
m_2'&=m_2+m_1'^2,\\
m_3'&=m_3+3m_2m_1'+m_1'^3,\\
m_4'&=m_4+4m_3m_1'+6m_2m_1'^2+m_1'^4.
\end{aligned}$$

Example 1.

(i) *If the first moment of a distribution is 2 about the value 2, find the mean.* [C.U. B.Com. 2006]

(ii) *Find the first two moments about zero for the set of numbers* 1, 3, 5, 7. [C.U. B.Com. 1998]

Solution: (i) The first moment of the distribution about the value $2=\frac{\Sigma f(x-2)}{N}$, where $N=\Sigma f$.

$$\therefore\ \frac{\Sigma f(x-2)}{N}=2,\quad \frac{\Sigma fx}{N}-\frac{\Sigma f2}{N}=2\ \text{or,}\ \frac{\Sigma fx}{N}=2+\frac{2\Sigma f}{N}=2+2\cdot1=4.$$

Hence, mean $(\bar{x})=\frac{\Sigma fx}{N}=4$.

Otherwise. If m_1' be the first moment about any value A, then mean $(\bar{x})=A+m_1'$.

Here $A=2$ and $m_1'=2$. Hence, mean $(\bar{x})=2+2=4$.

(ii) First moment about zero $=\dfrac{\Sigma x}{n}=\dfrac{1+3+5+7}{4}=\dfrac{16}{4}=4.$

Second moment about zero $=\dfrac{\Sigma x^2}{n}=\dfrac{1^2+3^2+5^2+7^2}{4}=\dfrac{1+9+25+49}{4}=\dfrac{84}{4}=21.$

Example 2. *Find the first, second and third moments about the origin* 4 *for the set of numbers* 2, 4, 6, 8.

Solution: rth moment about $4 = \frac{\Sigma(x-4)^r}{4}$, $r = 1,2,3$.

TABLE 6.1: CALCULATIONS FOR FIRST THREE MOMENTS

x	$x-4$	$(x-4)^2$	$(x-4)^3$
2	−2	4	−8
4	0	0	0
6	2	4	8
8	4	16	64
Total	$4=\Sigma(x-4)$	$24=\Sigma(x-4)^2$	$64=\Sigma(x-4)^3$

$$\therefore \quad \text{1st moment about } 4 = \frac{\Sigma(x-4)}{4} = \frac{4}{4} = 1.$$

$$\text{2nd moment about } 4 = \frac{\Sigma(x-4)^2}{4} = \frac{24}{4} = 6.$$

$$\text{3rd moment about } 4 = \frac{\Sigma(x-4)^3}{4} = \frac{64}{4} = 16.$$

Example 3. *Find the first, second, third and fourth central moments for the set of numbers* 2, 4, 6, 8.

Solution: $m_r = r$th central moment $= \dfrac{\Sigma(x-\bar{x})^r}{n}$, $r = 1,2,3,4$.

TABLE 6.2: CALCULATIONS FOR THE FIRST FOUR MOMENTS

x	$x-\bar{x}=x-5$	$(x-\bar{x})^2$	$(x-\bar{x})^3$	$(x-\bar{x})^4$
2	−3	9	−27	81
4	−1	1	−1	1
6	1	1	1	1
8	3	9	27	81
$20=\Sigma x$	$0=\Sigma(x-\bar{x})$	$20=\Sigma(x-\bar{x})^2$	$0=\Sigma(x-\bar{x})^3$	$164=\Sigma(x-\bar{x})^4$

$$\therefore \ \bar{x} = \frac{\Sigma x}{n} = \frac{20}{4} = 5.$$

$$\therefore \quad m_1 = \frac{\Sigma(x-\bar{x})}{4} = 0; \qquad m_2 = \frac{\Sigma(x-\bar{x})^2}{4} = \frac{20}{4} = 5;$$

$$m_3 = \frac{\Sigma(x-\bar{x})^3}{4} = \frac{0}{4} = 0; \quad m_4 = \frac{\Sigma(x-\bar{x})^4}{4} = \frac{164}{4} = 41.$$

Example 4. *The first two moments of a distribution about the value* 5 *of the variable are* 2 *and* 20. *Find the mean and the variance.*

Solution: We have by definition,

$$\frac{1}{N}\Sigma f(x-5) = 2 \tag{1}$$

$$\text{and} \quad \frac{1}{N}\Sigma f(x-5)^2 = 20. \tag{2}$$

From (1),
$$\frac{1}{N}(\Sigma fx - \Sigma f5) = 2 \quad \text{or,} \quad \frac{1}{N}\Sigma fx - \frac{5\Sigma f}{N} = 2$$
$$\text{or,} \quad \frac{\Sigma fx}{N} = 2 + 5 = 7 \; [\because \Sigma f = N]$$
$$\therefore \text{ Mean} = \frac{\Sigma fx}{N} = 7.$$

From (2),
$$\frac{1}{N}\Sigma f(x-5)^2 = 20 \quad \text{or,} \quad \frac{1}{N}\Sigma f\left(x^2 - 10x + 25\right) = 20$$
$$\text{or,} \quad \frac{1}{N}\Sigma fx^2 - \frac{1}{N}\cdot 10 \cdot \Sigma fx + 25 \cdot \frac{\Sigma f}{N} = 20$$
$$\text{or,} \quad \frac{1}{N}\Sigma fx^2 - 10 \times 7 + 25 \times 1 = 20$$
$$\text{or,} \quad \frac{1}{N}\Sigma fx^2 = 65.$$
$$\therefore \text{ Variance} = (\text{SD})^2 = \frac{\Sigma fx^2}{N} - \left(\frac{\Sigma fx}{N}\right)^2 = 65 - 49 = \mathbf{16}.$$

Otherwise: The mean and the variance can also be determined by using the relations $\bar{x} = A + m_1'$ and $\sigma^2 = m_2 = m_2' - m_1'^2$. Here $A = 5$, $m_1' = 2$, $m_2' = 20$.

Example 5. *The first* 3 *moments of a distribution about the value* 7 *calculated from a set of* 9 *observations are* 0.2, 19.4 *and* −41.0. *Find the measures of central tendency and dispersion, and also the third moment about origin.*

Solution: By definition, we have
$$\frac{\Sigma(x-7)}{9} = 0.2 \tag{1}$$
$$\frac{\Sigma(x-7)^2}{9} = 19.4 \tag{2}$$
$$\text{and} \quad \frac{\Sigma(x-7)^3}{9} = -41.0. \tag{3}$$

From (1),
$$\frac{\Sigma x - 7 \times 9}{9} = 0.2 \text{ or, } \frac{\Sigma x}{9} - 7 = 0.2 \text{ or, } \frac{\Sigma x}{9} = 7.2,$$

$\therefore$ Mean = 7.2, *which is a measure of central tendency.*

From (2),
$$\frac{\Sigma\left(x^2 - 14x + 49\right)}{9} = 19.4 \text{ or, } \frac{\Sigma x^2 - 14\Sigma x + 49 \times 9}{9} = 19.4$$
$$\text{or, } \frac{\Sigma x^2}{9} - \frac{14\Sigma x}{9} + 49 = 19.4 \text{ or, } \frac{\Sigma x^2}{9} - 14 \times 7.2 + 49 = 19.4$$
$$\text{or, } \frac{\Sigma x^2}{9} = 19.4 + 100.8 - 49 \text{ or, } \frac{\Sigma x^2}{9} = 71.2.$$
$$\therefore \text{ SD} = \sqrt{\frac{\Sigma x^2}{9} - \left(\frac{\Sigma x}{9}\right)^2} = \sqrt{71.2 - (7.2)^2} = \sqrt{71.2 - 51.84} = \sqrt{19.36} = \mathbf{4.4},$$

which is a measure of dispersion.

From (3), $$\frac{\Sigma(x^3 - 3x^2 \cdot 7 + 3 \cdot x \cdot 7^2 - 7^3)}{9} = -41.0$$

or, $$\frac{\Sigma x^3 - 21\Sigma x^2 + 147\Sigma x - 343 \times 9}{9} = -41.0$$

or, $$\frac{\Sigma x^3}{9} - 21 \times \frac{\Sigma x^2}{9} + 147 \times \frac{\Sigma x}{9} - \frac{343 \times 9}{9} = -41.0$$

or, $$\frac{\Sigma x^3}{9} - 21 \times 71.2 + 147 \times 7.2 - 343 = -41.0$$

or, $$\frac{\Sigma x^3}{9} = -41.0 + 343 - 1058.4 + 1495.2 \text{ or, } \frac{\Sigma x^3}{9} = 738.8.$$

$\therefore$ the third moment about the origin = 738.8.

Note: The problem can also be solved by applying the relations obtained in Section 6.2.1.

6.3 Skewness

A frequency distribution is said to be symmetrical when the values of the variable equidistant from their mean have equal frequencies.

If a frequency distribution is not symmetrical, it is said to be asymmetrical or skewed. Any deviation from symmetry is called *skewness.*

In the words of Riggleman and Frisbee:

"Skewness is the lack of symmetry. When a frequency distribution is plotted on a chart, skewness present in the items tends to be dispersed more on one side of the mean than on the other."

We discuss this important property of data in terms of symmetry or lack thereof without employing any particular measure.

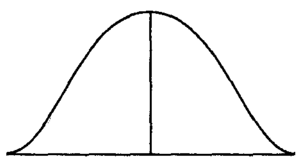


Fig. 6.1: Symmetrical Distribution.

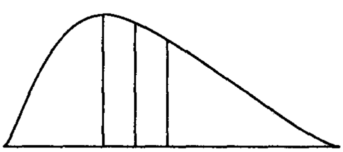


Fig. 6.2: Positively Skewed Distribution.

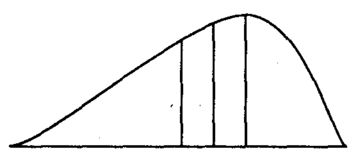


Fig. 6.3: Negatively Skewed Distribution.

Skewness may be positive or negative. A distribution is said to be positively skewed if the frequency curve has a longer tail towards the higher values of x, i.e., if the frequency curve gradually slopes down towards the high values of x. For a positively skewed distribution,

$$Mean\ (M) > Median\ (Me) > Mode\ (Mo).$$

A distribution is said to be negatively skewed if the frequency curve has a longer tail towards the lower values of x. For a negatively skewed distribution,

$$Mean < Median < Mode.$$

For a symmetrical distribution, Mean = Median = Mode.

6.3.1 Measures of Skewness

The degree of skewness is measured by its coefficient. The common measures of skewness are:

— Pearson's first measure:

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}.$$

Illustration 1. *If Mean = 25, Mode = 22 and Standard Deviation = 9, then*

$$Skewness = \frac{25-22}{9} = \frac{3}{9} = \frac{1}{3} \text{ or, } 0.33.$$

— Pearson's second measure:

$$\text{Skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{Standard Deviation}}.$$

Illustration 2. *If Mean = 18, Median = 16.5 and Standard Deviation = 6, then*

$$Skewness = \frac{3(18-16.5)}{6} = \frac{4.5}{6} = 0.75.$$

— Bowley's measure:

$$\text{Skewness} = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{(Q_3 - Q_2) + (Q_2 - Q_1)} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1},$$

where Q_1, Q_2, Q_3 are the first, second and third quartiles respectively.
For a symmetrical distribution, skewness = 0; $\therefore$ $Q_1 + Q_3 - 2Q_2 = 0$ or, $Q_1 + Q_3 = 2Q_2$.

— Moment-measure:

$$\text{Skewness} = \frac{m_3}{\sigma^3} = \frac{m_3}{m_2^{3/2}} = \frac{m_3}{m_2\sqrt{m_2}},$$

where m_2 and m_3 are the second and the third central moments and σ is the SD
All the four measures of skewness defined above are independent of the units of measurement.

Note: R.A. Fisher defined Skewness (γ_1) as $\gamma_1 = +\sqrt{\beta_1}$, where $\beta_1 = m_3^2/m_2^3$.

Example 6.

(i) *If $Q_1 = 26$, $Q_2 = 46$ and $Q_3 = 76$, find skewness.* **[C.U. B.Com. 1996]**

(ii) *For some symmetrical distribution, $Q_1 = 36$ and $Q_3 = 63$. Using Bowley's measure of skewness, find the median of the distribution.* **[C.U. B.Com. 1999]**

(iii) *If the second and third central moments of a distribution are 4 and 10, find the skewness of the distribution.* **[C.U. B.Com. 2001]**

(iv) *Compute Karl Pearson's coefficient of skewness of a distribution having 28.18, 28 and 2.64 as its mean, mode and SD respectively, and state the nature of skewness.* **[V.U. B.Com.(H) 2011]**

Solution: (i) Since $Q_1; Q_2$ and Q_3 are given, Bowley's measure of skewness is

$$\text{Skewness} = \frac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}, \text{ where } Q_1 = 26,\ Q_2 = 46 \text{ and } Q_3 = 76$$
$$= \frac{26 + 76 - 2 \times 46}{76 - 26} = \frac{10}{50} = 0.2.$$

(ii) Bowley's measure of skewness is

$$\text{Skewness} = \frac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}.$$

Since the distribution is symmetrical, Skewness = 0.

$$\therefore\ Q_1 + Q_3 - 2Q_2 = 0 \quad \text{or,} \quad 2Q_2 = Q_1 + Q_3 = 36 + 63 = 99$$
$$\text{or,} \quad Q_2 = \frac{99}{2} = 49.5.$$

(iii) m_2 = second central moment = 4, m_3 = third central moment = 10.

$\therefore$ Moment-measure of skewness is

$$\text{Skewness} = \frac{m_3}{(m_2)^{3/2}} = \frac{10}{(4)^{3/2}} = \frac{10}{8} = 1.15.$$

(iv) Coefficient of skewness $= \dfrac{\text{Mean}-\text{Mode}}{\text{SD}} = \dfrac{28.18 - 28}{2.64} = \dfrac{0.18}{2.64} = 0.068.$

Nature of skewness is *positive,* i.e., the frequency curve has a *longer tail towards the higher values of x.*

Example 7. *Calculate the Pearson's measure of skewness on the basis of Mean, Mode and Standard Deviation.*

x	14.5	15.5	16.5	17.5	18.5	19.5	20.5	21.5
f	35	40	48	100	125	87	43	22

Solution: Pearson's first measure of skewness is

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}.$$

Assuming a continuous series, we construct the following table:

TABLE 6.3: CALCULATIONS FOR MEAN AND SD

Class-intervals	x	f	$d = x - 18.5$	fd	fd^2
14–15	14.5	35	−4	−140	560
15–16	15.5	40	−3	−120	360
16–17	16.5	48	−2	−96	192
17–18	17.5	100	−1	−100	100
18–19	18.5	125	0	0	0
19–20	19.5	87	1	87	87
20–21	20.5	43	2	86	172
21–22	21.5	22	3	66	198
Total	—	$500 = N$	—	$-217 = \Sigma fd$	$1669 = \Sigma fd^2$

$$\text{Mean} = A + \frac{\Sigma fd}{N} = 18.5 - \frac{217}{500} = 18.5 - 0.43 = 18.07.$$

$$\text{SD} = \sqrt{\frac{1669}{500} - \left(\frac{-217}{500}\right)^2} = \sqrt{3.338 - 0.188} = \sqrt{3.150} = 1.775.$$

$$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i = 18 + \frac{125 - 100}{2 \times 125 - 100 - 87} \times 1 = 18 + \frac{25}{63} = 18 + 0.4 = 18.4.$$

$$\therefore \text{ Skewness} = \frac{18.07 - 18.4}{1.775} = -\frac{0.33}{1.775} = \mathbf{-0.186}.$$

Example 8. (i) *The Karl Pearson's coefficient of skewness of a distribution is* 0.32. *Its SD is* 6.5 *and the Mean is* 29.6. *Find the Mode.*

(ii) *In a distribution, Mean* = 65; *Median* = 70 *and Coefficient of skewness is* −0.6. *Find* (a) *Mode,* (b) *Coefficient of variation.* **[B.U. B.Com. 1990]**

(iii) *The median, mode and coefficient of skewness for a certain distribution are respectively* 17.4, 15.3 *and* 0.35. *Calculate the coefficient of variation.*

Solution: (i) Karl Pearson's first measure of skewness is

$$\text{Coefficient of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{SD}} \quad \text{or,} \quad 0.32 = \frac{29.6 - \text{Mode}}{6.5}$$

$$\text{or,} \quad 0.32 \times 6.5 = 29.6 - \text{Mode}$$

$$\text{or,} \quad \text{Mode} = 29.6 - 2.08 = \mathbf{27.52}.$$

(ii) Karl Pearson's Second measure of skewness is

$$\text{Coefficient of Skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{SD}} \quad \text{or,} \quad -0.6 = \frac{3(65 - 70)}{\text{SD}} = \frac{-15}{\text{SD}}$$

$$\text{or,} \quad \text{SD} = \frac{-15}{-0.6} = \frac{150}{6} = 25.$$

Again,

$$\text{Coefficient of Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{SD}} \quad \text{or,} \quad -0.6 = \frac{65 - \text{Mode}}{25}$$

$$\text{or,} \quad -15 = 65 - \text{Mode}$$

$$\text{or,} \quad \text{Mode} = 65 + 15 = \mathbf{80}.$$

$$\text{Coefficient of Variation} = \frac{\text{SD}}{\text{Mean}} \times 100 = \frac{25}{65} \times 100 = \frac{500}{13} = \mathbf{38.46\%}.$$

(iii) Mean − Mode = 3 (Mean−Median), or, Mean−15.3 = 3 (Mean−17.4) = 3 Mean−52.2, or, 2 (Mean) = 52.2 − 15.3 = 36.9 or, Mean = 18.45.

Again, skewness $= \dfrac{\text{Mean}-\text{Mode}}{\text{SD}}$ or, $0.35 = \dfrac{18.45 - 15.3}{\text{SD}}$ or, SD $= \dfrac{3.15}{0.35} = 9.$

Hence, the coefficient of variation $= \dfrac{\text{SD}}{\text{Mean}} \times 100\% = \dfrac{9}{18.45} \times 100\% = \mathbf{48.8\%}.$

Example 9. *Find the appropriate measure of skewness from the following distribution:*

Age (Years)	Below 20	20–25	25–30	30–35	35–40	40–45	45–55	55 and above
No. of Employees	13	29	46	60	112	94	45	21

[C.U. B.Com. 2004]

Solution: Since the frequency distribution has open-end classes, skewness based on Quartiles, i.e., Bowley's measure is the appropriate measure of skewness.

TABLE 6.4: CALCULATION OF CUMULATIVE FREQUENCY

Age (Years)	Cumulative Frequency (less than)
20	13
25	42
30	88
$Q_1 \longrightarrow$	$\longleftarrow 105 = N/4$
35	148
$Q_2 \longrightarrow$	$\longleftarrow 210 = N/2$
40	260
$Q_3 \longrightarrow$	$\longleftarrow 315 = 3N/4$
45	354
55	399
55 and above	$420 = N$

Here, $\frac{N}{4} = 105$, $\frac{N}{2} = 210$ and $\frac{3N}{4} = 315$;

$$\therefore Q_1 = l_1 + \frac{\frac{N}{4} - F_1}{f_1} \times c_1 = 30 + \frac{105-88}{60} \times 5 = 30 + \frac{17}{12} = 31.42,$$

$$Q_2 = l_2 + \frac{\frac{N}{2} - F_2}{f_2} \times c_2 = 35 + \frac{210-148}{112} \times 5 = 35 + \frac{62}{112} \times 5 = 35 + 2.8 = 37.8,$$

$$Q_3 = l_3 + \frac{\frac{3N}{4} - F_3}{f_3} \times c_3 = 40 + \frac{315-260}{94} \times 5 = 40 + \frac{55}{94} \times 5 = 40 + 2.9 = 42.9.$$

$$\therefore \text{Skewness} = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{(Q_3 - Q_2) + (Q_2 - Q_1)} = \frac{(42.9 - 37.8) - (37.8 - 31.4)}{(42.9 - 37.8) + (37.8 - 31.4)}$$

$$= \frac{5.1 - 6.4}{5.1 + 6.4} = \frac{-1.3}{11.5} = -0.11.$$

Example 10.

(i) *The first three moments of a distribution about the value* 1 *are* 2, 25 *and* 80. *Find its mean, SD and the moment-measure of skewness.* [C.U. B.Com. 1996 Type]

(ii) *If the first, second and third moments of a distribution about* 2 *are* 1, 16 *and* 40 *respectively, then find the first, second and third central moments and moment-measure of skewness of the distribution.* [C.U. B.Com. 1997]

Solution: (i) We have
$$\frac{1}{N}\Sigma f(x-1)=2 \tag{1}$$
$$\frac{1}{N}\Sigma f(x-1)^2=25 \tag{2}$$
and
$$\frac{1}{N}\Sigma f(x-1)^3=80. \tag{3}$$

From (1),
$$\frac{1}{N}(\Sigma fx-\Sigma f)=2 \quad \text{or,} \quad \frac{\Sigma fx}{N}-1=2 \;\; [\because \Sigma f=N]$$
$$\text{or,} \quad \frac{\Sigma fx}{N}=3; \; \therefore \text{ Mean}=3.$$

From (2),
$$\frac{1}{N}\Sigma f\left(x^2-2x+1\right)=25 \quad \text{or,} \quad \frac{\Sigma fx^2}{N}-\frac{2\Sigma fx}{N}+\frac{\Sigma f}{N}=25$$
$$\text{or,} \quad \frac{\Sigma fx^2}{N}-2\times 3+1=25$$
$$\text{or,} \quad \frac{\Sigma fx^2}{N}=30.$$
$$\therefore \text{ SD}=\sqrt{\frac{\Sigma fx^2}{N}-\left(\frac{\Sigma fx}{N}\right)^2}=\sqrt{30-3^2}=\sqrt{21}=\mathbf{4.58}.$$
$$m_3=\frac{\Sigma f(x-\bar{x})^3}{N}=\frac{\Sigma f(x-3)^3}{N}=\frac{\Sigma f\left(\overline{x-1}-2\right)^3}{N}$$
$$=\frac{\Sigma f\left\{(x-1)^3-3(x-1)^2\cdot 2+3(x-1)\cdot 2^2-2^3\right\}}{N}$$
$$=\frac{\Sigma f(x-1)^3}{N}-\frac{6\Sigma f(x-1)^2}{N}+\frac{12\Sigma f(x-1)}{N}-\frac{8\Sigma f}{N}$$
$$=80-6\times 25+12\times 2-8\times 1=80-150+24-8=-54.$$
$$\therefore \; \textit{Moment-measure of skewness}=\frac{m_3}{\sigma^3}=\frac{-54}{(4.58)^3}.$$

Let
$$z=\frac{54}{(4.58)^3};$$
$$\therefore \log z=\log 54-3\log 4.58=1.7324-3\times 0.6609=1.7324-1.9827=\bar{1}.7497;$$
$\therefore z=\text{antilog}\,(\bar{1}.7497)=0.5619.$

Hence, required moment-measure of skewness = **−0.5619.**

Otherwise. m_1' = 1st moment about 1 = 2, m_2' = 2nd moment about 1 = 25, $m_3'=80$.
$$\therefore \text{ Mean }(\bar{x})=A+m_1'=1+2=3.$$
$\sigma^2=m_2$ = 2nd central moment = $m_2'-\left(m_1'\right)^2=25-2^2=21; \therefore \sigma=\sqrt{21}=4.58.$

m_3 = 3rd central moment $=m_3'-3m_2'm_1'+2\left(m_1'\right)^3$
$$=80-3\times 25\times 2+2\cdot(2)^3$$
$$=80-150+16=-54.$$
$$\therefore \; \textit{Moment-measure of skewness}=\frac{m_3}{(m_2)^{3/2}}=\frac{m_3}{m_2\sqrt{m_2}}=\frac{-54}{21\sqrt{21}}=\mathbf{-0.5614}.$$

(ii) If the first three moments about 2 be m_1', m_2', m_3' respectively, then $m_1' = 1$, $m_2' = 16$ and $m_3' = 40$.

m_1 = 1st central moment = **0**, m_2 = 2nd central moment = $m_2' - (m_1')^2 = 16 - 1^2 = 15$ and m_3 = 3rd central moment $= m_3' - 3m_2'm_1' + 2(m_1')^3 = 40 - 3 \times 16 \times 1 + 2.(1)^3 = 40 - 48 + 2 = -6$.

$$\textit{The moment-measure of skewness} = \frac{m_3}{(m_2)^{3/2}} = \frac{-6}{(15)^{3/2}} = -\frac{6}{15\sqrt{15}} = -\frac{2}{5 \times 3.87} = -\frac{2}{19.35} = \mathbf{-0.103}.$$

6.4 Kurtosis

Kurtosis is the peakedness of the frequency curve. In two or more distributions having same average, dispersion and skewness, one may have high concentration of values near the mode; in this case its frequency curve will show a sharper peak than the other. This characteristic of frequency distribution is known as Kurtosis.

In the words of Clark and Schkade:

"Kurtosis is the property of a distribution which expresses its relative peakedness."

Kurtosis is measured by the coefficient β_2 which is defined by the formula:

$\beta_2 = m_4/m_2^2$ or by $\gamma_2 = \beta_2 - 3$, where m_4 is the 4th central moment and m_2 the 2nd central moment.

A distribution is said to be Platy-Kurtic, Meso-Kurtic and Lepto-Kurtic according as $\beta_2 < 3$, $\beta_2 = 3$ and $\beta_2 > 3$, i.e., according as $\frac{m_4}{\sigma_4} < 3$, $\frac{m_4}{\sigma_4} = 3$ and $\frac{m_4}{\sigma_4} > 3$.

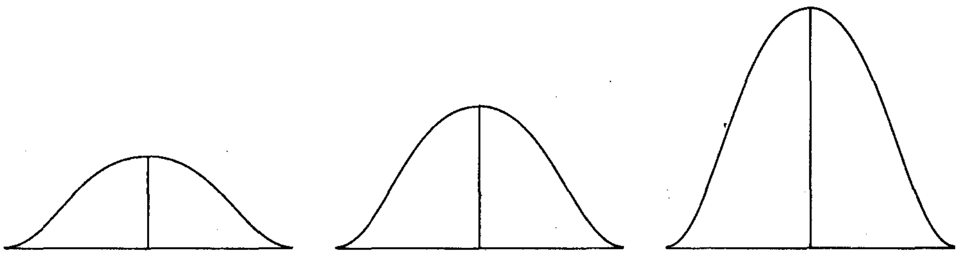


Fig. 6.4: Platy-Kurtic $b_2 < 3$

Fig. 6.5: Meso-Kurtic $b_2 = 3$

Fig. 6.6: Lepto-Kurtic $b_2 > 3$

Example 11. *The first four moments of a distribution about the value* 5 *are* 2, 20, 40 *and* 50. *Obtain, as far as possible, the various characteristics of the distribution on the basis of the information given. Comment upon the nature of the distribution.*

Solution: Here, in usual notations, we have $m_1' = 2$, $m_2' = 20$, $m_3' = 40$, $m_4' = 50$ and $A = 5$;

$\therefore$ Mean $(\bar{x}) = A + m_1' = 5 + 2 = 7$. $m_2 = m_2' - m_1'^2 = 20 - 2^2 = 20 - 4 = 16$;

$\therefore$ Variance $= m_2 = \mathbf{16}$.

$$m_3 = m_3' - 3m_2'm_1' + 2m_1'^3 = 40 - 3 \times 20 \times 2 + 2 \cdot 2^3 = -64;$$
$$m_4 = m_4' - 4m_3'm_1' + 6m_2'm_1'^2 - 3m_1'^4 = 50 - 4 \times 40 \times 2 + 6 \times 20 \times 2^2 - 3 \times 2^4 = 162.$$

$$\therefore \text{ Skewness} = \frac{m_3}{m_2^{3/2}} = \frac{-64}{16^{3/2}} = \frac{-64}{64} = \mathbf{-1}.$$

Since skewness is negative, the distribution is negatively skewed, i.e., the frequency curve has a longer tail towards the left.

$$\beta_2 = \frac{m_4}{m_2^2} = \frac{162}{16^2} = \frac{162}{256} = 0.63.$$

Since $\beta_2 < 3$, **the distribution is Platy-Kurtic.**

Example 12. *Calculate the first, the second, the third and the fourth central moments from the following frequency distribution:*

x	1	2	3	4	5	6	7
f	2	9	25	35	20	8	1

Solution:

TABLE 6.5: CALCULATIONS FOR THE FIRST FOUR CENTRAL MOMENTS

x	f	$y=x-4$	y^2	y^3	fy	fy^2	fy^3	fy^4
1	2	−3	9	−27	−6	18	−54	162
2	9	−2	4	−8	−18	36	−72	144
3	25	−1	1	−1	−25	25	−25	25
4	35	0	0	0	0	0	0	0
5	20	1	1	1	20	20	20	20
6	8	2	4	8	16	32	64	128
7	1	3	9	27	3	9	27	81
Total	$100=N$	—	—	—	$-10=\Sigma fy$	$140=\Sigma fy^2$	$-40=\Sigma fy^3$	$560=\Sigma fy^4$

$$m_1' = \text{1st moment about the value } 4 = \frac{\Sigma fy}{N} = \frac{-10}{100} = -0.1,$$

$$m_2' = \text{2nd moment about } 4 = \frac{\Sigma fy^2}{N} = \frac{140}{100} = 1.4,$$

$$m_3' = \text{3rd moment about } 4 = \frac{\Sigma fy^3}{N} = \frac{-40}{100} = -0.4,$$

$$m_4' = \text{4th moment about } 4 = \frac{\Sigma fy^4}{N} = \frac{560}{100} = 5.6.$$

$$\therefore\ m_1 = \text{1st central moment} = \Sigma f(x-\bar{x})/N = 0, \text{ where } \bar{x} = \text{Mean},$$

$$m_2 = \text{2nd central moment} = m_2' - m_1'^2 = 1.4 - (-0.1)^2 = 1.4 - 0.01 = 1.39.$$

$$m_3 = m_3' - 3m_2'm_1' + 2m_1'^3 = -0.4 - 3 \times 1.4 \times (-0.1) + 2(-0.1)^3$$
$$= -0.4 + 0.42 - 0.002 = 0.018,$$

and
$$m_4 = m_4' - 4m_3'm_1' + 6m_2'm_1'^2 - 3m_1'^4$$
$$= 5.6 - 4 \times (-0.4) \times (-0.1) + 6 \times 1.4 \times (-0.1)^2 - 3 \times (-0.1)^4$$
$$= 5.6 - 0.16 + 0.084 - 0.0003 = 5.5237.$$

Note: For a grouped frequency distribution x will represent the mid-values of the class-intervals.

EXERCISES ON CHAPTER 6(I)

Theory

1. Define raw and central moments of a frequency distribution.

   Express the second, third and fourth central moments in terms of raw moments.

2. What is meant by 'moment' for a distribution? Show how moments are used to describe the characteristics of a distribution, viz., central tendency, dispersion, skewness and kurtosis.

3. (a) Write a short note on Skewness.
   (b) Write a short note on Skewness and Kurtosis of a frequency distribution. [B.U. B.Com. 1990]
   (c) State Karl Pearson's first formula for measurement of Skewness. [C.U. B.Com. 1994]
   (d) Write a short note on Kurtosis. [V.U. B.Com.(H) 2010]

4. Explain the terms Skewness and Kurtosis used in connection with the frequency distribution of a continuous variable.

   Give the different measures of Skewness (any three of the measures to be given) and Kurtosis.

5. Give any three measures of skewness of a frequency distribution. Explain briefly (not exceeding ten sentences) with suitable diagrams the term 'Skewness' as mentioned above.

6. Distinguish between Skewness and Kurtosis and bring out their importance in describing the frequency distribution. [Utkal U. B.Com. 2000]

7. For a positively skewed distribution which one is true:

   (a) Mean > Median > Mode
   (b) Mean < Median < Mode.

   Put the correct statement in the form of a diagram. [C.U. B.Com. 1995]

8. (a) What do you mean by Lepto-Kurtic and Meso-Kurtic frequency curves? [C.U. B.Com. 1997]
   (b) What do you mean by 'Skewness' and 'Kurtosis'? State different formulae for measurement of skewness. [C.U. B.Com. 2005]
   (c) Write short notes on Measures of Kurtosis. [V.U. B.Com.(H) 2008]

Problems

1. (a) Find the first three moments about zero for the set of numbers 2, 4, 5, 8, 11.
   (b) For the numbers 1, 3, 5, 7, calculate first three moments about 3.
   (c) If the 1st moment of a distribution about the value 5 of the variable is 2, find the mean of the distribution. [C.U. B.Com. 2004]

   [Hints: $\frac{\Sigma f(x-5)}{N} = 2 \Rightarrow \frac{\Sigma fx - \Sigma f5}{N} = 2 \Rightarrow \frac{\Sigma fx}{N} - \frac{5\Sigma f}{N} = 2 \Rightarrow \frac{\Sigma fx}{N} = 2 + 5 = 7$, i.e., Mean of the distribution = 7.]

   (d) Find the first and second moments about the mean for the numbers 3, 4, 5, 8. [C.U. B.Com. 2005]

   [Hints: $\Sigma x = 20$, $n = 4$, $\bar{x} = 20/4 = 5$; $m_1 = \frac{\Sigma(x-\bar{x})}{4} = 0$ [$\because \Sigma(x-\bar{x}) = 0$]
   and $m_2 = \frac{\Sigma(x-\bar{x})^2}{4} = \frac{(3-5)^2+(4-5)^2+(5-5)^2+(8-5)^2}{4} = \frac{4+1+0+9}{4} = 3.5$.]

2. Find the first, second and third moments about the mean for the set of numbers 1, 3, 6, 7, 8.

3. Find the first, second and third moments about the value 4 for the set of numbers in Problem 2.

4. (a) The first two moments of a distribution about the value 4 of the variable are 3 and 34. Find the mean and the variance. [V.U. B.Com.(H) 2009]

   (b) If the first moment of a distribution is 2 about the value 2, find the mean. [C.U. B.Com. 2000]

   (c) The first two moments of a distribution about 1 are respectively 2 and 25. Find the mean and SD of the distribution. [C.U. B.Com. 2005]

5. (a) The first three moments of a distribution about the value 6 calculated from a set of 8 observations are 0.6, 18.2, −32.0. Find the measures of central tendency and dispersion and also the third moment about the origin.

   (b) If the second and third central moments of a distribution are 25 and −15.75, find the moment-measure of skewness. [C.U. B.Com. 2002]

   [Hints: Skewness = $\frac{m_3}{(m_2)^{3/2}} = -\frac{15.75}{(25)^{3/2}} = -\frac{15.75}{5^3} = -0.126$.]

   (c) The first three moments about the value 2 are 4, 65 and 325. Find its mean, SD and the moment-measure of skewness. [C.U. B.Com. 2001]

6. (a) For a group of 10 items, $\Sigma X = 452, \Sigma X^2 = 24270$ and Mode = 43.7. Find the Pearsonian coefficient of skewness.

   (b) For a moderately skewed distribution, arithmetic mean = 160, Mode = 157 and SD = 50. Find (a) coefficient of variation, (b) Pearsonian coefficient of skewness and (c) Median.

   (c) For a moderately skewed distribution, Mean = 172, Median = 167 and SD = 60. Find the coefficient of Skewness and Mode. [C.U. B.Com. 2000]

7. (a) The Karl Pearson's coefficient of skewness of a distribution is 0.32. Its SD is 6.5 and the mean is 29.6. Find the Mode and Median.

   (b) Given coefficient of Skewness = - 0.475, Mean = 64 and Median = 66. Find the value of the standard deviation.

   (c) If $Q_1 = 26$, $Q_3 = 76$ and coefficient of skewness = 0.2, find the median. [C.U. B.Com. 1996]

   [Hints: Bowley's measure of skewness is
   Skewness = $\frac{Q_1+Q_3-2Q_2}{Q_3-Q_1} = \frac{26+76-2Q_2}{76-26}$, or, $0.2 = \frac{102-2Q_2}{50}$ or, $2Q_2 = 102 - 10 = 92$.]

8. (a) Calculate the Karl Pearson's coefficient of skewness of the given distribution:

x	10–20	20–30	30–40	40–50	50–60	60–70	70–80
f	185	77	34	180	136	23	50

   (b) By using Bowley's formula find the coefficient of skewness for the following frequency distribution:

Marks	0–10	10–20	20–30	30–40	40–50
No. of Students	5	9	12	8	6

[C.U. B.Com. 1999]

9. Calculate an appropriate measure of skewness for the following data:

Income Group (₹)	No. of Students	Income Group (₹)	No. of Students
Below100	1	220–260	60
100–140	16	260–300	46
140–180	39	300–340	22
180–220	58	340 and above	9

10. Calculate from the undernoted data, the measure of skewness based on Mean, Median and SD:

x	100–200	200-300	300-400	400–500	500–600	600–700	700–800	800–900
f	45	88	146	206	79	52	30	14

11. Calculate Karl Pearson's and Bowley's measure of skewness for the following distribution:

Values	5–10	10–15	15–20	20–25	25–30	30–35	35–40	Total
Frequency	6	8	17	21	15	11	2	80

12. (a) The first four raw moments obtained from a frequency distribution are $-4, 22, -105$ and 444. Find the coefficient of Kurtosis. What is the nature of Kurtosis? [C.U. B.Com. 2006]

(b) Find out the Skewness and Kurtosis of the following series by the method of moments:

Measurement	0–10	10–20	20–30	30–40
Frequency	1	3	4	2

13. The AM of a certain distribution is 5. The second and the third moments about the mean are 20 and 140 respectively. Find the third moment of the distribution about 10. [C.U. B.Com. 2006]

14. The measure of skewness for a certain distribution is -0.8. If the lower and upper quartiles are 44.1 and 56.6 respectively, find the median.

15. Find the first, the second and the third Central moments of the frequency distribution given below:

Range of Expenditure in ₹ p.m.	No. of Families
3–6	28
6–9	292
9–12	389
12–15	212
15–18	59
18–21	18
21–24	2
	1000

16. Find the second, the third and the fourth central moments of the frequency distribution given below. Hence, find: (i) a measure of Skewness (γ_1) and (ii) a measure of Kurtosis (γ_2).

Class-limits	110.0–114.9	115.0–119.9	120.0–124.9	125.0–129.9	130.0–134.9	135.0–139.9	140.0–144.9
Frequency	5	15	20	35	10	10	5

ANSWERS

1. (a) $m_1' = 6$, $m_2' = 46$, $m_3' = 408$;
   (b) 1, 6, 16;
   (c) 7;
   (d) 0, 3.5.
2. $m_1 = 0$, $m_2 = 6.8$, $m_3 = -7.2$.
3. $m_1' = 1$, $m_2' = 7.8$, $m_3' = 14.2$.
4. (a) Mean = 7, Variance = 25;
   (b) 4;
   (c) 3, $\sqrt{21}$.
5. (a) $\bar{x} = 6.6$, $\sigma = 4.22$, $m_3' = 576.4$;
   (b) −0.126;
   (c) $\bar{x} = 6$, $\sigma = 7$ and sk = −0.95.
6. (a) +0.08;
   (b) CV = 31.25, Sk. = + 0.06, Median = 159;
   (c) 0.25; 157.
7. (a) Mode = 27.52, Median = 28.9;
   (b) 12.63;
   (c) 46.
8. (a) −0.46;
   (b) 0.029.
9. + 0.019.
10. + 0.1106.
11. 0, 0.014.
12. (a) $\beta_2 = \frac{108}{961} < 3$ (Platy-Kurtic);
    (b) −0.197, $\beta_2 = 2.26$ (Platy-Kurtic).
13. −285.
14. 55.35.
15. $m_1 = 0$, $m_2 = 9.3969$, $m_3 = 40.9914$.
16. $m_2 = 54.0$, $m_3 = 100.5$, $m_4 = 7827$, Sk. = 0.2533, Kurtosis = −0.319 (β_2 = 2.684) (Platy-Kurtic)

EXERCISES ON CHAPTER 6(II)

1. (a) Find the first four central moments for the set of numbers 2, 4, 6, 8.
   (b) Find the first, second and third moments about the origin 4 for the set of numbers 2, 4, 6, 8.
   (c) If the second and third central moments of a distribution be 4 and 12, find the skewness.
   [C.U. B.Com. 2003]

2. In a certain distribution the following results were obtained:
   Mean = 45, Median = 48, Coefficient of skewness = −0.4. The person who gave you the data failed to give the value of the standard deviation and you are required to estimate it with the help of the available information.

3. For a moderately skewed data, the arithmetic mean is 100, the coefficient of variation is 35 and the Karl Pearson's coefficient of skewness is 0.2. Find the mode and the median. **[V.U. B.Com.(H) 2007]**

[Hints: $\frac{\text{SD}}{\text{Mean}} \times 100\% = 35\%$ or, $\frac{\sigma}{100} \times 100 = 35$ or, $\sigma = 35$.

Now, $\frac{\text{Mean}-\text{Mode}}{\text{SD}} = \text{Skewness}$, or, $\frac{100-\text{Mode}}{35} = 0.2$ or, $100 - 7 = \text{Mode}$, i.e., Mode = 93.

Again, Mean−Mode = 3 (Mean−Median), or, 100 − 93 = 3 (100−Median), or, 3 (Median) = 300 − 7 = 293, or, Median = $\frac{293}{3} = 97.67$.]

4. In a frequency distribution the coefficient of skewness based upon the quartiles is 0.6. If the sum of upper and lower quartiles is 100 and the median is 38, find the value of the upper quartile.

5. Calculate Bowley's measure of skewness from the following data:

Payment of Commission in ₹	Nos. of Salesman
100–120	4
120–140	10
140–160	16
160–180	29
180–200	52
200–220	80
220–240	32
240–260	23
260–280	17
280–300	7

6. A frequency distribution gives the following results:

   (a) Coefficient of variation = 5

   (b) Standard deviation = 2

   (c) Karl Pearson's coefficient of skewness = 0.5.

   Find the mean and mode of the distribution.

7. (a) The first three moments of a distribution about 3 are respectively 4, 65, 134. Find the arithmetic mean, SD and moment-measure of skewness of the distribution.

   (b) In a distribution the first three moments about the point 10 are 0.3, 45.8 and 81.3 respectively. Calculate skewness.

   (c) Calculate the first three central moments and hence find the measure of skewness for the set of numbers 1, 3, 5, 7. [C.U. B.Com. 1995]

8. (a) Find the first four moments of the following distribution about the value 4:

x	1	3	5	7
f	2	3	4	1

[C.U. B.Com. 1991]

   (b) Find the first, second and third moments about 9 from the following data:

Values	2	4	6	8	10	12
Frequency	6	9	7	15	7	6

   Hence, find the first, second and third central moments and moment-measure of skewness of the distribution.

9. The first two moments of a distribution about the value 4 are −1.5 and 2.7. It is also known that the median of the distribution is 2.1. Comment on the shape of the distribution.

10. The median, mode and coefficient of skewness for a certain distribution are respectively 17.4, 15.3 and 0.35. Calculate the coefficient of variation.

11. If the first, second and third moments of a distribution about the value 2 are 1, 16 and 40 respectively, find the first, second and third central moments and moment-measure of skewness of the distribution. [C.U. B.Com. 1990]

12. (a) The first three moments about the value 2 are 4, 65 and 325. Find its mean, standard deviation and moment-measure of skewness. [C.U. B.Com. 2001]

    (b) If the first, second and third moments of a distribution about 2 are 1, 16 and 40 respectively, then find the first, second and third central moments and moment-measure of skewness of the distribution. [C.U. B.Com. 1997]

ANSWERS

1. (a) 0, 5, 0, 41;
   (b) 1, 6, 16;
   (c) 1.5.
2. 22.5.
3. 93, 97.67.
4. 70.
5. −0.035.
6. 40 and 39.
7. (a) 7, 7, −1.15;
   (b) 0.13;
   (c) 0, 5, 0; 0.
8. (a) −0.2, 3.4, −2.6, 25;
   (b) −1.96, 13.16, −64.36; 0, 9.32, −2.04, −0.072.
9. Positive skewness 1.8 shows that the frequency curve has a longer tail towards the right.
10. 49%.
11. 0, 15, −6, −0.1034.
12. (a) 6; 7,
    (b) 0, 15, −6; −0.1034.

Chapter 7

Correlation and Regression

7.1 Introduction

In the earlier chapters we have discussed the characteristics and shapes of distributions of a single variable, e.g., mean, SD and skewness of the distributions of income, height, weight, etc. We shall now study two (or more) variables simultaneously and try to find the quantitative relationship between them. *For example*, the relationship between two variables like (1) income and expenditure, (2) height and weight, (3) rainfall and yield of crops, (4) price and demand, etc., will be examined in this chapter. The methods of expressing the relationship between two variables are due mainly to Sir Francis Galton and Karl Pearson.

7.2 Correlation

Correlation is a statistical measure for finding out degree (or strength) of association between two (or more) variables. By 'association' we mean the tendency of the variables to move together. If two variables X and Y are so related that movements (or variations) in one, say X, tend to be accompanied by the corresponding movements (or variations) in the other Y, then X and Y are said to be *correlated*. The movements may be in the same direction (i.e., either both X, Y increase or both of them decrease) or in the opposite directions (i.e., one, say X, increases and the other Y decreases). Correlation is said to be positive or negative according as these movements are in the same or in the opposite directions. If Y is unaffected by any change in X, then X and Y are said to be *uncorrelated*.

In the words of L.R. Conner: "If two or more quantities vary in sympathy so that movements in the one tend to be accompanied by corresponding movements in the other, then they are said to be correlated."

Correlation may be *linear or non-linear*. If the amount of variation in X bears a constant ratio to the corresponding amount of variation in Y, then correlation between X and Y is said to be *linear*. Otherwise it is *non-linear* or *curvilinear*. In this case the amount of change in one variable does not bear a constant ratio to the amount of change in the other variable. As techniques of analysis of non-linear correlation are far more complicated than those of linear correlation, we shall consider linear correlation between the two variables in this book. *Correlation coefficient* or *Coefficient of correlation* (r) measures the **degree** of linear relationship (i.e., linear correlation) between two variables.

7.3 Determination of Correlation

Correlation between two variables may be determined by any one of the following methods:

- Two-way Frequency Table
- Scatter Diagram
- Co-variance Method or Karl Pearson's Method
- Rank Method.

Two-way Frequency Table (or Bivariate Frequency Table)

This is the simplest method of examining correlation between two variables. To construct a two-way frequency table we represent one variable, say X, along the vertical columns and the other Y along the horizontal rows. If there are m class-intervals for the X-series and n for the Y-series, we construct a table with m rows and n columns having $m \times n$ squares or cells. Each pair of observations is now shown by a tally mark (/) in the appropriate cell. When all the pairs of observations have been shown by tally marks in the two-way table, the cell frequencies are obtained by counting the tally marks of each cell. The type of correlation — positive or negative is roughly known by the nature of concentration of various cell frequencies. Two-way frequency table gives only a rough idea of the degree of correlation and, therefore, it is not generally used.

Method of construction of two-way frequency table will be clear from the example given below:

Example 1. *The ages of twenty husbands and wives are given below. Form a two-way frequency table showing the relationship between the ages of husbands and wives with class-intervals* 20–25, 25–30, *etc.*

Sl. Nos.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ages of Husband	28	37	42	25	29	47	37	35	23	41	27	39	23	33	36	32	22	29	38	48
Ages of Wife	23	30	40	26	25	41	35	25	21	38	24	34	20	31	29	35	23	27	34	47

Solution:

TABLE 7.1: TWO-WAY FREQUENCY TABLE SHOWING AGES OF HUSBANDS AND WIVES

		Ages of wife in years →						
Ages of husband in years ↓		20–25	25–30	30–35	35–40	40–45	45–50	Total
	20–25	/// 3						3
	25–30	// 2	/// 3					5
	30–35			/ 1	/ 1			2
	35–40		// 2	/// 3	/ 1			6
	40–45				/ 1	/ 1		2
	45–50					/ 1	/1	2
	Total	5	5	4	3	2	1	20

Note: In the above table class-intervals 20–25, 25–30, etc., represent respectively 20 and above but below 25, 25 and above but below 30, etc. For the Sl. No. 1, the age of husband, 28 years, falls in the 2nd class-interval 25–30 and the age of wife, 23, falls in the 1st class-interval 20–25. Therefore, we draw a tally mark (/) in the 1st cell of 3rd row. Similarly, all other pairs of observations can be shown by tally marks in the appropriate cells.

7.3.1 Scatter Diagram [C.U. B.Com. 2007]

The existence of correlation can be shown graphically by means of a *scatter diagram*. Statistical data relating to simultaneous movements (or variations) of two variables can be graphically represented by points. One of the two variables, say X, is shown along the horizontal axis OX and the other variable Y along the vertical axis OY. All the pairs of values of X and Y are now shown by points (or dots) on the graph paper. This diagrammatic representation of bivariate data is known as *scatter diagram*.

The scatter diagrams of these points as also the direction of the scatter reveal the nature and strength of correlation between the two variables. The following are some scatter diagrams showing different types of correlation between two variables.

In Figs 7.1 and 7.3, the movements (or variations) of the two variables are in the same direction and the scatter diagram shows a *linear path*. In this case, correlation is *positive* and association between the variables is *direct*.

In Figs 7.2 and 7.4, the movements of the two variables are in oppositive directions and the scatter shows a linear path. In this case correlation is *negative*, i.e., association between the variables is *indirect*.

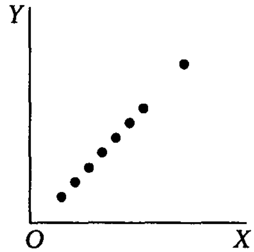


Fig. 7.1
($r = 1$)

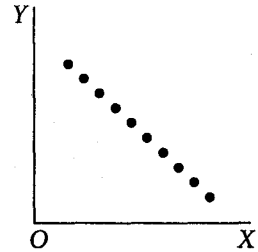


Fig. 7.2
($r = -1$)

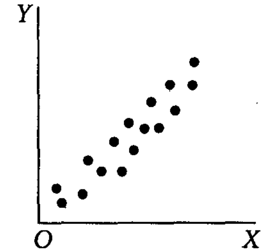


Fig. 7.3
(Positive correlation)

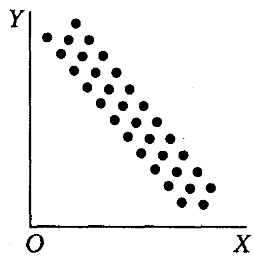


Fig.7 .4
(Negative correlation)

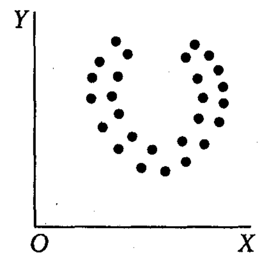


Fig. 7.5
($r = 0$)

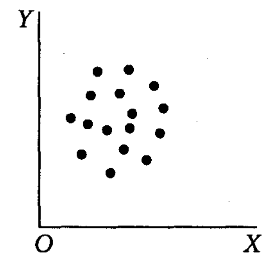


Fig. 7.6
($r = 0$)

In Figs 7.5 and 7.6, the points (or dots) instead of showing any linear path lie around a curve or form a swarm. In this case, correlation is very small and we can take $r = 0$.

In Figs 7.1 and 7.2, all the points lie on a straight line. In these cases, correlation is perfect and $r = +1$ or -1 according as the correlation is positive or negative.

7.3.2 Correlation Coefficient by Covariance Method

If $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be n pairs of observations on two variables X and Y, then the *covariance* of X and Y, written as cov (X, Y), is defined by

$$\text{cov}(X, Y) = \frac{1}{n}\Sigma(X - \bar{X})(Y - \bar{Y}).$$

Covariance indicates the joint variations between two variables.

The *correlation coefficient* or the *coefficient of correlation* (r) between X and Y is defined by

$$\boldsymbol{r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\Sigma[(X - \bar{X})(Y - \bar{Y})]}{n\sigma_X \sigma_Y}},$$

where σ_X, σ_Y are standard deviations of X and Y respectively.

The formula for the Correlation Coefficient r may be written in different forms.

(i) If $x = X - \bar{X}$ = deviation of X from its mean and $y = Y - \bar{Y}$ = deviation of Y from its mean, then

$$r = \frac{\Sigma xy}{n\sigma_X \sigma_Y}. \tag{1}$$

[$\because$ SD is independent of any change of origin.]

This formula was introduced by Karl Pearson.

Now,

$$\sigma_X = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{\Sigma x^2}{n}} \qquad [\because \Sigma x = \Sigma(X - \bar{X}) = 0]$$

Similarly,

$$\sigma_Y = \sqrt{\frac{\Sigma y^2}{n}}.$$

$\therefore$ from Eqn. (1),

$$r = \frac{\Sigma xy}{n\sqrt{\frac{\Sigma x^2}{n}} \times \sqrt{\frac{\Sigma y^2}{n}}} = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \times \sqrt{\Sigma y^2}}.$$

(ii)
$$\begin{aligned}
\text{cov}(X, Y) &= \frac{1}{n}\Sigma(X - \bar{X})(Y - \bar{Y}) = \frac{1}{n}\Sigma\{XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}\} \\
&= \frac{\Sigma XY}{n} - \bar{Y}\frac{\Sigma X}{n} - \bar{X}\frac{\Sigma Y}{n} + \frac{n\bar{X}\bar{Y}}{n} \\
&= \frac{\Sigma XY}{n} - \bar{X}\bar{Y} - \bar{X}\bar{Y} + \bar{X}\bar{Y} = \frac{\Sigma XY}{n} - \bar{X}\bar{Y} \\
&= \frac{\Sigma XY}{n} - \left(\frac{\Sigma X}{n}\right)\left(\frac{\Sigma Y}{n}\right);
\end{aligned}$$

$$\therefore \quad r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\Sigma XY}{n} - \left(\frac{\Sigma X}{n}\right)\left(\frac{\Sigma Y}{n}\right)\sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2} \times \sqrt{\frac{\Sigma Y^2}{n} - \left(\frac{\Sigma Y}{n}\right)^2}.$$

Since the correlation coefficient is independent of the choice of origin [for proof see **Section 7.4**], the corrected formula for correlation coefficient r, when the deviations are taken from the assumed mean, is given by

$$r = \frac{\frac{\Sigma xy}{n} - \frac{\Sigma x}{n} \times \frac{\Sigma y}{n}}{\sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} \times \sqrt{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2}},$$

where $x = X - a$ and $y = Y - b$, a and b being assumed means of X and Y respectively.

This formula can also be written as

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2} \times \sqrt{n\Sigma y^2 - (\Sigma y)^2}} = \frac{\Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\Sigma x^2 - \frac{(\Sigma x)^2}{n}} \times \sqrt{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}}.$$

(iii) Karl Pearson's formula for correlation coefficient r with frequencies of both the variables X and Y is given by

$$r = \frac{\dfrac{\Sigma f d_x d_y}{N} - \left(\dfrac{\Sigma f d_x}{N}\right)\left(\dfrac{\Sigma f d_y}{N}\right)}{\sqrt{\dfrac{\Sigma f d_x^2}{N} - \left(\dfrac{\Sigma f d_x}{N}\right)^2} \times \sqrt{\dfrac{\Sigma f d_y^2}{N} - \left(\dfrac{\Sigma f d_y}{N}\right)^2}}$$

$$= \frac{\Sigma f d_x d_y - \dfrac{(\Sigma f d_x)(\Sigma f d_y)}{N}}{\sqrt{\Sigma f d_x^2 - \dfrac{(\Sigma f d_x)^2}{N}} \times \sqrt{\Sigma f d_y^2 - \dfrac{(\Sigma f d_y)^2}{N}}},$$

where $d_x = \frac{X-a}{c}$ and $d_y = \frac{Y-b}{d}$.

Example 2. *Find the coefficient of correlation from the following data:*

X	1	2	3	4	5	6	7
Y	6	8	11	9	12	10	14

Solution:

TABLE 7.2: CALCULATIONS FOR COEFFICIENT OF CORRELATION

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
1	6	−3	−4	9	16	12
2	8	−2	−2	4	4	4
3	11	−1	1	1	1	−1
4	9	0	−1	0	1	0
5	12	1	2	1	4	2
6	10	2	0	4	0	0
7	14	3	4	9	16	12
$28 = \Sigma X$	$70 = \Sigma Y$	—	—	$28 = \Sigma x^2$	$42 = \Sigma y^2$	$29 = \Sigma xy$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{28}{7} = 4 \text{ and } \bar{Y} = \frac{\Sigma Y}{n} = \frac{70}{7} = 10.$$

Karl Pearson's coefficient of correlation (r) is given by

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \times \sqrt{\Sigma y^2}} = \frac{29}{\sqrt{28} \times \sqrt{42}}.$$

$$\therefore \log r = \log 29 - \left(\frac{1}{2}\log 28 + \frac{1}{2}\log 42\right) = 1.4624 - \left(\frac{1}{2} \times 1.4472 + \frac{1}{2} \times 1.6232\right)$$

$$= 1.4624 - (0.7236 + 0.8116) = 1.4624 - 1.5352 = \bar{1}.9272;$$

$\therefore$ **r = antilog($\bar{1}$.9272) = 0.8457 (approx.)**

Note: Student may use calculator instead of Log-table.

Example 3. (i) *Find the correlation coefficient, if* $\sigma_x^2 = 6.25$, $\sigma_y^2 = 4$, $\text{cov}(x, y) = 0.9$. [C.U. B.Com. 2000]

(ii) *Karl Pearson's coefficient of correlation between two variables X and Y is* 0.28, *their covariance is* +7.6. *If the variance of X is* 9, *find the standard deviation of Y-series.* [C.U. B.Com. 2004]

(iii) *If* $r = 0.4$, $\text{cov}(x, y) = 10$ *and* $\sigma_y = 5$, *then find the value of* σ_x.

Solution: (i) $r = \dfrac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y}$, where $\sigma_x^2 = 6.25$, $\sigma_y^2 = 4$ and $\text{cov}(x, y) = 0.9$

$$= \frac{0.9}{2.5 \times 2} \doteq \frac{0.9}{5} = \mathbf{0.18}, \text{ i.e., } \sigma_x = 2.5, \sigma = 2.$$

(ii) Karl Pearson's coefficient of correlation r is given by

$$r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}. \tag{1}$$

Here $r = 0.28$, $\text{cov}(X, Y) = 7.6$ and $\sigma_x^2 = 9$; $\therefore\ \sigma_X = 3$.

Using (1),

$$0.28 = \frac{7.6}{3\sigma_Y} \quad \text{or,} \quad 0.84\sigma_Y = 7.6 \quad \text{or,} \quad \sigma_Y = \frac{7.6}{0.84} = \frac{760}{84} = 9.048.$$

Hence, the standard deviation of Y-series is **9.048.**

(iii) $r = \dfrac{\text{cov}(x, y)}{\sigma_x, \sigma_y}$, or, $0.4 = \dfrac{10}{\sigma_x \cdot 5}$ or, $0.4\sigma_x = 2$, or, $\sigma_x = \dfrac{2}{0.4} = \mathbf{5}$.

Example 4. (i) *Calculate the coefficient of correlation between X and Y series from the following data:*

$$\sum_{i=1}^{15}\left(X_i - \bar{X}\right)^2 = 136, \sum_{i=1}^{15}\left(Y_i - \bar{Y}\right)^2 = 138 \text{ and } \sum_{i=1}^{15}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) = 122.$$

(ii) *From the following data, calculate the coefficient of correlation:* $n = 10$, $\Sigma x = 140$, $\Sigma y = 150$, $\Sigma(x - 10)^2 = 180$, $\Sigma(y - 15)^2 = 215$ and $\Sigma(x - 10)(x - 15) = 60$. [V.U. B.Com.(H) 2009]

Solution: (i) We have

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \times \sqrt{\Sigma y^2}},$$

where $x = X - \bar{X}$, $y = Y - \bar{Y}$.

Here $\Sigma xy = 122$, $\Sigma x^2 = 136$ and $\Sigma y^2 = 138$.

$$\therefore\ r = \frac{122}{\sqrt{136} \times \sqrt{138}};$$

$$\therefore\ \log r = \log 122 - \tfrac{1}{2}(\log 136 + \log 138) = 2.0864 - \tfrac{1}{2}(2.1335 + 2.1399) = 2.0864 - 2.1367 = \bar{1}.9497.$$

$$\therefore\ r = \text{antilog}\,(\bar{1}.9497) = 0.8910 = \mathbf{0.89\ (approx.)}$$

(ii) We have

$$r_{XY} = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{n\Sigma X^2 - (\Sigma X)^2} \times \sqrt{n\Sigma Y^2 - (\Sigma Y)^2}},$$

where $n = 10, X = x - 10, Y \neq y - 15$.

Then

$$\bar{x} = \frac{\Sigma x}{10} = \frac{140}{10} = 14,\ \bar{y} = \frac{\Sigma y}{10} = \frac{150}{10} = 15,\ \Sigma X = \Sigma(x-10) = \Sigma x - 10\times 10 = 140 - 100 = 40,$$

$$\Sigma Y = \Sigma(y-15) = \Sigma(y-\bar{y}) = 0,\ \Sigma X^2 = \Sigma(x-10)^2 = 180, \Sigma Y^2 = \Sigma(y-15)^2 = 215$$

and $\Sigma XY = \Sigma(x-10)(y-15) = 60$.

$$\therefore\ r_{XY} = \frac{10\times 60 - 40\times 0}{\sqrt{10\times 180-(40)^2}\times\sqrt{10\times 215-0}} = \frac{600}{\sqrt{1800-1600}\times\sqrt{2150}}$$

$$= \frac{600}{650.7} = \mathbf{0.92\ (approx.)}$$

Example 5. *Calculate Pearson's coefficient of correlation between advertisement cost and sales as per the data given below:*

Advertisement cost in '000 ₹	39	65	62	90	82	75	25	98	36	78
Sales in lac ₹	47	53	58	86	62	68	60	91	51	84

Solution: Karl Pearson's coefficient of correlation (r) is given by

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\times\sqrt{\Sigma y^2}},\ \text{where } x = X - \bar{X} \text{ and } y = Y - \bar{Y}.$$

TABLE 7.3: CALCULATIONS FOR CORRELATION COEFFICIENT

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
39	47	−26	−19	676	361	494
65	53	0	−13	0	169	0
62	58	−3	−8	9	64	24
90	86	25	20	625	400	500
82	62	17	−4	289	16	−68
75	68	10	2	100	4	20
25	60	−40	−6	1600	36	240
98	91	33	25	1089	625	825
36	51	−29	−15	841	225	435
78	84	13	18	169	324	234
$650 = \Sigma X$	$660 = \Sigma Y$	—	—	$5398 = \Sigma x^2$	$2224 = \Sigma y^2$	$2704 = \Sigma xy$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{650}{10} = 65;\ \bar{Y} = \frac{\Sigma Y}{n} = \frac{660}{10} = 66,\ r = \frac{2704}{\sqrt{5398}\times\sqrt{2224}};$$

$$\therefore\ \log r = \log 2704 - \tfrac{1}{2}(\log 5398 + \log 2224) = 3.4320 - \tfrac{1}{2}(3.7322 + 3.3472) = 3.4320 - 3.5397 = \bar{1}.8923;$$

$$\therefore\ r = \text{antilog}\,(\bar{1}.8923) = 0.7803 = \mathbf{0.78\ (approx.)}$$

Note: Student may use calculator instead of Log-table.

Example 6. *Calculate Pearson's coefficient of correlation from the following taking* 100 *and* 50 *as the assumed average of X and Y respectively*:

X	104	111	104	114	118	117	105	108	106	100	104	105
Y	57	55	47	45	45	50	64	63	66	62	69	61

Solution:

TABLE 7.4: CALCULATIONS FOR COEFFICIENT OF CORRELATION

X	Y	$x=X-a$	$y=Y-b$	x^2	y^2	xy
104	57	4	7	16	49	28
111	55	11	5	121	25	55
104	47	4	−3	16	9	−12
114	45	14	−5	196	25	−70
118	45	18	−5	324	25	−90
117	$50=b$	17	0	289	0	0
105	64	5	14	25	196	70
108	63	8	13	64	169	104
106	66	6	16	36	256	96
$100=a$	62	0	12	0	144	0
104	69	4	19	16	361	76
105	61	5	11	25	121	55
—	—	$96=\Sigma x$	$84=\Sigma y$	$1128=\Sigma x^2$	$1380=\Sigma y^2$	$312=\Sigma xy$

Pearson's coefficient of correlation (r) is given by

$$r=\frac{\Sigma xy-\frac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\Sigma x^2-\frac{(\Sigma x)^2}{n}}\times\sqrt{\Sigma y^2-\frac{(\Sigma y)^2}{n}}}=\frac{312-\frac{96\times84}{12}}{\sqrt{1128-\frac{96\times96}{12}}\times\sqrt{1380-\frac{84\times84}{12}}}$$

$$=312-672\sqrt{1128-768}\times\sqrt{1380-588}=\frac{-360}{\sqrt{360\times792}}.$$

Let $A=\dfrac{360}{\sqrt{360\times792}}$.

$\therefore\ \log A=\log360-\frac{1}{2}(\log360+\log792)=2.5563-\frac{1}{2}(2.5563+2.8987)=2.5563-2.7275=\bar{1}.8288;$

$$\therefore\ A=\text{antilog}\,(\bar{1}.8288)=0.6742;$$
$$\therefore\ r=-A=-0.6742=\mathbf{-0.67\ (approx.)}$$

Example 7. *A computer while calculating the correlation coefficient between two variables X and Y from* 25 *pairs of observations obtained the following results*: $n=25$, $\Sigma X=125$, $\Sigma Y=100$, $\Sigma X^2=650$, $\Sigma Y^2=460$ *and* $\Sigma XY=508$. *It was, however, discovered at the time of checking that two pairs of observations were not correctly copied. They were taken as* (6, 14) *and* (8, 6), *while the correct values were* (8, 12) *and* (6, 8). *Prove that the correct value of the correlation coefficient should be* $\frac{2}{3}$. [C.U. B.Com. 2000]

Solution: When the two incorrect pairs of observations are replaced by the correct pairs, the revised results for the whole series are

$$\Sigma X = 125 - (\text{Sum of two incorrect values of } X) + (\text{Sum of two correct values of } X)$$
$$= 125 - (6+8) + (8+6) = 125.$$

Similarly,
$$\Sigma Y = 100 - (14+6) + (12+8) = 100,$$
$$\Sigma X^2 = 650 - \left(6^2 + 8^2\right) + \left(8^2 + 6^2\right) = 650,$$
$$\Sigma Y^2 = 460 - \left(14^2 + 6^2\right) + \left(12^2 + 8^2\right) = 460 - 232 + 208 = 436$$
and
$$\Sigma XY = 508 - (6 \times 14 + 8 \times 6) + (8 \times 12 + 6 \times 8) = 508 - 132 + 144 = 520.$$

$\therefore$ correct value of the correlation coefficient (r) is

$$r = \frac{\Sigma XY - \dfrac{(\Sigma X)(\Sigma Y)}{n}}{\sqrt{\Sigma X^2 - \dfrac{(\Sigma X)^2}{n}} \times \sqrt{\Sigma Y^2 - \dfrac{(\Sigma Y)^2}{n}}} = \frac{520 - \dfrac{125 \times 100}{25}}{\sqrt{650 - \dfrac{(125)^2}{25}} \times \sqrt{436 - \dfrac{(100)^2}{25}}}$$
$$= \frac{20}{\sqrt{25 \times 36}} = \frac{20}{5 \times 6} = \frac{2}{3}.$$

Example 8. *Find the coefficient of correlation between the grouped frequency distribution of two variables (Profits and Sales) given below in the form of a two-way frequency table:*

		Sales (in '000 rupees) →					
Profits (in '000 rupees) →		80–90	90–100	100–110	110–120	120–130	Total
	50–55	1	3	7	5	2	18
	55–60	2	4	10	7	4	27
	60–65	1	5	12	10	7	35
	65–70	–	3	8	6	3	20
	Total	4	15	37	28	16	100

Solution:

TABLE 7.5: CALCULATIONS FOR CORRELATION COEFFICIENT

			Sales (in '000 ₹) ⟶								
		Class-intervals	80–90	90–100	100–110	110–120	120–130				
		Mid-values (x)	85	95	105	115	125				
Class-intervals	Mid-values (y)	d_y \ d_x	−2	−1	0	1	2	Total f	fd_y	fd_y^2	fd_xd_y
50–55	52.5	−1	1 (2)	3 (3)	7 (0)	5 (−5)	2 (−4)	18	−18	18	−4
55–60	57.5	0	2 (0)	4 (0)	10 (0)	7 (0)	4 (0)	27	0	0	0
60–65	62.5	1	1 (−2)	5 (−5)	12 (0)	10 (10)	7 (14)	35	35	35	17
65–70	67.5	2	0 (0)	3 (−6)	8 (0)	6 (12)	3 (12)	20	40	80	18
		Total f	4	15	37	28	16	100	57	133	31
		fd_x	−8	−15	0	28	32	37			
		fd_x^2	16	15	0	28	64	123			
		fd_xd_y	0	−8	0	17	22	31			

The correlation coefficient (r) is given by

$$r = \frac{\Sigma f d_x d_y - \dfrac{(\Sigma f d_x)(\Sigma f d_y)}{N}}{\sqrt{\Sigma f d_x^2 - \dfrac{(\Sigma f d_x)^2}{N}} \times \sqrt{\Sigma f d_y^2 - \dfrac{(\Sigma f d_y)^2}{N}}} = \frac{31 - \dfrac{37 \times 57}{100}}{\sqrt{123 - \dfrac{(37)^2}{100}} \times \sqrt{133 - \dfrac{(57)^2}{100}}}$$

$$= \frac{31 - 21.09}{\sqrt{123 - 13.69} \times \sqrt{133 - 32.49}} = \frac{9.91}{\sqrt{109.31 \times 100.51}}.$$

$$\therefore \log r = \log 9.91 - \tfrac{1}{2}(\log 109.31 + \log 100.51) = 0.9961 - \tfrac{1}{2}(2.0386 + 2.0021)$$
$$= 0.9961 - 2.0204 = \bar{2}.9757;$$

$$\therefore r = \text{antilog}\,(\bar{2}.9757) = 0.09456 = \mathbf{0.0946\ (approx.)}$$

Note: (1) $d_x = \frac{x-105}{10}$ and $d_y = y - 57.55$.

(2) Number within brackets in each cell is the product of that cell frequency and the corresponding values of d_x and d_y.

(3) fd_xd_y represents algebraic sum of the numbers within brackets in any row or column.

7.3.3 Rank Correlation (or Correlation by Rank Method)

Simple correlation coefficient (or product-moment correlation coefficient) is based on the magnitudes of the variable. But in many situations it is not possible to find the magnitude of the variable at all. *For example*, we cannot measure beauty or intelligence quantitatively. In this case, it is possible to rank the individuals in some order. Rank correlation is based on the rank or the order and not on the magnitude of the variable. It is more suitable if the individuals (or variables) can be arranged in order of merit or proficiency. If the ranks assigned to individuals range from 1 to N, then the correlation coefficient between two series of ranks is called *Rank Correlation Coefficient.* Edward Spearman's formula for Rank Coefficient of Correlation (R) is given by

$$R = 1 - \frac{6\Sigma D^2}{N(N^2-1)} \text{ or, } 1 - \frac{6\Sigma D^2}{N^3 - N},$$

where D is the difference between the ranks of the two series and N is the number of individuals in each series.

Deduction of Edward Spearman's Formula for Rank Correlation Coefficient

$$R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}.$$

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the ranks of n individuals in two characters (or series). Edward Spearman's Rank Correlation Coefficient R is the *product-moment correlation coefficient* between these ranks and, therefore, we can write,

$$R = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}, \text{ where } \text{cov}(x,y) = \frac{\Sigma\{(x_i - \bar{x})(y_i - \bar{y})\}}{n}. \tag{1}$$

But the ranks of n individuals are the natural numbers $1, 2, \ldots, n$ arranged in some orders depending on the qualities of the individuals.

$\therefore x_1, x_2, \ldots, x_n$ are the numbers $1, 2, \ldots, n$ in some order.

$$\therefore \Sigma x = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \text{ and } \Sigma x^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\therefore \bar{x} = \frac{\Sigma x}{n} = \frac{n+1}{2} \text{ and } \frac{\Sigma x^2}{n} = \frac{(n+1)(2n+1)}{6}.$$

$$\therefore \sigma_x^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \left(\frac{n+1}{12}\right)(4n+2-3n-3) = \frac{n^2-1}{12}.$$

Similarly,
$$\bar{y} = \frac{n+1}{2} \text{ and } \sigma_y^2 = \frac{n^2-1}{12}.$$

[$\because y_1, y_2, \ldots, y_n$ are also the numbers $1, 2, \ldots, n$ in some order.]

Let $d_i = x_i - y_i$; then $d_i = (x_i - \bar{x}) - (y_i - \bar{y})$ [$\because \bar{x} = \bar{y}$].

$$\therefore \frac{\Sigma d_i^2}{n} = \frac{\Sigma\{(x_i - \bar{x}) - (y_i - \bar{y})\}^2}{n} = \frac{\Sigma(x_i - \bar{x})^2}{n} + \frac{\Sigma(y_i - \bar{y})^2}{n} - \frac{2\Sigma\{(x_i - \bar{x})(y_i - \bar{y})\}}{n}$$
$$= \sigma_x^2 + \sigma_y^2 - 2\,\text{cov}(x,y)$$

$$\text{or, } 2\,\text{cov}(x,y) = \frac{n^2-1}{12} + \frac{n^2-1}{12} - \frac{\Sigma d_i^2}{n} = \frac{2(n^2-1)}{12} - \frac{\Sigma d_i^2}{n} \text{ or, } \text{cov}(x,y) = \frac{n^2-1}{12} - \frac{\Sigma d_i^2}{2n}.$$

Hence, from (1), we get

$$R = \frac{\frac{n^2-1}{12} - \frac{\Sigma d_i^2}{2n}}{\sqrt{\frac{n^2-1}{12}} \times \sqrt{\frac{n^2-1}{12}}} = \frac{\frac{n^2-1}{12} - \frac{\Sigma d_i^2}{2n}}{\frac{n^2-1}{12}} = 1 - \frac{6\Sigma d^2}{n(n^2-1)} \text{ [omitting } i\text{]}.$$

Example 9. *Find Spearman's Rank Correlation Coefficient R, when $\Sigma D^2 = 30$ and $N = 10$.* [C.U. B.Com. 2001]

Solution: We have

$$R = 1 - \frac{6\Sigma D^2}{N(N^2-1)} = 1 - \frac{6 \times 30}{10(10^2-1)} = 1 - \frac{18}{99} = 1 - \frac{2}{11} = \frac{9}{11} = \mathbf{0.82}.$$

Example 10. *Ten competitors in a beauty contest are ranked by two judges in the following order. Calculate Spearman's rank correlation coefficient:*

Competitors	A	B	C	D	E	F	G	H	I	J
1st Judge	1	6	5	10	3	2	4	9	7	8
2nd Judge	6	4	9	8	1	2	3	10	5	7

[C.U. B.Com. 1996]

Solution: Spearman's rank correlation coefficient (R) is given by

$$R = 1 - \frac{6\Sigma D^2}{N(N^2-1)},$$

where $D = x - y$ = (Rank by 1st Judge)−(Rank by 2nd Judge)
and N = no. of competitors = 10.

TABLE 7.6: CALCULATION FOR RANK CORRELATION COEFFICIENT

Competitors	Rank by		$D = x - y$ (Rank Diff.)	D^2
	1st Judge (x)	2nd Judge (y)		
A	1	6	−5	25
B	6	4	2	4
C	5	9	−4	16
D	10	8	2	4
E	3	1	2	4
F	2	2	0	0
G	4	3	1	1
H	9	10	−1	1
I	7	5	2	4
J	8	7	1	1
Total	—	—	—	$60 = \Sigma D^2$

$$\therefore\ R = 1 - \frac{6 \times 60}{10(10^2-1)} = 1 - \frac{36}{99} = 1 - \frac{4}{11} = \frac{7}{11} = \mathbf{0.636}.$$

Example 11. *Ten students got the following percentage of marks in Mathematics and Statistics:*

Student (Roll No.)	1	2	3	4	5	6	7	8	9	10
Marks in Mathematics	78	36	98	25	75	82	90	62	65	39
Marks in Statistics	84	51	91	60	68	62	86	58	53	47

Calculate the correlation coefficient.

Solution: In Mathematics, student with Roll No. 3 gets the highest mark 98 and is ranked 1; Roll No. 7 securing 90 marks has rank 2 and so on. Similarly, we can find the ranks of students in Statistics.

TABLE 7.7: CALCULATIONS FOR RANK CORRELATION

Roll No.	Mathematics		Statistics		Rank Diff.	D^2
	Marks	Rank (x)	Marks	Rank (y)	$D=x-y$	
1	78	4	84	3	1	1
2	36	9	51	9	0	0
3	98	1	91	1	0	0
4	25	10	60	6	4	16
5	75	5	68	4	1	1
6	82	3	62	5	−2	4
7	90	2	86	2	0	0
8	62	7	58	7	0	0
9	65	6	53	8	−2	4
10	39	8	47	10	−2	4
Total	—	—	—	—	—	$30=\Sigma D^2$

Applying Edward Spearman's formula,

$$R=1-\frac{6\Sigma D^2}{N(N^2-1)} \text{ [here } N=10,\ \Sigma D^2=30]$$

$$=1-\frac{6\times 30}{10\,(10^2-1)}=1-\frac{18}{99}=1-\frac{2}{11}=\frac{9}{11}=\mathbf{0.82}.$$

Edward Spearman's Formula for Tied Ranks

If in a series two or more individuals (or items) have the same score, then we find the average of the ranks of these individuals and allot this average rank to each of them. [In example 12, as the score 98 occurs three times, there is a tie for the 7th place and in this case, we find the average rank $\frac{7+8+9}{3}=8$ of 7, 8, 9 and assign it to each 98.]

In such cases, Edward Spearman's modified formula is

$$\boldsymbol{R=1-\frac{6[\Sigma D^2+\Sigma\,(t^3-t)\,/12]}{N(N^2-1)}},$$

where t is the number of individuals (or items) involved in a tie in the first or second series.

Example 12. *Find the rank correlation coefficient of the following data:*

Series A:	115	109	112	87	98	120	98	100	98	118
Series B:	75	73	85	70	76	82	65	73	68	80

Solution: In series A, the highest score is 120 and, therefore, its rank is 1; the rank of the second highest score 118 is 2. Proceeding in this manner, we find that the rank of 98 is 7 and 98 occurs three times. As there is a tie for the 7th place, we find the average rank $\frac{7+8+9}{3}$, i.e., 8, and assign 8 as rank to each of three 98. The score next to 98 is 87 and its rank is 10. Similarly, we can find the ranks of individuals (or items) in series B.

TABLE 7.8: CALCULATIONS FOR THE RANK CORRELATION COEFFICIENT

Series A		Series B		Rank Difference	D^2
Score	Rank (x)	Score	Rank (y)	$D = x - y$	
115	3	75	5	-2	4
109	5	73	6.5	-1.5	2.25
112	4	85	1	3	9
87	10	70	8	2	4
98	8	76	4	4	16
120	1	82	2	-1	1
98	8	65	10	-2	4
100	6	73	6.5	-0.5	0.25
98	8	68	9	-1	1
118	2	80	3	-1	1
Total	—	—	—	—	$42.50 = \Sigma D^2$

By Edward Spearman's formula,

$$R = 1 - \frac{6\left\{\sum D^2 + \sum\left(\frac{t^3 - t}{12}\right)\right\}}{N(N^2 - 1)},$$

where t = number of individuals involved in a tie.

Here $\Sigma D^2 = 42.50$, $N = 10$, $t = 3, 2$ and

$$\sum \frac{t^3 - t}{12} = \frac{3^3 - 3}{12} + \frac{2^3 - 2}{12} = \frac{24}{12} + \frac{6}{12} = 2 + 0.5 = 2.5.$$

$$\therefore\ R = 1 - \frac{6\{42.5 + 2.5\}}{10 \cdot (10^2 - 1)} = 1 - \frac{6 \times 45}{10 \times 99} = 1 - \frac{3}{11} = \frac{8}{11} = \mathbf{0.73}.$$

Example 13. *The coefficient of rank correlation between the marks in Statistics and Mathematics obtained by a certain group of students is $\frac{2}{3}$ and the sum of the squares of the differences in ranks is 55. Find the number of students in the group.*

Solution: Edward Spearman's formula for rank correlation coefficient is

$$R = 1 - \frac{6\Sigma D^2}{N(N^2 - 1)}.$$

Here $R = \frac{2}{3}$, $\Sigma D^2 = 55$, $N = ?$.

$$\therefore\ \frac{2}{3} = 1 - \frac{6 \times 55}{N(N^2 - 1)} = 1 - \frac{330}{N(N^2 - 1)}$$

$$\text{or, } \frac{330}{N(N^2 - 1)} = 1 - \frac{2}{3} = \frac{1}{3} \quad \text{or, } \quad N(N^2 - 1) = 990$$

$$\text{or, } N(N - 1)(N + 1) = 990 \quad \text{or, } \quad (N - 1)N(N + 1) = 9 \cdot 10 \cdot 11.$$

Hence, $N = 10$, i.e., the number of students in the group = **10**.

Example 14. *The coefficient of rank correlation of the marks obtained by 10 students in Mathematics and Statistics was found to be 0.5. It was then detected that the difference in ranks in the two subjects for one particular student was wrongly taken to be 3 in place of 7. What should be the correct rank correlation coefficient?*

[V.U. B.Com. 1995]

Solution: Edward Spearman's formula for the coefficient of rank correlation is

$$R = 1 - \frac{6\Sigma D^2}{N(N^2-1)}.$$

Here $R = 0.5$, $N = 10$, $\Sigma D^2 = ?$.

$$\therefore \quad 0.5 = 1 - \frac{6\Sigma D^2}{10(10^2-1)} = 1 - \frac{6\Sigma D^2}{10 \times 99} = 1 - \frac{\Sigma D^2}{165}$$

$$\text{or,} \quad \frac{\Sigma D^2}{165} = 1 - 0.5 = 0.5 = \frac{1}{2} \text{ or, } \Sigma D^2 = 82.5.$$

Now, correct value of ΣD^2

$= \text{Incorrect } \Sigma D^2 - (\text{incorrect difference in ranks})^2 + (\text{correct difference in ranks})^2$

$= 82.5 - (3)^2 + (7)^2 = 122.5.$

Hence, the correct rank correlation coefficient

$$= 1 - \frac{6 \times 122.5}{10(10^2-1)} = 1 - \frac{122.5}{165} = \frac{42.5}{165} = \mathbf{0.2576}.$$

Theorem 1. *Show that the correlation coefficient is independent of the origin of reference and the units of measurement.*

Proof. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be the n pairs of observations relating to two variables X and Y. Let us change the origins of X and Y to a and b, and the scales (or units measurement) to c and d respectively.

Let

$$x_i = \frac{X_i - a}{c} \text{ and } y_i = \frac{Y_i - b}{d},$$

where a, b, c, d are constants, c and d being positive.

The required results will be proved, if we can show that $r_{XY} = r_{xy}$.

Now

$$x_i = \frac{X_i - a}{c} \text{ or, } X_i = a + cx_i.$$

$$\therefore \bar{X} = \frac{\Sigma X_i}{n} = \frac{\Sigma(a + cx_i)}{n} = \frac{na + c\Sigma x_i}{n} = a + c\frac{\Sigma x_i}{n} = a + c\bar{x}.$$

$$\therefore X_i - \bar{X} = c(x_i - \bar{x}).$$

Similarly, $Y_i - \bar{Y} = d(y_i - \bar{y})$.

$$\sigma_X^2 = \frac{\Sigma(X_i - \bar{X})^2}{n} = \frac{\Sigma c^2(x_i - \bar{x})^2}{n} = c^2\sigma_x^2; \quad \therefore \sigma_X = c\sigma_x \; [\because c \text{ is positive}].$$

Similarly, $\sigma_Y = d\sigma_y$;

$$\therefore r_{XY} = \frac{\Sigma[(X_i - \bar{X})(Y_i - \bar{Y})]}{n\sigma_X\sigma_Y} = \frac{\Sigma[c(x_i - \bar{x}) \cdot d(y_i - \bar{y})]}{n \cdot c\sigma_x \cdot d\sigma_y} = \frac{\Sigma[(x_i - \bar{x})(y_i - \bar{y})]}{n\sigma_x\sigma_y} = r_{xy}.$$

Hence, *the theorem follows.*

Remark 1. *In general, $\sigma_X = |c|\sigma_x$ and $\sigma_Y = |d|\sigma_y$ so that $r_{XY} = \frac{cd}{|c|\cdot|d|} \cdot r_{xy}$. Now $\frac{cd}{|c|\cdot|d|} = 1$ or -1 according as c, d have the same sign or opposite signs. $\therefore$ $r_{XY} = \pm r_{xy}$ according as c, d have the same or opposite signs, i.e., the absolute value of the correlation coefficient is independent of any change of scale.*

Example 15. *If $r_{xy} = 0.6$, find r_{uv}, where* (i) $u = 3x + 5$, $v = 4y - 3$; (ii) $u = 3x + 5$, $v = -4y + 3$.

Solution: We know that if $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$, then $r_{uv} = r_{xy}$ or $-r_{xy}$ according as c, d are of the same or opposite signs.

(i) $u = 3x + 5 = 3\left(x + \frac{5}{3}\right) = \left(x + \frac{5}{3}\right)/\frac{1}{3}$ and $v = 4y - 3 = \left(y - \frac{3}{4}\right)/\frac{1}{4}$. Here $c = \frac{1}{3}$ and $d = \frac{1}{4}$ and c, d are both *positive*, i.e., they have the same sign.

$$\therefore\ r_{uv} = r_{xy} = 0.6.$$

(ii) $u = 3x + 5 = 3\left(x + \frac{5}{3}\right) = \left(x + \frac{5}{3}\right)/\frac{1}{3}$ and $v = -4y + 3 = -4\left(y - \frac{3}{4}\right) = \left(y - \frac{3}{4}\right)/-\frac{1}{4}$. Here $c = \frac{1}{3}$ and $d = -\frac{1}{4}$ and c, d have opposite signs.

$$\therefore\ r_{uv} = -r_{xy} = -0.6.$$

Theorem 2. *Prove that the correlation coefficient r lies between -1 and $+1$, i.e., $-1 \le r \le 1$.*

Proof. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be the n pairs of observations on two variables X and Y.

Let $x_i = \dfrac{X_1 - \bar{X}}{\sigma_X}$ and $y_i = \dfrac{Y_i - \bar{Y}}{\sigma_Y}$, where σ_X, σ_Y are the standard deviations of X, Y respectively.

We have
$$\Sigma x_i^2 = \sum\left(X_i - \bar{X}\sigma_X\right)^2 = \frac{\Sigma(X_i - \bar{X})^2}{\sigma_X^2} = \frac{n\sigma_X^2}{\sigma_X^2} = n.$$

Similarly, $\Sigma y_i^2 = n$.

$$\Sigma x_i y_i = \sum\left[\left(\frac{X_i - \bar{X}}{\sigma_X}\right)\left(\frac{Y_i - \bar{Y}}{\sigma_Y}\right)\right] = \frac{\Sigma[(X_i - \bar{X})(Y_i - \bar{Y})]}{\sigma_X\sigma_Y} = n\frac{\Sigma[(X_i - \bar{X})(Y_i - \bar{Y})]}{n\sigma_X\sigma_Y} = nr,$$

where r is the correlation coefficient between X and Y.

Since the square of a quantity is never negative, we have

$$\Sigma(x_i + y_i)^2 \ge 0 \tag{1}$$
and
$$\Sigma(x_i - y_i)^2 \ge 0. \tag{2}$$

From (1),

$$\Sigma(x_i^2 + y_i^2 + 2x_i y_i) \ge 0$$
or, $\Sigma x_i^2 + \Sigma y_i^2 + 2\Sigma x_i y_i \ge 0$

or, $n + n + 2nr \ge 0$

or, $2n + 2nr \ge 0$, [$\because$ n is positive]

or, $r \ge -1$, i.e., $-1 \le r$.

From (2),

$$\Sigma(x_i^2 + y_i^2 - 2x_i y_i) \ge 0$$
or, $\Sigma x_i^2 + \Sigma y_i^2 - 2\Sigma x_i y_i \ge 0$

or, $2n - 2nr \ge 0$

or, $2n(1 - r) \ge 0$

or, $1 - r \ge 0$

or, $1 \ge r$, i.e., $r \le 1$.

Combining $-1 \le r$ and $r \le 1$, we get $\mathbf{-1 \le r \le 1}$.

Hence, *the theorem follows.*

Uses of Correlation Coefficient

The correlation coefficient r is used to measure the strength (or degree) of relationship between two variables. If the scatter diagram of the values of the variables indicates a linear path, the correlation coefficient is considered as a useful measure for finding the strength of relationship between the variables. A positive value of r shows that the movements of the variables are in the same direction, i.e., high values of one variable are associated with high values of the other or low values with low values of the other. When r is negative, the movements are in the opposite directions, i.e., high values of one variable are associated with low values of the other.

7.4 Regression

By *regression* we mean average relationship between two variables, and this relationship is used to estimate or predict the most likely values of one variable for specified values of the other variable. One of the variables is called *independent* or the *explained variable* and the other is called *dependent* or the *explaining variable.*

In the words of M. M. Blair: "Regression is the measure of the average relationship between two or more variables in terms of the original units of the data."

Regression Equations

Prediction or estimation of most likely values of one variable for specified values of the other is done by using suitable equations involving the two variables. Such equations are known as *Regression Equations.*

Regression Equation of Y on X

In linear regression if we fit a straight line of the type $Y = a + bX$ to the given data by the method of least squares, we obtain the regression equation of Y on X.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be n pairs of observations and let the straight line to be fitted to these data be

$$Y = a + bX. \tag{1}$$

Applying the method of least squares, we get the following normal equations:

$$\Sigma Y = na + b\Sigma x \tag{2}$$

$$\text{and} \qquad \Sigma XY = a\Sigma X + b\Sigma X^2. \tag{3}$$

Dividing (2) by n,

$$\frac{\Sigma Y}{n} = a + b\frac{\Sigma X}{n} \quad \text{or,} \quad \bar{Y} = a + b\bar{X}. \tag{4}$$

Subtracting (4) from (1), we get

$$Y - \bar{Y} = b(X - \bar{X}). \tag{5}$$

Now multiplying (2) by ΣX and (3) by n and then subtracting, we have

$$(\Sigma X)(\Sigma Y) - n\Sigma XY = b(\Sigma X)^2 - nb\Sigma X^2$$

$$\text{or,} \quad b\{n\Sigma X^2 - (\Sigma X)^2\} = n\Sigma XY - (\Sigma X)(\Sigma Y)$$

$$\text{or,} \quad b = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{n\Sigma X^2 - (\Sigma X)^2} = \frac{\dfrac{\Sigma XY}{n} - \left(\dfrac{\Sigma X}{n}\right)\left(\dfrac{\Sigma Y}{n}\right)}{\dfrac{\Sigma X^2}{n} - \left(\dfrac{\Sigma X}{n}\right)^2} = b_{YX} \text{ (say)}. \tag{6}$$

Replacing b by b_{YX} in (5), *the regression equation of Y on X is*

$$Y - \bar{Y} = b_{YX}(X - \bar{X}), \tag{7}$$

where b_{XY} is given by (6).

Equation (7) is the regression equation of Y on X and it is used to estimate the most likely values of Y for given values of X.

Regression Equation of X on Y

The regression equation of X on Y is the equation of the best-fitted straight line of the type $X = a' + b'Y$ to the given data by the method of least squares.

Applying the principle of least squares, we get the following two normal equations for determining a' and b':

$$\Sigma X = na' + b'\Sigma Y \tag{8}$$

$$\text{and} \quad \Sigma XY = a'\Sigma Y + b'\Sigma Y^2. \tag{9}$$

Solving (8) and (9) for a' and b', and proceeding as before, *the regression equation of X on Y is*

$$X - \bar{X} = b_{XY}(Y - \bar{Y}), \tag{10}$$

where

$$b_{XY} = \frac{\frac{\Sigma XY}{n} - \left(\frac{\Sigma X}{n}\right)\left(\frac{\Sigma Y}{n}\right)}{\frac{\Sigma Y^2}{n} - \left(\frac{\Sigma Y}{n}\right)^2}. \tag{11}$$

Equation (10) *is used to estimate the most likely values of X for given values of Y.*

Regression Coefficients

b_{YX} and b_{XY} defined by (6) and (11) are called the *regression coefficients* of the two regression lines (7) and (10).

The formula for b_{YX} and b_{XY} may be written in different forms.

(i) We have

$$\text{cov}(X, Y) = \frac{\Sigma XY}{n} - \left(\frac{\Sigma X}{n}\right)\left(\frac{\Sigma Y}{n}\right) \quad \text{and} \quad r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} \quad \text{or,} \quad \text{cov}(X, Y) = r\sigma_X\sigma_Y.$$

$\therefore$ from (6) and (11), we get

$$b_{YX} = \frac{\text{cov}(X, Y)}{\sigma_X^2} = \frac{r\sigma_X\sigma_Y}{\sigma_X^2} = \frac{r\sigma_Y}{\sigma_X} \tag{12}$$

$$\text{and } b_{XY} = \frac{\text{cov}(X, Y)}{\sigma_Y^2} = \frac{r\sigma_X\sigma_Y}{\sigma_Y^2} = \frac{r\sigma_X}{\sigma_Y}. \tag{13}$$

In this case

$$b_{YX} \times b_{XY} = \frac{r\sigma_Y}{\sigma_X} \times \frac{r\sigma_X}{\sigma_Y} = r^2, \tag{14}$$

i.e., **the product of the two regression coefficients = r^2.**

From (14), we see that b_{YX} and b_{XY} are either both *positive* or both *negative*, otherwise r^2 will be *negative* which is *impossible*. Again, from (12) and (13), we see that b_{YX} and b_{XY} have the same sign as that of r, i.e., if b_{YX} and b_{XY} are both *positive*, then r is *positive*, or, if b_{YX} and b_{XY} are both *negative*, then r is *negative*.

(ii) Since σ_X, σ_Y, r are all independent of the choice of the origin of reference, *b_{YX} and b_{XY} are also independent of the choice of origin*. The formulae for b_{YX} and b_{XY}, when the deviations are taken from the assumed mean, are given by

$$\boldsymbol{b_{YX}} = \frac{\dfrac{\Sigma xy}{n} - \left(\dfrac{\Sigma x}{n}\right)\left(\dfrac{\Sigma y}{n}\right)}{\dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2} \text{ and } \boldsymbol{b_{XY}} = \frac{\dfrac{\Sigma xy}{n} - \left(\dfrac{\Sigma x}{n}\right)\left(\dfrac{\Sigma y}{n}\right)}{\dfrac{\Sigma y^2}{n} - \left(\dfrac{\Sigma y}{n}\right)^2},$$

where $x = X - a$ and $y = Y - b$, a and b being the assumed means of X and Y respectively.

(iii) If $x = X - \bar{X}$ = deviation of X from its mean and $y = Y - \bar{Y}$ = deviation of Y from its mean, then

$$\boldsymbol{b_{YX}} = \frac{\dfrac{\Sigma xy}{n}}{\dfrac{\Sigma x^2}{n}} = \frac{\boldsymbol{\Sigma xy}}{\boldsymbol{\Sigma x^2}} \quad [\because \Sigma x = 0 \text{ and } \Sigma y = 0].$$

Similarly,

$$\boldsymbol{b_{XY}} = \frac{\boldsymbol{\Sigma xy}}{\boldsymbol{\Sigma y^2}}.$$

Remarks: We have two regression lines:

$$Y - \bar{Y} = b_{XY}(X - \bar{X}) \tag{15}$$

$$\text{and} \quad X - \bar{X} = b_{XY}(Y - \bar{Y}). \tag{16}$$

Equation (15) is used to find the best estimates of Y for given values of X and (16) is used to obtain the best estimates of X for given values of Y. As the equations (15) and (16) have been derived under *two different assumptions*, one equation cannot serve both the purposes.

Equations (15) and (16) will be *identical* if their gradients b_{YX} and $\frac{1}{b_{XY}}$ are equal, i.e., if

$$b_{XY} = \frac{1}{b_{YX}} \text{ or, } b_{YX} \cdot b_{XY} = 1 \text{ or, } r^2 = 1, \text{ i.e., if } r = \pm 1.$$

Example 16.

(i) *The regression coefficients of y on x and x on y are* 1.2 *and* 0.3 *respectively. Find the coefficient of correlation.*

(ii) *If* $\sigma_x = 10$, $\sigma_y = 12$, $b_{xy} = -0.8$, *find the value of* r. [C.U. B.Com. 1998]

(iii) *If* $\bar{x} = 6$, $\bar{y} = 7$, $b_{yx} = 0.45$ and $b_{xy} = 0.65$, *then find the regression equations.* [C.U. B.Com. 2002]

Solution: (i) We have $b_{yx}.b_{xy} = r^2$ or, $1.2 \times 0.3 = r^2$, or, $r = \pm\sqrt{0.36} = \pm 0.6$.

Since b_{yx} and b_{xy} are both *positive*, r must be *positive*; $\therefore$ $\boldsymbol{r = 0.6}$.

(ii) We have

$$b_{xy} = r.\frac{\sigma_x}{\sigma_y} \text{ or, } -0.8 = r.\frac{10}{12} \text{ or, } 10r = -9.6 \text{ or, } r = -\frac{9.6}{10} = \boldsymbol{-0.96}.$$

(iii) The regression equation of y on x is

$$y-\bar{y}=b_{yx}(x-\bar{x})$$
or, $$y-7=0.45(x-6)$$
or, $$y=7+0.45x-2.7$$
or, $$\mathbf{y=0.45x+4.3.}$$

The regression equation of x on y is

$$x-\bar{x}=b_{xy}(y-\bar{y})$$
or, $$x-6=0.65(y-7)$$
or, $$x=6+0.65y-4.55$$
or, $$\mathbf{x=0.65y+1.45.}$$

Example 17. *From the following data, find the two regression equations:*

X	1	2	3	4	5	6	7
Y	2	4	7	6	5	6	5

Solution:

TABLE 7.9: CALCULATIONS FOR REGRESSION EQUATIONS

X	Y	$x=X-\bar{X}=X-4$	$y=Y-\bar{Y}=Y-5$	x^2	y^2	xy
1	2	−3	−3	9	9	9
2	4	−2	−1	4	1	2
3	7	−1	2	1	4	−2
4	6	0	1	0	1	0
5	5	1	0	1	0	0
6	6	2	1	4	1	2
7	5	3	0	9	0	0
$28=\Sigma X$	$35=\Sigma Y$	—	—	$28=\Sigma x^2$	$16=\Sigma y^2$	$11=\Sigma xy$

$$\bar{X}=\frac{\Sigma X}{n}=\frac{28}{7}=4,\quad \bar{Y}=\frac{\Sigma Y}{n}=\frac{35}{7}=5.$$

$$\therefore \quad b_{YX}=\frac{\Sigma xy}{\Sigma x^2}=\frac{11}{28}=0.39 \text{ (approx.)}$$

and $$b_{XY}=\frac{\Sigma xy}{\Sigma y^2}=\frac{11}{16}=0.69 \text{ (approx.)}$$

Hence, the regression equation of Y on X is

$$Y-\bar{Y}=b_{YX}(X-\bar{X})$$
or, $$Y-5=0.39(X-4)=0.39X-1.56$$
or, $$\mathbf{Y=0.39X+3.44}$$

and the regression equation of X on Y is

$$X-\bar{X}=b_{XY}(Y-\bar{Y})$$

or, $$X-4=0.69(Y-5)=0.69Y-3.45$$

or, $$\mathbf{X=0.69Y+0.55.}$$

Example 18. *From the following results, obtain the two regression equations and estimate the yield of crops when the rainfall is* 29 cm *and the rainfall when the yield is* 600 kg.

	Y (Yield in kg)	X (Rainfall in cm)
Mean	508.4	26.7
SD	36.8	4.6

Coefficient of correlation between yield and rainfall = 0.52.

Solution: We have $\bar{X}=26.7$, $\bar{Y}=508.4$, $\sigma_X=4.6$, $\sigma_Y=36.8$, $r=0.52$.

The regression equation of Y on X is

$$Y-\bar{Y}=b_{YX}(X-\bar{X}),\text{ where } b_{YX}=r\frac{\sigma_Y}{\sigma_X}$$

or, $$Y-508.4=0.52\times\frac{36.8}{4.6}(X-26.7)$$

or, $$Y-508.4=4.16(X-26.7)$$

or, $$Y=508.4+4.16X-111.072$$

or, $$\mathbf{Y=4.16X+397.328.}$$

When $X=29$, $Y=4.16\times 29+397.328$ = **517.968 kg.**

The regression equation of X on Y is

$$X-\bar{X}=b_{XY}(Y-\bar{Y}),\text{ where } b_{XY}=r\frac{\sigma_X}{\sigma_Y}$$

or, $$X-26.7=0.52\times\frac{4.6}{36.8}(Y-508.4)$$

or, $$X-26.7=0.065(Y-508.4)=0.065Y-33.046$$

or, $$X=26.7+0.065Y-33.046$$

or, $$\mathbf{X=0.065Y-6.346.}$$

When $Y=600$, $X=0.065\times 600-6.346$ = **32.654 cm.**

Example 19. *You are given the data relating to purchases and sales. Obtain the two regression equations by the method of least squares and estimate the likely sales when the purchases equal* 100:

Purchases	62	72	98	76	81	56	76	92	88	49
Sales	112	124	131	117	132	96	120	136	97	85

Solution:

TABLE 7.10 CALCULATIONS FOR REGRESSION EQUATIONS

Purchases X	Sales Y	$x = X - \bar{X}$ $= X - 75$	$y = Y - \bar{Y}$ $= Y - 115$	x^2	y^2	xy
62	112	−13	−3	169	9	39
72	124	−3	9	9	81	−27
98	131	23	16	529	256	368
76	117	1	2	1	4	2
81	132	6	17	36	289	102
56	96	−19	−19	361	361	361
76	120	1	5	1	25	5
92	136	17	21	289	441	357
88	97	13	−18	169	324	−234
49	85	−26	−30	676	900	780
$750 = \Sigma X$	$1150 = \Sigma Y$	—	—	$2240 = \Sigma x^2$	$2690 = \Sigma y^2$	$753 = \Sigma xy$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{750}{10} = 75 \text{ and } \bar{Y} = \frac{\Sigma Y}{n} = \frac{1150}{10} = 115,$$

$$b_{YX} = \frac{\Sigma xy}{\Sigma x^2} = \frac{1753}{2240} = 0.78$$

$$\text{and } b_{XY} = \frac{\Sigma xy}{\Sigma y^2} = \frac{1753}{2690} = 0.652.$$

The regression equation of Y on X is

$$Y - \bar{Y} = b_{YX}(X - \bar{X})$$

or, $Y - 115 = 0.78(X - 75) = 0.78X - 58.50$

or, $\mathbf{Y = 0.78X + 56.5.}$ (1)

The regression equation of X on Y is

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

or, $X - 75 = 0.652(Y - 115) = 0.652Y - 74.98$

or, $\mathbf{X = 0.652Y + 0.02.}$

When $X = 100$, from (1), Y is given by $Y = 0.78 \times 100 + 56.5 = \mathbf{134.5}$.

Example 20. *From the following data find the two regression equations and predict the value of y for x* $= 2.5$:

x	1	2	3	4	5
y	2	2	5	4	6

[C.U. B.Com. 2001]

Solution:

TABLE 7.11: CALCULATIONS FOR REGRESSION EQUATIONS

x	y	x^2	y^2	xy
1	2	1	4	2
2	2	4	4	4
3	5	9	25	15
4	4	16	16	16
5	6	25	36	30
$15=\Sigma x$	$19=\Sigma y$	$55=\Sigma x^2$	$85=\Sigma y^2$	$67=\Sigma xy$

$$\bar{x}=\frac{\Sigma x}{n}=\frac{15}{5}=3 \text{ and } \bar{y}=\frac{\Sigma y}{n}=\frac{19}{5}=3.8.$$

$$b_{yx}=\frac{n\Sigma xy-(\Sigma x)(\Sigma y)}{n\Sigma x^2-(\Sigma x)^2}=\frac{5\times 67-15\times 19}{5\times 55-(15)^2}=\frac{335-285}{275-225}=\frac{50}{50}=1$$

$$\text{and} \quad b_{xy}=\frac{n\Sigma xy-(\Sigma x)(\Sigma y)}{n\Sigma y^2-(\Sigma y)^2}=\frac{50}{5\times 85-(19)^2}=\frac{50}{425-361}=\frac{50}{64}=0.78.$$

The regression equation of y on x is

$$y-\bar{y}=b_{yx}(x-\bar{x})$$

or, $y-3.8=1(x-3)$

or, $y=3.8+x-3=x+0.8$

or, $\mathbf{y=x+0.8}$. (1)

The regression equation of x on y is

$$x-\bar{x}=b_{xy}(y-\bar{y})$$

or, $x-3=0.78(y-3.8)=0.78y-2.964$

or, $\mathbf{x=0.78y+0.036}$.

Now, if $x=2.5$, from (1), we get

$$y=2.5+0.8=\mathbf{3.3}.$$

Example 21. *The regression equations calculated from a given set of observations are $x=-0.4y+6.4$ and $y=-0.6x+4.6$. Calculate $\bar{x},\bar{y}$ and r_{xy}.*

Solution: We have

$$x=-0.4y+6.4 \quad (1)$$

and $$y=-0.6x+4.6. \quad (2)$$

Putting $x=-0.4y+6.4$ in (2), we get

$$y=-0.6(-0.4y+6.4)+4.6=0.24y-3.84+4.6$$

or, $y(1-0.24)=0.76$

or, $0.76y=0.76; \quad \therefore\ y=1.$

From (1),

$$x = -0.4 \times 1 + 6.4 = -0.4 + 6.4 = 6.$$

$\therefore$ $\bar{\boldsymbol{x}} = \mathbf{6}$ and $\bar{\boldsymbol{y}} = \mathbf{1}$ [$\because (\bar{x}, \bar{y})$ is the point of intersection of (1) and (2)].

Clearly, (1) is the regression equation of x on y and (2) is the regression equation of y on x.

$$\therefore\ b_{xy} = -0.4 \text{ and } b_{yx} = -0.6.$$

$$\therefore\ r_{xy}^2 = b_{xy} \cdot b_{yx} = -0.4 \times (-0.6) = 0.24;$$

$$\therefore\ r_{xy} = \pm\sqrt{0.24}.$$

Since b_{xy} and b_{yx} are *both negative*, r_{xy} is *negative.*

$$\boldsymbol{\therefore\ r_{xy} = -\sqrt{0.24} = -0.49.}$$

Example 22. *You are given variance of $X = 9$. The regression equations are*

$$8X - 10Y + 66 = 0$$

$$\textit{and} \quad 40X - 18Y = 214.$$

Find (i) *Average values of X and Y,* (ii) *Correlation coefficient between the two variables,* (iii) *Standard deviation of Y.*

Solution: The regression equations are

$$8X - 10Y = -66 \tag{1}$$

$$\text{and} \quad 40X - 18Y = 214. \tag{2}$$

Since the point $(\bar{X}, \bar{Y})$ lies on both the lines of regression (1) and (2), the means $\bar{X}$ and $\bar{Y}$ of X and Y are given by

$$8\bar{X} - 10\bar{Y} = -66 \tag{3}$$

$$\text{and} \quad 40\bar{X} - 18\bar{Y} = 214. \tag{4}$$

Multiplying (3) by 5, we get

	$40\bar{X} - 50\bar{Y}$	=	-330
From (4),	$40\bar{X} - 18\bar{Y}$	=	214
Subtracting, we get	$-32\bar{Y}$	=	-544

$$\therefore\ \bar{\boldsymbol{Y}} = \mathbf{17}.$$

Substituting $\bar{Y} = 17$ in (3), we get

$$8\bar{X} - 10 \times 17 = -66$$

$$\text{or,} \quad 8\bar{X} = 170 - 66 = 104; \quad \therefore\ \bar{\boldsymbol{X}} = \mathbf{13}.$$

We know that

$$r = \pm\sqrt{b_{YX} \cdot b_{XY}},$$

where b_{YX} and b_{XY} are the regression coefficients of the two lines of regression.

From (1),
$$8X+66=10Y \text{ or, } Y=\frac{4}{5}X+\frac{33}{5}; \therefore b_{YX}=\frac{4}{5}.$$

From (2),
$$40X=18Y+214 \text{ or, } X=\frac{9}{20}Y+\frac{107}{20}; \therefore b_{XY}=\frac{9}{20}.$$

Since the regression coefficients are *positive*, the correlation coefficient r is also *positive*.
Hence
$$r=+\sqrt{\frac{4}{5}\times\frac{9}{20}}=\sqrt{\frac{9}{25}}=\frac{3}{5}=0.6.$$

Again, σ_X^2 = variance of $X=9$; $\therefore \sigma_X=3$.
We have
$$b_{YX}=r\cdot\frac{\sigma_Y}{\sigma_X} \text{ or, } \frac{4}{5}=\frac{3}{5}\times\frac{\sigma_Y}{3}=\frac{\sigma_Y}{5}; \therefore \sigma_Y=4.$$

Note: Both the regression coefficients obtained in the above problem are less than 1 and, therefore, their product $b_{YX}\cdot b_{XY}=r^2<1$. If we choose (1) as the regression equation of X on Y and (2) as the regression equation of Y on X, we would get $b_{XY}=\frac{5}{4}$ and $b_{YX}=\frac{20}{9}$ so that $b_{XY}\cdot b_{YX}>1$, which is impossible.

Example 23. *In trying to evaluate the effectiveness in its advertising campaign, a firm compiled the following information:*

Year	1968	1969	1970	1971	1972	1973	1974	1975
Advertising Expenditure ('000 ₹)	12	15	15	23	24	38	42	48
Sales (lac ₹)	5.0	5.6	5.8	7.0	7.2	8.8	9.2	9.5

Calculate the regression equation of sales on advertising expenditure. Estimate the probable sales when advertisement expenditure is ₹ 60 thousand.

Solution:

TABLE 7.12: CALCULATIONS FOR REGRESSION EQUATION

Advertising Expenditure (in '000 ₹) X	Sales (in lac ₹) Y	$x=X-a$ $=X-25$	$y=Y-b$ $=Y-7$	x^2	y^2	xy
12	5.0	−13	−2.0	169	4.00	26.0
15	5.6	−10	−1.4	100	1.96	14.0
15	5.8	−10	−1.2	100	1.44	12.0
23	7.0	−2	0	4	0	0
24	7.2	−1	0.2	1	0.04	−0.2
38	8.8	13	1.8	169	3.24	23.4
42	9.2	17	2.2	289	4.84	37.4
48	9.5	23	2.5	529	6.25	57.5
Total	—	$17=\Sigma x$	$2.1=\Sigma y$	$1361=\Sigma x^2$	$21.77=\Sigma y^2$	$170.1=\Sigma xy$

We have
$$\bar{X}=a+\frac{\Sigma x}{n}=25+\frac{17}{8}=25+2.125=27.125,$$
$$\bar{Y}=b+\frac{\Sigma y}{n}=7+\frac{2.1}{8}=7+0.26=7.26.$$
$$b_{XY}=\frac{\Sigma xy-\frac{(\Sigma x)(\Sigma y)}{n}}{\Sigma x^2-\frac{(\Sigma x)^2}{n}}=\frac{170.1-\frac{17\times 2.1}{8}}{1361-\frac{(17)^2}{8}}=\frac{170.1-4.463}{1361-36.125}=\frac{165.637}{1324.875}=0.125.$$

The regression line of sales (Y) on advertising expenditure (X) is

$$Y-\bar{Y}=b_{YX}(X-\bar{X})$$
or, $$Y-7.26=0.125(X-27.125)$$
or, $$Y=7.26+0.125X-3.39=0.125X+3.87$$
i.e., $$Y=0.125X+3.87.$$

When $X=60$, $Y=0.125\times 60+3.87=11.37$, i.e., the probable sales = ₹ 11.37 lac, when the advertisement expenditure is ₹ 60 thousand.

EXERCISES ON CHAPTER 7(I)

(Correlation and Regression)

Theory

1. (a) Define the term correlations. [C.U. B.Com.(H) 2011]
   (b) Define positive and negative correlations. [V.U. B.Com.(H) 2011]
2. What is a scatter diagram? Indicate by means of suitable scatter diagrams different types of correlation that may exist between the variables in bivariate data.
3. (a) Briefly explain the concept of regression and its use. [C.U. B.Com. 1997]
   (b) Explain the terms: Coefficient of Correlation and Regression Coefficients.
4. (a) Define Rank correlation. Write down Spearman's Formula for rank correlation coefficient R. What are the limits of R? Interpret the case when R assumes the minimum value.
   (b) What do you mean by linear and non-linear correlation? [V.U. B.Com.(H) 2010]
5. Write short notes on:
   (a) Scatter diagram, [C.U. B.Com. 2007; V.U. B.Com.(H) 2007]
   (b) Rank correlation coefficient,
   (c) Regression equations.
6. (a) What are regression coefficients? Show that $r^2=b_{XY}\cdot b_{YX}$, where the symbols have their usual meanings. What can you say about the angle between the regression lines when (i) $r=0$, (ii) $r=\pm 1$, (iii) r increases from 0 to 1?

(b) What are the coefficients of linear regression? Prove that the coefficient of correlation is the geometric mean of the coefficients of regression. [C.U. B.Com.(H) 1991]

7. What are regression lines? Why is it necessary to consider two lines of regression? In case the two lines are identical, prove that the correlation coefficient is either +1 or −1.

8. Explain the concept of 'correlation' between two variables. Prove that r is independent of the origin of reference and the units of measurement.

9. Prove that the correlation coefficient r lies between −1 and +1.

10. Obtain the equations of the two lines of regression for a bivariate distribution.

Problems
A: Correlation

1. (a) Find the coefficient of correlation from the following data:

X	1	2	3	4	5
Y	3	2	5	4	6

[C.U. B.Com. 2003]

(b) Calculate the coefficient of correlation from the following data:

(i)

X	1	2	3	4	5
Y	6	8	11	8	12

[C.U. B.Com. 1995]

(ii)

x	1	3	5	6	7
y	4	2	1	3	5

[C.U. B.Com. 2007]

(c) Calculate the correlation coefficient of the following data:

x	11	12	13	14	18	15
y	13	12	15	14	12	11

[C.U. B.Com. 1997]

2. (a) Karl Pearson's coefficient of correlation between two variables X and Y is 0.52, their covariance is +7.8. If the variance of X is 16, find the standard deviation of Y series.

(b) If $\text{cov}(x, y) = -40$, $\sigma_x = 8$, $\sigma_y = 15$, obtain the correlation coefficient of x and y.

[C.U. B.Com. 2002]

[Hints: $r = \frac{\text{cov}(x,y)}{\sigma_x \cdot \sigma_y} = \frac{-40}{8\times 15}$.]

(c) Find the correlation coefficient from the data: $\sigma_x^2 = 2.25$, $\sigma_y^2 = 1$ and $\text{cov}(x, y) = 0.9$.

[C.U. B.Com. 1999]

(d) $r = 0.4$, $\text{cov}(x, y) = 10$ and $\sigma_y = 5$, then find σ_x. [C.U. B.Com.(H) 2006]

3. Calculate the correlation coefficient between X and Y series from the following data:

(a) $\sum_{i=1}^{12}(X_i - \bar{X})^2 = 360, \sum_{i=1}^{12}(Y_i - \bar{Y})^2 = 250$ and $\sum_{i=1}^{12}(X_i - \bar{X})(Y_i - \bar{Y}) = 225;$

(b) From the following data, calculate the coefficient of correlation: $n = 10$, $\Sigma x = 140$, $\Sigma y = 150$, $\Sigma(x-10)^2 = 180$, $\Sigma(y-15)^2 = 215$, and $\Sigma(x-10)(y-15) = 60$. [V.U. B.Com.(H) 2009]

[Hints: If $X = x - 10$ and $Y = y - 15$, then

$$r_{xy} = r_{XY} = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{n\Sigma X^2 - (\Sigma X)^2} \times \sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} \quad (1)$$

Now

$$\bar{x} = \frac{\Sigma x}{n} = \frac{140}{10} = 14,\ \bar{y} = \frac{\Sigma y}{n} = \frac{150}{10} = 15.$$

$\Sigma X = \Sigma(x-10) = \Sigma x - \Sigma 10 = 140 - 100$, $\Sigma Y = \Sigma(y-15) = 0$ [$\because \bar{y} = 15$]

$\Sigma X^2 = \Sigma(x-10)^2 = 180$, $\Sigma Y^2 = \Sigma(y-15)^2 = 215$ and $\Sigma XY = \Sigma(x-10)(y-15) = 60$.

Hence

$$r_{xy} = \frac{10 \times 60 - 40 \times 0}{\sqrt{10 \times 180 - (40)^2} \times \sqrt{10 \times 215 - 0}} = \frac{600}{\sqrt{1800 - 1600} \times \sqrt{2150}}$$

$$= \frac{600}{\sqrt{200} \times \sqrt{2150}} = \frac{600}{\sqrt{43000}} = \frac{600}{\sqrt{655.7}} = 0.915.]$$

4. (a) The heights and weights of 15 persons are given below. Form a two-way frequency table with class-intervals 60″ - 62″, 62″ - 64″ and so on and 50–55 kg, 55–60 kg and so on:

Sl. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Height	61	65	68	62	60	63	67	69	61	64	63	65	66	64	68
Weight	62	55	70	60	58	50	53	65	54	73	64	66	68	73	67

(b) Draw a scatter diagram for the following bivariate data:

X	1	2	3	4	5	6
Y	6	4	3	5	4	2

5. State in each case whether you would expect to find a positive correlation, a negative correlation or no correlation:

(a) The ages of husbands and wives,
(b) Shoe size and intelligence,
(c) Years of education and income,
(d) Insurance companies' profits and the number of claims they have to pay,
(e) Amount of rainfall and yield of crop.

6. (a) Calculate Pearson's coefficient of correlation between advertisement cost and sales as per the data given below:

Advertisement Cost in '000 ₹	40	65	60	90	85	75	35	90	34	76
Sales in lac ₹	45	56	58	82	65	70	64	85	50	85

(b) Calculate the coefficient of correlation from the given data:

X	9	8	7	6	5	4	3	2	1
Y	15	16	14	13	11	12	10	8	9

[V.U. B.Com.(H) 2011]

[Hints: (a) See worked-out Exs 5 and 6 in Section 7.3.2. Here $n = 10$, $\Sigma X = 650$, $\Sigma Y = 660$, $\bar{X} = \frac{650}{10} = 65$, $\bar{Y} = \frac{660}{10} = 66$; $r_{XY} = r_{xy} = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \times \sqrt{\Sigma y^2}} = \frac{2225}{\sqrt{4382} \times \sqrt{1860}}$, i.e., $r_{XY} = \frac{2225}{2855} = 0.78$ ($\because \sqrt{4382} = 66.2$ and $\sqrt{1860} = 43.13$.)
(Log-table may be used for calculation.)]

7. (a) Find the coefficient of correlation from the following data:

X	10	12	13	16	17	20	25
Y	19	22	24	27	29	33	37

(b) Find the correlation coefficient between x and y from the following data:

x	45	55	56	58	60	65	68	70
y	56	50	48	60	62	64	65	70

[C.U. B.Com. 1999]

(c) Calculate the correlation coefficient of the following data:

X	63	60	67	61	69	70
Y	61	65	64	63	68	63

[C.U. B.Com. 2003]

(d) From the following data, find the coefficient of correlation:

X	65	63	67	64	68	62	70	66
Y	68	66	68	65	69	66	68	65

[C.U. B.Com. 2008]

8. Calculate the Pearson's coefficient of correlation from the followng data using 44 and 26 respectively as the origin of X and Y:

X	43	44	46	40	44	42	45	42	38	40	42	57
Y	29	31	19	18	19	27	27	29	41	30	26	10

9. (a) In order to find the correlation coefficient between two variables X and Y from 12 pairs of observations, the following calculations were made: $\Sigma X = 30$, $\Sigma Y = 5$, $\Sigma X^2 = 670$, $\Sigma Y^2 = 285$, $\Sigma XY = 334$.

On subsequent verification it was found that the pair $(X = 11,\ Y = 4)$ was copied wrongly, the correct value being $(X = 10,\ Y = 14)$. Find the correct value of correlation coefficient.

[C.U. B.Com. 2001]

(b) In order to determine the correlation coefficient between x and y from 25 pairs of observations, following data were available: $N = 25$, $\Sigma x = 125$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.

Subsequently, it was found that the pair $(x = 6,\ y = 8)$ was copied wrongly, the correct value being $(x = 8,\ y = 6)$. Find the correct value of the correlation coefficient. [C.U. B.Com. 2000]

10. Find the coefficient of correlation between the ages of husbands and the ages of wives given below in the form of a two-way frequency table:

		Ages of husbands (in years) →				
Ages of wives (in years) ↓		20–25	25–30	30–35	35–40	Total
	15–20	20	10	3	2	35
	20–25	4	28	6	4	42
	25–30	—	5	11	—	16
	30–35	—	—	2	—	2
	35–40	—	—	—	—	0
	Total	24	43	22	6	95

11. (a) Compute the correlation coefficient between the corresponding values of X and Y in the following table:

X	2	4	5	6	8	11
Y	18	12	10	8	7	5

(b) Multiply each X-value in the table by 2 and add 6. Multiply each value of Y in the table by 3 and subtract 15. Find the correlation coefficient between the two new sets of values. Explain why you do or do not obtain the same result as in (a).

12. (a) Write down the Spearman's formula for determining rank correlation coefficient R. For the two series, we have $\Sigma D^2 = 30$ and $N = 10$; find R. [C.U. B.Com. 2001]

(b) The rankings of 10 individuals at the start and on the finish of a course of training are as follows:

Individuals	A	B	C	D	E	F	G	H	I	J
Rank Before	1	6	3	9	5	2	7	10	8	4
Rank After	6	8	3	7	2	1	5	9	4	10

Calculate Spearman's coefficient of correlation. **[Bombay U. B.Com. 2001]**

13. (a) In a contest, two judges ranked eight candidates A, B, C, D, E, F, G and H in order of their performances, as shown in the following table. Find the rank correlation coefficient:

Candidates	A	B	C	D	E	F	G	H
First Judge	5	2	8	1	4	6	3	7
Second Judge	4	5	7	3	2	8	1	6

[ICWAI June 1998 Type]

(b) In a contest, judges assessed the performances of eight candidates as shown below:

Candidates	A	B	C	D	E	F	G	H
First Judge	5	3	2	1	4	7	8	6
Second Judge	4	2	1	3	5	8	6	7

Using Spearman's formula obtain the rank correlation coefficient. [C.U. B.Com. 1998]

14. (a) Calculate the coefficient of rank correlation for the following data of marks obtained by 10 students in Book-keeping and Commercial Arithmetic at an examination:

Student Nos.	1	2	3	4	5	6	7	8	9	10
Marks in Book-keeping (out of 60)	43	29	35	18	40	11	49	10	5	22
Marks in Commercial Arithmetic (out of 40)	36	6	17	14	25	10	32	0	3	20

(b) Find the Rank Correlation Coefficient for the following data of marks obtained by 10 students in Mathematics and Statistics:

Student (Roll Nos.)	1	2	3	4	5	6	7	8	9	10
Marks in Mathematics	80	38	95	30	74	84	91	60	66	40
Marks in Statistics	85	50	92	58	70	65	88	56	52	46

[C.U. B.Com. 2004]

(c) In a contest, two judges ranked seven competitors in order of their performances in the following order. For the data, find the rank correlation coefficient:

Competitors	A	B	C	D	E	F	G
Judge-I	2	1	4	5	3	7	6
Judge-II	3	4	2	5	1	6	7

[C.U. B.Com. 2008]

(d) Find the rank correlation coefficient between poverty and overcrowding from the table given below:

Town	A	B	C	D	E	F	G	H	I	J
Below min standard of living	17	13	15	16	6	11	14	9	7	12
Overcrowding	36	46	35	24	12	18	27	22	2	8

B: Regression

1. (a) The regression coefficients of y on x and x on y are -1.2 and -0.3 respectively. Find the correlation coefficient. [C.U. B.Com. 1997; V.U. B.Com.(H) 2010]

(b) If $\bar{x} = 6$, $\bar{y} = 7$, $b_{yx} = 0.45$, $b_{xy} = 0.65$, find the equations of regression lines. [C.U. B.Com. 2008]

(c) From the following data, find the two regression equations:

X	1	2	3	4	5
Y	2	3	5	4	6

Find also the most probable value of Y, when $X = 2.5$. [C.U. B.Com. 1992, 2003]

(d) Find $\vec{v}, \vec{y}$ if the regression equations are $5x - 2y - 4 = 0$ and $7y - 4x = 13$. [C.U. B.Com. 2003]

2. (a) From the following data, find the two regression equations:

x	1	2	3	4	5
y	2	2	5	4	6

Predict the value of y for $x = 2.5$. [C.U. B.Com. 2001]

(b) Find the two regression equations from the following data:

x	6	2	10	4	8	12	14	16
y	9	11	5	8	7	11	16	18

[C.U. B.Com. 1998]

(c) From the following data, obtain the two regression equations:

Sales	91	97	108	121	67	124	51	73	111	57
Purchases	71	75	69	97	70	91	39	61	80	47

(d) With the help of a suitable regression line determine the estimated value of x when $y = 22$ by using the following data:

x	4	5	8	9	11	12	14
y	16	10	8	7	6	5	4

[C.U. B.Com. 2002]

(e) Find the regression equations from the following data and hence find the value of correlation coefficient:

X	41	82	62	37	58	96	127	74	123	100
Y	28	56	35	17	42	85	105	61	98	73

[C.U. B.Com. 2005]

Find the value of Y, when $X = 80$.

(f) Find the two linear regression equations from the following observations:

X	12	23	37	46	57	76
Y	36	42	57	64	68	82

[C.U. B.Com. 2007]

3. (a) If two regression coefficients are 0.9 and 0.2, what would be the value of the correlation coefficient r?

(b) Find out the value of r and σ_Y from the following data: $3X = Y$, $4Y = 3X$ and $\sigma_X = 2$.

(c) Find a bivariate data, the mean value of x is 20 and that of y is 45. The regression coefficient of y on x is 4 and that of x on y is 0.0625. Obtain the coefficient of correlation and the standard deviation (SD) of x, when the SD of y is 16. [C.U. B.Com. 2002]

[**Hints:** $r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{4 \times 0.0625} = \sqrt{0.25} = 0.5$, taking positive sign, as b_{yx} and b_{xy} are both positive. Again, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} \Rightarrow 4 = 0.5 \cdot \frac{16}{\sigma_x}$ or, $\sigma_x = \frac{8.0}{4} = 2$.]

4. For some bivariate data, the following results were obtained:

The mean value of $X = 53.2$, the mean value of $Y = 27.9$, the regression coefficient of Y on $X = -1.5$ and the regression coefficient of X on $Y = -0.2$.

Find (a) the most probable value of Y, when $X = 60$, (b) the coefficient of correlation between X and Y.

5. (a) Find the regression equation of X on Y from the following data: $\Sigma X = 24$, $\Sigma Y = 44$, $\Sigma XY = 306$, $\Sigma X^2 = 164$, $\Sigma Y^2 = 574$, $N = 4$. [C.U. B.Com. 2008]

(b) From the following data, find the two regression equations:

Age (years)	1	3	4	5	7
Weight (kg)	3	5	8	12	17

What will be the most probable weight of a baby at the age of 8 years? [C.U. B.Com. 1990]

(c) To study the relationship between expenditure on accommodation,₹, X and expenditure on food and entertainment, ₹, Y on enquiry into 50 families gave the following results:

$$\Sigma x = 8500,\ \Sigma y = 9600, \sigma_x = 60,\ \sigma_y = 20,\ r = 0.6.$$

Estimate the expenditure on food and entertainment when expenditure on accommodation is ₹200. [V.U. B.Com.(H) 2007]

[**Hints:** Given, $n = 50, \Sigma x = 8500, \Sigma y = 9600, \sigma_x = 60, \sigma_y = 20, r = 0.6$.
We have to first find the regression equation of y on x.

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} = 0.6 \times \frac{20}{60} = 0.2,\ \bar{x} = \frac{8500}{50} = 170,\ \bar{y} = \frac{9600}{50} = 192.$$

The regression equation of y on x is $y - \bar{y} = b_{yx} = (x - \bar{x})$ or, $y - 192 = 0.2(x - 170)$.
Now, if $x = 200, y = 192 + 0.2(200 - 170) = 192 + 0.2 \times 30 = ₹198$.]

6. (a) You are given the following data:

	X	Y
Arithmetic Mean	20	25
Standard Deviation	5	4

Correlation coefficient between X and Y is 0.6. Find the two regression equations.

(b) You are given the following data:
Correlation coefficient between X and $Y = 0.66$.

	X	Y
Arithmetic Mean	36	85
Standard Deviation	11	8

(i) Find the two regression equations.
(ii) Estimate the value of X, when $Y = 75$ and that of Y, when $X = 73.3$.

[C.U. B.Com. 2000; V.U. B.Com.(H) 2008]

(c) You are given the following data:

	X	Y
Arithmetic Mean	20	25
Standard Deviation	5	4

Correlation coefficient between X and $Y = 0.6$. Find the two regression equations.

[V.U. B.Com. 1995]

[**Hints**: Given: $b_{yx} = r\frac{\sigma_y}{\sigma_x}$ and $b_{xy} = r\frac{\sigma_x}{\sigma_y}$.]

7. (a) For certain X and Y series, which are correlated, the two lines of regression are

$$5X - 6Y + 90 = 0$$

and $$15X - 8Y - 130 = 0.$$

Find which is the regression equation of Y on X and which is that of X on Y. Find the means of the two series and the correlation coefficient between them.

(b) Two lines of regression are given by $x + 2y = 5$ and $2x + 3y = 8$ and $\sigma_x^2 = 12$. Calculate the values of $\bar{x}$, $\bar{y}$, σ_x^2 and r.

8. You are given variance of $Y = 16$. The regression equations are $4X - 5Y + 33 = 0$ and $20X - 9Y = 107$; find (a) the average values of X and Y, (b) correlation coefficient between X and Y, and (c) standard deviation of X, if SD of Y is 4. **[C.U. B.Com. 2006; V.U. B.Com.(H) 2011]**

[**Hints**: See worked-out Ex. 22 in **Section 7.4.** If $4x - 5y + 33 = 0$ be the regression equation of y on x, then $5y = 4x + 33$ or, $y = \frac{4}{5}x + \frac{33}{5}$ gives $b_{yx} = \frac{4}{5}$. If $20x - 9y = 107$ be the regression equation of x on y, then $20x = 9y + 107$ or, $x = \frac{9}{20}y + \frac{107}{20}$ gives $b_{xy} = \frac{9}{20}$; $\therefore$ $r_{xy}^2 = b_{yx} \times b_{xy} = \frac{4}{5} \times \frac{9}{20}$ or, $r_{xy} = \frac{3}{5} < 1$ and b_{yx} and b_{xy} are both positive. Hence $\boldsymbol{r_{xy} = 0.6}$. Solve the two equations to obtain. $\bar{x} = 13$ and $\bar{y} = 17$. Now, $b_{yx} = r_{xy} \times \frac{\sigma_y}{\sigma_x}$ or, $\frac{4}{5} = \frac{3}{5} \times \frac{4}{\sigma_x}$ or, $\boldsymbol{\sigma_x = 3}$.]

9. The following data are given for marks in English (X) and Mathematics (Y) at a certain examination:

	English	Mathematics
Mean Marks	39.5	47.5
SD Marks	10.8	16.8

Coefficient of correlation between marks in English and Mathematics = +0.42. Find the two regression equations.

Using these regression equations, estimate the value of Y for $X = 50$ and the value of X for $Y = 30$.

10. A department store gives in-service training to its salesmen which is followed by a test. It is considering whether it should terminate the services of any salesman who does not do well in the test. The following data give the test scores and sales made by nine salesmen during a certain period:

Test Scores	14	19	24	21	26	22	15	20	19
Sales ('00 ₹)	31	36	48	37	50	45	33	41	39

Calculate the Coefficient of correlation between the test scores and the sales. Does it indicate that the termination of services of low test scores is justified? If the firm wants a minimum sales volume of ₹ 3000.00, what is the minimum test score that will ensure continuation of service?

11. While calculating the coefficient of correlation between two variables X and Y, the following results were obtained:

The number of observations = 25; $\Sigma X = 125$; $\Sigma Y = 100$; $\Sigma X^2 = 650$; $\Sigma Y^2 = 460$; $\Sigma XY = 508$.

It was, however, later discovered at the time of checking that two pairs of observations (X, Y) were copied (6, 14) and (8, 6) while the correct values were (8, 12) and (6, 8) respectively. Determine the correct value of the coefficient of correlation.

Hence, find the correct equations of the two lines of regression.

ANSWERS

A: Correlation

1. (a) +0.8;
   (b) (i) 0.775,
   (ii) 0.13;
   (c) −0.34.
2. (a) 3.75;
   (b) −1/3 or −0.33;
   (c) +0.6;
   (d) 5.
3. (a) +0.75;
   (b) 0.915.
5. (a) Positive correlation;
   (b) No correlation;
   (c) Positive correlation;
   (d) Negative correlation;
   (e) Positive correlation.
6. (a) +0.78;
   (b) 0.95;
7. (a) +0.99;
   (b) 0.74;
   (c) 0.32;
   (d) 0.64;
   (e) 0.73.
8. −0.73.
9. (a) +0.78;
   (b) 0.26.

10. +0.61.

11. (a) −0.92;
 (b) −0.92.

12. (a) +0.82;
 (b) +0.394.

13. (a) +2/3;
 (b) +0.83.

14. (a) +0.87;
 (b) +0.82;
 (c) 0.64;
 (d) 0.73.

B: Regression

1. (a) −0.6;
 (b) $y-7=0.45(x-6)$, $x-6=0.65(y-7)$;
 (c) $Y=0.9X+1.3$, $X=0.9Y-0.6$; 3.55;
 (d) $\bar{x}=2, \bar{y}=3$.

2. (a) $y=x+0.8$, $x=0.78y+0.036$; 3.3;
 (b) $y=0.55x+5.675$, $x=0.67y+1.88$;
 (c) $Y=0.61X+15.1$ and $X=1.36Y-5.2$;
 (d) −2.57;
 (e) $Y=0.93X-14.4, X=1.03Y+18.2, 60$;
 (f) $y=0.73x+27.63, x=1.35y-36.7$.

3. (a) +0.4;
 (b) +0.5, 3;
 (c) +0.5, 2.

4. (a) 17.7;
 (b) −0.55.

5. (a) $X=0.467Y+0.863$;
 (b) $Y=2.45X-0.8$, $X=0.39Y+0.49$; 18.8 kg.;
 (c) ₹198.

6. (a) $Y=0.48X+15.4$ and $X=0.75Y+1.25$;
 (b) $Y=0.48X+67.72, X=0.9075Y-41.1375$; and 26.925, −102.9;
 (c) $4X-3Y=5$, $12X-25Y+385=0$.

7. (a) $5X-6Y+90=0$, $10X-8Y-130=0$ and $\bar{X}=30$, $\bar{Y}=40$, $r=2/3$;
 (b) $\bar{x}=1$, $\bar{y}=2$, $r=-\sqrt{3}/2=-0.866$ and $\sigma_y^2=4$.

8. (a) $\bar{X}=13$, $\bar{Y}=17$;
 (b) $r=+0.6$;
 (c) $\sigma_X=3$.

9. $X=0.27Y+26.675$, $Y=0.653X+21.693$, $X=34.775$ and $Y=54.343$.

10. $r=0.947$; the value 0.947 of r is high enough to justify the proposal; $Y=1.61X+7.8$, $X=0.557Y-2.4$, minimum test score that will ensure continuation of service = 14.3 or 14.

11. 2/3, $Y=0.8X$ and $9X-5Y=25$.

EXERCISES ON CHAPTER 7(II)

(Correlation and Regression)

1. Mention the correct value of correlation coefficient with reasons:

(a)

X	2	3	5	8	9
Y	4	6	10	16	18

(b)

Rank of X	1	2	3	4	5
Rank of Y	5	4	3	2	1

2. Draw the scatter diagram of the following bivariate data:

X	2	3	5	7	9	12	13
Y	3	2	7	8	11	10	16

State the nature of correlation, if any, from the scatter diagram.

3. (a) If $N=10$ and $\Sigma d^2=280$, what is the value of rank correlation?

(b) If $R = 0.6$ and $N = 10$, find the value of ΣD^2, where D is the difference in ranks of the two series.

(c) If $b_{xy} = -0.4$ and $b_{yx} = -0.9$, find r_{xy}. [C.U. B.Com. 2007]

(d) If $r_{xy} = 0.6, \sigma_Y = 4$ and $b_{YX} = 0.48$, find the value of σ_X. [C.U. B.Com. 2003]

4. From the data given below, find the number of items: $r = 0.5$, $\Sigma xy = 120$, $\Sigma x^2 = 90$, $\sigma_y = 8$. (Here x, y are the deviations from arithmetic average.)

5. Calculate the correlation coefficient between the heights of father and son from the given data:

Heights of Father (in inches)	64	65	66	67	68	69	70
Heights of Son (in inches)	66	67	65	68	70	68	72

6. Consider the following data:

X	1	2	3	4	5	6
Y	6	4	3	5	4	2

(a) Draw a scatter diagram from the data.

(b) From the scatter diagram, guess whether the coefficient of correlation is closer to +1, −1 or zero.

(c) Calculate r by using Pearson product-moment correlation coefficient formula.

(d) Find the linear regression of Y on X by the method of least squares. Graph this line on the scatter diagram. Predict the value of Y, if $X = 4$.

7. The marks secured by a group of 10 students in the Written Selection Test (X) and in the Aptitude Test (Y) are given below:

X	44	42	40	52	39	32	24	46	41	50
Y	24	25	28	29	32	35	36	41	45	50

Calculate product-moment correlation coefficient and rank correlation coefficient. Why the two co-efficients differ?

8. In the estimation of regression equations of two variables X and Y the following results were obtained: $\bar{X} = 90$, $\bar{Y} = 70$, $N = 10$, $\Sigma x^2 = 6360$, $\Sigma y^2 = 2860$, $\Sigma xy = 3900$.

Obtain the two equations.

9. The marks obtained by 25 students in Statistics and Economics are given below. The first figure in brackets indicates the marks in Statistics and the second marks in Economics.

(14, 12), (0, 2), (1, 5), (7, 3), (15, 9), (2, 8), (12, 18), (9, 11),
(5, 3), (17, 13), (19, 18), (11, 7), (10, 13) (13, 16), (16, 14), (6, 10),
(4, 1), (11, 14), (8, 3), (9, 15), (13, 11), (14, 17), (10, 10), (11, 7),
(15, 15)

Prepare a two-way table taking the magnitude of each class-interval as 4 marks, the first being equal to 0 and less than 4.

10. Calculate coefficient of rank correlation of the following data of marks of eight students in Accountancy and Statistics:

Student's Number	1	2	3	4	5	6	7	8
Marks in Statistics	52	63	45	36	72	65	45	25
Marks in Accountancy	62	53	51	25	79	43	60	33

11. Ten competitors in a beauty contest are ranked by three judges in the following order:

1st Judge	1	6	5	10	3	2	4	9	7	8
2nd Judge	3	5	8	4	7	10	2	1	6	9
3rd Judge	6	4	9	8	1	6	3	10	5	7

Use the rank correlation coefficient to determine which pair of judges has the nearest approach to common tastes in beauty.

12. Find the coefficient of rank correlation between marks obtained in Mathematics and Statistics:

(a)

Marks in Stat.	50	88	90	55	64	34	25	44
Marks in Math.	35	72	85	44	70	30	28	38

[C.U. B.Com. 2007]

(b)

Students	A	B	C	D	E	F	G	H	I	J
Marks in Mathematics	30	20	40	50	30	20	30	50	10	0
Marks in Statistics	15	40	40	45	20	30	15	50	20	10

13. The coefficient of rank correlation of the ranks obtained by 10 students in Statistics and Accountancy was found to be 0.5. It was later discovered that the difference in ranks in the two subjects obtained by one student was wrongly taken as 3 instead of 7. Find the correct coefficient of rank correlation.

[Hints: Using **Edward Spearman's formula**, find $\Sigma D^2 = 82.5$ and then find the correct value of $\Sigma D^2 = 82.5 - 3^2 + 7^2 = 122.5$ and then find the correct value of R.]

14. Write down a formula for finding the coefficient of correlation between two variables X, Y and obtain the coefficient from the following data:

	X	Y
No. of Observations	15	15
Arithmetic Means	25	18
Sum of the Squares of the Deviations from the Mean	136	138

Sum of the product of the deviations of X and Y series from their respective means is 122.

15. (a) If $U = 2X + 11$ and $V = 3Y + 7$, what will be the correlation coefficient between U and V? Justify your statement. [C.U. B.Com. 2008]

(b) Calculate Pearson's coefficient of correlation from the following data:

X	45	55	56	58	60	65	68	70	75	80	85
Y	56	50	48	60	62	64	65	70	74	82	90

Take 65 and 70 as the assumed averages of X, Y.

16. With the following data in 6 cities, calculate the coefficient of correlation by Pearson's method between the density of population and the death rate:

Cities	Area in Square Miles	Population in '000	No. of Deaths
A	150	30	300
B	180	90	1440
C	100	40	560
D	60	42	840
E	120	72	1224
F	80	24	312

[**Hints**: Density of population = Population ÷ Area = 200, 500, 400, 700, 600, 300 and Death rate per thousand = Death ÷ Population = 10, 16, 14, 20, 17, 13. Now find r.]

17. (a) Two variables gave the following data: $\bar{x} = 20$, $\bar{y} = 15$, $\sigma_x = 4$, $\sigma_y = 3$, $r = 0.7$. Obtain the two regression equations and find the most likely value of y, when $x = 24$.

    (b) The correlation coefficient between X and Y is 0.60. If the variance of $X = 2.25$, the variance of $Y = 4.00$, mean of $X = 10$ and mean of $Y = 20$, find the equations of the regression lines of (i) Y on X, (ii) X on Y.

18. (a) If the average prices of a commodity (per kg) at Bombay and Madras are ₹ 2.40 and ₹ 2.79 respectively, estimate the most likely price at Bombay corresponding to the price of ₹ 2.33 at Madras. Given SD of prices at Bombay and at Madras are 0.326 and 0.207 respectively; coefficient of correlation between the prices = 0.774.

    (b) The equations of two lines of regression obtained in a correlation analysis are the following: $2x + 3y - 8 = 0$ and $x + 2y - 5 = 0$. Obtain the value of the correlation coefficient and the variance of y, given that the variance of x is 12.

19. (a) The equations of two regression lines in a correlation analysis are as follows: $3x + 2y = 26$ and $6x + y = 31$.

    A student obtains the mean values $\bar{x} = 7$, $\bar{y} = 4$ and the value of the correlation coefficient $r = 0.5$. Do you agree with him? If not, suggest your results.

    (b) In a partially destroyed record the following data are available: Variance of $x = 25$, Regression equation of x upon y is $5x - y = 22$ and regression equation of y upon x is $64x - 45y = 25$. Find (i) Mean values of x and y, (ii) SD of y, (iii) Coefficient of correlation between x and y.

ANSWERS

1. (a) 1;
   (b) -1.
3. (a) -0.7;
   (b) 66;
   (c) -0.6;
   (d) 5.
4. 10.
5. 0.81.
6. (c) $r=-0.68$;
   (d) $Y+0.514X=5.8$, $X+0.9Y=7.1$, 7.856.
7. 0.081, 0.018; difference due to some loss of information.
8. $Y=0.6132X-14.812$; $X=1.3636Y-5.45$.
10. 0.64.
11. $R_{12}=-0.212$, $R_{23}=-0.297$, $R_{13}=0.636$; The first and the 3rd Judges have the nearest approach to common tastes in beauty.
12. (a) 0.976;
    (b) 0.63.
13. 0.258.
14. 0.89.
15. (a) $r_{UV}=r_{XY}=0.5$, since the correlation coefficient is independent of any change of origin and scale;
    (b) 0.92.
16. 0.99.
17. (a) $y=0.52x+4.5$, $x=0.93y+6.05$, 17.1;
    (b) $Y=0.8X+12$ and $X=0.45Y+1$.
18. (a) ₹ 1.90;
    (b) $-\sqrt{3}/2$, 4.
19. (a) No; $\bar{x}=4$, $\bar{y}=7$ and $r=-0.5$;
    (b) $\bar{x}=6$, $\bar{y}=8$, $\sigma_y=40/3$ and $r=8/15$.

Chapter 8

Elements of Set Theory

8.1 Introduction

Set theory is the most basic concept in any discipline of studies — be it Science or Humanities or Commerce. It is, therefore, quite natural that we begin with the discussions on elements of sets.

What is a Set? Not easy to give a precise definition of a set. Our purpose will be served if we state:

A set is a well defined collection of distinct objects. Objects may be of any kind, e.g., Numbers, Letters of the English/Bengali alphabets, Rivers, People, etc. We shall be more interested with Sets of Numbers. The objects which form a set are called *elements* or *members* of the set.

The term well defined collection in the definition of a set means a collection made according to some characteristic property: Given any object whatsoever, using that characteristic property, one can decide whether the given object should belong to that collection or it should not belong to the collection.

The objects of the collection should be written within the braces { }.

e.g., (i) $\{1,2,3,4,5\}$ is a set of first five natural numbers.

(ii) $\{a,e,i,o,u\}$ is a set of five vowels of the English alphabets.

It is usual to denote a set by capital letters like A, B, C, …, S, T, X, …, and the elements of a set by small letters a, b, c, …, x, y, z, …, or by Greek letters α, β, γ, ….

The following symbols are used in Mathematics:

Symbols	*Their meanings*
$\in$	belongs to or an element of e.g., $a \in A$ means the element a belongs to the set A.
$\notin$	does not belong to or not an element of e.g., $a \notin A$ means the element a does not belong to the set A.
s.t.	such that
w.r.t.	with respect to
iff	if and only if
$\exists$	there exists
$\forall$	for all (e.g., $\forall x \in S$, for every element x belonging to the set S)
$\wedge$	and (e.g., $a \wedge b$ means a and b)
$\vee$	or (e.g., $a \vee b$ means a or b)
$\Rightarrow$ or $\rightarrow$	implies that; example is given below.

e.g., let P and Q be two statements, say, $P: x+2=4$ and $Q: x=2$. When we write, $P \Rightarrow Q$, it will mean that if $P(x+2=4)$, then Q (i.e., $x=2$).

8.2 How to Write a Set?

There are usually two methods of writing a set —

- **Tabular or Roster Method:** In this method we define a particular set by listing its members, e.g., let A consist of the numbers 1, 3, 5 and 10, then we write
$$A = \{1, 3, 5, 10\},$$
i.e., the elements are separated by commas and enclosed in braces { }. We call this method **Tabular method** or **Tabular form** of a set or **Roster method** *of writing a set.*
- **Property** or **Set-builder Method:** We may define a particular set B by stating properties which its elements must satisfy: e.g., let B be the set of all even positive integers, then we use a letter x, to represent an arbitrary element and we write,
$B = \{x \mid x \text{ is an even positive integer}\}$ or $B = \{x : x \text{ is an even positive integer}\}$.
[read as: B consists of x, where x is an even positive integer.]
Notice that the vertical line " | " or ":" is read *such that* (s.t).
In Roster Method, $B = \{2, 4, 6, 10, 12, \ldots\}$.
Similarly, $B = \{x \mid x \text{ is a positive integer divisible by } 6\}$. In tabular form, $B = \{6, 12, 18, \ldots\}$.

8.3 Finite and Infinite Sets: Other Important Terms

A set is *finite* if it contains a definite (or specific) number of different elements, i.e., if the process of counting the different elements of the set comes to an end. Otherwise, a set is *infinite.* For an infinite set the process of counting its elements would never come to an end.

Illustration 1. $\{1, 2, 3, 4, 5\}$ *is a finite set.*

Illustration 2. $\{a, e, i, o, u\}$ *is a finite set.*

Illustration 3. $I^+ : \{1, 2, 3, 4, \ldots, n, \ldots\}$ *is an infinite set.*

Illustration 4. R: *The set of all real numbers is an infinite set.*

Illustration 5. $A = \{x : 3 < x < 5\}$ *is an infinite set.*

Illustration 6. *Let* $P = \{x \mid d \text{ is a river on the earth}\}$ *is a finite set.*

Although it may be difficult to count the number of rivers in the world, P is still a finite set.

1. **Equality of Two Sets:** Two sets are said to be equal if they contain the same elements. Thus, two sets A and B are equal if and only if every element of A is an element of B and every element of B is an element of A.
Symbolically, $A = B$ iff $(a \in A \Rightarrow a \in B \wedge b \in B \Rightarrow b \in A)$.
Here '$\wedge$' means 'and'; '$\Rightarrow$' means 'implies'.

Illustration 7. *The set* $A = \{1, 4, 3, 5\}$ *is equal to* $B = \{1, 3, 4, 5\}$.

2. **Null Set or Empty Set or Void Set:** Sometimes it happens that while making collection of objects by a well defined property, no definite element occurs. We call such a collection a *Null set* or *Empty set* or *Void set*. Thus, a null set has no element in it. We denote it by ϕ, or $\{\quad\}$.

   Examples of Null Sets:

   Illustration 8. *The set of positive integers between 2 and 3 is a null set.*

   Illustration 9. *The set of real roots of the equation* $x^2 + 4 = 0$ *is a null set.*

   Illustration 10. $\{x : x \neq x\}$ *is a set with no element.*

3. **Singleton Set:** A set having only one element in it is called a *Singleton set* or a *Unit set*. *For example,* {2} or the set {President of India} are both singleton sets.

4. **Subsets:** If A and B are two sets such that every element of A is also an element of B, then A is called a *subset* of B (or A is contained in B) (or B contains A) and we write, $A \subseteq B$ (or $B \supseteq A$).
   Symbolically, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

   Illustration 11. *The set* $A = \{2, 3, 7\}$ *is a subset of* $B = \{1, 2, 3, 4, 5, 7\}$.

   Illustration 12. *The set* $A = \{2, 5, 7\}$ *is a subset (improper subset) of* $B = \{5, 7, 2\}$.

   Illustration 13. *Any set A is a subset of itself* $A \subseteq A$.

   Illustration 14. *The null set* ϕ *is a subset of every set A* ($\phi \subseteq A$).

5. **Proper Subsets:** A set A is said to be a proper subset of B, if

   (a) A is a subset of B, i.e., every element of A is also an element of B and

   (b) $A \neq B$, i.e., there is an element in B which is not in A.

   'A is a proper subset of B' is denoted by $A \subset B$ or $A \subsetneq B$. In this case, B is called a *superset* of A.

   Illustration 15. *The set* $A = \{2, 4, 7\}$ *is a proper subset of the set* $B = \{2, 3, 4, 6, 7\}$, *and B is a superset of A.*

6. **Universal Set:** All the sets being discussed in a particular investigation are considered to be subsets of a fixed set. We call this fixed set the *Universal set* or *Universe* for that particular discussion and we denote this set by U or S. Thus the Universal set is the set which contains all the elements of all subsets under investigation in a particular context.

   Illustration 16. *Human beings of the world form a Universal set while the population of India, England, U.S.A., etc., are its subsets.*

   Illustration 17. *Suppose, we have discussed in a particular investigation all possible sets of vowels, i.e.,* $\{a\}, \{e\}, \{i\}, \{o\}, \{u\}, \{a, e\}, \ldots, \{a, e, i, o, u\}$; *all these sets are subsets of the set* $\{a, e, i, o, u\}$. *Hence, in this case,* $U = \{a, e, i, o, u\}$ *is the Universal set.*

7. **Power Set:** The set of all possible subsets of a set A is called the *Power set* of A and it is denoted by the symbol $P(A)$ or 2^A.
   If $A=\{1\}$, $P(A)=[\{\phi,\{1\}]$, i.e., $P(A)$ contains 2 elements, i.e., 2^1 elements.
   If $A=\{1,2\}$, $P(A)=[\{\phi,\{1\},\{2\},\{1,2\}]$, i.e., $P(A)$ contains 4 elements, i.e., 2^2 elements.
   If $A=\{1,2,3\}$, $P(A)=[\phi,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}]$, i.e., $P(A)$ contains $2^3=8$ elements.
   [C.U. B.Com. 2008]

   Note: In writing the set $P(A)$ we are to include ϕ and A as subsets of A.
   Proceeding in this way, we see that if A contains n elements, then $P(A)$ contains 2^n elements.
   Thus, if a finite set A has n elements, its power set contains 2^n elements.

   Illustration 18. *If* $A=\{2,3\}$, *then* $P(A)=2^A=[\phi,\ \{2\},\ \{3\},\ \{2,3\}]$, *where* ϕ *is the null set.*

   Illustration 19. *If* $A=\{1,2,3\}$, *then* $P(A)=2^A=[\phi,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}]$, *where* ϕ *is the null set.* [C.U. B.Com.(H) 2001]

   Note: A power set is a set of sets.

8.4 Operations on Sets

By operations on sets, we mean the process of constructing new sets from given sets by combining the latter in some suitably well defined fashion as explained below:

Venn Diagram: Pictorial way of explaining operations on sets

Operations on sets or any property or theorem relating to sets can be well understood with the help of a diagram known as *Venn-Euler diagram* or simply *Venn diagram.* Venn diagram was first introduced by Euler and subsequently this was developed by John Venn, a British Mathematician. In this diagram the universal set U is denoted by a rectangular region and any subset of U by a region enclosed by a closed curve (or a circle) lying within the rectangular region. These closed curves (or circles) representing the subsets of U will intersect each other if they have some common elements among them. We give below five operations on sets:

Operation I: Union or Join of Sets

The union or join of two sets A and B, written as $A \cup B$, is the set of all elements which belong either to A or to B or to both A and B.

Symbolically, $A \cup B=\{x: x \in A \vee x \in B)$. Here, '$\vee$' means 'or'.

$A \cup B$ is read 'A union B' or 'A cup B'.

Illustration 1. *If* $A=\{1,3,4,5\}$ *and* $B=\{2,3,4,6,8\}$, *then*

$$A \cup B=\{1,2,3,4,5,6,8\}.$$

Venn Diagram for $A \cup B$

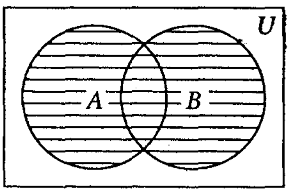


Fig. 8.1

In Venn diagram Fig. 8.1, we have shaded $A \cup B$, i.e., the area of A and the area of B.

It follows from definition that $A \cup B=B \cup A$ and both A and B are always subsets of $A \cup B$, i.e., $A \subset A \cup B$ and $B \subset A \cup B$.

Operation II: Intersection or Meet of Sets

The intersection or meet of two sets A and B, written as $A \cap B$, is the set of all elements which are Common to both A and B.

Symbolically, $A \cap B = \{x : x \in A \wedge x \in B\}$. Here, '$\wedge$' means 'and'.

$A \cap B$ is read 'A intersection B' or 'A cap B'.

Illustration 2. *If* $A = \{2,3,5,7\}$ *and* $B = \{1,3,4,6,7,9\}$, *then* $A \cap B = \{3,7\}$.

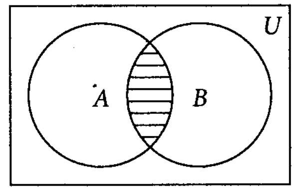


Fig. 8.2

Venn Diagram for $A \cap B$

In Venn diagram we have shaded $A \cap B$, i.e., the area common to both A and B Fig. 8.2.

It follows from definition that $A \cap B = B \cap A$ and each of A and B contains $A \cap B$ as a subset, i.e., $A \supset (A \cap B)$ and $B \supset (A \cap B)$.

Disjoint Sets. If two sets A and B have no elements in common, i.e., if no element of A is in B and no element of B is in A, then A and B are said to be disjoint or mutually exclusive sets. Clearly, $A \cap B = \phi$, when A and B are disjoint.

[C.U. B.Com.(H) 1991]

Illustration 3. *Two sets* $A = \{2,5,7\}$ *and* $B = \{1,3,6,8\}$ *are disjoint, since they have no common elements. Here,* $A \cap B = \phi$.

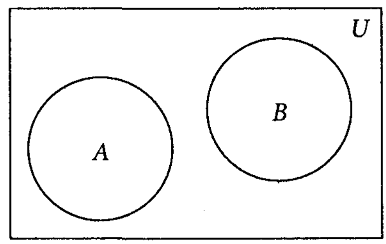


Fig. 8.3

Venn Diagram for Disjoint Sets

Two disjoint sets A and B having no common elements among them are shown in the Venn diagram [Fig. 8.3].

Operation III: Difference of Two Sets

The difference of two sets A and B is the set of elements which belong to A but do not belong to B. We denote the difference of A and B by $A - B$ or, $A \sim B$.

Symbolically, $A - B = \{x : x \in A \wedge x \notin B\}$.

Illustration 4. *If* $A = \{1, 2, 3, 5, 7\}$ *and* $B = \{2, 3, 4, 5, 6\}$, *then* $A - B = \{1,7\}$ *and* $B - A = \{4,6\}$.

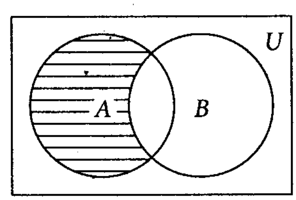


Fig. 8.4

Venn Diagram for Difference of Two Sets

In Venn diagram of Fig. 8.4, we have shaded $A - B$, i.e., the area in A which does not include any part of B.

It follows from the definition that $A - B \subset A$ and $B - A \subset B$.

Operation IV: Complement of a Set (or Negation of a Set)

[C.U. B.Com.(H) 1991]

It is always defined w.r.t. the Universal set U. The complement of a set A is the set of all the elements of the Universal set U which do not belong to A, i.e., it is the difference of the Universal set U and the set A. We shall denote the complement of the set A by A' or A^c or $\sim A$.

Symbolically, $A' = \{x : x \in U \wedge x \notin A\}$, $\wedge$ means 'and'.

Clearly, $A \cap A' = \phi, A \cup A' = U, U' = \phi, \phi' = U$.

Illustration 5. *Let* $U=\{a,e,i,o,u\}$ *and* $A=\{e,o\}$. *Then* $A'=U-A=\{a,i,u\}$.

Note: The complement of the complement of a set A is the set A itself, i.e., $(A')'=A$. Also, $A-B=A\cap B'$ and $B-A=B\cap A'$.

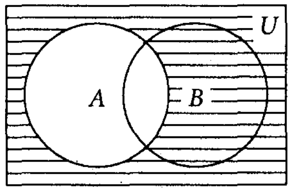


Fig. 8.5

Venn Diagram for the Complement of a Set

In the Venn diagram, we have shaded the complement of A, i.e., the area outside A.

It can be easily verified by Venn diagram that $\boldsymbol{A-B=A\cap B'}$.

Operation V: Cartesian Product of Two Sets

We know that the two sets $\{x,y\}$ and $\{y,x\}$ are equal, i.e., $\{x,y\}=\{y,x\}$. But when we speak of ordered pairs, $\boldsymbol{(x,y)\neq(y,x)}$ if x,y are distinct; this is because if we plot the ordered pairs (x,y) and (y,x) on a graph paper taking x along OX and y along OY (perpendicular to OX), they will represent two distinct points.

If A and B are two sets, then the set of all ordered pairs (x,y) such that $x\in A$ and $y\in B$ is called the *Cartesian Product* of A and B, and it is denoted by $A\times B$ (read "A cross B").

Symbolically, $A\times B=\{(x,y): x\in A\wedge y\in B\}$.

Illustration 6. *If* $A=\{1,2\}$ *and* $B=\{3,4,7\}$, *then* $A\times B=\{(1,3),(1,4),(1,7),(2,3),(2,4),(2,7)\}$ *and* $B\times A=\{(3,1),(3,2),(4,1),(4,2),(7,1),(7,2)\}$.

Note: Clearly, $A\times B\neq B\times A$, if A and B are two different sets.

8.5 Illustrative Examples

Example 1. *Write down the following statements in set-theoretic notations:* (i) *3 is an element of a set A*; (ii) *4 does not belong to a set B*, (iii) *C is a subset of D*, (iv) *P and Q are disjoint sets.*

Solution: (i) $3\in A$; (ii) $4\notin B$; (iii) $C\subseteq D$; (iv) $P\cap Q=\phi$.

Example 2. *State with reasons which of the following statements is true or false:*

(i) $\{a\}\in\{a,b,c\}$; (ii) $a\in\{a,b,c\}$; (iii) $a\subset\{a,b,c\}$; (iv) $a\notin\{a,b,c\}$. [C.U. B.Com.(H) 1992]

Solution: (i) The statement $\{a\}\in\{a,b,c\}$ is *false;* because $\{a\}$ is a set and it does not belong to $\{a,b,c\}$; it is a proper subset of $\{a,b,c\}$.

(ii) The statement $a\in\{a,b,c\}$ is *true;* because a is an element of $\{a,b,c\}$.

(iii) The statement $a\subset\{a,b,c\}$ is *false;* because an element cannot be a subset. Here, $\{a\}\subset\{a,b,c\}$ is true.

(iv) $a\notin\{a,b,c\}$ is *false;* because $a\in\{a,b,c\}$.

Example 3. (i) *Find the power set of* $\{1,2,3\}$. [C.U. B.Com.(H) 2008]

(ii) *Given,* $A=\{2,3,8\}$ *and* $B=\{6,4,3\}$; *find* $A\times B$. [C.U. B.Com.(H) 2001]

(iii) *If* $A=\{1,2,3,4,5,6,7,8,9,10\}$, *find a subset B of A such that* $B=\{x: x^2\in A\}$. [B.U. B.Com. 2008]

Solution: (i) Let $A=\{1,2,3\}$. Then by definition, the power set $P(A)$ of set A is given by $P(A)=[\phi,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}]$, where ϕ is the null set.

(ii) $A\times B=\{2,3,8\}\times\{6,4,3\}=\{(2,6),(2,4),(2,3),(3,6),(3,4),(3,3),(8,6),(8,4),(8,3)\}$.

(iii) Clearly, if $x^2\in A$, then $x^2=1^2,2^2,3^2=1,4,9\in A$ and $x=\pm1,\pm2,\pm3$.

But $-1,-2,-3\notin A$; $\therefore x=1,2,3$ and $\boldsymbol{B=\{x:x^2\in A\}=\{1,2,3\}}$.

Example 4. *If* $A=\{1,2,3,4\}$, $B=\{2,4,5,8\}$, $C=\{3,4,5,6,7\}$, *find* (i) $A\cap B$; (ii) $B\cup C$; (iii) $A\cap(B\cup C)$; (iv) $A\cup(B\cap C)$. (v) *Also verify that* $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Solution: (i) $A\cap B=\{2,4\}$; (ii) $B\cup C=\{2,3,4,5,6,7,8\}$; (iii) $A\cap(B\cup C)=\{2,3,4\}$; (iv) $(B\cap C)=\{4,5\}$; $\therefore A\cup(B\cap C)=\{1,2,3,4,5\}$; (v) $A\cup(B\cap C)=\{1,2,3,4,5\}$.

$A\cup B=\{1,2,3,4,5,8\}$ and $A\cup C=\{1,2,3,4,5,6,7\}$;

$\therefore (A\cup B)\cap(A\cup C)=\{1,2,3,4,5\}$.

Hence, $\boldsymbol{A\cup(B\cap C)=(A\cup B)\cap(A\cup C)}$.

Example 5. *If* $A=\{1,2,3\}$, $B=\{a,b\}$, *find* $A\times B$, *and* $B\times A$ *and hence verify that* $A\times B\neq B\times A$.

Solution: $A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$ and

$$B\times A=\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}.$$

Clearly, $\boldsymbol{A\times B\neq B\times A}$.

Example 6. *Let the sets A and B be given by* $A=\{1,2,3,4\}$, $B=\{2,4,6,8,10\}$ *and the universal set* $S=\{1,2,3,4,5,6,7,8,9,10\}$. *Find* $(A\cup B)'$ *and* $(A\cap B)'$. [C.U. B.Com.(H) 1991]

Solution: $A\cup B=\{1,2,3,4\}\cup\{2,4,6,8,10\}=\{1,2,3,4,6,8,10\}$;

$\therefore (A\cup B)'=S-(A\cup B)=\{1,2,3,4,5,6,7,8,9,10\}-\{1,2,3,4,6,8,10\}=\{5,7,9\}$.

Again, $(A\cap B)=\{1,2,3,4\}\cap\{2,4,6,8,10\}=\{2,4\}$;

$\therefore (A\cap B)'=S-(A\cap B)=\{1,2,3,4,5,6,7,8,9,10\}-\{2,4\}=\boldsymbol{\{1,3,5,6,7,8,9,10\}}$.

Example 7. *If* $A=\{1,2\}$, $B=\{2,3\}$, $C=\{3,4\}$; *verify that*

$$A\times(B\cup C)=(A\times B)\cup(A\times C).$$

Solution: $B\cup C=\{2,3,4\}$.

$$\therefore\quad A\times(B\cup C)=\{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\};\qquad(1)$$

$$A\times B=\{(1,2),(1,3),(2,2),(2,3)\}$$

and $$A\times C=\{(1,3),(1,4),(2,3),(2,4)\}.$$

$$\therefore\quad (A\times B)\cup(A\times C)=\{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}.\qquad(2)$$

Hence, from (1) and (2), we have $\boldsymbol{A\times(B\cup C)=(A\times B)\cup(A\times C)}$.

Example 8. *If* $A = \{1,2,3\}$; $B = \{2,3,4\}$; $S = \{1,3,4\}$; $T = \{2,4,5\}$, *verify that*

$$(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T).$$

Solution:

$$A \times B = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$$

and $$S \times T = \{(1,2),(1,4),(1,5),(3,2),(3,4),(3,5),(4,2),(4,4),(4,5)\}.$$

$$\therefore (A \times B) \cap (S \times T) = \{(1,2),(1,4),(3,2),(3,4)\}. \quad (1)$$

Again, $(A \cap S) = \{1,3\}$ and $(B \cap T) = \{2,4\}$;

$$\therefore (A \cap S) \times (B \cap T) = \{(1,2),(1,4),(3,2),(3,4)\}. \quad (2)$$

Hence, from (1) and (2), we get $\boldsymbol{(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)}$.

Example 9. *If* $A = \{1,2,3,4\}$, $B = \{2,3,4,5\}$, $C = \{1,3,4,5,6,7\}$, *verify that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Solution: $B \cup C = \{1,2,3,4,5,6,7\}$;

$$\therefore A \cap (B \cup C) = \{1,2,3,4\}. \quad (1)$$

Again, $A \cap B = \{2,3,4\}$ and $A \cap C = \{1,3,4\}$;

$$\therefore (A \cap B) \cup (A \cap C) = \{1,2,3,4\}. \quad (2)$$

Hence, from (1) and (2), we get $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$.

Example 10. *Let* $A = \{1,2,3,4,\ldots,8,9\}$, $B = \{2,4,6,8\}$, $C = \{1,3,5,7,9\}$, $D = \{3,4,5\}$ *and* $E = \{3,5\}$. *Which set can equal X if we are given the following information?*

(i) *X and B are disjoint;* (ii) $X \subset A$ *but* $X \not\subset C$;
(iii) $X \subset D$ *but* $X \not\subset B$; (iv) $X \subset C$ *but* $X \not\subset A$.

Solution: (i) 'X and B are disjoint' means X and B have no common elements. From the given sets, we see that only B and C, and B and E have no common elements. Hence, $X = C, E$.

(ii) $A \subset A$ but $A \not\subset C$, $B \subset A$ but $B \not\subset C$; and $D \subset A$ but $D \not\subset C$; $\therefore X = A$, B and D.

(iii) $D \subset D$ but $D \not\subset B$, and $E \subset D$ but $E \not\subset B$; $\therefore X = D, E$.

(iv) Since each of A, B, C, D and E is a subset of A, none of the sets A, B, C, D, E can equal X.

Example 11. *In the Venn diagram below, shade:* (i) B', (ii) $(B - A)'$ *and* (iii) $A' \cap B'$.

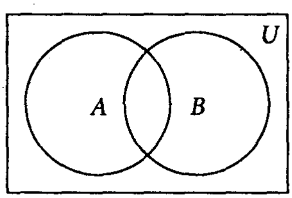


Fig. 8.6

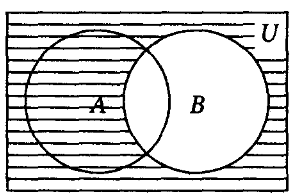


Fig. 8.7

Solution: In Fig. 8.6 we have shown two subsets A, B of the universal set U.

(i) B' is the complement of B and, therefore, B' consists of elements which do not belong to B. Hence, we shade the area outside B [See Fig. 8.7].

(ii) First we shade the area $B - A$ with upward slanted strokes (///) in Fig. 8.8; then $(B - A)'$ is the area outside $B - A$ which is shaded with horizontal lines in Fig. 8.9.

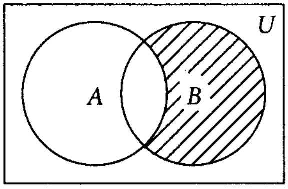


Fig. 8.8

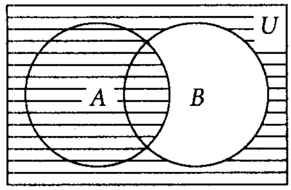


Fig. 8.9

(iii) We first shade A', the area outside A, with strokes (///) slanted in the right side and then shade B' with strokes (\\\) slanted in the left side; $A' \cap B'$ is the cross-shaded (or cross-hatched) area, i.e., the area common to A' and B' which is shown in Fig. 8.11.

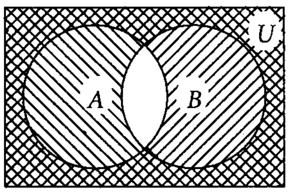


Fig. 8.10

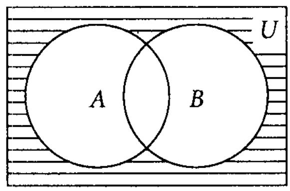


Fig. 8.11

8.6 Laws of Algebra of Sets

Three main operations on sets, viz., Intersection ($\cap$), Union ($\cup$) and Complement ($'$) satisfy certain laws of Algebra. These laws are stated below:

I. *Idempotent Law:* For any set A, we have:
(i) $A \cup A = A$ and (ii) $A \cap A = A$.

II. *Associative Law:* For any three sets A, B and C, we have:
(i) $A \cup (B \cup C) = (A \cup B) \cup C$ and (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

III. *Commutative Law:* For a pair of sets A and B, we have:
(i) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$.

IV. *Distributive Law:* For any three sets A, B and C, we have:
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

V. *De Morgan's Laws:* For any two sets A and B, we have:
(i) $(A \cup B)' = A' \cap B'$ and (ii) $(A \cap B)' = A' \cup B'$. [B.U. B.Com.(H) 2005]

VI. *Identity Law:*
(i) $A \cup \phi = A$, (ii) $A \cap U = A$, (iii) $A \cap \phi = \phi$ and (iv) $A \cup U = U$.

VII. *Complement Law:*

(i) $A \cup A' = U$, (ii) $A \cap A' = \phi$, (iii) $(A')' = A$ and (iv) $U' = \phi$, $\phi' = U$.

The proofs of the Idempotent Law, Commutative Law, Identity Law and Complement Law are obvious. Let us verify the Associative Law, Distributive Law and De Morgan's Law by using Venn diagrams. Analytical proofs using just definitions of these three laws will be given later.

Note: If we shade A with strokes slanted in the right side and then shade B with strokes slanted in the left side, then $A \cap B$ is the cross-shaded area and $A \cup B$ is the total shaded area.

8.7 Explanations of Laws of Algebra of Sets

Associative Laws (Verification by using Venn Diagram)

Using Venn diagram, verify that:

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ and (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof. (i) LHS $= A \cup (B \cup C)$

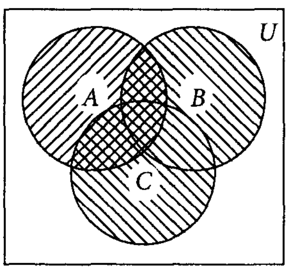


Fig. 8.12

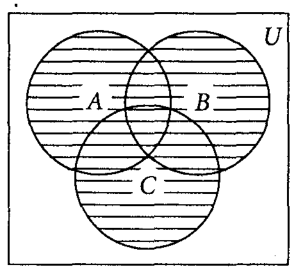


Fig. 8.13

In Fig. 8.12, we first shade A with strokes (///) slanted towards right and then shade $B \cup C$ with strokes (\\\\\\) slanted towards left. $A \cup (B \cup C)$ is the total shaded area which is shown in Fig. 8.13.

RHS $= (A \cup B) \cup C$

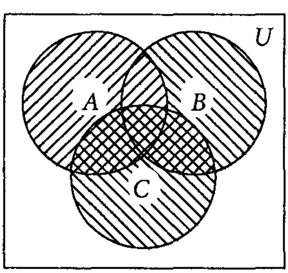


Fig. 8.14

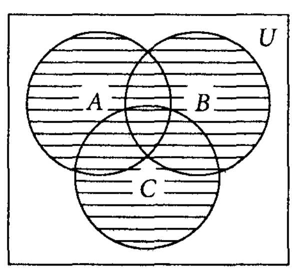


Fig. 8.15

In Fig. 8.14, we first shade $(A \cup B)$ with strokes (///) slanted towards right and then shade C with strokes (\\\\\\) slanted towards left. $(A \cup B) \cup C$ is the total shaded area which is shown in Fig. 8.15.

From Figures 1.13 and 1.15, we see that $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) LHS $= A \cap (B \cap C)$.

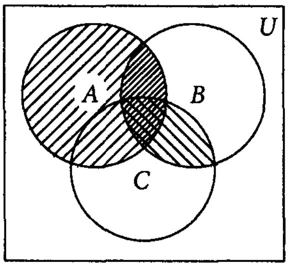


Fig. 8.16

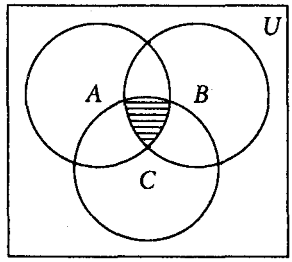


Fig. 8.17

In Fig. 8.16, first we shade A with strokes (///) slanted towards right and then shade $B \cap C$ with strokes (\\\\\\) slanted towards left. $A \cap (B \cap C)$ is the cross-shaded area which is shown in Fig 8.17.

RHS $= (A \cap B) \cap C$.

In Fig. 8.18, first we shade $A \cap B$ with strokes (///) slanted towards right and then shade C with strokes (\\\\\\) slanted towards left. $(A \cap B) \cap C$ is the cross-shaded area which is shown in Fig. 8.19.

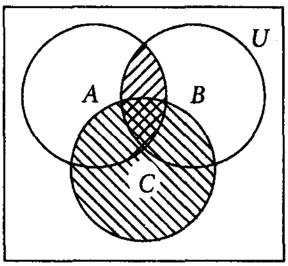


Fig. 8.18

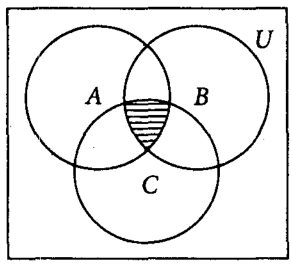


Fig. 8.19

Hence, from Fig. 8.17 and Fig. 8.19, we obtain $A \cap (B \cap C) = (A \cap B) \cap C$.

Distributive Laws (Using Venn Diagram)

Using Venn diagram, verify that:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

[C.U. B.Com. 2001; B.U. B.Com.(H) 2003]

Proof. (i) LHS $= A \cup (B \cap C)$.

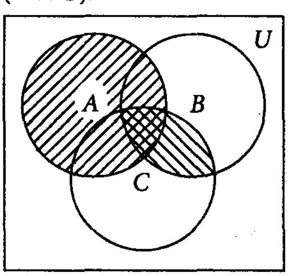


Fig. 8.20

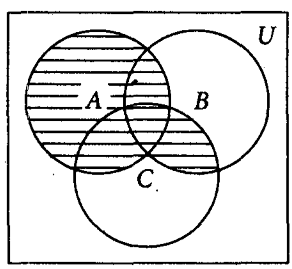


Fig. 8.21

We first shade A with strokes slanted towards right and $(B \cap C)$ with strokes slanted towards left. Then

$A \cup (B \cap C)$ is the total shaded area which is shown in Fig. 8.21.

RHS $= (A \cup B) \cap (A \cup C)$.

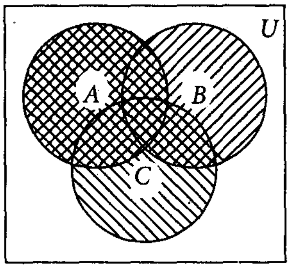


Fig. 8.22

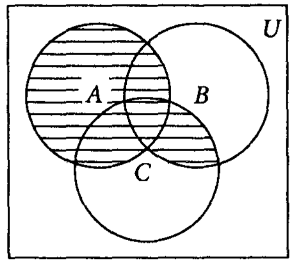


Fig. 8.23

First we shade $A \cup B$ with strokes (///) slanted towards right and shade $A \cup C$ with strokes (\\\) slanted towards left; then $(A \cup B) \cap (A \cup C)$ is the cross-shaded area which is shown in Fig. 8.23.

From Fig. 8.21 and Fig. 8.23, we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) LHS $= A \cap (B \cup C)$.

We first shade A with strokes (///) slanted towards right and then $B \cup C$ with strokes (\\\) slanted towards left. $A \cap (B \cup C)$ is the cross-shaded area which is shown in Fig. 8.25.

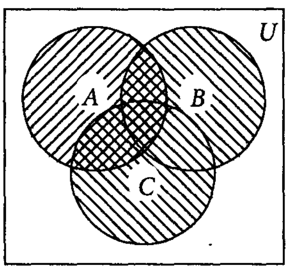


Fig. 8.24

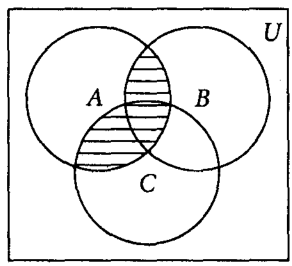


Fig. 8.25

RHS $= (A \cap B) \cup (A \cap C)$.

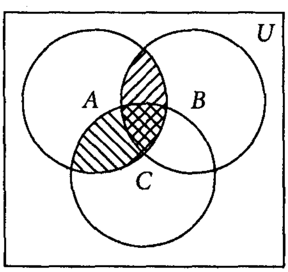


Fig. 8.26

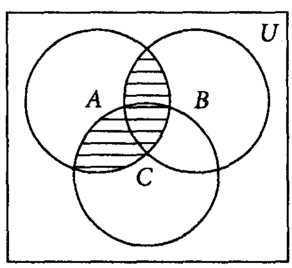


Fig. 8.27

We first shade $A \cap B$ with strokes (///) slanted towards right and shade $A \cap C$ with strokes (\\\) slanted towards left. Then $(A \cap B) \cup (A \cap C)$ is the total shaded area which is shown in Fig. 8.27.

From Fig. 8.25 and Fig. 8.27, it follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

De Morgan's Laws (Using Venn Diagram)

Using Venn diagram, verify that: (i) $(A \cup B)' = A' \cap B'$; (ii) $(A \cap B)' = A' \cup B'$. [C.U. B.Com.(H) 2008]

Proof. (i) To prove: $(A \cup B)' = A' \cap B'$.

LHS $= (A \cup B)'$.

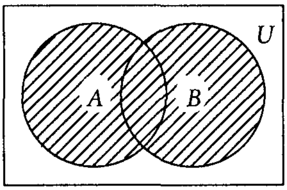


Fig. 8.28

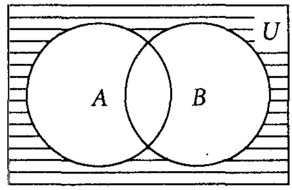


Fig. 8.29

In Fig. 8.28, $A \cup B$ is shaded with strokes slanted towards right. $(A \cup B)'$ is the area outside $A \cup B$ which is shown in Fig. 8.29.

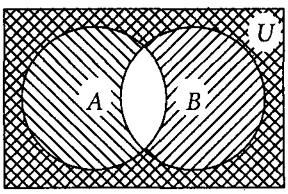


Fig. 8.30

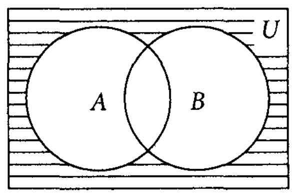


Fig. 8.31

RHS $= A' \cap B'$.

We first shade A', i.e., the area outside A with strokes (///) slanted towards right and then shade B', the area outside B, with strokes (\\\) slanted towards left; $A' \cap B'$ is the cross-hatched area, i.e., the area common to A' and B' which is shown in Fig. 8.31.

Hence, from Fig. 8.29 and Fig. 8.31, we have $(A \cup B)' = A' \cap B'$.

(ii) To prove: $(A \cap B)' = A' \cup B'$.

LHS $= (A \cap B)'$.

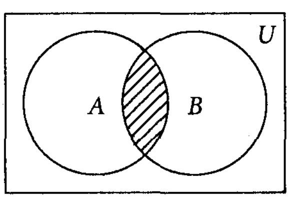


Fig. 8.32

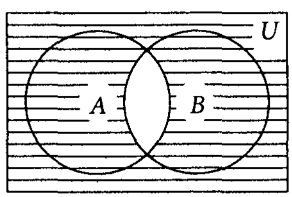


Fig. 8.33

In Fig. 8.32, we have shaded $A \cap B$, i.e., the area common to A and B. $(A \cap B)'$ is the area outside $A \cap B$ which is shaded in Fig. 8.33.

RHS $= A' \cup B'$.

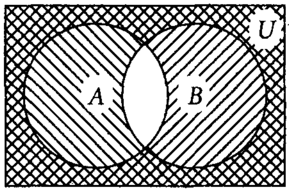


Fig. 8.34

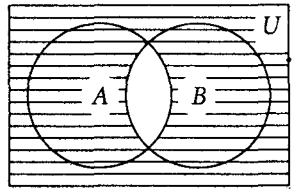


Fig. 8.35

First we shade A', the area outside A with strokes slanted towards right and then B' with strokes slanted towards left. $A' \cup B'$ is the total shaded area which is shown in Fig. 8.35.

From Fig. 8.33 and Fig. 8.35, it follows that $(A \cap B)' = A' \cup B'$.

8.8 Analytical Proofs of Laws of Algebra of Sets using Definitions

We now deduce the Laws of Algebra of set from the definitions. The proofs of the Idempotent law, Commutative law, Identity law and Complement law are obvious. To prove the other three laws, i.e., Associative law, Distributive law and De Morgan's law, we shall use the following results which are evident from the definition of subsets:

(i) If $x \in A \Rightarrow x \in B$, then $A \subseteq B$, where '$\Rightarrow$' means 'implies'.

(ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Associative Law: Prove that (i) $\boldsymbol{A \cup (B \cup C) = (A \cup B) \cup C}$

and (ii) $\boldsymbol{A \cap (B \cap C) = (A \cap B) \cap C}$.

Proof. (i) Let $x \in A \cup (B \cup C)$. Then, by definition of union, we write,

$$\begin{aligned} x \in A \cup (B \cup C) &\Rightarrow x \in A \text{ or } x \in (B \cup C) \quad \text{[Here 'or' means 'or/and']} \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \cup C. \end{aligned}$$

Thus, $x \in A \cup (B \cup C) \Rightarrow x \in (A \cup B) \cup C$;

$$\therefore\ A \cup (B \cup C) \subseteq (A \cup B) \cup C. \qquad (1)$$

Next, let $y \in (A \cup B) \cup C$. Then, by definition,

$$\begin{aligned} y \in (A \cup B) \cup C &\Rightarrow y \in (A \cup B) \text{ or } y \in C \\ &\Rightarrow (y \in A \text{ or } y \in B) \text{ or } y \in C \\ &\Rightarrow y \in A \text{ or } (y \in B \text{ or } y \in C) \\ &\Rightarrow y \in A \text{ or } y \in (B \cup C) \\ &\Rightarrow y \in A \cup (B \cup C); \end{aligned}$$

$$\therefore\ (A \cup B) \cup C \subseteq A \cup (B \cup C). \qquad (2)$$

Hence, from (1) and (2), we get $\boldsymbol{A \cup (B \cup C) = (A \cup B) \cup C}$.

(ii) Using the definition of intersection and proceeding as above, we can prove this result. This is left as an exercise for the reader.

Distributive Law: Prove that

(i) $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$; [C.U. B.Com.(H) 2007; V.U. B.Com.(H) 2007, 08, 09, 10]

(ii) $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$. [C.U. B.Com.(H) 1997]

Proof. (i) Let $x \in A \cup (B \cap C)$. Then, by definition of union and intersection, we get

$$\begin{aligned}
x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in (B \cap C) \quad \text{[Here 'or' means 'or/and']} \\
&\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\
&\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
&\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\
&\Rightarrow x \in (A \cup B) \cap (A \cup C).
\end{aligned}$$

Thus, $x \in A \cup (B \cap C) \Rightarrow x \in (A \cup B) \cap (A \cup C)$.

$$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \tag{1}$$

Next let $y \in (A \cup B) \cap (A \cup C)$. Then, by definitions

$$\begin{aligned}
y \in (A \cup B) \cap (A \cup C) &\Rightarrow y \in (A \cup B) \text{ and } y \in (A \cup C) \\
&\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C) \\
&\Rightarrow y \in A \text{ or } (y \in B \text{ and } y \in C) \\
&\Rightarrow y \in A \text{ or } y \in (B \cap C) \\
&\Rightarrow y \in A \cup (B \cap C).
\end{aligned}$$

$$\therefore (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C). \tag{2}$$

Hence, from (1) and (2), we get $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$.

(ii) Using the definitions of Union and Intersection and proceeding as in (i), we can prove this result. This is left as an exercise.

De Morgan's Law: From any two sets A and B, prove that

(i) $\boldsymbol{(A \cup B)^c = A^c \cap B^c}$; [C.U. B.Com.(H) 2006; V.U. B.Com.(H) 2008]

(ii) $\boldsymbol{(A \cap B)^c = A^c \cup B^c}$. [C.U. B.Com.(H) 2003; B.U. B.Com.(H) 2007]

Proof. (i) Let $x \in (A \cup B)^c$. Then by definition of complement

$$\begin{aligned}
x \in (A \cup B)^c &\Rightarrow x \notin (A \cup B) \\
&\Rightarrow x \notin A \text{ and } x \notin B \quad \text{[i.e., } x \in \text{ neither to } A \text{ nor to } B.] \\
&\Rightarrow x \in A^c \text{ and } x \in B^c \\
&\Rightarrow x \in (A^c \cap B^c).
\end{aligned}$$

Thus, $x \in (A \cup B)^c \Rightarrow x \in (A^c \cap B^c)$;

$$\therefore (A \cup B)^c \subseteq A^c \cap B^c. \tag{1}$$

Next Let $y \in (A^c \cap B^c)$. Then, by definition,

$$\begin{aligned} y \in (A^c \cap B^c) &\Rightarrow y \in A^c \text{ and } y \in B^c \\ &\Rightarrow y \notin A \text{ and } y \notin B \\ &\Rightarrow y \notin (A \cup B) \\ &\Rightarrow y \in (A \cup B)^c; \end{aligned}$$

$$\therefore A^c \cap B^c \subseteq (A \cup B)^c. \qquad (2)$$

Hence, from (1) and (2), we get $(A \cup B)^c = A^c \cap B^c$.

(ii) Using definition of complement and proceeding as above, we can prove this result.

Alternative proof using (i) is given below:

We have by (i), $(A \cup B)^c = A^c \cap B^c$.

Replacing A by A^c and B by B^c, we get $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$ $[\because (A^c)^c = A]$.

Taking complement of both sides, we get

$$\{(A^c \cup B^c)^c\}^c = (A \cap B)^c \text{ or, } \mathbf{A^c \cup B^c = (A \cap B)^c}.$$

Absorption Law: For any two sets A and B, we have

(i) $A \cup (A \cap B) = A$; (ii) $A \cap (A \cup B) = A$.

Proof is left as an exercise.

Some Important Results based on Laws of Algebra of Sets

Given any three sets A, B, C; prove that

(i) Most helpful result: $\mathbf{A - B = A \cap B'}$, (dash denotes 'complement').
(ii) $(A \cup B) - C = (A - C) \cup (B - C)$.
(iii) $(A \cap B) - C = (A - C) \cap (B - C)$.
(iv) $A - (B \cup C) = (A - B) \cap (A - C)$.
(v) $A - (B \cap C) = (A - B) \cup (A - C)$. [C.U. B.Com.(H) 2002]
(vi) $A \times (B \cup C) = (A \times B) \cup (A \times C)$, '$\times$' denotes Cartesian product. [C.U. B.Com. 2006]
(vii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. [C.U. B.Com.(H) 1988]

The proofs are applications of Laws of Algebra of sets:

Proof. (i) Let $x \in A - B$.

Then $x \in A$ and $x \notin B$, i.e., $x \in A$ and $x \in B'$ or, $x \in A \cap B' \Rightarrow A - B \subseteq A \cap B'$.

Again, let $y \in A \cap B'$.

Then $y \in A$ and $y \in B'$, i.e., $y \in A$ and $y \notin B$, i.e., $y \in A - B \Rightarrow A \cap B' \subseteq A - B$.

Hence, $A - B = A \cup B'$ **(Proved).**

Proof. (ii) $(A \cup B) - C = (A \cup B) \cap C'$ [using (i)]

$= (A \cap C') \cup (B \cap C')$ [Distributive Law]

$= (A - C) \cup (B - C)$ [using (i) again]

Hence (ii) is **proved.**

(iii) $(A \cap B) - C = (A \cap B) \cap C'$ [using (i)]

$= (A \cap B) \cap (C' \cap C')$ [Clearly $C' \cap C' = C'$]

$= \{(A \cap B) \cap C'\} \cap C'$ [Associative Law]

$= \{A \cap (B \cap C')\} \cap C'$ [Associative Law]

$= \{(B \cap C') \cap A\} \cap C'$ [Commutative Law]

$= (B \cap C') \cap (A \cap C')$

$= (A \cap C') \cap (B \cap C')$ [Commutative Law]

$= (A - C) \cap (B - C)$ **(Proved)** [using (i)]

(iv) $A - (B \cup C) = A \cap (B \cup C)'$ [using (i)]

$= A \cap \{B' \cap C'\}$ [De Morgan's Law]

$= (A \cap A) \cap (B' \cap C')$

$= A \cap \{A \cap (B' \cap C')\}$ [Associative Law]

$= A \cap \{(A \cap B') \cap C'\}$

$= A \cap \{C' \cap (A \cap B')\}$ [Commutative Law]

$= (A \cap C') \cap (A \cap B')$

$= (A \cap B') \cap (A \cap C')$ [Commutative Law].

$\therefore\ A - (B \cup C) = (A - B) \cap (A - C)$ **(Proved)** [using (i)].

Otherwise: Let $x \in A - (B \cup C)$.

Then $x \in A$ and $x \notin (B \cup C)$.

$\Rightarrow\ x \in A \wedge \{x \notin B \wedge x \notin C\}$ [$\wedge$ is the symbol for "and"]

$\Rightarrow\ (x \in A \wedge x \notin B) \wedge \{x \in A \wedge x \notin C\}$

$\Rightarrow\ \{x \in A - B\} \wedge x \in (A - C)$

$\Rightarrow\ x \in \{(A - B) \cap (A - C)\}$.

$$\therefore\ A - (B \cup C) \subseteq (A - B) \cap (A - C). \quad (1)$$

Again, let $y \in (A - B) \cap (A - C)$.

Then $y \in (A - B) \wedge y \in (A - C)$.

$\Rightarrow\ \{y \in A \wedge y \notin B\} \wedge \{y \in A \wedge y \notin C\}$

$\Rightarrow\ \{y \in A\} \wedge y \notin (B \cup C)$

$\Rightarrow\ y \in A - (B \cup C)$.

$$\therefore\ (A - B) \cap (A - C) \subseteq A - (B \cup C). \quad (2)$$

$\therefore$ from (1) and (2), we get $A - (B \cup C) = (A - B) \cap (A - C)$, **(Proved)**.

(v) $A - (B \cap C) = A \cap (B \cap C)'$ [using (i)]

$= A \cap (B' \cup C')$ [De Morgan's Law]

$= (A \cap B') \cup (A \cap C')$ [Distributive Law]

$= (A - B) \cup (A - C)$ [using (i)].

(vi) Let $(x,y)\in A\times(B\cup C)$.
Then $x\in A\wedge y\in(B\cup C)$.

$$\begin{aligned}
&\Rightarrow\ x\in A\wedge\{y\in B\vee y\in C\}\ [\vee \text{ stands for "or"}]\\
&\Rightarrow\ \{x\in A\wedge y\in B\}\vee\{x\in A\wedge y\in C\}\\
&\Rightarrow\ \{(x,y)\in A\times B\}\vee\{(x,y)\in A\times C\}\\
&\Rightarrow\ (x,y)\in(A\times B)\cup(A\times C). \qquad (1)
\end{aligned}$$

Exactly in a similar way:
Let $(p,q)\in(A\times B)\cup(A\times C)$.

$$\Rightarrow (p,q)\in A\times(B\cup C). \qquad (2)$$

Hence, equations (1) and (2) together $\Rightarrow$ $A\times(B\cup C)=(A\times B)\cup(A\times C)$ **(Proved).**
(vii) Let $(x,y)\in A\times(B\cap C)$.
Then $x\in A\wedge y\in(B\cap C)$.

$$\begin{aligned}
&\Rightarrow\ x\in A\wedge\{y\in B\wedge y\in C\}\\
&\Rightarrow\ \{x\in A\wedge y\in B\}\wedge\{x\in Ay\in C\}\\
&\Rightarrow\ (x,y)\in(A\times B)\wedge(x,y)\in(A\times C)\\
&\Rightarrow\ (x,y)\in(A\times B)\cap(A\times C).
\end{aligned}$$

$$\therefore\ A\times(B\cap C)\subseteq(A\times B)\cap(A\times C) \qquad (1)$$

Again, let $(p,q)\in(A\times B)\cap(A\times C)$
Then $(p,q)\in(A\times B)\wedge(p,q)\in(A\times C)$

$$\begin{aligned}
&\Rightarrow\ \{p\in A\wedge q\in B\}\wedge\{p\in A\wedge q\in C\}\\
&\Rightarrow\ p\in A\wedge q\in(B\cap C)\\
&\Rightarrow\ (p,q)\in A\times(B\cap C).
\end{aligned}$$

$$\therefore\ (A\times B)\cap(A\times C)\subseteq A\times(B\cap C) \qquad (2)$$

$\therefore$ equations (1) and (2) together $\Rightarrow$ $A\times(B\cap C)=(A\times B)\cap(A\times C)$ **(Proved).**

8.9 Illustrative Examples (Continuation of Section 8.5)

Example 12. *If A and B are two given sets, then show that $A\cap(B-A)=\phi$.*

Solution: If possible, let $A\cap(B-A)\neq\phi$, where ϕ is the null set and A, B are not null sets. Then there is at least one element, say x, such that $x\in A\cap(B-A)$.

$$\begin{aligned}
\therefore\ x\in A\cap(B-A)\ &\Rightarrow\ x\in A \text{ and } x\in(B-A)\\
&\Rightarrow\ x\in A \text{ and } (x\in B \text{ and } x\notin A)\\
&\Rightarrow\ x\in A \text{ and } x\in B \text{ and } x\notin A,
\end{aligned}$$

which is absurd, since $x\in A$ and $x\notin A$ cannot hold simultaneously.

Hence, $A \cap (B - A) = \phi$.

If A, B are null sets, the result is obvious.

Otherwise:
$$\begin{aligned} A \cap (B - A) &= A \cap (B \cap A^c) \\ &= A \cap (A^c \cap B) = (A \cap A^c) \cap B \text{ [by Associative Law]} \\ &= \phi \cap B = \phi \quad \text{(by Complement and Identity Laws)} \end{aligned}$$

Example 13. *If $U = \{1,2,3,4,5,6\}$ be the universal set and A, B, C are three subsets of U, where $A = \{1,3,4\}$ and $B \cup C = \{1,3,5,6\}$, find* (i) $(A \cap B) \cup (A \cap C)$ *and* (ii) $B' \cap C'$. [C.U. B.Com.(H) 2007 Type]

Solution: (i) By Distributive Law, we get

$$(A \cap B) \cup (A \cap C) = A \cap (B \cup C) = \{1,3,4\} \cap \{1,3,5,6\} = \mathbf{\{1,3\}}.$$

(ii) By De Morgan's Law, we get

$$B' \cap C' = (B \cup C)' = U - (B \cup C) = \{1,2,3,4,5,6\} - \{1,3,5,6\} = \mathbf{\{2,4\}}.$$

Example 14. *Three sets A, B, C be such that $A - B = \{2,4,6\}$, $A - C = \{2,3,5\}$; find $A - (B \cup C)$ and $A - (B \cap C)$.* [C.U. B.Com.(H) 1997]

Solution: By Algebra of sets, we have

$$A - (B \cup C) = (A - B) \cap (A - C) = \{2,4,6\} \cap \{2,3,5\} = \mathbf{\{2\}}$$
$$\text{and} \quad A - (B \cap C) = (A - B) \cup (A - C) = \{2,4,6\} \cup \{2,3,5\} = \mathbf{\{2,3,4,5,6\}}.$$

8.10 Counting the Number of Elements in a Finite Set (or Cardinal Number)

We shall denote the number of elements (or members) in any finite set A by the symbol $\boldsymbol{n(A)}$. *For example,* if $A = \{a, e, i, o, u\}$, then $n(A) = 5$.

Example 15. *With the help of Venn diagram, prove that*

$$\boldsymbol{n(A \cup B) = n(A) + n(B) - n(A \cap B).}$$

Hence (or otherwise) prove that

$$\boldsymbol{n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).}$$

Solution:

Case I. *If A and B are disjoint sets, then $A \cap B = \phi$.*

$$\therefore \; n(A \cap B) = n(\phi) = 0.$$

Clearly, then

$$n(A \cup B) = n(A) + n(B) = n(A) + n(B) - n(A \cap B).$$

Case II. *If A and B are not disjoint sets, then they have some elements common to them. From the Venn diagram, we see that while counting the elements of A and those of B, the elements of $A \cap B$ (i.e., the common elements) are counted twice.*

$$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

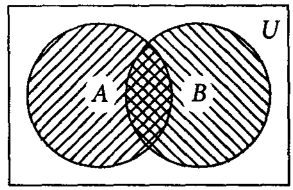


Fig. 8.36

Hence, for any two sets A and B, we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B). \qquad (1)$$

Otherwise: From the Venn diagram, we see that $A \cup B$ is the Union of three mutually disjoint sets $A - B$, $A \cap B$ and $B - A$.

$$\therefore\ n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A).$$

If $n(A) = x$, $n(B) = y$ and $n(A \cap B) = z$, then $n(A - B) = x - z$, $n(B - A) = y - z$.

$$\therefore\ n(A \cup B) = x - z + z + y - z = x + y - z = n(A) + n(B) - n(A \cap B).$$

2nd Part: $n(A \cup B \cup C) = n\{A \cup (B \cup C)\} = n(A \cup D)$, where $D = B \cup C$

$$\begin{aligned}
&= n(A) + n(D) - n(A \cap D) \text{ [using equations (1)]}\\
&= n(A) + n(B \cup C) - n\{A \cap (B \cup C)\}\\
&= n(A) + n(B) + n(C) - n(B \cap C) - n\{(A \cap B) \cup (A \cap C)\}\\
&= n(A) + n(B) + n(C) - n(B \cap C) - [n(A \cap B) + n(A \cap C)\\
&\qquad\qquad - n\{(A \cap B) \cap (A \cap C)\}]\\
&= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C)\\
&\qquad\qquad + n\{(A \cap B) \cap (A \cap C)\}.
\end{aligned}$$

With the help of Venn diagram, we see that $(A \cap B) \cap (A \cap C)$ represents the cross-shaded area which is common to all the three sets A, B and C.

$$\therefore\ (A \cap B) \cap (A \cap C) = A \cap B \cap C.$$

$$\therefore\ n[(A \cap B) \cap (A \cap C) = n(A \cap B \cap C).$$

Hence, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

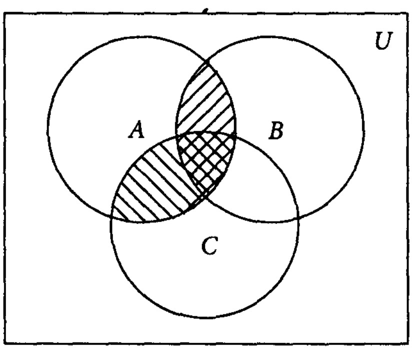


Fig. 8.37(a)

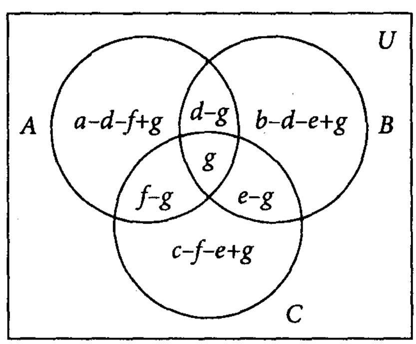


Fig. 8.37(b)

[**Otherwise:** Let $n(A)=a$, $n(B)=b$, $n(C)=c$, $n(A\cap B)=d$, $n(B\cap C)=e$, $n(A\cap C)=f$ and $n(A\cap B\cap C)=g$. Then from the Venn diagram, we have n (only A) $=a-d-f+g$, n (only B) $=b-d-e+g$ and n (only C) $=c-f-e+g$. [See Fig. 8.37(b).]

$$\begin{aligned}\therefore\ n(A\cup B\cup C)&=(a-d-f+g)+(b-d-e+g)+(c-f-e+g)+(d-g)+g+(f-g)+(e-g)\\&=a+b+c-d-e-f+g\\&=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C).]\end{aligned}$$

Example 16. *In a city, three daily newspapers A, B, C are published. 42% of the people on that city read A, 51% read B, 68% read C; 30% read A and B; 28% read B and C; 36% read A and C; 8% do not read any of the three newspapers. Find the percentage of persons who read all the three papers, using the result of Ex.* 1.

[B.U. B.Com.(H) 2008 Type]

Solution: Let the no. of persons in the city = 100. Then we have $n(A)=42$, $n(B)=51$, $n(C)=68$; $n(A\cap B)=30$, $n(B\cap C)=28$, $n(A\cap C)=36$, $n(A\cup B\cup C)=100-8=92$.

By the result of Ex. 1, we get

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C).$$

Substituting the above values, we have $92=42+51+68-30-28-36+n(A\cap B\cap C)$, or, $92=161-94+n(A\cap B\cap C)=67+n(A\cap B\cap C)$.

$\therefore\ n(A\cap B\cap C)=92-67=25$.

Hence, **25% of the people read all the three papers.**

Example 17. *In an Engineering College,* 80 *students get chance for Computer Science,* 75 *for Information Technology,* 72 *for Electronics. If* 60 *students get chance in 1st and 2nd,* 50 *in* 2nd *and* 3rd, 40 *in* 1st *and* 3rd *and* 30 *get chance in all* 3 *branches, how many seats are then in the Engineering College? (The College has only* 3 *disciplines.)*

[C.U. B.Com.(H) 2001]

Solution: Let A be the set of students who get chance in Computer Science. Similarly, B and C are the sets of students getting chance in Information Technology and Electronics. There $n(A)=80$, $n(B)=75$, $n(C)=72$, $n(A\cap B)=60$, $n(B\cap C)=50$, $n(A\cap C)=40$, $n(A\cap B\cap C)=30$.

$$\begin{aligned}\therefore\ n(A\cup B\cup C)&=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C)\\&=80+75+72-60-50-40+30=107.\end{aligned}$$

Since the college has only 3 disciplines, the required number of seats in the Engineering College = the no. of the students who get chance in at least one of Computer Science, Information Technology and Electronics = **107.**

8.11 Miscellaneous Examples

Example 18. *Let* $A=\{a,b,c\}$, $B=\{a,b\}$, $C=\{a,b,d\}$, $D=\{c,d\}$ *and* $E=\{d\}$. *State which of the following statements are correct and give reasons:* (i) $B\subset A$, (ii) $D\not\supset E$, (iii) $D\subset B$, (iv) $\{a\}\subset A$.

Solution: We have $A=\{a,b,c\}$, $B=\{a,b\}$, $C=\{a,b,d\}$, $D=\{c,d\}$, $E=\{d\}$.

(i) The statement $B\subset A$ is correct; because all the elements of B belong to A.

(ii) The statement $D\not\supset E$ is not correct; because the only element d of E belongs to D and hence $D\supset E$.

(iii) The statement $D \subset B$ is not correct; because no element of D belongs to B. Clearly, B and D are two disjoint sets.

(iv) The statement $\{a\} \subset A$ is correct; since $a \in A$, the set $\{a\}$ is a proper subset of A.

Example 19. *Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ be the universal set. Suppose $A = \{1,2,3, 4,5,6\}$ and $B = \{5,6,7\}$ are its two subsets. Write down the elements of $A - B$ and $A \cap B'$. [Here B' is the complement of B.]*

Solution: We have,

$$A - B = \{1,2,3,4,5,6\} - \{5,6,7\} = \mathbf{\{1,2,3,4\}}.$$

$$B' = U - B = \{1,2,3,4,5,6,7,8,9,10\} - \{5,6,7\} = \{1,2,3,4,8,9,10\}.$$

$$\therefore\ A \cap B' = \{1,2,3,4,5,6\} \cap \{1,2,3,4,8,9,10\} = \mathbf{\{1,2,3,4\}}.$$

Note: $A - B = A \cap B'$.

Example 20. (i) *If a set A has 4 elements and a set B has 6 elements, what can be the minimum number of elements in the set $A \cup B$? What conclusion can then be drawn about A and B?* [C.U. B.Com.(H) 1995]

(ii) *If $A = \{x : x$ is an integer and $1 \le x \le 10\}$ and $B = \{x : x$ is a multiple of 3 and $5 \le x \le 30\}$, find $A \cup B$, $A \cap B$, $A - B$, $B - A$.* [C.U. B.Com.(H) 1996]

Solution: (i) We know that $A \subset A \cup B$ and $B \subset A \cup B$. Given $n(A) = 4$ and $n(B) = 6$.

$\therefore$ the number of elements in $A \cup B$ will be minimum when $A \subset B$, i.e., when A is a proper subset of B. In this case, $n(A \cup B) = n(B) = 6$.

(ii) $A = \{x : x$ is an integer and $1 \le x \le 10\} = \{1,2,3,4,5,6,7,8,9,10\}$ and

$B = \{x : x$ is a multiple of 3 and $5 \le x \le 30\} = \{6,9,12,15,18,21,24,27,30\}$.

$$\begin{aligned}
\therefore\ A \cup B &= \{1,2,3,4,5,6,7,8,9,10\} \cup \{6,9,12,15,18,21,24,27,30\} \\
&= \mathbf{\{1,2,3,4,5,6,7,8,9,10,12,15,18,21,24,27,30\}}. \\
A \cap B &= \mathbf{\{6,9\}}, \\
A - B &= \{x : x \in A \wedge x \notin B\} = \mathbf{\{1,2,3,4,5,7,8,10\}} \\
\text{and}\quad B - A &= \{x : x \in B \wedge x \notin A\} = \mathbf{\{12,15,18,21,24,27,30\}}.
\end{aligned}$$

Example 21. *If $S = \{1,2,3,4,5,6\}$ be the universal set and A, B, C be three subsets of S, where $A = \{1,3,4,6\}$ and $B \cap C = \{1,2,6\}$, find $(A \cup B) \cap (A \cup C)$ and $B' \cup C'$.*

Solution: By Distributive Law of sets, we have

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C) = \{1,3,4,6\} \cup \{1,2,6\} = \mathbf{\{1,2,3,4,6\}}.$$

By De Morgan's Law, we have

$$B' \cup C' = (B \cap C)' = S - (B \cap C) = \{1,2,3,4,5,6\} - \{1,2,6\} = \mathbf{\{3,4,5\}}.$$

Example 22. *If $A = \{1,2,3,4\}$, $B = \{2,3,5,6\}$ and $C = \{3,4,6,7\}$, find $A - B$, $A - C$ and verify that $A - (B \cap C) = (A - B) \cup (A - C)$ and $A - (B \cup C) = (A - B) \cap (A - C)$.*

Solution: We have $A - B = \{1,2,3,4\} - \{2,3,5,6\} = \{1,4\}$ and

$$A - C = \{1,2,3,4\} - \{3,4,6,7\} = \{1,2\}. \ B \cap C = \{2,3,5,6\} \cap \{3,4,6,7\} = \{3,6\}.$$

$\therefore A - (B \cap C) = \{1,2,3,4\} - \{3,6\} = \{1,2,4\}$ and

$$(A - B) \cup (A - C) = \{1,4\} \cup \{1,2\} = \{1,2,4\}.$$

Hence, $\boldsymbol{A - (B \cap C) = (A - B) \cup (A - C)}$.

Again, $B \cup C = \{2,3,5,6\} \cup \{3,4,6,7\} = \{2,3,4,5,6,7\}$.

$\therefore A - (B \cup C) = \{1,2,3,4\} - \{2,3,4,5,6,7\} = \{1\}$

and $(A - B) \cap (A - C) = \{1,4\} \cap \{1,2\} = \{1\}$.

Hence, $\boldsymbol{A - (B \cap C) = (A - B) \cap (A - C)}$.

Example 23. *List the sets A, B and C, given that* $A \cup B = \{p,q,r,s\}$; $A \cup C = \{q,r,s,t\}$, $A \cap B = \{q,r\}$ *and* $A \cap C = \{q,s\}$. [C.U. B.Com. 2008]

Solution: We have

$$A \cup B = \{p,q,r,s\} \qquad (1)$$

$$A \cup C = \{q,r,s,t\} \qquad (2)$$

$$A \cap B = \{q,r\} \qquad (3)$$

$$A \cap C = \{q,s\}. \qquad (4)$$

From (3) and (4), we see that $q, r, s \in A; q, r \in B$ and $q, s \in C$.

From equations (1) and (2), we see that $p \notin A$ and $t \notin A$. $\therefore$ $p \in B$ and $t \in C$.

Hence, $\boldsymbol{A = \{q,r,s\}}$, $\boldsymbol{B = \{p,q,r\}}$ and $\boldsymbol{C = \{q,s,t\}}$.

Example 24. *Using Set Operations, find the HCF of the three numbers* 15, 40, 105. [V.U. B.Com.(H) 2010 Type]

Solution: If A, B, C be the sets of all possible factors of 15, 40 and 105 respectively, then $A = [1,3,5,15]$, $B = [1,2,4,5,8,10,20,40]$ and $C = [1,3,5,7,15,21,35,105]$.

Clearly, the required HCF is the greatest element of the set $A \cap B \cap C$.

Now, $A \cap B \cap C = \{1,5\}$.

Hence, the required HCF = the greatest element in $A \cap B \cap C = 5$.

Otherwise. If A, B, C be the sets of prime factors of 15, 40, 105 respectively, then $A = \{3,5\}$, $B = \{2_a, 2_b, 2_c, 5\}$, $C = \{3,5,7\}$, and $A \cap B \cap C = \{5\}$.

HCF = product of the factors of $A \cap B \cap C = 5$.

Example 25. *Using Set Operations, find the LCM (i.e., elements) of the three numbers* 12, 15, 20. [C.U. B.Com. 2006 Type]

Solution: Let A, B, C be the sets of all positive integral multiples of 12, 15, 20 respectively. Then, we have

$$A = \{12,24,36,48,60,72,84,96,108,120,\ldots\}$$

$$B = \{15,30,45,60,75,90,105,120,\ldots\}$$

$$\text{and} \quad C = \{20,40,60,80,100,120,140,\ldots\}.$$

Clearly, the required LCM is the least element common to all the three sets A, B, C, i.e., it is the least element of $A \cap B \cap C$.

Now, $A \cap B \cap C = \{60, 120, \ldots\}$.

Hence, the required LCM = The least element in $A \cap B \cap C = \mathbf{60}$.

Otherwise: If A, B, C be the set of prime factors of 12, 15, 20 respectively, then since $12 = 2^2 \cdot 3$, $15 = 3.5$, $20 = 2^2 \cdot 5$, we have $A = \{2_a, 2_b, 3\}$, $B = \{3, 5\}$, $C = \{2_a, 2_b, 5\}$. Since 2 occurs twice in A and C, we represent them as 2_a and 2_b.

$\therefore$ $A \cup B \cup C = (2_a, 2_b, 3, 5)$. The required LCM = product of the factors of $A \cup B \cup C = 2^2 \times 3 \times 5 = \mathbf{60}$.

Example 26. *Use a Venn diagram to solve the following problem. In a statistical investigation of* 1003 *families of Kolkata, it was found that* 63 *families had neither a radio nor a* TV, 794 *families had a radio and* 187 *a* TV. *How many families in that group had both a radio and a* TV? [C.U. B.Com.(H) 1999 Type]

Solution: Let R be the set of families having a radio and T the set of families having a TV. Then $n(R \cup T)$ = the no. of families having at least one of radio and TV = $1003 - 63 = 940$, $n(R) = 794$ and $n(T) = 187$.

Let $n(R \cap T) = x$, i.e., x families had both a radio and a TV. Then the no. of families who have only Radio $= 794 - x$ and the no. of families who have only TV $= 187 - x$.

$\therefore$ from Venn diagram, $794 - x + x - 187 - x = 940$ or, $981 - x = 940$ or, $x = 41$.

Hence, the required no. of families having both a radio and a TV = **41**.

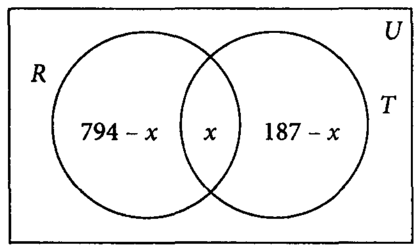


Fig. 8.38

Example 27. *In a class of* 42 *students, each play at least one of the three games — cricket, hockey and football. It is found that* 14 *play cricket,* 20 *play hockey and* 24 *play football;* 3 *play both cricket and football,* 2 *play both hockey and football, and none plays all the three games. Find the number of students who play cricket but not hockey.*

Solution:

First Method: (Using Venn Diagram): Let C be the set of students who play cricket. Similarly, the sets H and F for the students playing hockey and football respectively. Then $n(C \cup H \cup F) = 42$, $n(C) = 14$, $n(H) = 20$, $n(F) = 24$, $n(C \cap F) = 3$, $n(H \cap F) = 2$, $n(C \cap H \cap F) = 0$.

Let x be the no. of students who play cricket and hockey, i.e., $n(C \cap H) = x$.

Then the no. of students who play only cricket $= 14 - (x + 3) = 11 - x$.

Then the no. of students who play only hockey $= 20 - (x + 2) = 18 - x$.

Then the no. of students who play only football $= 24 - (3 + 2) = 19$.

$\therefore$ from the Venn diagram, we have

$$11 - x + x + 18 - x + 2 + 3 + 19 = 42 \text{ or, } 53 - x = 42 \text{ or, } x = 53 - 42 = 11.$$

$\therefore$ the required no. of students who play cricket but not hockey $= 14 - x = 14 - 11 = 3$.

Second Method: (Using Formula): We have

$n(C \cup H \cup F) = n(C) + n(H) + n(F) - n(C \cap H) - n(H \cap F) - n(C \cap F) + n(C \cap H \cap F)$
or, $42 = 14 + 20 + 24 - x - 2 - 3 + 0$ or, $x = 53 - 42 = 11$.

Again, $n(C) = n(C \cap H) + n(C - H)$ or, $14 = 11 + n(C - H)$ or, $n(C - H) = 3$.

Hence, the required no. of students who play cricket but not hockey $= n(C \cap H') = n(C - H) = 3$.

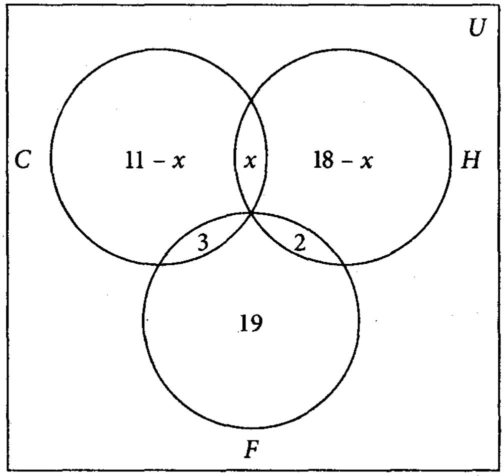


Fig. 8.39

Example 28. *In a survey of* 100 *families, the number of families that read the most recent issues of various magazines were found to be: India Today* 42, *Sunday* 30, *New Delhi* 28, *India Today and Sunday* 10, *India Today and New Delhi* 5, *New Delhi and Sunday* 8; *all the three magazines* 3. *Find the number of families that read none of the magazines. Find also the number of families who read only one magazine.* [C.U. B.Com.(H) 2000]

Solution: Let A be the set of families who read India Today. Similarly, the sets B and C for the families reading Sunday and New Delhi respectively. Then $n(A) = 42$, $n(B) = 30$, $n(C) = 28$, $n(A \cap B) = 10$, $n(A \cap C) = 5$, $n(B \cap C) = 8$, $n(A \cap B \cap C) = 3$.

From the Venn diagram, we see that n (only A) $= 42 - (7 + 3 + 2) = 30$,

n (only B) $= 30 - (7 + 3 + 5) = 15$.

n (only C) $= 28 - (2 + 3 + 5) = 18$.

$\therefore$ $n(A \cup B \cup C) = 30 + 15 + 18 + 7 + 3 + 2 + 5 = 80$.

If U be the set of all 100 families, then $n(U) = 100$.

Hence, required no. of families that read none of the magazines

$= n(U) - n(A \cup B \cup C) = 100 - 80 = \mathbf{20}$;

the no. of families who read only one magazine

$= 30 + 15 + 18 = \mathbf{63}$.

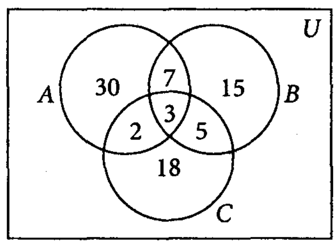


Fig. 8.40

Example 29. *In a survey of* 1000 *families in a city, it was found that* 458 *families had TV sets,* 620 *had Radiograms,* 600 *had Tape Recorders;* 294 *families had both TV set and Radiogram,* 277 *had both Radiogram and Tape Recorder,* 190 *both Tape Recorder and TV set. How many families had all the three appliances? How many had only a Radiogram? (Assume that each family had at least one of the three appliances.)*

Solution: Let A = set of families having a TV set, B = set of families having a Radiogram, C = set of families having a Tape Recorder. Then $n[A \cup B \cup C) = 1000$, $n(A) = 458$, $n(B) = 620$, $n(C) = 600$, $n(A \cap B) = 294$, $n(B \cap C) = 277$, $n(A \cap C) = 190$, $n(A \cap B \cap C) = $?.

Substituting these values in $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$, we get $n(A \cap B \cap C) = 83$, i.e., 83 families had all the three appliances.

2nd Part: If $n(A \cap B \cap C) = X$, from the adjoining Venn diagram Fig. 8.41, the no. of families who had only a Radiogram $= n(B) - (294 - X + X + 277 - X) = 620 - (571 - X) = X + 49 = 83 + 49 = 132$.

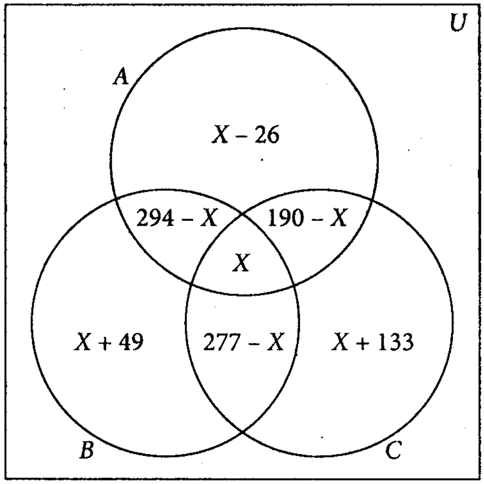


Fig. 8.41

Otherwise. We have to find $n(B \cap A' \cap C')$.

Clearly, $n(B) = n(B \cap A \cap C) + n(B \cap A' \cap C) + n(B \cap A \cap C') + n(B \cap A' \cap C')$. (1)

Now, $n(B \cap C) = n(B \cap A \cap C) + n(B \cap A' \cap C)$ gives $n(B \cap A' \cap C) = 277 - 83 = 194$ and $n(B \cap A) = n(B \cap A \cap C) + n(B \cap A \cap C')$ gives $n(B \cap A \cap C') = 211$.

$\therefore$ from equations (1), $620 = 83 + 194 + 211 + n(B \cap A' \cap C')$

$$\text{or, } n(B \cap A' \cap C') = 620 - 488 = 132.$$

Example 30. *Prove that* $(A - C) \cap (B - C) = (A \cap B) - C$.

Solution: Let $x \in (A - C) \cap (B - C)$; then

$$\begin{aligned} x \in (A - C) \cap (B - C) &\Rightarrow x \in (A - C) \text{ and } x \in (B - C) \\ &\Rightarrow (x \in A \text{ but } x \notin C) \text{ and } (x \in B \text{ but } x \notin C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ but } x \notin C \\ &\Rightarrow x \in (A \cap B) \text{ but } x \notin C \\ &\Rightarrow x \in (A \cap B) - C. \end{aligned}$$

$$\therefore (A - C) \cap (B - C) \subseteq (A \cap B) - C. \tag{1}$$

Next let $y \in (A \cap B) - C$; then

$$\begin{aligned} y \in (A \cap B) - C \;&\Rightarrow\; y \in A \cap B \text{ but } y \notin C \\ &\Rightarrow\; (y \in A \text{ and } y \in B) \text{ but } y \notin C \\ &\Rightarrow\; (y \in A \text{ but } y \notin C) \text{ and } (y \in B \text{ but } y \notin C) \\ &\Rightarrow\; y \in (A - C) \text{ and } y \in (B - C) \\ &\Rightarrow\; y \in (A - C) \cap (B - C) \end{aligned}$$

$$\therefore (A \cap B) - C \subseteq (A - C) \cap (B - C). \qquad (2)$$

Hence, from equations (1) and (2), we get $(A - C) \cap (B - C) = (A \cap B) - C$ **(Proved).**

EXERCISES ON CHAPTER 8

Theory

1. Define the terms: Set; finite and infinite set; null set; proper subset; complement at a set; equal sets and disjoint sets. Give one example for each. **[V.U. B.Com.(H) 2010]**

2. Define subsets and proper subsets. Give an example for each.

3. Draw a Venn diagram of three non-empty sets A, B and C so that A, B and C have the property:

   (a) $A \subset B$, $C \not\subset B$, $A \cap C = \phi$, where ϕ is the null set;

   (b) $A \subset B$, $B \cap C \neq \phi$, $A \cap C = \phi$, $C \not\subset B$.

4. Define the following:

   (a) Intersection of two sets,

   (b) Union of two sets,

   (c) Difference of two sets,

   (d) Complement of a set.

   Give one example of each. **[C.U. B.Com.(H) 1991; V.U. B.Com.(H) 2010]**

5. When are two sets A and B said to be disjoint sets? Define the complement of a set B.

   [C.U. B.Com.(H) 1991]

6. Define Cartesian product of two sets. Give an example.

7. Explain the following terms:

   (a) Universal set,

   (b) Power set, **[V.U. B.Com.(H) 2010]**

   (c) Venn diagram.

8. State the Laws of algebra of sets.

9. Define power set of a set A. Find the power set of $A = \{a, b, c\}$. If B be the power set of A, state with reasons which of the following statements is correct: $A \supset B$, $A \in B$, $A \subset B$, $A = B$, $A \not\subset B$.

   [C.U. B.Com.(H) 1991]

10. If X' and Y' be subsets of X and Y respectively and $X \subset Y$, then prove that $Y' \subset X'$.

11. State De Morgan's laws of sets. Verify the laws in terms of Venn diagram. [C.U. B.Com.(H) 1992]

12. For any three sets A, B, C, prove that $A \cap (B - C) = (A \cup B) - (A \cap C)$. [B.U. B.Com.(H) 2006]

EXERCISES ON CHAPTER 8(I)

Problems: Sections A–D

A

1. Express the following set notations in words:

(a) $x \notin A$, (b) $A \subseteq B$, (c) $A \subset B$, (d) $A \not\subset B$.

2. (a) Let $A = [2, 3, 4, 5]$. State which of the following statements are correct. Give reasons for your answer:

(i) $4 \in A$;
(ii) $\{3, 5\} \subset A$;
(iii) $\{5, 2, 4\} \subset A$;
(iv) $\{2, 3, 4, 5\} \subsetneqq A$;
(v) $\{1, 3, 5\} \subset A$.

(b) Which of the following statements are correct/incorrect?

(i) $3 \subseteq \{1, 3, 5\}$;
(ii) $3 \in \{1, 3, 5\}$;
(iii) $\{3\} \subseteq \{1, 3, 5,\}$;
(iv) $\{3\} \in \{1, 3, 5\}$.

3. (a) Which of the following sets are finite? Which are infinite?

(i) $\{1, 3, 5, 8, 9\}$;
(ii) The rivers in India;
(iii) The set of even positive integers;
(iv) $\{x : x \text{ is divisible by } 3\}$;
(v) $\{2, 5, 8, \ldots, 152\}$.

(b) Write down all the subsets of $A = [1, 2, 3]$.

(c) Which of the following sets is the null set ϕ? Briefly say why.

(i) $A = \{x : x \text{ is} > 1 \text{ and } x \text{ is} < 1\}$;
(ii) $B = \{x : x + 3 = 3\}$;
(iii) $C = \{\phi\}$.

4. (a) If $A = [a, b, c]$, name

(i) the subsets of A; [V.U. B.Com.(H) 2007]
(ii) the proper subsets of A.

(b) If $a \in A$ and $a \in B$, does it follow that $A \subseteq B$? Give reasons.

(c) Show the relationships among the following three sets in respect of subsets and supersets.

(i) $N = \{x : x \text{ is a positive integer}\}$;
(ii) $Z = \{x : x \text{ is an integer}\}$;
(iii) $R = \{x : x \text{ is a real number}\}$.

5. If $A=\{1,3,5,7\}$; $B=\{2,3,4,5\}$; $C=\{2,4,6,8\}$; $D=\{4,5,6\}$; write down:
   (a) $A \cap B$; (b) $C \cup D$; (c) $B \cup (C \cup D)$; (d) $A \cap (B \cup C)$.

6. If $A=\{1,2,3,4\}$; $B=\{2,4,6,8\}$; $C=\{3,4,5,6\}$; prove that
   (a) $A \cap (B \cap C)=(A \cap B) \cap C$; (b) $A \cup (B \cup C)=(A \cup B) \cup C$.

7. (a) If $A=\{1,2,3,4\}$; $B=\{3,4,5,6\}$; $C=\{1,5,6,7,8\}$; verify that
$$A \cup (B \cap C)=(A \cup B) \cap (A \cup C).$$
   (b) If $U=\{1,2,3,4,5,6,7,8,9\}$ and $A=\{2,4,6,8\}$, find A and hence verify that $A \cap A'=\phi$.
[B.U. B.Com.(H) 2002]

8. (a) If $A=\{1,2,3,4\}$, $B=\{2,4,5,6\}$, $C=\{1,3,4,6,8\}$, verify that
$$A \cap (B \cup C)=(A \cap B) \cup (A \cap C).$$
   (b) If $A=\{1,2,3\}$, $B=\{1,2,3,4\}$ and $C=\{3,4,5,6,7\}$, prove that $A \cup B \subset (A \cap C) \cap (B \cup C)$.
[B.U. B.Com.(H) 2002]

9. (a) If $A=\{2,3,8\}$ and $B=\{6,4\}$, then find $A \times B$. [C.U. B.Com. 2005]
   (b) If $A=\{1,2,3\}$, $B=\{4,5\}$, find $A \times B$ and $B \times A$ and hence prove that $A \times B \neq B \times A$.

10. If $U=\{a,e,i,o,u\}$, $A=\{a,e,u\}$ and $B=\{e,o,u\}$, prove that
   (a) $(A \cup B)^c=A^c \cap B^c$, (b) $(A \cap B)^c=A^c \cup B^c$.

11. State which of the following are null sets:
   (a) $\{x: 3x^2-4=0, x \text{ is an integer}\}$;
   (b) $\{x: (x+3)(x+3)=9, x \text{ is a real number}\}$;
   (c) $(A \cap B)-A$.

12. Let $S=\{1,2,3,4,5,6\}$ be the universal set. Let $A \cup B=\{2,3,4\}$; find $A^c \cap B^c$, where A^c, B^c, are complements of A and B respectively. Also show that $A \cup B$ and $A^c \cap B^c$ are disjoint sets. [C.U. B.Com.(H) 2007]

13. If $A=\{1,2,7\}$ and $B=\{3,5,7\}$ are the subset of the universal set
$$S=\{1,2,3,4,5,6,7,8,9,10\},$$
then show that $(A \cup B)'=A' \cap B'$. [C.U. B.Com. 2005]

14. (a) Find the power set of $\{2,3,5\}$. [C.U. B.Com. 2005]
   (b) Given $A=\{1,2,3\}$ and $B=\{4,6,8\}$. Find $A \times B$. [C.U. B.Com. 2005]

15. Find $A \cap B$ if $A=$ {letters of ASSASSINATION} and $B=$ {letters of POSSESSION}.

16. If $A=\{1,2,3,4,5\}$, $B=\{2,3,5,6,7\}$, $C=\{2,4,7,8,9\}$, show that $A \cup B=B \cup A$ and $A \cup (B \cup C)=(A \cup B) \cup C$. [N.B.U. B.Com.(H) 2007]

17. If $A=\{x : x$ is a natural number and $x \leq 6\}$, $B=\{x : x$ is the natural number and $3 \leq x \leq 8\}$, find $A-B$ and $A \cap B$. [B.U. B.Com.(H) 2005]

18. Let $S=\{a,b,c,d,e\}$ be the universal set and let $A=\{a,b,d\}$ and $B=\{b,d,e\}$ be two of its subsets. Find $(A \cap B)'$ and $(A \cup B)'$, where dash denotes compliment.

B

1. If $A=\{a,b,c,d,e\}$, $B=\{a,c,e,g\}$ and $C=\{b,e,f,g\}$, verify that
(a) $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$;
(b) $(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$.

2. If $A=\{1,4\}$; $B=\{4,5\}$; $C=\{5,7\}$; find
(a) $(A\times B)\cup(A\times C)$; (b) $(A\times B)\cap(A\times C)$.

3. (a) If $A=\{1,4\}$, $B=\{4,5\}$, $C=\{5,7\}$, prove that $A\times(B\cap C)=(A\times B)\cap(A\times C)$. [V.U. B.Com.(H) 2007]
(b) If $A=\{1,2\}$ and $B=\{1,2,3\}$, find $(A\times B)\cap(B\times A)$. [V.U. B.Com.(H) 2007]

[**Hints:** $A\times B=\{1,2\}\times\{1,2,3\}=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$ and $B\times A=\{1,2,3\}\times\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$; $\therefore$ $(B\times B)\cap(B\times A)=\{(1,1),(1,2),(2,1),(2,2)\}$.]

4. If $A=\{1,4\}$, $B=\{4,3\}$, $C=\{3,6\}$, prove that $A\times(B\cup C)=(A\times B)\cup(A\times C)$.

5. If $A=\{1,2\}$, $B=\{2,3\}$, $C=\{1,3\}$, $D=\{3,5\}$, prove that

$$(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D).$$

6. (a) Let $U=1,2,3,\ldots,8\}$, $A=\{1,2,3,4\}$, $B=\{2,4,5,7\}$ and $C=\{2,4,6,8\}$. Find
(i) B^c; (ii) $(A\cap B)^c$; (iii) $A^c\cup B^c$
and hence show that $(A\cap B)^c=A^c\cup B^c$.
(b) Determine two sets P and Q such that $P\cup Q$ is a universal set and $P\cap Q$ is a null set.
(c) If A and B are respectively the solution sets of the two equations

$$x^2-5x+6=0$$
$$\text{and}\quad x^2-6x+8=0,$$

then show that $A\cap B\subseteq A\cup B$.

[**Hints:** $A\times B=\{12\}\times\{1,2,3\}=\{(1,1),(1,2),(1,3),(2,1),(2,2)\}$ and $B\times A=\{1,2,3\}\times\{1,2\}=\{(1,1),(1,2)\},\{(3,1),(2,2),(3,1)\}$; $\therefore$ $\{(B\times B)\cap(B\times A)=\{(1,1),(1,2),(2,1),(2,2)\}$.]

7. If $A=\{1,2,3,4,7\}$, $B=\{2,4,5,6\}$ and $C=\{1,3,4,6,8\}$, verify that
(a) $A-(B\cup C)=(A-B)\cap(A-C)$;
(b) $A-(B\cap C)=(A-B)\cup(A-C)$.

8. (a) Using set theory find the Highest Common Factor (HCF) of the numbers 12 and 15. [C.U. B.Com.(H) 2003]
(b) Find the HCF and LCM of the numbers 30, 35, 105 by using set operations.

[**Hints:** If A, B, C be the sets of prime factors of 30, 35, 105, then HCF = the product of the common elements of A, B, C, i.e., the product of the elements of $A\cap B\cap C$ and LCM = the product of the elements of $A\cup B\cup C$.]

9. Find the HCF and LCM of the numbers 21, 45, 105. [C.U. B.Com. 2007; V.U. B.Com.(H) 2010]

[**Hints:** See worked-out Exs 7 and 8, Section 1.11.]

10. In a Venn diagram shade the following:

(a) $A \cup B$; (c) A^c; (e) $A \cup (A \cap B)$; (g) $(A - B)^c$; (i) $A \cap (B \cup C)$.
(b) $A \cap B$; (d) $B - A$; (f) $A \cap (A \cup B)$; (h) $A \cup (B \cap C)$;

11. Using set operation find the HCF of the numbers 30, 105, 165. [C.U. B.Com. 2005]

C

1. Using Venn diagram, verify that
(a) $A \cup (B \cup C) = (A \cup B) \cup C$; (b) $A \cap (B \cap C) = (A \cap B) \cap C$.

2. Using Venn diagram, verify that
(a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

3. Using Venn diagram verify that
(a) $(A \cup B)^c = A^c \cap B^c$; (b) $(A \cap B)^c = A^c \cup B^c$.

4. Using Venn diagram prove that
(a) $A - (B \cup C) = (A - B) \cap (A - C)$; (b) $A - (B \cap C) = (A - B) \cup (A - C)$.

5. Without using Venn diagram, prove that

(a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
(b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. [C.U. B.Com.(H) 2000]

6. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$. Verify also this relation by a Venn diagram.

7. Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$. Verify also this relation by using Venn diagram.

8. Prove that $(A - C) \cap (B - C) = (A \cap B) - C$.

9. If A, B, C are any three sets, then prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

10. Let $A = \{1,2,3,\ldots,6,7\}$, $B = \{2,4,6\}$, $C = \{1,3,5,7\}$, $D = \{2,3,6\}$ and $E = \{2,6\}$; which set can equal X if we are given the following information?
(a) X and C are disjoint; (c) $X \subseteq B$ but $X \not\subset C$;
(b) $X \subseteq A$ but $X \not\subset B$; (d) $X \subseteq D$ but $X \not\subset A$.

11. If $U = \{x : x < 20, x \in N\}$, $A = \{x : x \text{ is a prime number less than } 23\}$, $B = \{x : x \text{ is divisible by } 3, x \leq 15\}$ and $C = \{x : x \text{ is an odd number less than } 18\}$, find the set $(A \cap B) \cup (A \cap C)$.

D

1. (a) In a class of 65 students, 35 students have taken Mathematics, 40 have taken Statistics. Find the number of students who have taken both Mathematics and Statistics. Find also the number of those who have taken Mathematics but not Statistics.

(Assume that every student has to take at least one of the two subjects.)

(b) In a class of 50 students, 20 students play football and 16 students play hockey. It is found that 10 students play both the games. Use algebra of sets to find out the number of students who play neither.

[**Hints:** Find $n(F \cup H)$ and subtract from 50.]

(c) In a class of 100 students, 55 students read History, 41 students read Philosophy and 25 students read both the subjects. Find the number of students who study neither of the subjects.

[C.U. B.Com. 2005]

2. Out of 450 students who appeared at the B.Com. Examination from a centre, 135 failed in Accountancy, 150 failed in Mathematics and 137 in Costing. Those who failed in both Accountancy and Mathematics were 93, in Mathematics and Costing were 98 and in Accountancy and Costing 106. The number of students who failed in all the three subjects were 75. Find the number of those who failed in at least one of the three subjects and the number of those who passed in all the three subjects. Assume that each student appeared in all the three subjects.

3. In a survey of college students it was found that 40% use their own books, 50% use library books, 30% use borrowed books, 20% use both their own books and library books, 15% use their own books and borrowed books, 10% use library books and borrowed books and 4% use their own books, library books and borrowed books. Calculate the percentage of students who do not use a book at all.

4. In the city, three daily newspapers X, Y, Z are published. 65% of the people of the city read X, 54% read Y, 45% read Z; 38% read X and Y; 32% read Y and Z; 28% read X and Z; 12% do not read any of the three papers. If 10,00,000 persons live in the city, find the number of persons who read all the three newspapers.

5. Out of 1600 students in a college, 390 played cricket, 450 played hockey and 580 played basketball; 90 played both cricket and hockey, 125 played hockey and basketball, and 155 played cricket and basketball; 50 played all the three games. How many students did not play any game? How many played only cricket? How many played only one game? How many played only two games? [C.U. B.Com.(H) 1998]

ANSWERS

A

1. (a) x does not belong to A (or x is not an element of A),
   (b) A is a subset of B,
   (c) A is a proper subset of B,
   (d) A is not a proper subset of B (or A is not contained in B).

2. (a) (i) correct,
   (ii) correct,
   (iii) correct,
   (iv) incorrect (it is an improper subset of A),
   (v) incorrect.
   (b) Incorrect; correct; incorrect; incorrect.

3. (a) (i) finite,
   (ii) finite,
   (iii) infinite,
   (iv) infinite,
   (v) finite.
   (b) $[\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}]$
   (c) The set A is the null set out of the three given sets.

4. (a) (i) $\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}$;
   (ii) $\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}$.
   (b) No.
   (c) $N \subset Z \subset R$ and $R \supset Z \supset N$.

5. (a) $A \cap B = \{3,5\}$,
   (b) $C \cup D = \{2,4,5,6,8\}$,
   (c) $B \cup (C \cup D) = \{2,3,4,5,6,8\}$,
   (d) $A \cap (B \cup C) = \{3,5\}$.

7. (b) $\{1,3,5,7,9\}$.

9. (a) $A \times B = \{(2,6),(2,4),(3,6),(3,4),(8,6),(8,4)\}$;

   (b) $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$ and $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$.

11. Sets (a) and (c) are null sets.

12. $\{1,5,6\}$.

14. (a) $\{\phi,\{2\},\{3\},\{5\},\{2,3\},\{3,5\},\{2,5\},\{2,3,5\}\}$,

    (b) $\{(1,4),(1,6),(1,8),(2,4),(2,6),(2,8),(3,4),(3,6),(3,8)\}$.

15. $\{S,I,N,O\}$.

17. $\{1,2\}$; $\{3,4,5,6\}$.

18. $\{a,c,e\}$ and $\{c\}$.

B

2. (a) $\{(1,4),(1,5),(4,4),(4,5),(1,7),(4,7)\}$;

   (b) $\{(1,5),(4,5)\}$;

   (c) $\{(1,1),(1,2),(2,1),(2,2)\}$.

6. (a) (i) $\{1,3,6,8\}$,

   (ii) $\{1,3,5,6,7,8\}$,

   (iii) $\{1,3,5,6,7,8\}$;

   (b) If $U=\{1,2,3,4\}$, then $P=\{1,3\}$ and $Q=\{2,4\}$ or, $P=\{1,2\}$ and $Q=\{3,4\}$ or etc. Similarly, we can find many other sets for P and Q when universal set is given.

8. (a) 3;

   (b) 5; 210.

9. 3; 315.

11. 15.

C

10. (a) $X = B, E$;

    (b) $X = A, C, D$;

    (c) $X = B, E$;

    (d) None of the sets can equal X.

11. $\{3,5,7,11,13,17\}$.

D

1. (a) 10, 25;

   (b) 24;

2. 200, 250.

3. 21%;

4. 2,20,000;

5. 500; 195; 830; 220.

FURTHER MISCELLANEOUS EXERCISES ON CHAPTER 8(II)

1. (a) Verify the relation $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for the set $A = \{1,2,3,5\}$, $B = \{2,3,4,6\}$ and $C = \{1,2,4,5,7\}$.

   (b) Let $S = \{a,b,c,d,e\}$ be the Universal set and let $A = \{a,b,d\}$ and $B = [b,d,e]$ be two of its subsets. Find $(A \cap B)'$ and $(A \cup B)'$.

   (c) Let $S = \{1,2,3,4,5\}$ be the Universal set and let $A = \{3,4,5\}$ and $B = \{1,4,5\}$ be two of its subsets. Verify: $(A \cup B)' = A' \cap B'$ (dash denotes *complement*).

2. (a) If $A = \{x : x$ is a multiple of 3 in $1 \le x \le 12\}$, $B = \{x : x$ is an even number in $1 \le x \le 12\}$ and $C = \{x : x$ is an odd number in $1 \le x \le 12\}$, find

   (i) $A \cap B$; (ii) $B \cap C$; (iii) $(A \cup B) \cap C$.

   (b) If $U = \{x : x$ is a positive integer and $8 \le x \le 24\}$ be the universal set and A, B be the sets of odd and prime numbers respectively, prove that $(A \cup B)' = A' \cap B'$.

   (c) If $n(A) = 15$, $n(B) = 24$ and $A \subset B$, then find the value of $n(A \cup B)$ and $n(A \cap B)$.

3. If $A = \{1,4\}$, $B = \{4,5\}$ and $C = \{5,7\}$, find

(a) $(A\times B)\cup(B\times C)$; (b) $(A\times B)\cap(B\times C)$.

4. (a) Let $A=\{1,2,3\}$, $B=\{1,2\}$, $C=\{1,2,4\}$, $D=\{3,4\}$ and $E=\{4\}$. State which of the following statements are correct and give reasons:
   (i) $B\subset C$; (ii) $D\not\supset E$; (iii) $D\supset C$; (iv) $\{3\}\subset A$.

   (b) Three sets A, B, C be such that $A-B=\{2,4,6\}$, $A-C=\{2,3,5\}$; find $A-(B\cup C)$ and $A-(B\cap C)$.

5. Given $A=\{2, B, 4\}$, where $B=\{4,5\}$; which of the following statements are not correct and why?
   (a) $5\in A$; (b) $[5]\in A$; (c) $4\in A$.

6. (a) If $S=\{a,b,c,d,e,f\}$ be the Universal set and A, B, C are three subsets of S, where $A=\{a,c,d,f\}$, $B\cap C=\{a,b,f\}$, find $(A\cup B)\cap(A\cup C)$ and $B'\cup C'$.

   (b) If $S=\{1,2,3,4,5,6\}$ be the Universal set and A, B, C are the three subsets of S, where $A\cup B=\{1,3,4,6\}$ and $A\cup C=\{1,2,4,5\}$, find $A\cup(B\cap C)$ and $A'\cap B'$.

7. If $U=\{1,2,3,4,5,6,7,8\}$ be the Universal set and A,B,C be the three subsets of U, where $A=\{1,3,5,7,8\}$ and $B\cup C=\{2,3,4,6,7,8\}$, find $(A\cap B)\cup(A\cap C)$ and $B'\cap C'$. [C.U. B.Com.(H) 1999]

8. (a) List the sets A, B and C, given that $A\cup B=[a,b,c,d]$; $A\cup C=\{b,c,d,e\}$; $A\cap B=\{b,c\}$ and $A\cap C=\{b,d\}$.

   (b) If $A\cup B=\{a,b,c,d\}$, $A\cap B=\{b,c\}$, $A\cup C=\{a,b,c,f\}$ and $A\cap C=\{a,b\}$, then find A, B and C. [C.U. B.Com. 2005]

9. Find the sets P, Q, R if $P\cap Q=\{2,3\}$, $P\cap R=\{2,4\}$, $P\cup Q=\{1,2,3,4\}$ and $P\cup R=\{2,3,4,5\}$.

10. (a) Draw a Venn diagram of three non-empty sets A, B, C so that A, B, C have the properties $A\subset B, C\not\subset B, A\cap C\neq\phi$.

    (b) There are three categories of workers in a factory: Shift workers, workers who are members of a trade union, and skilled workers.

    Assuming that these are overlapping categories, draw a Venn diagram describing the situation and now shade clearly that part of the diagram which represents unskilled workers who are not members of a trade union.

    (c) Using a Venn diagram or otherwise, solve the following problem:

    In a class of 70 students, each student has taken either English or Hindi or both. 45 students have taken English and 30 students have taken Hindi. How many students have taken both English and Hindi?

11. In a class test of 45 students, 23 students passed in the first paper, 15 passed in the first paper but did not pass in the second paper. Using set theory results, find the number of students who passed in both the papers and who passed in the second paper but did not pass in the first paper.

    [Hints: Assume that each student passed at least in one of the two papers.]

12. Three daily newspapers E, B, H, are published in a certain city. 62% of the citizens read E, 59% read B, 41% read H, 40% read both E and B, 28% read both B and H, 24% read both E and H. Find the percentage of citizens who read all the three papers. [C.U. B.Com.(H) 1992]

    [Hints: See worked-out Ex. 16. Assume that each citizen reads at least one newspaper.]

13. In a survey of 100 students it was found that 50 used the college library books, 40 had their own books, and 30 used borrowed books; 20 used both college library books and their own books, 15 used their own books and borrowed books, whereas 10 used college library books and borrowed books. Assuming that each student uses either college library books or their own or borrowed books, find the number of students using books from all the three sources.

14. A company studies the product of 300 consumers. It was found that 226 liked product A, 51 liked product B and 54 liked product C, 21 liked products A and B, 54 liked products A and C, 39 liked products B and C and 9 liked all the three products. Prove that the study results are not correct.

[**Hints**: Assume that each consumer likes at least one of the three products.]

15. In a survey of 150 students, it was found that 40 students studied Economics, 50 students studied Mathematics, 60 students studied Accountancy and 15 students studied all the three subjects. It was also found that 27 students studied Economics and Accountancy, 35 students studied Accountancy and Mathematics and 25 students studied Economics and Mathematics. Find the number of students who studied only Economics and the number who studied none of these subjects. [C.U. B.Com.(H) 1994]

16. It is known that in a group of people, each of whom speaks at least in one of the languages English, Hindi and Bengali, 31 speak in English, 36 speak in Hindi and 27 speak in Bengali. 10 speak in both in English and Hindi, 9 both in English and Bengali, 11 both in Hindi and Bengali. Prove that the group contains at least 64 people and not more than 73 people. [C.U. B.Com.(H) 1990]

[**Hints**: If x be the no. of people who can speak in English, Hindi and Bengali, then the total no. of people $= 64 + x$. As $x \geq 0$, the least value of $n(E \cup H \cup B) = 64 + 0 = 64$. Again, $n(E \cup H \cup B)$ will be greatest when x is greatest, i.e., when x is minimum of $n(E \cap H), n(H \cap B), n(E \cap B)$, i.e., when $x = 9$. Hence, the group contains not more than $64 + 9$, i.e., 73 people.]

17. The production manager of Sen, Sarkar and Lahiri Company examined 100 items produced by the workers and furnished the following report to his superior: Defect in measurement 50, defect in colouring 30, defect in quality 23, defect in quality and colouring 10, defect in measurement and colouring 8, defect in measurement and quality 20, and 5 are defective in all respects. The manager was penalized for the report. Using appropriate result of set theory, explain the reason for the penal measure. [C.U. B.Com.(H) 2002]

[**Hints**: The study results are not correct, since $n(M \cup C \cup Q) = 70$, which contradicts the given report $n(M \cup C \cup Q) = 100$.]

ANSWERS

1. (a) $\{a, c, e\}$; $\{c\}$.
2. (a) (i) $\{6, 12\}$;
   (ii) ϕ;
   (iii) $\{3, 9\}$;
   (c) $24, 15$.
3. (a) $\{(1,4), (1,5), (4,4), (4,5), (4,7), (5,5), (5,7)\}$;
   (b) $\{(4,5\}$.
4. (a) (i) correct;
   (ii) incorrect;
   (iii) incorrect;
   (iv) correct.
   (b) $\{2\}$ and $\{2,3,4,5,6\}$.
5. (a) incorrect [$\because 5 \in B$ but $\notin A$];
   (b) incorrect, since $\{5\} \notin A$;
   (c) correct.
6. (a) $\{a, b, c, d, f\}$ and $\{c, d, e\}$;
   (b) $\{1, 4\}$ and $\{2, 5\}$.
7. $\{3, 7, 8\}$ and $\{1, 5\}$.
8. (a) $A = \{b, c, d\}$, $B = \{a, b, c\}$ and $C = \{b, d, e\}$;
   (b) $A = \{q, b, c\}$, $B = \{b, c, d\}$ and $c = \{a, b, f\}$.
9. $P = \{2, 3, 4\}$, $Q = \{1, 2, 3\}$ and $R = \{2, 4, 5\}$.
10. (c) 5.
11. 8; 22.
12. 30%
13. 25.
14. The study results are not correct. [$\because$ $n(A \cup B \cup C) = 226$ and the given value of $n(A \cup B \cup C)$ is 300.]
15. 3, 72.

Chapter 9

Probability Theory

9.1 Introduction

The first foundation of the *Theory of Probability* was laid in the middle of the seventeenth century. It originated in problems dealing with games of chance played by tossing coins, drawing cards, throwing dice, etc. The notion of Probability is now used in almost all disciplines — Statistics, Economics, Business, Industry, Engineering, etc. The Probability Theory deals with uncertain situations regarding the occurrence of given phenomena, i.e., it is the study of events which are neither absolutely certain nor impossible. In this chapter, we will study the definition of Probability, Set Theoretic Approach, and Total and Compound Probability. We first explain some important terms which will be used in defining Probability.

Experiment (or Random Experiment): It is an operation that results in two or more outcomes. Although we know the set of all different outcomes of an experiment, it is not possible to predict which one of the set will occur at any particular execution of the experiment. For example: (i) '*Tossing a fair coin*' is an *experiment*, because if a coin is tossed, either head or tail will turn up, i.e., we have two possible outcomes. (ii) *Casting an unbiased die* is an *experiment*, because if a die is cast, then there are six possible outcomes, i.e., appearance of 1 or 2 or 3 or 4 or 5 or 6 on the die. (iii) Drawing a card from a well-shuffled pack of 52 cards in an experiment. (iv) Drawing a ball from a bag containing, say, 5 red and 4 white balls is an experiment.

9.2 Events

The results (or outcomes) of random experiments (or observations) are called events. Events may be either **elementary** (i.e., *simple*) or **composite**. Elementary events cannot be decomposed further, while composite events can be decomposed into elementary events. *For example*, 'getting Head on tossing a coin' is an elementary event while 'throwing 8 points by a single cast of two dice' is a composite event. Here 8 points may be obtained by throwing $(2,6)$ or $(3,5)$ or $(4,4)$ or $(5,3)$ or $(6,2)$, each of which is an elementary event.

Equally likely events: A number of events are said to be *equally likely* (or equally probable) if, having taken into account all the conditions, we have no reason to expect any one of the events in preference to the others. *For example*, (i) as a result of drawing a card at random from a well-shuffled pack of 52 cards, any card may appear in the draw so that the 52 different cases (elementary events) are equally likely; (ii) If a fair die (i.e., unbiased die) is cast, all the six possible outcomes 1, 2, 3, 4, 5, 6 are *equally likely*, i.e., we have no reason to expect any one of these in preference to the others; (iii) In tossing a perfect coin (i.e., unbiased coin), the two possible outcomes, *Head and Tail, are equally likely*, i.e., we have no reason to expect that Heads will appear more often than Tails or vice versa.

Mutually exclusive (or incompatible) events: Events are said to be *mutually exclusive* (*or incompatible*) when only one of the events can occur and two (or more) events cannot happen simultaneously, i.e., occurrence of one of the events prevents the occurrence of others. On the contrary, events are compatible (i.e., not mutually exclusive) if it is possible for them to happen simultaneously. The events of 'getting Head and Tail' in the case of tossing a coin are mutually exclusive, because if Head turns up, we cannot get Tail and vice versa, i.e., Head and Tail cannot occur simultaneously. Similarly, when a die (unbiased) is cast, then the outcomes 1, 2, 3, 4, 5, 6 are all *mutually exclusive,* because the occurrence of any one of these six outcomes implies non-occurrence of the others. But if A be the event that an *even number* will appear on top, and B that a *number less than* 4 will appear, then clearly A and B are not *mutually exclusive,* since when 2 appears on top, both A and B occur.

Exhaustive: Events are said to be *exhaustive* when they include all possible cases. The events of getting Head and Tail in the case of tossing a coin are exhaustive. In casting a die, there are 6 exhaustive events (or cases), since any one of all the six numbers 1, 2, 3, 4, 5, 6 may appear on top.

At random: *At random* means without giving any preference or priority to any (i.e., without bias for any) case so that every one may get an equally likely possibility of occurrence.

9.3 Definition of Probability

We shall now give three definitions of Probability: (i) **Classical or Mathematical definition.** (ii) **Empirical or Statistical definition.** (iii) **Subjective Approach to Probability.**

9.3.1 Classical or Mathematical Definition of Probability

Definition 1. *If there are n mutually exclusive, exhaustive and equally likely cases, and m of them are favourable to a certain event A, then the probability (or chance) of the occurrence of A is defined as the ratio m/n.*

Symbolically,
$$P(A)=\frac{m}{n}=\frac{\text{No. of favourable cases}}{\text{Total no. of all equally likely cases}},$$

where $P(A)$ denotes the probability of occurrence of the event A.

The probability (or chance) of non-occurrence of the event A

$$=\frac{n-m}{n}=\frac{\text{No. of unfavourable cases}}{\text{Total no. of all equally likely cases}}.$$

Symbolically, if $\bar{A}$ or A^c denotes the non-occurrence of A, then

$$P(\bar{A})=\frac{n-m}{n}=\frac{n}{n}-\frac{m}{n}=1-\frac{m}{n}=1-P(A) \quad \left[\because \frac{m}{n}=P(A)\right],$$

i.e., $P(A^c)=1-P(A)$ or, $P(A)+P(A^c)=1$.

Odds. (i) The **odds in favour** of event A are $m:(n-m)$

= (no. of favourable cases) : (no. of unfavourable cases),

and (ii) The **odds against** the event A are $(n-m):m$

= (no. of unfavourable cases) : (no. of favourable cases).

If the *odds in favour of an event A* be $m:n$, then $P(A)=\frac{m}{m+n}$ and $P(A^c)=\frac{n}{m+n}$.

If the *odds against the event B* be $u : v$, then $P(B) = \frac{v}{u+v}$ and $P(B^c) = \frac{u}{u+v}$.
Thus *odds in favour* of the event $A = P(A) : P(A^c)$ and

$$\textit{odds against the event } A = P(A^c) : P(A).$$

Note 1. The probability as defined above is clearly a number lying between 0 and 1. When the occurrence of the event is an absolute certainty, $m = n$, and therefore, the probability of its occurrence, i.e., $P(A) = n/n = 1$. When the occurrence of the event is an absolute impossibility, $m = 0$, and therefore, the probability of its occurrence $= 0/n = 0$. Thus for a certain event $p = 1$ and for an impossible event $p = 0$, where p = probability (or chance).

Again, $0 \leq m \leq n$ or, $0 \leq m/n \leq 1$ or, $\mathbf{0 \leq P(A) \leq 1}$.

Note 2. The *classical definition of Probability* is based on two assumptions, viz., (1) all the outcomes (i.e., all the n mutually exclusive cases) are **equally likely** and (2) the total number of exhaustive cases (i.e., n) is **finite**. If any one of these two assumptions does not hold, *classical definition* fails, i.e., we cannot apply classical definition of Probability.

9.3.2 Illustrative Examples

Example 1. *A bag contains three white and five black balls. What is the chance that a ball drawn at random will be black? Also find the odds in favour of the event and against the event.*

Solution: 1st Part. The total number of balls = 3 + 5 = 8. 1 ball can be drawn out of these 8 balls in $^8C_1$ ways = 8 ways.

$\therefore$ the total number of possible cases for the event = 8.

Again, one black ball can be drawn out of 5 black balls in $^5C_1 = 5$ ways;

$\therefore$ the total number of cases favourable to the event = 5.

$\therefore$ the chance that the ball drawn is black = $\frac{5}{8}$.

2nd Part. Total number of cases unfavourable to the event $= 8 - 5 = 3$.

$\therefore$ the *odds* in favour of the event are **5 : 3** and the *odds* against the event are **3 : 5**.

Note. The chance that the ball drawn is white = 1− (the chance that the ball drawn is not black) = $1 - \frac{5}{8} = \frac{3}{8}$. This result can also be obtained by the direct method.

The odds in favour of the ball being white = 3 : 5 and the odds against the ball being white = 5 : 3.

Example 2. *A (six-faced) die is thrown; find the chance that* (i) *an ace turns up;* (ii) *any one of* 1, 2, 3 *turns up.*

Solution: A die has six faces; if it is thrown, then either 1 or 2 or 3 or 4 or 5 or 6 will appear on the uppermost face of the die.

$\therefore$ total number of all possible cases = **6**.

(i) Ace (i.e., 1) will turn up in 1 way only.

$\therefore$ no. of cases favourable to the event of getting an ace = **1**.

Hence, *the required chance that an ace turns up* = $\frac{1}{6}$.

(ii) Any one of 1, 2, 3 turns up in 3 ways.

$\therefore$ no. of cases favourable to any one of 1, 2, 3 is 3.

Hence, *the required chance that any one of* 1, 2, 3 *turns up* = $\frac{3}{6} = \frac{1}{2}$.

Example 3. *If a coin is tossed, what is the chance of a 'head'?*

Solution: Either 'head' or 'tail' may turn up when a coin is tossed.

$\therefore$ there are 2 equally likely cases, of which only 1 is in favour of showing a 'head'.

Hence, *the required chance* = $\frac{1}{2}$.

Example 4. *Three coins are tossed. What is the probability that they all fall heads?*

Solution: Each coin can fall in 2 different ways, showing 'head' or 'tail'.

$\therefore$ 3 coins can fall in $2 \times 2 \times 2$ different ways.

$\therefore$ the total number of possible cases for the event $= 2 \times 2 \times 2 = 8$.

Now 3 coins can fall to show all heads in only 1 way.

$\therefore$ out of 8 possible cases only 1 is in favour of the event of showing all heads.

Hence, *the required probability that they all fall heads* $= \frac{1}{8}$.

Example 5. *If one card is drawn at random from a well-shuffled pack of* 52 *cards, find the chance that the card is* (i) *a diamond,* (ii) *not a diamond,* (iii) *an ace,* (iv) *neither a spade nor a heart.* [C.U.B.Com.(H) 1998]

Solution: One card can be drawn out of 52 in ${}^{52}C_1$ ways = 52 ways.

$\therefore$ total no. of all possible equally likely cases = 52.

(i) One diamond can be drawn out of 13 in ${}^{13}C_1$ ways = 13 ways.

$\therefore$ total no. of favourable cases = 13.

Hence, *the required chance that the card is a diamond* $= \frac{13}{52} = \frac{1}{4}$.

(ii) The required chance that the card is not a *diamond* $= 1 - \frac{1}{4} = \frac{3}{4}$.

(iii) An ace can be drawn out of 4 in ${}^{4}C_1$ ways = 4 ways.

$\therefore$ no. of favourable cases = 4.

Hence, *required chance that the card is an ace* $= \frac{4}{52} = \frac{1}{13}$.

(iv) There are 13 spades and 13 hearts in a pack of 52 cards.

$\therefore$ either a spade or a heart can be drawn in ${}^{26}C_1$ ways = 26 ways.

$\therefore$ the chance that the card is either a spade or a heart $= \frac{26}{52} = \frac{1}{2}$.

Hence, *the required chance that the card is neither a spade nor a heart* $= 1 - \frac{1}{2} = \frac{1}{2}$.

Example 6. *What is the probability of drawing either a spade or an ace from a pack of* 52 *cards?*

[C.U.B.Com.(H) 2001]

Solution: 1 card can be drawn out of 52 cards in ${}^{52}C_1$ ways = 52 ways.

$\therefore n(S) =$ total no. of all equally likely cases = 52.

Let A be event of drawing either a spade or an ace from the pack. There are 13 spades and 3 aces not of spade, and 1 card can be drawn out of $13 + 3$, i.e., 16 in ${}^{16}C_1$ ways = 16 ways; $\therefore n(A) = 16$.

Hence, *the required probability of drawing a spade or an ace* $= \frac{n(A)}{n(S)} = \frac{16}{52} = \frac{4}{13}$.

Example 7. (i) *A man draws at random* 3 *balls from a bag containing* 6 *red and* 5 *white balls. What is the chance of getting the balls all red?*

(ii) *A bag contains* 4 *white and* 3 *black balls. If* 4 *balls are drawn at random, what is the probability that* 2 *are white and* 2 *black?* [C.U.B.Com.(H) 1994]

Solution: Total no. of balls $= 6 + 5 = 11$.

3 balls can be drawn out of 11 in ${}^{11}C_3$ ways $= \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$ ways.

$\therefore$ total number of possible cases for the event = 165.

Again, 3 red balls can be drawn out of 6 in ${}^{6}C_3$ ways $= \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ ways.

$\therefore$ total number of cases favourable to the event = 20.

Hence, the required chance of getting the balls all red $= \frac{20}{165} = \frac{4}{33}$.

(ii) Total no. of balls $= 4+3 = 7$; 4 balls can be drawn out of 7 in ${}^7C_4$ ways.

$\therefore$ total no. of possible cases $= {}^7C_4 = \frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1} = 35$.

2 white balls can be drawn out of 4 white balls in ${}^4C_2$ ways and 2 black balls out of 3 in ${}^3C_2$ ways.

$\therefore$ no. of favourable cases $= {}^4C_2 \times {}^3C_2 = \frac{4\times3}{2\times1} \times \frac{3\times2}{2\times1} = 18$.

Hence, *the required probability that* 2 *are white and* 2 *black balls* $= \frac{18}{25}$.

Example 8. *Three balls are drawn at random from a bag containing* 6 *blue and* 4 *red balls. What is the chance that two balls are blue and one is red?*

Solution: Total no. of balls $= 6+4 = 10$.

3 balls can be drawn out of 10 in ${}^{10}C_3$ ways $= \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 120$ ways.

$\therefore$ total no. of cases for the event = 120.

Again, 2 blue balls can be drawn out of 6 in ${}^6C_2$ ways and 1 red ball can be drawn out of 4 in ${}^4C_1$ ways.

$\therefore$ 2 blue balls and 1 red ball jointly can be drawn in ${}^6C_2 \times {}^4C_1$ ways $= \frac{6\cdot5}{2\cdot1} \times \frac{4}{1} = 60$ ways.

$\therefore$ total no. of favourable cases for the event = 60.

Hence, *the required chance of drawing* 2 *blue and* 1 *red balls* $= \frac{60}{120} = \frac{1}{2}$.

Example 9. 8 *men in a company of* 25 *are graduates. If* 3 *men picked out of the* 25 *at random, what is the probability that* (i) *they are all graduates,* (ii) *at best one graduate?* [C.U. B.Com.(H) 1999]

Solution: No. of graduates = 8 and the no. of non-graduates $= 25 - 8 = 17$.

3 men can be selected out of 25 in ${}^{25}C_3$ ways.

$\therefore$ total no. of all equally likely cases $= {}^{25}C_3 = \frac{25\times24\times23}{3\times2\times1} = 2300$.

(i) 3 graduates can be selected out of 8 in ${}^8C_3 = \frac{8\times7\times6}{3\times2\times1} = 56$ ways.

$\therefore$ the no. of cases favourable to the selection of 3 graduates = 56.

Hence, by *the classical definition of probability, the required probability that* 3 *men picked out at random are all graduates* $= \frac{56}{2300} = \frac{14}{575}$.

(ii) Let A be the event that the 3 men selected at random are all non-graduates. Then 3 non-graduates can be selected out of 17 in ${}^{17}C_3$ ways $= \frac{17\times16\times15}{3\times2\times1} = 680$ ways.

$\therefore$ no. of cases favourable to the selection of 3 non-graduates = 680.

$\therefore$ $P(A)$ = the probability of selecting 3 non-graduates $= \frac{680}{2300} = \frac{34}{115}$.

Hence, the *required probability that at least one of the* 3 *men selected is a graduate*

$$= 1 - P(A) = 1 - \frac{34}{115} = \mathbf{\frac{81}{115}}.$$

Example 10. *What is the chance that a leap year selected at random will contain*

(i) 53 *Sundays;* [C.U. B.Com.(H) 1996]

(ii) 53 *Thursdays or* 53 *Fridays?*

Solution: A leap year consists of 366 days in which there are 52 complete weeks and 2 more consecutive days. 52 weeks contain 52 Sundays, 52 Thursdays and 52 Fridays.

(i) A leap year will contain 53 Sundays if these two consecutive days contain one Sunday. 2 *consecutive* days can be selected from the 7 days in a week in the following possible ways:

(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday), (Saturday, Sunday). (1)

We see that out of the above 7 possibilities, 2 (the first and the last) contain 'Sunday', i.e., no. of favourable cases = 2.

Hence, *the required chance that a leap year will contain* 53 *Sundays* $= \frac{2}{7}$.

(ii) Again, from (1) we see that *out of the* 7 *outcomes*, (i.e., possibilities) 3 *are favourable* to the event of 53 Thursdays or 53 Fridays and they are (Wednesday, Thursday), (Thursday, Friday) and (Friday, Saturday).

[In Set Theory, 'or' means or/and Note that by "either 53 Thursdays or 53 Fridays", we mean that either Thursday or Friday or *both Thursday and Friday.*]

Hence, *the required chance that a leap year will contain either* 53 *Thursdays or* 53 *Fridays* $= \frac{3}{7}$.

Note. If by the event "either 53 Thursdays or 53 Fridays" we mean "Thursdays or Fridays" and exclude the case Thursday and Friday, then the required chance will be $\frac{2}{7}$.

Example 11. *Find the probability that in a game of bridge a hand of* 13 *cards will contain all the* 4 *aces.*

Solution: There are 52 cards in all. A hand of 13 cards can be made up out of 52 in ${}^{52}C_{13}$ ways.

∴ total number of equally likely cases = ${}^{52}C_{13}$.

Now 4 aces can be selected out of 4 in ${}^{4}C_{4}$ ways and 9 other cards can be selected out of the remaining 48 in ${}^{48}C_{9}$ ways.

∴ A hand of 13 cards will contain all the 4 aces in ${}^{4}C_{4} \times {}^{48}C_{9}$ ways.

∴ no. of favourable cases $= {}^{4}C_{4} \times {}^{48}C_{9} = {}^{48}C_{9}$.

Hence, *the required probability* $= \dfrac{{}^{48}C_9}{{}^{52}C_{13}} = \dfrac{\frac{48\cdot47\cdot46\cdot45\cdot44\cdot43\cdot42\cdot41\cdot40}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}}{\frac{52\cdot51\cdot50\cdot49\cdot48\cdot47\cdot46\cdot45\cdot44\cdot43\cdot42\cdot41\cdot40}{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}} = \dfrac{13\cdot12\cdot11\cdot10}{52\cdot51\cdot50\cdot49} = \dfrac{\mathbf{11}}{\mathbf{4165}}$.

Example 12. *If* 20 *dates are named at random, what is the probability that* 5 *of them will be Sundays?*

[C.U. B.Com.(H) 1999]

Solution: Since a week consists of 7 days, 1 day may be named in 7 different ways. For each way of the first date, second date may be also named in 7 different ways.

∴ first and second dates can be named in 7×7, i.e., 7^2 different ways.

Proceeding in the same way, 20 dates can be named in 7^{20} different ways.

∴ total no. of all possible equally likely cases = 7^{20}.

Now if 5 dates are Sundays, then 5 dates can be selected out of 20 in ${}^{20}C_5$ ways and each of the remaining 15 dates can be named in 6 different ways, viz., Monday, Tuesday, Wednesday, Thursday, Friday and Saturday.

∴ no. of all favourable cases for 5 Sundays = ${}^{20}C_5 \times 6^{15}$.

Hence, *the required probability of getting* 5 *Sundays* $= \dfrac{{}^{20}C_5 \times 6^{15}}{7^{20}} = \dfrac{\mathbf{15504 \times 6^{15}}}{\mathbf{7^{20}}}$.

9.3.3 Empirical or Statistical Definition of Probability

If a trial (random experiment) is repeated a (large) number of times under essentially the same conditions, then the limiting value of the ratio of the number of times that an event happens, to the total number of trials, *as the number of trials become indefinitely large, is called the probability of happening of the event, provided that the limit is a definite finite number.*

Symbolically, if in N trials, an event A occurs f times, then the probability p of the happening of A is given by $p = P(A) = \lim_{N \to \infty} \left(\frac{f}{N}\right)$, provided the limit exists and is finite.

Note 1. If (i) the elementary events are not equally likely (or equally probable) or (ii) the exhaustive number of cases in a trial is infinite, we cannot use the *classical definition of probability.* In these two cases, we may use the *empirical definition of probability* which is based on the limit of relative frequency f/N as $N \to \infty$. To find this limit, a large number of experiments has to be performed under identical conditions.

Note 2. The *empirical definition of probability* can never be obtained in practice. The experimental conditions may not remain the same when a random experiment is repeated a large number of times and the relative frequency f/N may not attain a unique limiting value when $N \to \infty$.

Illustration 1. *From the past records of attendance in an office it was found that an employee came late on* 73 *days in the last* 365 *days of the year. The probability of his coming late today* $= \frac{73}{365} = \frac{1}{5}$.

Example 13. *The marks obtained by* 1000 *students are given below:*

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
No. of Students	20	50	90	160	240	200	140	80	15	5

If a student is selected at random from the whole group of 1000 *students, find the probability that his marks is* (i) *under* 30; (ii) 60 *or above and* (iii) *either between* 40 *to* 50 *or* 50 *to* 60.

Solution: N = total number of students = 1000.

(i) No. of students securing marks below $30 = 20 + 50 + 90 = 160$.

$\therefore$ the required probability $= \frac{160}{1000} = \frac{4}{25}$.

(ii) No. of students who secure marks 60 or above $= 140 + 80 + 15 + 5 = 240$.

$\therefore$ the required probability $= \frac{240}{1000} = \frac{6}{25}$.

(iii) No. of students who secure marks either between 40 to 50 or 50 to 60 = 240 + 200 = 440.

$\therefore$ the required probability $= \frac{440}{1000} = \frac{11}{25}$.

9.3.4 Subjective Probability

By *classical* and *empirical* definitions, we can find objective probability values which indicate the relative rate of occurrence of the event in the long run. In the former, the probability of occurrence of an event is based upon a *prior knowledge* of the process involved, i.e., the number of favourable cases and the total number of outcomes were known from the process involved. But in the latter, i.e., in the *Empirical Definition of Probability,* these outcomes are based on actual observations, not on a prior knowledge of the process, and this definition involves the examination of a fairly large number of outcomes. However, in many cases, either only a small number of past outcomes of an event may be available, or there may not be any past outcomes to examine (as in the case of 'marketing of a new product'). In such a case, probability of success that depends on personal judgement is called **Subjective Probability** or **Personal Probability.**

The **Subjective Probability** *of an event is the* **degree of belief** *or* **degree of confidence** *placed in the occurrence of an event by an individual based on all evidence available to him. This evidence may be relative frequency of occurrence of data or any other quantitative (or qualitative) information. As the probability value is a personal judgement, the subjective probability is also called the* **personalistic** (*or* **personal**) **probability.**

Thus, *subjective probability* refers to the chance of occurrence placed on an event by a particular individual. This chance may be different for different individuals. The assignment of probabilities to various events is based on a combination of an individual's past experience, personal opinion and analysis of a particular situation.

Illustration 2. *The probability that there will be a recession next year is a subjective probability.*

Illustration 3. *The probability that a new product on the market will be successful is a subjective probability.*

The state of economy at the time of marketing of the new product is thought to affect the likely success of the product. Suppose

A is the event that national income next year will rise by more than 6%;

B is the event that national income will rise by 6% or less;

C is the event that national income will be the same as this year;

D is the event that national income will fall.

After considering the various forecasts of the economy for the next year and taking into account the past knowledge of the reliability (or unreliability) of such forecasts, the marketing manager may think that the probability that the event A will occur (i.e., national income next year will rise by more than 6%) is $1/4$, i.e., $P(A) = 1/4$. He may also think that $P(B) = 3/8$, $P(C) = 1/8$, $P(D) = 1/4$.

Roughly speaking subjective probability may be considered as the odds one would give in betting on an event.

The mathematical theory of probability deals with objective probabilities, its origin being in the prediction of results in gambling. Business and administration deals with both objective and subjective probabilities. We shall now discuss *objective probabilities in detail.*

9.4 Set Theoretic Approach

We know that the results of an experiment which cannot be decomposed further are called *elementary* events or outcomes. 'Getting a head' on tossing a coin is an elementary event. The set S of all possible elementary events (or outcomes) of a given experiment is called the **Sample Space** of experiment. A particular elementary event or outcome which is an element of S is called a *Sample Point* or simply a *Sample.* A *composite event* A is a set of elementary events (or outcomes) of S, and A is a subset of the sample space S. We shall call the null set ϕ, the *impossible event,* and the sample space S, the *certain or sure event.* We shall now explain some set operations combining two or more events $A, B, \ldots$

(i) The event $A \cup B$ will denote the occurrence of either A or B or both A and B, i.e., occurrence of at least one of A and B.

(ii) The event $A \cap B$ will denote the simultaneous occurrence of A and B. Similarly, $A \cap B \cap C \cap \cdots$ will denote the simultaneous occurrence of $A, B, C, \ldots$

(iii) The event A^c (or A') will denote the non-occurrence of A.

(iv) The event $A \cap B^c$ will denote the occurrence of A and the non-occurrence of B. Clearly, $A - B = A \cap B^c$.

(v) Two events of A and B are said to be mutually exclusive if they are disjoint, i.e., if $A \cap B = \phi$.

Example (Experiment). *Toss a die and observe the number appearing on top. Notice that the possible outcomes or sample points are* 1, 2, 3, 4, 5, 6 *and the sample space S consists of these six sample points, i.e.,* $S = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event that an *even number* appears and B that an *odd number* appears and C that a *number less than* 4 appears. Then $A = \{2, 4, 6\}, B = \{1, 3, 5\}, C = \{1, 2, 3\}$.

$A \cup B = \{1,2,3,4,5,6\} = S$ which is the certain event, since either an even number or an odd number must appear.

$A \cap B = \phi$, i.e., an even number and an odd number cannot occur simultaneously. In other words, A and B are mutually exclusive events.

$A^c = \{1,3,5\}$ = the event that an even number does not occur.

$B \cup C = \{1,2,3,5\}$ = the event that an odd number or a number less than 4 occurs.

$B \cap C = \{1,3\}$ = the event that an odd number less than 4 appears.

Note. The number of all possible sample points in the sample space S will be denoted by the symbol $n(S)$ and the number of sample points in any event A by $n(A)$.

Definition of Probability (Classical Definition). Set Theoretic Treatment of Probability Theory will be based on two assumptions, viz., (1) the total no. of elementary events in the sample space S is finite, say N, and (2) the experiment is such that the N elementary events are equally likely. Then, if m is the number of elementary events favourable to an event A (i.e., A occurs when any one of the m elementary events occurs and conversely), the probability $P(A)$ of the event A is defined by $P(A) = \frac{m}{N} = \frac{n(A)}{n(S)}$, where $n(A)$ = no. of sample points in A and $n(S)$ = total no. of sample points in S.

Properties of $P(A)$:

(i) Since $0 \le m \le N$, dividing by N we have $0 \le m/N \le 1$, i.e., $0 \le P(A) \le 1$;

(ii) If A is an impossible event, then $m = 0$. $\therefore P(A) = 0/N = 0$;

(iii) If A is a certain event, then $m = N$. $\therefore P(A) = N/N = 1$;

(iv) If the occurrence of A implies the occurrence of B (the occurrence of B may not imply the occurrence of A), then $P(A) \le P(B)$.

Example 14. (a) *What is the probability of getting at least one head in the experiment of tossing of two unbiased coins simultaneously?* [B.U. B.Com.(H) 2006]

(b) *Three fair coins are tossed once. Construct the sample space of the outcomes of the random experiment. Find the probability of* (i) *at least one head,* (ii) *exactly one tail.*

Solution: (a) If S be the sample connected with the tossing of two unbiased coins, then

$$S = \{HH, HT, TH, TT\} \text{ and } n(S) = 4.$$

If A be event of getting at least one head, then $A = \{HH, HT, TH\}$ and $n(A) = 3$.

Hence, *the required probability* $= P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$.

(b) If S be the required sample space, H represents head and T represents tail, then S is the set of all possible outcomes given by

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}. \quad \therefore n(S) = 8.$$

(i) If A be the event that at least one head appears, then

$$A = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}. \therefore\ n(A) = 7;$$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{7}{8}$.

(ii) If B be the event that exactly one tail appears, then

$$B = \{HHT,\ HTH,\ THH\}. \quad \therefore n(B) = 3;$$

$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$.

Example 15. *If a die is tossed, what is the probability that the number appearing on top is* (i) *even,* (ii) *less than* 4, (iii) *not an even number,* (iv) *either an even or an odd number,* (v) *an odd number less than* 4?

Solution: If a die is tossed, the possible sample points are $1,2,3,4,5,6$ and, therefore, the sample space consists of these 6 sample points, i.e., $S=\{1,2,3,4,5,6\}$; $\therefore n(S)=6$.

Let A be the event that an even number appears, B that an odd number appears and C that a number less than 4 appears. Then $A=\{2,4,6\}$, $B=\{1,3,5\}$, $C=\{1,2,3\}$, $A^c=\{1,3,5\}$.

$A\cup B=\{1,2,3,4,5,6\}=S, B\cap C=\{1,3\}$.

$\therefore n(A)=3, n(C)=3, n(A^c)=3, n(A\cup B)=6, n(B\cap C)=2$.

(i) *The probability that the number appearing on top is even, is*

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\mathbf{\frac{1}{2}}.$$

(ii) *The probability that the number appearing on top is less than* 4, *is*

$$P(C)=\frac{n(C)}{n(S)}=\frac{3}{6}=\mathbf{\frac{1}{2}}.$$

(iii) *The probability that the number appearing on top is not an even number, is*

$$P(A^c)=\frac{3}{6}=\mathbf{\frac{1}{2}}. \quad \left[\text{Also } P(A^c)=1-P(A)=1-\frac{1}{2}=\frac{1}{2}.\right]$$

(iv) *The probability that the number appearing on top is either an even or an odd number, is*

$$P(A\cup B)=\frac{n(A\cup B)}{n(S)}=\frac{6}{6}=\mathbf{1}.$$

(v) *The probability that the number appearing on top is an odd number less than* 4, *is*

$$P(B\cap C)=\frac{n(B\cap C)}{n(S)}=\frac{2}{6}=\mathbf{\frac{1}{3}}.$$

Example 16. *What is the probability of getting* 3 *white balls in a draw of* 3 *balls from a box containing* 5 *white and* 4 *black balls?*

Solution: Total no. of balls is $5+4=9$.

3 balls can be drawn out of 9 in ${}^9C_3$ ways = $\frac{9.8.7}{3.2.1}=84$ ways.

$\therefore$ total no. of possible outcomes = 84.

$\therefore$ the sample space S consists of 84 sample points, i.e., $n(S)=84$.

Let A be the event that 3 white balls are drawn. Then

$n(A)=$ the possible no. of ways in which 3 white balls can be drawn out of 5 is

$${}^5C_3=\frac{5.4.3}{3.2.1}=10.$$

Hence, *the required probability that* 3 *white balls are drawn from the box is*

$$\frac{n(A)}{n(S)}=\frac{10}{84}=\mathbf{\frac{5}{42}}.$$

Example 17. *A pair of unbiased dice is thrown. If the two numbers appearing be different, then find the probabilities that* (i) *the sum is six, and* (ii) *the sum is* 5 *or less.* [C.U. B.Com.(H) 1995]

Solution: If the two numbers appearing be different when a pair of unbiased dice is thrown, then the sample space S is given by $S = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5)\}$.

[Note that $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$ are excluded from S.]

$\therefore n(S) = 30.$

(i) If A be the event that the sum is 6, then $A = \{(1,5),(2,4),(4,2),(5,1)\}$; $\therefore n(A) = 4$.

Hence, by definition, *the required probability* $= P(A) = \frac{n(A)}{n(S)} = \frac{4}{30} = \frac{2}{15}$.

(ii) If B be the event that the sum is 5 or less, then

$$B = \{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}.$$

$\therefore n(B) = 8$ and $P(B) = \frac{n(B)}{n(S)} = \frac{8}{30} = \mathbf{\frac{4}{15}}$.

Example 18. *A sample of* 3 *items is selected at random from a box of* 12 *items of which* 3 *are defective. Find the possible number of defective combinations of the said* 3 *selected items along with probability of a defective combination.*

Solution: 3 items can be selected from the box containing 12 items in ${}^{12}C_3$ ways

$$= \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220 \text{ ways.}$$

$\therefore$ total no. of cases = 220.

No. of non-defective items in the box = $12 - 3 = 9$.

$\therefore$ 3 non-defective items can be selected out of 9 in ${}^9C_3$ ways = $\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

$\therefore$ possible number of non-defective combinations = 84.

$\therefore$ possible number of defective combinations = $220 - 84 = \mathbf{136}$.

Hence, *the required probability of a defective combination* $= \frac{136}{220} = \mathbf{\frac{34}{55}}$.

[**Otherwise.** 1 defective and 2 non-defective items can be selected in ${}^3C_1 \times {}^9C_2$ ways, 2 defective and 1 non-defective items can be selected in ${}^3C_2 \times {}^9C_1$ ways, and 3 defective items can be selected in ${}^3C_3$ ways.

$\therefore$ no. of defective combinations = ${}^3C_1 \times {}^9C_2 + {}^3C_2 \times {}^9C_1 + {}^3C_3 = \mathbf{136}$.]

Example 19. *In a family there are* 3 *children. Find the probability that all of them will have different birthdays.*

Solution: The birthday of the first child may fall on any one of the 365 days, the second child's birthday may fall on any one of the 365 days and similarly for the third child.

$\therefore$ the total no. of all possible ways in which the 3 children can have their birthdays

$$= 365 \times 365 \times 365, \quad \text{i.e.,} \quad n(S) = 365 \times 365 \times 365.$$

For the number of favourable cases, we see that the first child can have his birthday on any one of the 365 and then the second child has a different birthday on any one of remaining 364 days in the year, and similarly, the third child can have a different birthday on any of the remaining 363 days.

$\therefore$ the number of cases favourable to the event A ('different birthdays') is $365 \times 364 \times 363$, i.e., $n(A) = 365 \times 364 \times 363$.

Hence, *the required probability that all the 3 children will have different birthdays* is

$$\frac{n(A)}{n(S)} = \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = 0.99179 = \mathbf{0.99\ (approx.)}.$$

9.5 Theorems of Total Probability (The Rule of Addition)

We shall now prove *two theorems on Total Probability* which will enable us to find the probability of the union of a set of events. First we consider the case of a set of mutually exclusive events.

Theorem 1. *If A and B be two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$, i.e., the probability that either A or B occurs is the sum of the probabilities of the events A and B.* **[B.U. B.Com.(H) 2007]**

Extension. *For r mutually exclusive events $A_1, A_2, \ldots, A_r$*

$$P(A_1 \cup A_2 \cup \cdots \cup A_r) = P(A_1) + P(A_2) + \cdots + P(A_r).$$

Proof. Let the total number of all possible elementary events be N out of which m_1 are favourable to A and m_2 are favourable to B. Then the number of elementary events that are favourable to either A or B is $m_1 + m_2$. [since A and B are two mutually exclusive events]

Hence, $P(A \cup B)$ = probability that either A or B occurs

$$= \frac{m_1 + m_2}{N} = \frac{m_1}{N} + \frac{m_2}{N} = P(A) + P(B) \quad \left[\because \frac{m_1}{N} = P(A) \text{ and } \frac{m_2}{N} = P(B), \text{ by definition}\right],$$

$$\text{i.e.,} \quad \mathbf{P(A \cup B) = P(A) + P(B).}$$

Extension. $P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_r) = P(A_1 \cup [A_2 \cup A_3 \cup \cdots \cup A_r])$

$$= P(A_1) + P(A_2 \cup A_3 \cup \cdots \cup A_r) \quad \text{[using the result (1)]}$$
$$= P(A_1) + P(A_2) + P(A_3 \cup A_4 \cup \cdots \cup A_r) \quad \text{[proceeding as before]}$$
$$= \cdots\cdots\cdots\cdots\cdots\cdots$$
$$= P(A_1) + P(A_2) + \cdots + P(A_r).$$

Hence, *the theorem follows.*

Corollary 1. *If $A_1, A_2, \ldots, A_r$ are exhaustive, i.e., one of them must occur, then $A_1 \cup A_2 \cup \cdots \cup A_r$ is a certain event and, therefore, $P(A_1 \cup A_2 \cup \cdots \cup A_r) = 1$*
or, $P(A_1) + P(A_2) + \cdots + P(A_r) = 1$ [$\because A_1, A_2, \ldots, A_r$ are mutually exclusive events.]

In particular, the event A and its complement, i.e., A and A^c are exhaustive and mutually exclusive.
$\therefore P(A) + P(A^c) = 1$ or, $\mathbf{P(A) = 1 - P(A^c)}$ or, $\mathbf{P(A^c) = 1 - P(A)}$.

Corollary 2. *If A and B are two events, not mutually exclusive, connected with a random experiment, then A will occur if and only if any one of the mutually exclusive events $A \cap B$ and $A \cap B^c$ occurs. Therefore, by Theorem 1,* $\mathbf{P(A) = P(A \cap B) + P(A \cap B^c)}$.

Similarly, $\mathbf{P(B) = P(A \cap B) + P(A^c \cap B)}$. [See Fig. 9.1]

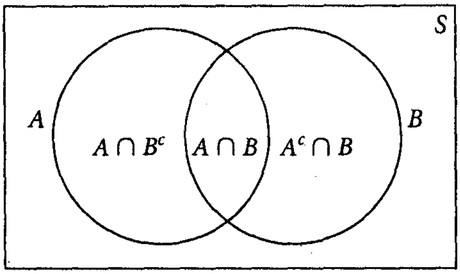


Fig. 9.1

Theorem 2. *If A and B be any two events (not necessarily mutually exclusive), then $P(A \cup B)$, i.e., the probability that at least one of the two events A and B occurs is given by*

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$ [C.U. B.Com.(H) 2002; B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2008]

Proof. Let S be the sample space and A, B be any two events of S.

We have
$$P(A \cup B) = \frac{n(A \cup B)}{n(S)}, \tag{1}$$

where $n(A \cup B)$ = the number of sample points (or elements) in $A \cup B$ and $n(S)$ = the total no. of all possible sample points in S.

From *Set Theory* we know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

From (1),
$$P(A \cup B) = \frac{n(A) + n(B) - n(A \cap B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} = P(A) + P(B) - P(A \cap B).$$

Hence, *the theorem follows.*

Corollary. *If A and B are two mutually exclusive events, then $A \cap B = \phi$:*

$\therefore n(A \cap B) = 0$; $\therefore P(A \cap B) = 0$. Hence, $\boldsymbol{P(A \cup B) = P(A) + P(B)}$.

Generalisation. We know that for any three events, A, B, C,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$ and $P(A \cup B \cup C) = \frac{n(A \cup B \cup C)}{n(S)}$.

Hence,
$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).}$$

Similarly, we can write the formulae for the probabilities of four or more events.

Note: $P(A \cup B)$ may be written as $P(A + B)$, i.e., $P(A \cup B) = P(A + B)$.

Example 20. *From a set of* 17 *balls marked* $1, 2, 3, \ldots, 16, 17$, *one is drawn at random.* (a) *What is the chance that its number is a multiple of* 3 *or of* 7? (b) *What is the probability that its number is an even number greater than* 9? [C.U. B.Com.(H) 2000]

Solution: Let S be the sample space connected with the drawing of one ball.

Then $S = \{1, 2, 3, \ldots, 16, 17\}$; $\therefore n(S) = 17$.

(a) Let A be the event that the number of the ball is a multiple of 3, and B be the event that its number is a multiple of 7.

Then $A=\{3,6,9,12,15\}$ and $B=\{7,14\}$; $\therefore n(A)=5$ and $n(B)=2$. Also $A\cap B=\phi$.

$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{5}{17} \text{ and } P(B)=\frac{n(B)}{n(S)}=\frac{2}{17}.$$

Since A and B are mutually exclusive events, by the theorem of total probability, we have $P(A\cup B)=P(A)+P(B)=\frac{5}{17}+\frac{2}{17}=\frac{7}{17}$.

Hence, *the required chance that its number is a multiple of* 3 *or of* 7 *is* $P(A\cup B)=\frac{7}{17}$.

(b) Let C be the event that the number of the ball is an even number greater than 9. Then $C=\{10,12,14,16\}$; $\therefore\ n(C)=4$.

Hence, *the required probability that its number is an even number greater than* $9=\frac{4}{17}$.

Example 21. *A fair die is thrown. Write down the sample space. What is the chance that either an odd number or a number greater than 4 will turn up?*

Solution: Let S be the sample space connected with the throwing of the die. Then

$$S=\{1,2,3,4,5,6\}; \quad \therefore n(S)=6.$$

If A be the event that an odd number will turn up and B, the event that a number greater than 4 will turn up, then $A=\{1,3,5\}$ and $B=\{5,6\}$. Also $A\cap B=\{5\}$.

$\therefore A$ and B are not mutually exclusive events.

Now $n(A)=3$, $n(B)=2$ and $n(A\cap B)=1$.

$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\frac{1}{2},\quad P(B)=\frac{n(B)}{n(S)}=\frac{2}{6}=\frac{1}{3} \text{ and } P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{6}.$$

$\therefore$ by the theorem of total probability, we have

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{3+2-1}{6}=\frac{4}{6}=\frac{2}{3}.$$

Hence, *the required chance* $=\frac{2}{3}$.

Example 22. *The probability that a contractor will get a plumbing contract is* 2/3 *and the probability that he will not get an electric contract is* 5/9. *If the probability of getting at least one contract is* 4/5, *what is the probability that he will get both the contracts?* **[B.U. B.Com.(H) 2007]**

Solution: Let A be the event that the contractor will get the plumbing contract and B, that he will get the electric contract. Then $P(A)=\frac{2}{3}, P(B^c)=\frac{5}{9}$ and $P(A\cup B)=\frac{4}{5}$.

We have to find the value of $P(A\cap B)$. We have $P(B)=1-P(B^c)=1-\frac{5}{9}=\frac{4}{9}$.

By the theorem of total probability, we have $P(A\cup B)=P(A)+P(B)-P(A\cap B)$;

$\therefore \frac{4}{5}=\frac{2}{3}+\frac{4}{9}-P(A\cap B)=\frac{10}{9}-P(A\cap B)$ or, $P(A\cap B)=\frac{10}{9}-\frac{4}{5}=\frac{50-36}{45}=\frac{14}{45}$.

Hence, *the required probability* $=\frac{14}{45}$.

Example 23. *A and B are two events, not mutually exclusive, connected with a random experiment E. If* $P(A)=\frac{1}{4}$, $P(B)=\frac{2}{5}$ *and* $P(A\cup B)=\frac{1}{2}$, *find the values of the following probabilities:* (i) $P(A\cap B)$; (ii) $P(A\cap B^c)$; (iii) $P(A^c\cup B^c)$, *where c stands for the complement.*

Solution: We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where A and B are two events, not mutually exclusive.

$$\therefore \ \frac{1}{2} = \frac{1}{4} + \frac{2}{5} - P(A \cap B) \text{ or, } P(A \cap B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{2} = \frac{5+8-10}{20} = \mathbf{\frac{3}{20}}.$$

(ii) Event A will occur if and only if any one of the two mutually exclusive events $A \cap B$ and $A \cap B^c$ occurs.

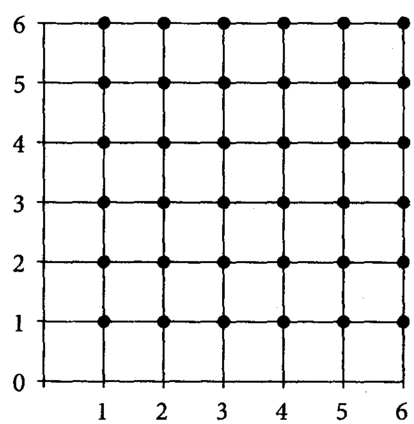
$$\therefore P(A) = P(A \cap B) + P(A \cap B^c) \text{ or, } \frac{1}{4} = \frac{3}{20} + P(A \cap B^c) \text{ or, } P(A \cap B^c) = \frac{1}{4} - \frac{3}{20} = \frac{5-3}{20} = \mathbf{\frac{1}{10}}.$$

(iii) By De Morgan's Law, $(A \cap B)^c = A^c \cup B^c$.
$\therefore P(A^c \cup B^c) = P[(A \cap B)^c] = 1 - P(A \cap B) = 1 - \frac{3}{20} = \mathbf{\frac{17}{20}}$.

Example 24. *If a pair of dice is thrown, find the probability that the sum is neither* 7 *nor* 11.

Solution: If S be the sample space connected with the throwing of the pair of dice, then $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), \ldots, (6,6)\}$.

$\therefore n(S) = 6 \times 6 = 36$.



If A be the event that the sum is 7 and B the event that the sum is 11, then $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ and $B = \{(5,6), (6,5)\}$. $\therefore n(A) = 6$ and $n(B) = 2$.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6} \quad \text{and} \quad P(B) = \frac{n(B)}{n(S)} = \frac{2}{36} = \frac{1}{18}.$$

$\therefore$ the probability that the sum is either 7 or 11

$$= P(A \cup B) = P(A) + P(B) \quad [\because A \text{ and } B \text{ are mutually exclusive events}]$$
$$= \frac{1}{6} + \frac{1}{18} = \frac{4}{18} = \frac{2}{9}.$$

Hence, *the required probability that the sum is neither* 7 *nor* 11

$$= 1 - P(A \cup B) = 1 - \frac{2}{9} = \mathbf{\frac{7}{9}}.$$

9.6 Compound Probability: Conditional Probability

The probability of the simultaneous occurrence of two or more events is called **compound probability** and is denoted by $P(A\cap B)$ for two events A and B, and by $P(A_1\cap A_2\cap\cdots\cap A_r)$ for r events $A_1, A_2, \ldots, A_r$.

The probability of the occurrence of the event B assuming that the event A has actually occurred is called the **conditional probability** of B and is denoted by $P(B/A)$ or $P(B,A)$, which is read as the probability of B given A.

Theorem of Compound Probability (The Rule of Multiplication)

The probability of simultaneous occurrence of A and B is given by the product of the unconditional probability of the event A and the conditional probability of B, assuming that A has actually occurred.

Symbolically, $\boldsymbol{P(A\cap B)=P(A)\cdot P(B/A)}$. [C.U. B.Com.(H) 1997]

Proof. Let A be a subset of the sample space S such that $n(A)\neq 0$, i.e., the event A occurs.

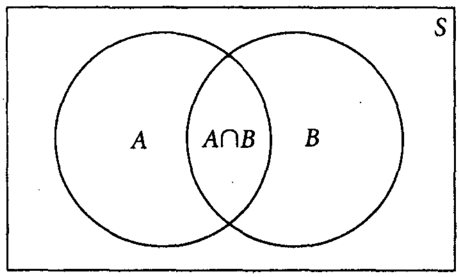


Fig. 9.2

Let us consider, together with $n(A)$, the number of elementary events $n(A\cap B)$ which are favourable to both A and B. Then $n(A)$ is the number of all possible elementary events for the conditional event B/A, assuming that A has actually occurred, out of which $n(A\cap B)$ are favourable to B/A. [In this case, A is the reduced sample space, due to which occurrence of B is restricted to $A\cap B$.]

$\therefore \frac{n(A\cap B)}{n(A)}$ = the conditional probability of B assuming that A has actually occurred $= P(B/A)$

$$\text{or,}\quad P(B/A)=\frac{n(A\cap B)}{n(A)}=\frac{n(A\cap B)/n(S)}{n(A)/n(S)}=\frac{P(A\cap B)}{P(A)}$$

$$\text{or,}\quad \boldsymbol{P(A\cap B)=P(A)\cdot P(B/A)}. \tag{1}$$

Hence, *the theorem follows.*

[**Otherwise.** Let N be the total number of all equally likely cases and m of them are favourable to the event A. Then the cases favourable to both A and B are all included in the m cases favourable to A. Let their number be m_1. The probability $P(A\cap B)$ that both A and B will occur is given by

$$P(A\cap B)=\frac{m_1}{N}=\frac{m}{N}\cdot\frac{m_1}{m}. \tag{1}$$

Now,

$$\frac{m}{N}=\frac{\text{No. of cases favourable to } A}{\text{Total no. of all possible cases}}=P(A).$$

Also assuming the occurrence of A, there are only m equally likely cases left for B (the remaining $N-m$ cases becoming impossible to B), out of which m_1 cases are favourable to B. Hence, the ratio $\frac{m_1}{m}$ represents the conditional probability $P(B/A)$ of B, assuming that A has actually occurred, i.e., $\frac{m_1}{m}=P(B/A)$.

$\therefore$ from (1), we get $P(A \cap B) = P(A) \cdot P(B/A)$. Hence, *the theorem follows.*]

Generalisation. For three events, A, B and C, we have

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B).$$ [C.U. B.Com.(H) 1992]

Proof. $P(A \cap B \cap C) = P(A \cap D)$, where $D = B \cap C$
$= P(A) \cdot P(D/A) = P(A) \cdot P\{(B \cap C)/A\}$
$= P(A) \cdot P(B/A) \cdot P(C/A \cap B)$.

Similarly, for four events A, B, C, D,

$$P(A \cap B \cap C \cap D) = P(A) \cdot P(B/A) \cdot P(C/A \cap B) \cdot P(D/A \cap B \cap C).$$

For r events $A_1, A_2, \ldots, A_r$, the formula becomes

$$P(A_1 \cap A_2 \cap \cdots \cap A_r) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2) \cdots P(A_r/A_1 \cap A_2 \cap \cdots \cap A_{r-1}).$$

Note. The theorem of Compound Probability enables us to find a formula for the calculation of Conditional Probability $P(B/A)$. Since $A \cap B = B \cap A$, interchanging A and B in (1), we get $P(B \cap A) = P(B) \cdot P(A/B)$, and hence, $P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$.

Corollary. If two events are *independent,* the occurrence of one of them will not affect the probability of occurrence of the other. If A and B are independent events, then $P(B/A) = P(B)$. Thus for two independent events A and B, the theorem of Compound Probability becomes $\mathbf{P(A \cap B) = P(A) \cdot P(B)}$.

For r mutually independent events $A_1, A_2, \ldots, A_r$, we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_r) = P(A_1) \cdot P(A_2) \cdots P(A_r).$$ [C.U. B.Com.(H) 1992]

Illustration 1. *If A, B are two independent events with $P(A) = 0.5$ and $P(B) = 0.3$, then* $P(AB) = 0.5 \times 0.3 =$ **0.15.** [$\because$ $P(A \cap B) = P(AB)$]

Dependent and Independent Events

Two events A and B are said to be *independent* if the occurrence of one of them, say A, does not affect the probability of occurrence of the other event B. In other words, A and B are *independent,* if and only if $P(B/A) = P(B)$ and $P(A/B) = P(A)$,

i.e., $P(A \cap B) = P(A) \cdot P(B)$.

Otherwise, they are said to be *dependent* (or interdependent) events. For dependent events, the occurrence or non-occurrence of one of them affects the probability of occurrence of the other events. If A and B are two *dependent events,* then $P(A \cap B) = P(A)P(B/A)$ or $P(B)P(A/B)$.

Illustration 2. *When a coin is tossed twice in succession, the result (say, getting a head) of the first tossing does not affect the result (say, getting a tail) of the second tossing. Hence, the two events are independent.*

Illustration 3. *When a die is thrown twice, the result of the first throw does not affect the result (and hence, probability) of the second throw. Hence the two events are independent.*

Illustration 4. *If a card is drawn from a pack of 52 cards and replaced, and then a card is again drawn from the pack, the two events are independent. But, if the first card is not replaced before the second drawing, the probability of drawing the second card is affected by the first event and hence, in this case the two events are dependent.*

Note. When events are *not independent*, we have to use *conditional probability*, since the result of the second event is conditional, depending on the result of the first event.

Example 25. (i) *If* $P(A)=2/5$, $P(B)=1/3$ *and* $P(A\cup B)=1/2$, *find* $P(A/B)$ *and* $P(B/A)$.

(ii) *If* A *and* B *are two independent events and* $P(A)=2/3$, $P(B)=3/5$, *find* $P(A\cup B)$ *and* $P(A\cap B)$.

[C.U. B.Com.(H) 2005]

Solution: (i) We have
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$\text{or,}\quad \frac{1}{2}=\frac{2}{5}+\frac{1}{3}-P(A\cap B)\quad\text{or,}\quad P(A\cap B)=\frac{2}{5}+\frac{1}{3}-\frac{1}{2}=\frac{7}{30}.$$

We know that $P(A\cap B)=P(A)\cdot P(B/A)=P(B)\cdot P(A/B)$.
$$\therefore\quad P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{7}{30}\Big/\frac{1}{3}=\mathbf{\frac{7}{10}}$$
$$\text{and}\quad P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{7}{30}\Big/\frac{2}{5}=\frac{7}{30}\times\frac{5}{2}=\mathbf{\frac{7}{12}}.$$

(ii) Since A and B are two independent events,
$$\therefore P(A\cap B)=P(A)\cdot P(B)=\frac{2}{3}\times\frac{3}{5}=\mathbf{\frac{2}{5}}.$$

We have
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{2}{3}+\frac{3}{5}-\frac{2}{5}=\frac{10+9-6}{15}=\mathbf{\frac{13}{15}}.$$

Example 26. *In an examination, 30% of the students have failed in Mathematics, 20% of the students have failed in Chemistry and 10% have failed in both Mathematics and Chemistry. A student is selected at random.* (i) *What is the probability that the student has failed in Mathematics if it is known that he has failed in Chemistry?* (ii) *What is the probability that the student has failed either in Mathematics or in Chemistry?*

Solution: Let M and C be the events that a student selected at random 'fails in Mathematics' and 'fails in Chemistry' respectively.

Then
$$P(M)=\frac{30}{100}=0.3,\quad P(C)=\frac{20}{100}=0.2\quad\text{and}\quad P(M\cap C)=\frac{10}{100}=0.1.$$

(i) The required probability that the student selected has *failed in Mathematics given that he has failed in Chemistry* $=P(M/C)=\frac{P(M\cap C)}{P(C)}=\frac{0.1}{0.2}=\frac{1}{2}=0.5$.

(ii) The required probability that the student selected at random has failed *either in Mathematics or in Chemistry* $=P(M\cup C)=P(M)+P(C)-P(M\cap C)=0.3+0.2-0.1=\mathbf{0.4}$.

Example 27. *The personnel department of a company has records which show the following analysis of its 200 engineers:*

Age	Bachelor's degree only	Master's degree	Total
Under 30	90	10	100
30 to 40	20	30	50
Over 40	40	10	50
Total	150	50	200

If one engineer is selected at random from the company, find: (a) *The probability that he has only a bachelor's degree;* (b) *The probability that he has a master's degree, given that he is over* 40; (c) *The probability that he is under* 30, *given that he has only a bachelor's degree.*

Solution: Let A be the event that an engineer has only a bachelor's degree; B, that he is over 40 years of age; C, that he has master's degree and D, that he is under 30 years of age.

(a) The probability that he has only a bachelor's degree is $P(A)=\frac{150}{200}=\mathbf{\frac{3}{4}}$.

(b) The probability that an engineer has a master's degree, given that he is over 40, is

$$P(C/B)=\frac{P(C\cap B)}{P(B)}=\frac{\frac{10}{200}}{\frac{50}{200}}=\frac{10}{50}=\mathbf{\frac{1}{5}}.$$

(c) The probability that an engineer under 30, given that he has only a bachelor's degree, is

$$P(D/A)=\frac{P(D\cap A)}{P(A)}=\frac{\frac{90}{200}}{\frac{150}{200}}=\frac{90}{150}=\mathbf{\frac{3}{5}}.$$

Example 28. *An unbiased coin is tossed twice in succession. Write down the sample space. Show that the events "first toss is a head" and the event "second toss is a head" are independent events.* [C.U. B.Com.(H) 1995]

Solution: If S be the sample space, then $S=\{HH, HT, TH, TT\}$, where H denotes "head" and T denotes "Tail", when the coin is tossed.

Let A be the event "first toss is a head" and B, that "second toss is a head". Then $A=\{HH, HT\}$, $B=\{HH, TH\}$ and $A\cap B=\{HH\}$; $\therefore n(S)=4, n(A)=2, n(B)=2$.

$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{2}{4}=\frac{1}{2},\quad P(B)=\frac{n(B)}{n(S)}=\frac{2}{4}=\frac{1}{2}\quad\text{and}\quad P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{4}.$$

Now, $P(A)\cdot P(B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}=P(A\cap B)$, i.e., $\mathbf{P(A\cap B)=P(A)\cdot P(B)}$.

Hence, *the events A and B are independent.*

Example 29. *If A and B be two independent events, prove that* (i) *A and* B^c, (ii) A^c *and* B^c *are also independent.* [B.U. B.Com.(H) 2008; C.U. B.Com.(H) 2004; V.U. B.Com.(H) 2010]

Solution: (i) For any two events A and B, we have

$$P(A)=P(A\cap B)+P(A\cap B^c)\quad\text{or,}\quad P(A\cap B^c)=P(A)-P(A\cap B). \tag{1}$$

Since A and B are two independent events, $P(A\cap B)=P(A)\cdot P(B)$.

$\therefore$ from (1), $P(A\cap B^c)=P(A)-P(A)\cdot P(B)=P(A)\{1-P(B)\}$

$$\text{or,}\quad P(A\cap B^c)-P(A)\cdot P(B^c)\quad [\because 1-P(B)=P(B^c)].$$

This shows that A and B^c are also *independent*.

(ii) By De Morgan's Law, $A^c\cap B^c=(A\cup B)^c$;

$$\begin{aligned}\therefore P(A^c\cap B^c)&=P\{(A\cup B)^c\}=1-P(A\cup B)=1-\{P(A)+P(B)-P(A\cap B)\}\\ &=1-P(A)-P(B)+P(A)\cdot P(B)\ [\because A \text{ and } B \text{ are two independent events}]\\ &=\{1-P(A)\}-P(B)\{1-P(A)\}=\{1-P(A)\}\{1-P(B)\}=P(A^c)\cdot P(B^c).\end{aligned}$$

i.e., $P(A^c\cap B^c)=P(A^c)\cdot P(B^c)$.

This shows that A^c and B^c are also *independent events.*

Example 30. *The probabilities of n independent events are $p_1, p_2, \ldots, p_n$. Find an expression for the probability that at least one of the events will happen.* [C.U. B.Com.(H) 2002]

Solution: Probability that the first event will not happen $= (1 - p_1)$.

Probability that the second event will not happen $= (1 - p_2)$ and so on.

$\therefore$ probability that none of the n events will happen $= (1 - p_1)(1 - p_2)\cdots(1 - p_n)$.

[$\because p_1, p_2, \ldots, p_n$ are independent events, $1 - p_1, 1 - p_2, \ldots, 1 - p_n$ are also independent events.]

Hence, the required probability that at least one of the events will happen

$$= 1 - (1 - p_1)(1 - p_2)\cdots(1 - p_n).$$

Example 31. (a) *The probability that A speaks the truth is 0.4 and that B speaks the truth is 0.7. What is the probability that they will contradict each other?* [C.U. B.Com.(H) 2001]

(b) *A speaks the truth 3 out of 4 times and B speaks the truth 5 out of 6 times. What is the probability that they will contradict each other in stating the same fact.* [B.U. B.Com.(H) 2006]

Solution: (a) Let A be the event that A speaks the truth. Similarly, B the event that B speaks the truth. Then $P(A) = 0.4, P(B) = 0.7, P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$ and $P(B^c) = 1 - P(B) = 1 - 0.7 = 0.3$.

A and B will contradict each other in the following two *mutually exclusive events:*

(i) A will speak the truth and B will not speak the truth, i.e., $A \cap B^c$;

(ii) A will not speak the truth and B will speak the truth, i.e., $A^c \cap B$.

Hence, *the required probability* $= P(A \cap B^c) + P(A^c \cap B) = P(A) \cdot P(B^c) + P(A^c) \cdot P(B)$ [$\because A$ and B are two independent events] $= 0.4 \times 0.3 + 0.6 \times 0.7 = 0.12 + 0.42 = \mathbf{0.54}$.

[Hints: (b) $P(A) = \frac{3}{4}, P(B) = \frac{5}{6}; \therefore P(A^c) = 1 - \frac{3}{4} = \frac{1}{4}$ and $P(B^c) = 1 - \frac{5}{6} = \frac{1}{6}$.

Required Probability $= P(A) \times P(B^c) + P(A^c) \times P(B) = \frac{3}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{5}{6} = \frac{3+5}{24} = \frac{8}{24} = \frac{1}{3}$.]

Example 32. *One purse contains 1 dollar and ₹3, a second purse contains 2 dollars and ₹4 and a third contains 3 dollars and 1. If a coin is taken out of one of the purses selected at random, find the chance that it is a dollar.* [C.U. B.Com.(H) 2000]

Solution: Let A, B, C be the events of selecting the first, second and third purses respectively. Let D be the event that a coin drawn from the selected purse is a dollar.

Then $P(A) = \frac{1}{3}, P(B) = \frac{1}{3}, P(C) = \frac{1}{3}, P(D/A) = \frac{1}{4}, P(D/B) = \frac{2}{6} = \frac{1}{3}, P(D/C) = \frac{3}{4}$.

Probability of selecting 1 dollar from the first purse

$$= P(A \cap D) = P(A) \cdot P(D/A) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}.$$

Probability of selecting 1 dollar from the second purse $= P(B \cap D) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ and probability of selecting 1 dollar from the third purse $= P(C \cap D) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$.

$\therefore$ by *the theorem of total probability,* the required probability

$$= P(A \cap D) + P(B \cap D) + P(C \cap D) = \frac{1}{12} + \frac{1}{9} + \frac{1}{4} = \frac{3+4+9}{36} = \frac{16}{36} = \mathbf{\frac{4}{9}}.$$

Example 33. *A problem in Statistics is given to three students, A, B, C, whose chances of solving it are* $\frac{1}{2}, \frac{1}{3}$ *and* $\frac{1}{4}$ *respectively. If they try it independently, what is the probability that the problem will be solved? Find also the probability that the problem could not be solved.* [C.U. B.Com.(H) 2006]

Solution: Let A be the event that student A will solve the problem. Similarly, B and C be the events for the students B and C respectively, to solve the problem.

Then
$$P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(C)=\frac{1}{4}.$$

$\therefore P(A^c)=$ probability that A will not solve the problem $=1-P(A)=1-\frac{1}{2}=\frac{1}{2}$.

Similarly, $P(B^c)=1-\frac{1}{3}=\frac{2}{3}$, and $P(C^c)=1-\frac{1}{4}=\frac{3}{4}$.

$\therefore$ the probability that none of the students A, B, C will solve the problem

$$=P(A^c)\cdot P(B^c)\cdot P(C^c)=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}.$$

[$\because A, B, C$ are independent events, A^c, B^c, C^c are also independent events]

Hence, *the required probability that the problem will be solved (i.e., at least one of A, B, C will solve the problem)* $=1-\frac{1}{4}=\frac{3}{4}$.

[**Otherwise.**

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$
$$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\left(\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}\right)+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}$$

($\because A, B, C$ are independent events)

$$=\frac{13}{12}-\frac{9}{24}+\frac{1}{24}=\frac{18}{24}=\frac{3}{4}.]$$

The probability that the problem could not be solved $=1-\frac{3}{4}=\frac{1}{4}$.

Example 34. *A bag contains* 8 *red balls and* 5 *white balls. Two successive draws of* 3 *balls are made without replacement. Find the probability that the first drawing will give* 3 *white balls and the second,* 3 *red balls.*

Solution: Let A be the event that the 3 balls in the first drawing are all white, and B, that the 3 balls in the second drawing are all red. Total number of balls in the bag is $8+5=13$.

3 balls can be drawn out of 13 in ${}^{13}C_3$ ways $=\frac{13\cdot 12\cdot 11}{3\cdot 2\cdot 1}=286$ ways.

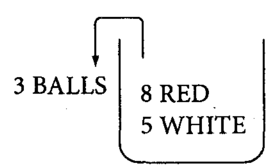
$\therefore n(S)=286$.

3 white balls can be drawn out of 5 in ${}^{5}C_3$ ways $=\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}=10$ ways.

$\therefore n(A)=10$;

$\therefore P(A)=$ probability of getting 3 white balls in the first drawing

$$=\frac{n(A)}{n(S)}=\frac{10}{286}=\frac{5}{143}.$$

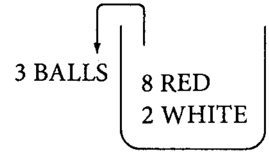


First drawing

If the 3 white balls are not replaced, we are left with 8 red and 2 white balls. We have to find the conditional probability $P(B/A)$ where A has actually occurred.

In this case, total no. of balls is $8 + 2 = 10$.

3 balls can be drawn out of 10 in ${}^{10}C_3$ ways $= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ ways
and 3 red balls can be drawn out of 8 in ${}^{8}C_3$ ways $= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$
$= 56$ ways.



Second drawing

$\therefore P(B/A) = \frac{56}{120} = \frac{7}{15}$ [$\because$ the 3 white balls are not replaced after the first drawing.]

Hence, *by the theorem of compound probability, the required probability*

$$= P(A \cap B) = P(A) \cdot P(B/A) = \frac{5}{143} \times \frac{7}{15} = \mathbf{\frac{7}{429}}.$$

Example 35. *Two sets of candidates are competing for the positions on the Board of Directors of a company. The probabilities that the first and second sets will win are* 0.6 *and* 0.4 *respectively. If the first set wins, the probability of introducing a new product is* 0.8, *and the corresponding probability, if the second set wins, is* 0.3. *What is the probability that the new product will be introduced?*

Solution: Let A be the event that the first set will win; B, that the second set will win and C, that the new product will be introduced.

Then C occurs, if and only if one of the two mutually exclusive events $A \cap C$ and $B \cap C$ occurs. Therefore, $P(C) = P(A \cap C) + P(B \cap C)$,

$$\text{i.e.,} \quad P(C) = P(A) \cdot P(C/A) + P(B) \cdot P(C/B). \qquad (1)$$

Given $P(A) = 0.6$, $P(B) = 0.4$. $P(C/A) = 0.8$ and $P(C/B) = 0.3$.

$\therefore$ from (1), $P(C) = 0.6 \times 0.8 + 0.4 \times 0.3 = 0.48 + 0.12 = 0.60$.

Hence, *the required probability is* **0.60.**

Example 36. *The odds against student X solving a Business Statistics problem are* 8 *to* 6, *and odds in favour of student Y solving the same problem are* 14 *to* 16. (i) *What is the chance that the problem will be solved if they both try independently of each other?* (ii) *What is the probability that neither solves the problem?*

Solution: Let A be the event that the first student X will solve the problem and B, that the second student Y will solve the problem.

Then clearly,

$$P(A) = \frac{6}{8+6} = \frac{6}{14} = \frac{3}{7} \quad \text{and} \quad P(B) = \frac{14}{14+16} = \frac{14}{30} = \frac{7}{15}.$$

[$\because$ odds against an event = (no. of unfavourable cases) : (no. of favourable cases).
and odds in favour of the event = (no. of favourable cases) : (no. of unfavourable cases).]

(i) The required probability that the problem will be solved if they both try independently of each other is $P(A \cup B)$.

$P(A \cap B) = P(A) \cdot P(B)$ [$\because A$ and B are independent events] $= \frac{3}{7} \times \frac{7}{15} = \frac{1}{5}$.

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{7} + \frac{7}{15} - \frac{1}{5} = \frac{45 + 49 - 21}{105} = \mathbf{\frac{73}{105}}.$$

(ii) The probability that X cannot solve the problem is

$$P(A^c) = 1 - P(A) = 1 - \frac{3}{7} = \frac{4}{7}.$$

Similarly, $$P(B^c) = 1 - \frac{7}{15} = \frac{8}{15}.$$

Hence, *the required probability that neither solves the problem is*
$P(A^c \cap B^c) = P(A^c) \times P(B^c)$ [$\because A^c$ and B^c are independent events] $= \frac{4}{7} \times \frac{8}{15} = \mathbf{\frac{32}{105}}$.

[**Otherwise.** The required probability that neither solves the problem $= 1-$ (Probability that at least one of X and Y will solve the problem) $= 1 - P(A \cup B) = 1 - \frac{73}{105} = \frac{32}{105}$.]

Example 37. *There are three men aged* 60, 65 *and* 70 *years. The probability to live* 5 *years more is* 0.8 *for a* 60-*year old,* 0.6 *for a* 65-*year old and* 0.3 *for a* 70-*year old persons. Find the probability that at least two of the three persons will remain alive* 5 *years hence.*

Solution: Let A be the event that the man aged 60 years will live 5 years more. Similarly, events B and C denote the same thing for the other two men aged 65 and 70 years respectively. Then $P(A) = 0.8, P(B) = 0.6, P(C) = 0.3; \therefore P(A^c) = 1 - P(A) = 1 - 0.8 = 0.2, P(B^c) = 0.4$ and $P(C^c) = 0.7$.

Clearly, A, B, C are three *independent events*. $\therefore A^c, B^c, C^c$ are also independent events.

The required probability that at least two of the three persons will remain alive 5 *years hence*

$$= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) + P(A \cap B \cap C)$$
$$= P(A) \cdot P(B) \cdot P(C^c) + P(A) \cdot P(B^c) \cdot P(C) + P(A^c) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C)$$

[$\because A, B, C, A^c, B^c, C^c$ are all *independent events*]

$$= 0.8 \times 0.6 \times 0.7 + 0.8 \times 0.4 \times 0.3 + 0.2 \times 0.6 \times 0.3 + 0.8 \times 0.6 \times 0.3$$
$$= 0.336 + 0.096 + 0.036 + 0.144 = \mathbf{0.612}.$$

9.7 Bayes' Theorem

We can revise probabilities when new (or additional) information concerning a random experiment is available. The procedure for revising probabilities due to a specific cause is known as *Bayes' theorem* and it was originally developed by Rev. Thomas Bayes. It gives a probability law relating *a posteriori probability to a priori probability.*

Statement. *An event A can occur, only if any one of the set of exhaustive and mutually exclusive events* $B_1, B_2, \ldots, B_n$ *occurs. The probabilities* $P(B_1), P(B_2), \ldots, P(B_n)$ *and the conditional probabilities* $P(A/B_i)$; $i = 1, 2, \ldots, n$, *for A to occur are known. Then the conditional probability* $P(B_i/A)$, *when A has actually occurred, is given by*

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + \cdots + P(B_n) \cdot P(A/B_n)}.$$

Alternative Statement. *If* $B_1, B_2, \ldots, B_n$ *are mutually disjoint events with* $P(B_i) \neq 0$; $i = 1, 2, \ldots, n$, *then for any event A which is a subset of* $B_1 \cup B_2 \cup \cdots \cup B_n$ *such that* $P(A) > 0$, *we have*

$$\mathbf{P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}}.$$

Proof. Since $B_1, B_2, \ldots, B_n$ are mutually exclusive events and since A occurs only when any one of $B_1, B_2, \ldots, B_n$ occurs, therefore, A occurs in any one of the mutually exclusive events $A \cap B_1, A \cap B_2, \ldots, A \cap B_n$.

$\therefore$ by the *theorem of total probability,* we have

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n) \\ &= P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_n \cap A) \\ &= P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + \cdots + P(B_n) \cdot P(A/B_n) \end{aligned}$$

[By the Theorem of compound probability]

Again, by the theorem of compound probability, we have $P(A \cap B_i) = P(A) \cdot P(B_i/A)$ and $P(B_i \cap A) = P(B_i) \cdot P(A/B_i)$.

$$\therefore P(A) \cdot P(B_i/A) = P(B_i) \cdot P(A/B_i) \quad [\because P(A \cap B_i) = P(B_i \cap A)]$$

$$\begin{aligned} \text{or,} \quad P(B_i/A) &= \frac{P(B_i) \cdot P(A/B_i)}{P(A)} \\ &= \frac{P(B_i) \cdot P(A/B_i)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + \cdots + P(B_n) \cdot P(A/B_n)}. \end{aligned}$$

Hence, *the theorem follows.*

Example 38. *There are two identical boxes containing respectively* 4 *white and* 3 *red balls, and* 3 *white and* 7 *red balls. A box is chosen at random and a ball is drawn from it. If the ball is white, what is the probability that it is from the first box?* [C.U. B.Com. 2002]

Solution: Let A be the event that the ball drawn is white and let B_1, B_2 be the events of choosing the first and the second boxes respectively.

Then $P(B_1)$ = probability of selecting the first box $= \frac{1}{2}$

and $P(B_2)$ = probability of selecting the second box $= \frac{1}{2}$.

$P(A/B_1)$ = probability of selecting 1 white ball from the first box $= \frac{{}^4C_1}{{}^7C_1} = \frac{4}{7}$.

Similarly, $P(A/B_2) = \frac{3}{10}$.

By Bayes' theorem, we get

$$\begin{aligned} P(B_1/A) &= \text{the required probability that if the ball is white, then it is from the first box} \\ &= \frac{P(B_1) \cdot P(A/B_1)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)} \\ &= \frac{\frac{1}{2} \times \frac{4}{7}}{\frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{10}} = \frac{4}{14} \times \frac{140}{61} = \mathbf{\frac{40}{61}}. \end{aligned}$$

Note. $P(B_2/A)$ = probability that if the ball is white, it is from the second box

$$= \frac{P(B_2) \cdot P(A/B_2)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)} = \frac{\frac{1}{2} \times \frac{3}{10}}{\frac{61}{140}} = \mathbf{\frac{21}{61}}.$$

Example 39. *In a bolt factory, machines M_1, M_2, M_3 manufacture respectively* 25, 35 *and* 40 *per ceht of the total output. Of their output,* 5, 4 *and* 2 *per cent respectively are defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured in the machine M_2?*

Solution: Let B_1, B_2, B_3 be the events that the bolt drawn at random is manufactured by machines M_1, M_2, M_3 respectively, and let A be the event that the bolt drawn is defective.

Then $P(B_1)$ = probability that the bolt drawn was manufactured by $M_1 = \frac{25}{100} = \frac{1}{4}$.

Similarly, $P(B_2) = \frac{35}{100} = \frac{7}{20}$ and $P(B_3) = \frac{40}{100} = \frac{2}{5}$.

$P(A/B_1)$ = the probability of drawing a defective bolt from the lot of bolts manufactured by $M_1 = \frac{5}{100} = \frac{1}{20}$. Similarly, $P(A/B_2) = \frac{4}{100} = \frac{1}{25}$ and $P(A/B_3) = \frac{2}{100} = \frac{1}{50}$.

By *Bayes' Theorem*, we have

$$P(B_2/A) = \frac{P(B_2)\cdot P(A/B_2)}{P(B_1)\cdot P(A/B_1) + P(B_2)\cdot P(A/B_2) + P(B_3)\cdot P(A/B_3)}$$

$$= \frac{\frac{7}{20}\times\frac{1}{25}}{\frac{1}{4}\times\frac{1}{20}+\frac{7}{20}\times\frac{1}{25}+\frac{2}{5}\times\frac{1}{50}} = \frac{\frac{7}{500}}{\frac{25+28+16}{2000}} = \frac{7}{500}\times\frac{2000}{69} = \frac{28}{69}.$$

Hence, *the required probability that the defective bolt drawn is manufactured by machine* $M_2 = \frac{28}{69}$.

Note. The probability that the defective bolt drawn is manufactured by machine M_1

$$= P(B_1/A) = \frac{P(B_1)\cdot P(A/B_1)}{\sum_{i=1}^{3} P(B_i)\cdot P(A/B_i)} = \frac{\frac{1}{80}}{\frac{69}{2000}} = \frac{25}{69}$$

and the probability that the defective bolt drawn is manufactured by machine M_3

$$= P(B_3/A) = \frac{P(B_3)\cdot P(A/B_3)}{\sum_{i=1}^{3} P(B_i)\cdot P(A/B_i)} = \frac{\frac{1}{125}}{\frac{69}{2000}} = \frac{16}{69}.$$

9.8 Probability: Axiomatic Definition

Let S be a sample space and A be any event in S (i.e., $A \subseteq S$). We define a function P on S [for any event A, the functional value $P(A)$ is a real number] and call it a probability function on the sample space S if the following three axioms hold:

1. *For any event A in S, $P(A) \geq 0$.*
2. *$P(S) = 1$.*
3. *If $A_1, A_2, \ldots$ be any sequence of mutually exclusive events in S, then*

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

We can now prove a number of theorems directly from the axioms.

Theorem 1. *For every event A, $P(A^c) = 1 - P(A)$, where A^c is the complement of A.* [B.U. B.Com.(H) 2006]

Proof. For every event A, the sample space S can be decomposed into the two mutually exclusive events A and A^c, i.e., $S = A \cup A^c$.

Therefore, $P(S) = P(A \cup A^c) = P(A) + P(A^c)$, by *Axiom* 3.

Again, by *Axiom* 2, $P(S) = 1$. $\therefore P(A) + P(A^c) = 1$. Hence, $\boldsymbol{P(A^c) = 1 - P(A)}$.

Theorem 2. *For every event A in S,* $0 \leq P(A) \leq 1$.

Proof. By *Axiom* 1, $P(A) \geq 0$ and $P(A^c) \geq 0$.

Since $P(A^c) = 1 - P(A)$; $\therefore 1 - P(A) \geq 0$ or, $1 \geq P(A)$.

Thus, $0 \leq P(A)$ and $P(A) \leq 1$.

Hence, $\boldsymbol{0 \leq P(A) \leq 1}$.

Theorem 3. *If* ϕ *is the null set, then* $P(\phi) = 0$.

Proof. For any event A, $A \cup \phi = A$, where ϕ is the null set (i.e., impossible event).

$\therefore P(A) = P(A \cup \phi) = P(A) + P(\phi)$, by *Axiom* 3 [$\because$ A and ϕ are two mutually exclusive events]. Hence, $\boldsymbol{P(\phi) = 0}$.

Theorem 4. *If* $A \subseteq B$, *then* $P(A) \leq P(B)$.

Proof. If $A \subseteq B$, then B can be decomposed into the two mutually exclusive events A and $(B - A)$, i.e., $B = A \cup (B - A)$.

$\therefore P(B) = P[A \cup (B - A)] = P(A) + P(B - A)$, by *Axiom* 3.

Since by *Axiom* 1, $P(B - A) \geq 0$, we get $P(B) \geq P(A)$, i.e., $P(A) \leq P(B)$.

Theorem 5. *If A and B are any two events, then* $P(A - B) = P(A) - P(A \cap B)$.

Proof. If A and B are any two events, then A can be decomposed into the mutually exclusive events $(A - B)$ and $A \cap B$, i.e., $A = (A - B) \cup (A \cap B)$.

By *Axiom* 3, $P(A) = P[(A - B) \cup (A \cap B)] = P(A - B) + P(A \cap B)$.

Hence, $\boldsymbol{P(A - B) = P(A) - P(A \cap B)}$.

Theorem 6. *If A and B be any two events, then* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. Event $A \cup B$ can be decomposed into the mutually exclusive events $A - B$ and B, i.e., $A \cup B = (A - B) \cup B$.

By *Axiom* 3, $P(A \cup B) = P(A - B) + P(B)$,

and by *Theorem* 5, $P(A - B) = P(A) - P(A \cap B)$.

Hence, $P(A \cup B) = P(A) - P(A \cap B) + P(B) = \boldsymbol{P(A) + P(B) - P(A \cap B)}$.

Corollary. Using the result of *Theorem* 6, we can prove that for any three events A, B, C, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

9.8.1 Finite Probability Space : Equiprobable Space

Let S be a finite sample space consisting of N elementary events (called sample points), i.e., $S = \{a_1, a_2, \ldots, a_N\}$. Then a finite probability space is obtained by assigning to each sample point a_i of S a real number p_i, called the probability of a_i satisfying (i) $p_i \geq 0$ and (ii) $p_1 + p_2 + \cdots + p_N = 1$. The probability $P(A)$ of any event A of S is defined as the sum of the probabilities of the sample points in A. If A is made up of m elementary events $A_1, A_2, \ldots, A_m$, then $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and, therefore,

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_m), \quad \text{by } \textit{Axiom } 3.$$

In particular, if all elementary events, say $A_1, A_2, \ldots, A_N$, of the sample space S be assigned *equal probabilities,* then the sample space S is said to be **equiprobable** or **uniform space** in which the probability of each elementary event is $1/N$. In this case,

$$\begin{aligned} P(A) &= \frac{1}{N} + \frac{1}{N} + \cdots \text{to } m \text{ terms} = \frac{m}{N} = \frac{\text{no. of elements in } A}{\text{no. of elements in } S} \\ &= \frac{n(A)}{n(S)}, \quad \text{which is the } \textit{classical definition of probability.} \end{aligned}$$

9.9 Miscellaneous Problems

Example 40. *Three horses A, B and C are in a race: A is twice as likely to win as B and B is twice as likely to win as C. What are the respective probabilities of winning?*

Solution: Let the probability of the event that C will win = p, i.e., $P(C) = p$.

Then the probability that B will win $= 2p$, i.e., $P(B) = 2p$

and the probability that A will win $= 2 \times 2p = 4p$, i.e., $P(A) = 4p$.

Since either A or B or C will win the race; $\therefore P(A) + P(B) + P(C) = 1$.

$\therefore p + 2p + 4p = 1$ or, $7p = 1$ or, $p = \frac{1}{7}$.

Hence, $P(A) = 4p = \frac{4}{7}$, $P(B) = 2p = \frac{2}{7}$ and $P(C) = p = \frac{1}{7}$.

Example 41. *There are* 100 *students in a college class of which* 36 *are boys studying Statistics and* 13 *are girls not studying Statistics. If there are* 55 *girls in all, find the probability that a boy picked up at random is not studying Statistics.*

Solution: Total number of boys in the college class $= 100 - 55 = 45$.

1 boy can be selected out of 45 boys in ${}^{45}C_1$ ways = 45 ways.

$\therefore$ total number of all possible cases for the event = 45.

Now 1 boy studying Statistics can be selected out of 36 in ${}^{36}C_1$ ways = 36 ways.

$\therefore$ no. of cases favourable to the event A that the boy picked up at random is studying Statistics = 36.

$\therefore P(A) =$ probability that the boy picked up is *studying Statistics* $= \frac{36}{45} = \frac{4}{5}$.

Hence, *the required probability that the boy picked up at random is not studying Statistics* $= P(A^c) = 1 - P(A) = 1 - \frac{4}{5} = \frac{1}{5}$.

[**Otherwise.** Total number of boys not studying Statistics $= 45 - 36 = 9$.

1 boy can be selected out of 9 boys in ${}^{9}C_1$ ways = 9 ways.

$\therefore$ no. of favourable cases for the event that a boy picked up at random is not studying Statistics = 9. Hence, the required probability $= \frac{9}{45} = \frac{1}{5}$.]

Example 42. *A packet of* 10 *electronic components is known to include* 3 *defectives. If* 4 *components are randomly chosen and tested, what is the probability of finding among them not more than one defective?*

Solution: 4 components can be chosen out of 10 in ${}^{10}C_4$ ways.

$\therefore$ total no. of all possible cases $= {}^{10}C_4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$, i.e., $n(S) = 210$.

Now among 4 components drawn, not more than one defective can occur in the following two mutually exclusive events:

(i) A: getting 1 defective component and 3 non-defective components.

(ii) B: getting no defective component, i.e., 4 non-defective components.

1 defective component can be drawn out of 3 in ${}^3C_1$ ways and 3 non-defective components can be drawn out of 7 in ${}^7C_3$ ways.

$\therefore n(A) =$ no. of cases favourable to $A = {}^3C_1 \times {}^7C_3 = 3 \times 35 = 105$.

Similarly, $n(B) =$ no. of cases favourable to $B = {}^7C_4 = 35$.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{105}{210} = \frac{1}{2} \quad \text{and} \quad P(B) = \frac{n(B)}{n(S)} = \frac{35}{210} = \frac{1}{6}.$$

Hence, by the theorem of total probability, *the required probability*

$$= P(A \cup B) = P(A) + P(B) = \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{2}{3}.$$

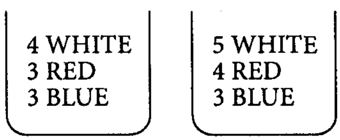
Example 43. *Boxes* 1 *and* 2 *contain respectively* 4 *white,* 3 *red and* 3 *blue balls; and* 5 *white,* 4 *red and* 3 *blue balls. If one ball is drawn at random from each box, what is the probability that both the balls are of the same colour?* [C.U. B.Com.(H) 2000; V.U. B.Com.(H) 2011]

Solution: Both the balls will be of the same colour if they are either (i) both white, or (ii) both red, or (iii) both blue.

Let A, B, C be the events that both the balls are white, red and blue, respectively.

1 ball can be drawn out of 10 from box 1 in ${}^{10}C_1$ ways = 10 ways and 1 white ball can be drawn out of 4 in ${}^4C_1$ ways = 4 ways.

$\therefore P(A_1) =$ probability of drawing 1 white ball from box 1 $= \frac{{}^4C_1}{{}^{10}C_1} = \frac{4}{10} = \frac{2}{5}$.



Box 1 Box 2

Similarly, $P(A_2) =$ probability of drawing 1 white ball from box 2 $= \frac{{}^5C_1}{{}^{12}C_1} = \frac{5}{12}$.

Since A_1 and A_2 are two independent events,

$$\therefore P(A) = P(A_1) \times P(A_2) = \frac{2}{5} \times \frac{5}{12} = \frac{1}{6}.$$

Similarly, $P(B) =$ probability that both the balls are red $= \frac{{}^3C_1}{{}^{10}C_1} \times \frac{{}^4C_1}{{}^{12}C_1} = \frac{3}{10} \times \frac{4}{12} = \frac{1}{10}$

and $P(C) =$ probability that both the balls are blue $= \frac{{}^3C_1}{{}^{10}C_1} \times \frac{{}^3C_1}{{}^{12}C_1} = \frac{3}{10} \times \frac{3}{12} = \frac{3}{40}$.

Since A, B and C are three mutually exclusive events, by the theorem of total probability, *the required probability that both the balls are of the same colour* $= \frac{1}{6} + \frac{1}{10} + \frac{3}{40} = \frac{20+12+9}{120} = \mathbf{\frac{41}{120}}$.

Example 44. *If $P(A\cap B^c)=1/3$ and $P(A\cup B)=2/3$, find $P(B)$. What is $P(A)$, if $P(A\cap B)=1/6$?*

Solution: By De Morgan's Law, $(A\cup B)^c = A^c\cap B^c$.

$\therefore P(A^c\cap B^c)=P[(A\cup B)^c]=1-P(A\cup B)=1-2/3=1/3$.

Event B^c will occur, if and only if any one of the two mutually exclusive events $A\cap B^c$ and $A^c\cap B^c$ occurs.

$$\therefore P(B^c)=P(A\cap B^c)+P(A^c\cap B^c)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}.$$

Hence, $P(B)=1-P(B^c)=1-\frac{2}{3}=\frac{1}{3}$.

2nd Part. Event A will occur, if and only if any one of the two mutually exclusive events $A\cap B$ and $A\cap B^c$ occurs.

$$\therefore P(A)=P(A\cap B)+P(A\cap B^c)=\frac{1}{6}+\frac{1}{3}=\frac{1+2}{6}=\mathbf{\frac{1}{2}}.$$

Example 45. *A candidate is selected for interview for three posts. For the first post there are 3 candidates, for the second there are 4, and for the third there are 2. What is the chance of his getting at least one post?*

Solution: Let A be the event that the candidate gets the first post, B be the event that he gets the second post, and C, that he gets the third post.

Then $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$ and $P(C)=\frac{1}{2}$.

$\therefore P(A^c)=1-\frac{1}{3}=\frac{2}{3}$, $P(B^c)=1-\frac{1}{4}=\frac{3}{4}$, $P(C^c)=1-\frac{1}{2}=\frac{1}{2}$.

Since A, B, C are independent, therefore, A^c, B^c, C^c are also independent.

$\therefore$ the chance that the candidate does not get any post $=P(A^c)\cdot P(B^c)\cdot P(C^c)$
$=\frac{2}{3}\times\frac{3}{4}\times\frac{1}{2}=\frac{1}{4}$.

Hence, *the required chance that he gets at least one post* $=1-\frac{1}{4}=\mathbf{\frac{3}{4}}$.

[**Otherwise.** The required chance $=P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$. Here $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$, $P(C)=\frac{1}{2}$; $P(A\cap B)=P(A)\cdot P(B)=\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}$, $P(B\cap C)=P(B)\cdot P(C)=\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$, $P(A\cap C)=P(A)\cdot P(C)=\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$ and $P(A\cap B\cap C)=\frac{1}{3}\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{24}$, since A, B, C are independent.

Hence, *the required chance* $=\frac{1}{3}+\frac{1}{4}+\frac{1}{2}-\frac{1}{12}-\frac{1}{8}-\frac{1}{6}+\frac{1}{24}=\mathbf{\frac{3}{4}}$.]

Example 46. *A, B, C and D are four mutually exclusive and exhaustive events. If the odds against the events B, C, D are respectively 7 : 2, 7 : 5 and 13 : 5, find the odds in favour of the event A.*

Solution: Odds against an event = (no. of unfavourable cases) : (no. of favourable cases).

Since the odds against the events B, C and D are $7:2, 7:5$ and $13:5$ respectively,

$$\therefore P(B)=\frac{2}{7+2}=\frac{2}{9},\quad P(C)=\frac{5}{7+5}=\frac{5}{12},\quad P(D)=\frac{5}{13+5}=\frac{5}{18}.$$

Again, since A, B, C and D are four mutually exclusive and exhaustive events,

$$\therefore\quad P(A\cup B\cup C\cup D)=P(A)+P(B)+P(C)+P(D)\quad\text{and}\quad P(A\cup B\cup C\cup D)=1.$$

$$\therefore\quad P(A)+P(B)+P(C)+P(D)=1\quad\text{or,}\quad P(A)+\frac{2}{9}+\frac{5}{12}+\frac{5}{18}=1;$$

$$\therefore\quad P(A)=1-\left(\frac{2}{9}+\frac{5}{12}+\frac{5}{18}\right)=1-\frac{11}{12}=\frac{1}{12}\quad\text{and}\quad P(A^c)=1-\frac{1}{12}=\frac{11}{12}.$$

Hence, *the required odds in favour of the event* $A=P(A):P(A^c)=\mathbf{1:11}$.

Example 47. *Two persons A and B stand in a line with* 10 *other persons. What is the probability that there are* 3 *persons between A and B?*

Solution: Total no. of persons = 2 + 10 = 12. 12 persons can be arranged in a line in 12! ways. ∴ total no. of all possible equally likely cases = 12!.

To find the no. of favourable cases, let the 1st, 2nd, 3rd, ..., 12th places be represented by 1, 2, 3, ..., 12. There will be 3 persons between A and B if they occupy the following 8 pairs of places: $(1,5)$, $(2,6)$, $(3,7)$, $(4,8)$, $(5,9)$, $(6,10)$, $(7,11)$, $(8,12)$. When A and B occupy places 1 and 5 respectively, the remaining 10 places can be filled up by the other 10 persons in 10! ways. But A and B can be interchanged in their positions in 2! ways.

∴ the no. of arrangements when A and B occupy the 1st and 5th places = $2! \times 10!$.

Similarly, the no. of arrangements when A and B occupy any one of the other 7 pairs of places = $2! \times 10!$.

∴ the no. of favourable cases = $8 \times 2! \times 10!$.

Hence, *the required probability that there are* 3 *persons between A* and *B*

$$= \frac{8\times 2!\times 10!}{12!} = \frac{\mathbf{4}}{\mathbf{33}}.$$

[**Otherwise.** A and B can occupy any two of the 12 places in ${}^{12}C_2$ ways = $\frac{12\cdot 11}{2\cdot 1} = 66$ ways.

∴ total no. of all possible equally likely cases = 66.

There will be 3 persons between A and B, if A and B occupy the pairs of places $(1,5)$, $(2,6)$, $(3,7)$, $(4,8)$, $(5,9)$, $(6,10)$, $(7,11)$, $(8,12)$, i.e., the number of favourable cases = 8.

Hence, the required probability that there will be 3 persons between A and $B = \frac{8}{66} = \frac{\mathbf{4}}{\mathbf{33}}$.]

Example 48. *A box contains* 20 *tickets of identical appearance, the tickets being numbered* 1, 2, 3, ..., 20. *If* 3 *tickets are chosen at random, find the probability that the numbers on the drawn tickets are in arithmetic progression.*

Solution: 3 tickets can be chosen out of 20 in ${}^{20}C_3$ ways.

∴ the total number of all possible equally likely cases $= {}^{20}C_3 = \frac{20\times 19\times 18}{3\times 2\times 1} = 1140$.

The three numbers on the three drawn tickets will be in A.P. if there is a common difference of either 1, or 2, or 3, or ..., or 9 in the three numbers. (If the common difference is 9, the three numbers are 1, 10, 19 or 2, 11, 20).

With a common difference 1, the sets of three numbers are $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, ..., $\{18,19,20\}$, i.e., there are 18 such sets.

With common difference 2, the sets of three numbers are $\{1,3,5\}$, $\{2,4,6\}$, $\{3,5,7\}$, ..., $\{16,18,20\}$, i.e., there are 16 such sets.

With common difference 3, the sets of three numbers are $\{1,4,7\}$, $\{2,5,8\}$, $\{3,6,9\}$, ..., $\{14,17,20\}$, i.e., there are 14 such sets and so on.

Finally, with a common difference 9, the sets of three numbers are $\{1,10,19\}$, $\{2,11,20\}$, i.e., there are 2 such sets.

∴ the number of favourable cases $= 18 + 16 + 14 + \cdots + 2 = \frac{9}{2}\{2\times 18 + (9-1)\times(-2)\} = 90$ [$\because 18 + (n-1)\times(-2) = 2$ gives $n = 9$].

Hence, *the required probability* $= \frac{90}{1140} = \frac{\mathbf{3}}{\mathbf{38}}$.

Example 49. *Four different objects* 1, 2, 3, 4 *are distributed at random on four places marked* 1, 2, 3, 4. *What is the probability that none of the objects occupies the place corresponding to its number?*

Solution: 4 objects can be distributed on the four places in 4! ways = 24 ways.

$\therefore$ total number of all possible equally likely cases = 24.

Number of ways in which all the four objects can occupy the places corresponding to their numbers = 1.

If three objects occupy their places, then the 4th object will also occupy its place and this is already considered.

If two objects occupy their places, the remaining two can go wrong by occupying each other's place, i.e., in only 1 way, and 2 objects can be selected out of 4 in ${}^4C_2$ ways.

$\therefore$ the number of ways in which only two objects can occupy their places $= 1 \times {}^4C_2 = 6$.

If one object occupies its place, any one of the remaining 3 objects can go wrong in 2 ways by occupying the positions of the other two, and 1 object can be chosen out of 4 in ${}^4C_1$ ways.

$\therefore$ the number of ways in which only one object can occupy its place $= 2 \times {}^4C_1 = 8$.

$\therefore$ total no. of ways in which at least one object can occupy its place = 1 + 6 + 8 = 15.

$\therefore$ probability that at least one object occupies its place $= \frac{15}{24} = \frac{5}{8}$.

Hence, *the required probability that none of the objects occupies its place* $= 1 - \frac{5}{8} = \mathbf{\frac{3}{8}}$.

[**Otherwise.** The total number of cases $= 4! = 24$.

The number of ways in which none of the objects occupies its position

$$= 4!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\} = 24\left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\} = 24 \times \frac{12-4+1}{24} = 9.$$

[$\because$ the number of ways in which none of the n objects numbered $1, 2, 3, \ldots, n$ occupies the place corresponding to its number $= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}\right\}$.]

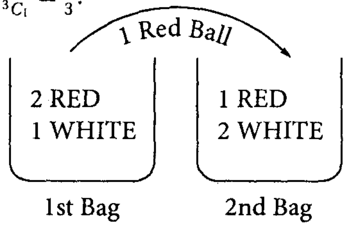
Hence, *the required probability* $= \frac{9}{24} = \mathbf{\frac{3}{8}}$.]

Example 50. *There are two bags. The first contains 2 red and 1 white balls whereas the second bag has only 1 red and 2 white balls. One ball is taken out at random from the first bag and is put in the second bag. Then a ball is chosen at random from the second bag. What is the probability that this last ball is red?*

Solution: The ball transferred from the first to the second bag may either be red or white.

Case 1. 1 *red ball is transferred from the first to the second bag.*

Probability of transferring 1 red ball from the first to the second bag $= \frac{{}^2C_1}{{}^3C_1} = \frac{2}{3}$.



After 1 red ball is transferred, the no. of red balls in the second bag $= 1 + 1 = 2$ and the total no. of balls in the second bag $= 2 + 2 = 4$.

$\therefore$ probability of drawing 1 red ball from the second bag $= \frac{{}^2C_1}{{}^4C_1} = \frac{2}{4} = \frac{1}{2}$.

$\therefore$ probability of first transferring 1 red ball and then drawing 1 red ball from the second bag $= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

Case 2. 1 *white ball is transferred from the first to the second bag.*

Probability of transferring 1 white ball from the first to the second bag $= \frac{{}^1C_1}{{}^3C_1} = \frac{1}{3}$.

As in Case 1, probability of drawing 1 red ball from the second bag $= \frac{{}^1C_1}{{}^4C_1} = \frac{1}{4}$.

$\therefore$ probability of first transferring 1 white ball and then drawing 1 red ball from the second bag $= \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$.

Hence, by the theorem of total probability (mutually exclusive events), *the required probability* $= \frac{1}{3} + \frac{1}{12} = \frac{4+1}{12} = \mathbf{\frac{5}{12}}$.

Example 51. *What is the probability that over a two-day period the number of requests would either be* 11 *or* 12, *if at a motor garage, the records of service requests with their probabilities, are as given below?*

Daily Demand:	5	6	7
Probability:	0.25	0.65	0.10

Solution: Let A, B, C be the events that the daily demands (i.e., the number of requests) are 5, 6, 7 respectively. Then $P(A) = 0.25$, $P(B) = 0.65$ and $P(C) = 0.10$.

The number of requests over a two-day period will be 11 if either (i) the demand in the first day is 5 and in the second day is 6 or (ii) the demand in the first day is 6 and in the second day is 5.

$\therefore$ probability of 11 service requests over a two-day period = $P(A) \times P(B) + P(B) \times P(A) = 0.25 \times 0.65 + 0.65 \times 0.25 = 0.1625 + 0.1625 = 0.3250$.

Again, the number of requests over a two-day period will be 12 if the demands on the first and second days respectively, are (i) 5 and 7 or, (ii) 7 and 5 or, (iii) 6 and 6.

$\therefore$ probability of 12 service requests over a two-day period = $P(A) \times P(C) + P(C) \times P(A) + P(B) \times P(B) = 0.25 \times 0.10 + 0.10 \times 0.25 + 0.65 \times 0.65 = 0.025 + 0.025 + 0.4225 = 0.4725$.

Hence, *the required probability of* 11 *or* 12 *service requests over a two-day period* = $0.3250 + 0.4725 =$ **0.7975**.

EXERCISES ON CHAPTER 9(I)

(Probability Theory)

Theory

1. Explain the following terms with a suitable example for each: (a) Events, (b) Elementary and compound events, (c) Equally likely events, (d) Mutually exclusive events, (e) Sample space, (f) Outcome, (g) Exhaustive set of events. **[B.U. B.Com.(H) 2007; V.U. B.Com.(H) 2011]**

2. Write down the Classical Definition of Probability and point out its limitations.
[C.U. B.Com.(H) 1992; B.U. B.Com.(H) 2007; V.U. B.Com.(H) 2010]

3. (a) State and prove the theorem of total probability for two mutually exclusive events. How is the result modified when the events are not mutually exclusive.
[C.U. B.Com.(H) 1999; V.U. B.Com.(H) 2010]

   (b) State the theorem of total probability of two events. **[C.U. B.Com.(H) 1996]**

4. (a) Define Conditional Probability. State and prove the theorem of compound probability.

   (b) State the theorem of compound probability for two events A and B. Deduce the case when A and B are independent. **[C.U. B.Com.(H) 1997]**

5. Explain with examples the rules of Addition and Multiplication in theory of Probability.

6. (a) Define (i) mutually exclusive events and (ii) independent events.
[B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2008, 11]

   (b) State Bayes' theorem of conditional probability.

7. Define independent and mutually exclusive events. Can the two events be mutually exclusive and independent simultaneously? Support your answer with examples. [C.U. B.Com.(H) 1994]

8. A and B are two given events; what conclusion can be made in each of the following cases: (a) $P(A) = P(B)$; (b) $P(A \cup B) = 1$; (c) $P(A \cup B) = P(A) + P(B)$; (d) $P(A \cap B) = P(A) \cdot P(B)$; (e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$? [C.U. B.Com.(H) 1999]

9. Let A, B, C be three events connected with an experiment. Under what conditions will the events be exhaustive and mutually exclusive?

10. If A and B be two events, prove that

    (a) $P(A \cup B) \leq P(A) + P(B)$;

    (b) $P(A \cap B) \geq P(A) + P(B) - 1$. [ICWAI Dec. 1991]

    [Hints: (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$ [$\because P(A \cap B) \geq 0$];
    (b) $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$ or, $P(A) + P(B) - 1 \leq P(A \cap B)$.]

Problems

A

1. A bag contains 3 green and 8 white balls. If one ball is drawn from it at random, find the chance that the ball drawn is green.

2. (a) If a coin is tossed, what is the chance of a 'Tail'? If three coins are tossed, find the chance that they are all Tails.

   (b) From a pack of 52 cards, 1 card is drawn at random. Find the chance of drawing a spade and also not drawing a spade. [C.U. B.Com.(H) 1998]

   (c) Two unbiased dice are thrown together. Find the probability of obtaining 2 in both the dice. [C.U. B.Com.(H) 2005]

3. If a die is tossed, what is the chance of getting (a) an even number, (b) a number greater than 2 and (c) an even number greater than 2.

4. From a pack of 52 cards, 1 card is drawn at random. Find the chance that the card is (a) red, (b) an ace, (c) a spade and (d) either a king or an ace.

5. (a) What is the chance of picking a spade, or an ace not of spade, from a pack of 52 cards? [C.U. B.Com.(H) 2001]

   (b) Out of 5 players of which two are members of a certain club, three are to be selected to represent the country at an international tournament. Find the probability that less than two of those selected to represent are members of the club.

6. (a) From a well-shuffled pack of 52 cards, one card is drawn at random. Find the chance that the card is (i) either a king or a knave, (ii) neither a king nor a knave, (iii) neither a heart nor a diamond, (iv) neither an ace, nor a king, nor a queen, nor a knave.

   (b) Two letters are drawn at random from the word HOME. Write down the sample space. Now find the probability that (i) both the letters are vowels; (ii) at least one is a vowel; (iii) one of the letters chosen is M.

7. From a bag containing 7 white and 5 red balls, 4 balls are drawn at random. What is the probability that they are all white?

8. (a) In a single cast with two dice, find the chance of throwing 7 (i.e., of throwing two numbers whose sum is 7).

   (b) If 4 unbiased coins are tossed, find the probability that there should be two heads.

   (c) Find the chance of throwing at least 8 in a single cast with two dice. [C.U. B.Com.(H) 1997]

   [Hints: $n(S)=36$. If A be the event of getting at least 8, then $A=\{(6,2),(5,3),(4,4),(3,5),(2,6),(6,3),(5,4),(4,5),(3,6),(6,4),(5,5),(4,6),(6,5),(5,6),(6,6)\}$. $\therefore n(A)=15$.]

9. Two balls are drawn at random from a bag containing 6 white balls and 4 black balls. Find the chance that one is white and the other is black.

10. Four balls are drawn at random from a bag containing 5 red and 7 white balls. Compute the probability of getting (a) 4 red balls, (b) 2 red and 2 white balls, (c) 3 white balls and 1 red ball.

11. Four balls are drawn at random from a bag containing 5 black and 8 white balls. What are the odds in favour of the balls being all white?

12. (a) Find the chance of not throwing an ace, two or three, in a single throw with a die.

    (b) There are 17 balls numbered from 1 to 17 in a bag. If a person selects one at random, what is the probability that the number printed on the ball will be an even number greater than 9?

13. What are the odds against throwing ace or six in a single throw with a die? And what are the odds in favour?

14. Given below are the weekly wages (in ₹) of six workers in a factory: 62, 90, 78, 85, 79 and 68.

    If two of these workers are selected at random to serve as representatives, what is the probability that at least one will have a wage lower than the average.

15. (a) Four men in a company of 10 employees are engineers. If 2 men are selected at random, then find the probability that at least one of them will be an engineer. [C.U. B.Com.(H) 2006]

    (b) Five men in a company of 20 are graduates. If 3 men are picked out of the 20 at random, what is the probability that (i) they are all graduates, (ii) there is no graduate? What is the probability of at least one graduate?

16. A bag contains 10 rupee-coins, 7 fifty-paise coins and 4 twenty-five paise coins. Find the probability of drawing (a) a rupee-coin; (b) three rupee-coins; and (c) three coins, one of each type.

17. Consider four events: A, NOT A, B, NOT B. It is given that $P(B)=0.4$, $P(A$ but NOT $B)=0.3$, P (NOT $A)=0.7$. Prove that A and B are mutually exclusive and hence, find $P(A\cup B)$.

18. Let E denote the experiment of tossing a coin three times in succession. Construct the sample space S. Write down the elements of the 2 events E_1 and E_2, where E_1 is the event that the number of Heads exceeds the number of Tails and E_2 is the event of getting Head in the first trial. Find the probabilities $P(E_1)$ and $P(E_2)$ assuming that all the elements of S are equally likely to occur.

19. A, B and C are three mutually exclusive and exhaustive events. Find $P(B)$, if $\frac{1}{3}P(C)=\frac{1}{2}P(A)=P(B)$.

    [B.U. B.Com.(H) 2007; V.U. B.Com.(H) 2007]

20. The probability that the teacher in Statistics will take a sudden test on any day of a week is $\frac{2}{5}$. A student abstains for 2 days in a week. What is the probability that he will miss at least one test?

[**Hints:** Probability that he will not miss a test on any of two days $= \frac{3}{5} \times \frac{3}{5}$.

$\therefore$ *required probability* $= 1 - \frac{9}{25} = \frac{16}{25}$.]

21. An urn contains 13 balls numbering from 1 to 13. Find the probability that a ball selected at random is a ball with number that is a multiple of 3 or 4.

[**Hints:** If S be sample space connected with the selection of a ball at random, then

$$S = \{1, 2, 3, 4, \ldots, 12, 13\}; \therefore n(S) = 13.$$

Let A be the event that the number of the ball is a multiple of 3 and B, the event that its number is a multiple of 4. Then $A = \{3, 6, 9, 12\}$ and $B = \{4, 8, 12\}$.

$\therefore n(A) = 4, n(B) = 3$ and $n(A \cap B) = 1; P(A) = \frac{4}{13}, P(B) = \frac{3}{13}, P(A \cap B) = \frac{1}{13}$.

$\therefore$ *required probability* $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$, etc.]

22. Find the chance of throwing at least 9 in a single case with two dice.

23. Two fair dice are thrown simultaneously. The two scores are then multiplied together. Calculate the probability that the product is (a) 12 and (b) even. [C.U. B.Com.(H) **1999**]

[**Hints:** 2 fair dice can fall in 6×6, i.e., 36 different ways, i.e., $n(S) = 36$.

(i) If A be the event that the product of scores on the 2 dice is 12, then

$$A = \{(2,6), (3,4), (4,3), (6,2)\}, \quad \text{i.e.,} \quad n(A) = 4; \quad \therefore \quad P(A) = 4/36 = 1/9.$$

(ii) If B be the event that the product of the scores on 2 dice is *odd*, then $B = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)\}$. $\therefore n(B) = 9$.

$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{9}{36} = \frac{1}{4}$. Hence, required probability $= 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$.]

24. In a lot of 20 articles, 5 are defective; 4 articles are drawn at random from the lot. Find the probability that exactly 2 of the drawn articles are defective.

[**Hints:** Let A be the event that exactly 2 of the articles drawn are defective.

No. of non-defective articles $= 20 - 5 = 15$. 4 non-defective articles can be drawn out of 20 in ${}^{20}C_4$ ways. $\therefore n(S) = {}^{20}C_4$.

2 defective articles can be drawn out of 5 in ${}^{5}C_2$ ways and 2 non-defective out of 15 in ${}^{15}C_2$ ways. $\therefore n(A) = {}^{5}C_2 \times {}^{15}C_2$ and $P(A) = \dfrac{{}^{5}C_2 \times {}^{15}C_2}{{}^{20}C_2} =$ etc.]

B

1. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability that (a) both are spades, (b) both are kings, (c) one is a spade and the other heart.

2. A sub-committee of 6 members is to be formed out of a group consisting of 7 gentlemen and 4 ladies. Calculate the probability that the sub-committee will consist of (a) exactly 2 ladies, and (b) at least 2 ladies.

3. (a) An article manufactured by a company consists of two parts, A and B. In the process of manufacture of part A, 9 out of 100 are likely to be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part B. Calculate the probability that the assembled article will not be defective.

(b) The manufacturing process of an article consists of two parts X and Y. The probabilities of defect in parts X and Y are 10% and 15% respectively. What is the probability that the assembled product will not have any defect? [C.U. B.Com.(H) 2007]

4. (a) There are four balls in a bag — white, red, green and blue. If a ball is drawn out at random three times in succession, what is the chance that all the three would be white? (Assume that the ball is replaced after each drawing.)

   (b) A box contains 9 red and 6 white balls. Two balls are drawn at random one after another. Find the probability that both the balls are white, when the first drawn ball is not replaced before the second drawing. [C.U. B.Com.(H) 2005]

5. (a) A coin is tossed three times in succession. What is the probability of the tosses coming in the order: (i) Head, Tail, Head; (ii) Head, Head, Head; (iii) Two Heads and one Tail?

   (b) Find the chance of throwing Head and Tail alternately in three successive tossing of a coin.

6. From a bag containing 5 green, 6 blue and 4 red balls, 7 balls are drawn at random. Find the chance that 2 green, 3 blue and 2 red balls are drawn. What are the odds against drawing 2 green, 3 blue and 2 red balls?

7. (a) If a die is thrown twice, what is the chance that the first throw does not show less than 4 and the second does not show more than 4?

   (b) A pair of dice is thrown. Find the probability of getting a sum of 7, when it is known that the digit in the first die is greater than that of the second. [C.U. B.Com.(H) 2003]

   [Hints: (b) $n(S) = 6 \times 6 = 36$. If A be the event of getting a sum of 7, when the digit in the first die > the digit in the second die, then $A = \{(6,1), (5,2), (4,3)\}$ and $n(A) = 3$. Hence, $P(A) = \frac{3}{36} = \frac{1}{12}$.]

8. In a single cast with 2 dice (a) what are the odds against throwing 8, i.e., against two numbers the sum of which is 8, (b) what is the chance of throwing more than 8, and what is the chance of throwing less than 8? (c) Find the chance of throwing at least 8 in a single cast with two dice.

9. In a single cast with 3 dice, what is the chance of throwing (a) three fives, (b) two-four-six, (c) five-six-six? What are the odds against the events?

10. Three cards are drawn from a pack of 52 cards, each card being replaced before the next one is drawn. Compute the probability that all are (a) spades; (b) aces; and (c) red.

11. (a) One bag contains 4 red and 2 black balls; another bag contains 3 red and 5 black balls. If one ball is drawn from each bag, determine the probability that (a) both are red, (b) both are black, (c) one is red and one is black.

    (b) Box A contains 4 white and 3 red balls while box B contains 5 white and 4 red balls. If one ball is drawn at random from each box, what is the probability that both the balls are of the same colour? [V.U. B.Com.(H) 2009]

12. (a) One counter-foil is drawn at random from a bag containing 70 counter-foils marked with the first 70 numerals. Find the chance that it is multiple of 8 or 9.

    (b) From a set of 18 balls marked $1, 2, 3, \ldots, 17, 18$, one is drawn at random. What is the chance that its number is either a multiple of 3 or of 4?

13. An investment consultant predicts that odds against the price of a certain stock will go up during the next week are 2 : 1 and the odds in favour of the price remaining the same are 1 : 3. What is the probability that the price of the stock will go down during the next week?

[**Hints:** If A be the event that 'price will go up', and B be the event that 'price will remain the same', then $P(A)=\frac{1}{2+1}=\frac{1}{3}$ and $P(B)=\frac{1}{1+3}=\frac{1}{4}$; $\therefore P(A\cup B)=P(A)+P(B)=\frac{7}{12}$.
$\therefore$ probability that the price will go down $= 1-P(A\cup B)=$ etc.]

14. A bag contains 4 green and 6 red balls. A ball is drawn at random and then, without replacing it, a second ball is drawn. What is the chance that a green ball is drawn each time?

15. Two drawings, each of 4 balls, are made from a bag containing 6 red and 5 black balls, the balls not being replaced before the second trial; find the chance that the first drawing gives 4 red balls and the second gives 1 red and 3 black balls.

16. (a) A problem in Statistics is given to three students, A, B, C, whose chances of solving it are $\frac{1}{3},\frac{1}{4},\frac{1}{5}$ respectively. What is the probability that the problem will be solved?

(b) The probability of solving a problem by three students A, B and C are $\frac{2}{7},\frac{3}{8}$ and $\frac{1}{2}$ respectively. If each of them tries independently, find the probability that the problem is solved. Find also the probability that the problem could not be solved. [C.U. B.Com.(H) 1992]

17. The independent probabilities that the three sections of a costing department will encounter a computer error are 0.1, 0.3 and 0.3, each week respectively. Calculate the probability that there will be: (a) at least one computer error; (b) one and only one computer error encountered by the costing department next week.

18. A bag contains 4 red and 3 blue balls. Two drawings of 2 balls are made. Find the probability of drawing first 2 red balls and second 2 blue balls (a) if the balls are returned to the bag after the first draw; (b) if the balls are not returned after the first draw. [ICWAI Dec. 1990]

19. If 20 dates are named at random, find the chance that 5 of them will be Sundays.

20. Two urns contain 3 white, 7 red, 15 black and 10 white, 6 red, 9 black balls respectively. One ball is drawn from each urn. Find the probability that both the balls are of the same colour. [C.U. B.Com.(H) 2000]

[**Hints:** See worked-out Ex. 4 in **Section 9.9**. Both the balls will be of the same colour when they are (i) both white or (ii) both red or (iii) both black.
Hence, *required probability* $=\frac{^3C_1}{^{25}C_1}\times\frac{^{10}C_1}{^{25}C_1}+\frac{^7C_1}{^{25}C_1}\times\frac{^6C_1}{^{25}C_1}+\frac{^{15}C_1}{^{25}C_1}\times\frac{^9C_1}{^{25}C_1}=\frac{30+42+135}{625}=\frac{\mathbf{207}}{\mathbf{625}}$.]

21. 10 balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls?

22. If $P(A)=\frac{1}{4}, P(B)=\frac{2}{5}$, find $P(A\cup B)$, when A and B are two independent events.

[B.U. B.Com.(H) 2008]

23. If $P(A)=\frac{1}{3}, P(B)=\frac{1}{2}, P(A\cap B)=\frac{1}{4}$, find $P(B/A)$ and $P(A^c\cap B^c)$, where A^c and B^c are the complements of events A and B respectively. [B.U. B.Com.(H) 2008]

24. If the two events A and B are mutually independent and $P(A)=0.3, P(B)=0.1$, find $P(A\cup B)$.

[B.U. B.Com.(H) 2008]

25. A speaks the truth 3 out of 5 times and B speaks the truth 5 out of 7 times. What is the probability that they will contradict each other in stating the same fact? [B.U. B.Com.(H) 2008]

[Hints: See worked-out Ex. 31.]

C

1. If $P(A)=\frac{1}{4}, P(B)=\frac{2}{5}, P(A\cup B)=\frac{1}{2}$, find $P(A\cap B), P(A/B)$ and $P(B/A)$.

2. (a) If A and B are two independent events and $P(A)=\frac{3}{5}, P(B)=\frac{2}{3}$, find $P(A\cap B)$ and $P(A\cup B)$.

   (b) If A and B be events with $P(A+B)=\frac{3}{4}$, $P(A)=\frac{2}{3}$ and $P(AB)=\frac{1}{4}$, find (i) $P(A)$; (ii) $P(B)$ and (iii) $P(AB)$.

3. (a) If $P(A)=\frac{1}{2}, P(B)=\frac{2}{5}, P(A\cap B)=\frac{1}{3}$, find $P(A\cup B), P(A^c\cap B^c), P(A^c\cup B^c), P(A\cap B^c)$ and $P(A^c\cap B)$.

   (b) If $P(A)=\frac{2}{3}, P(B)=\frac{1}{2}, P(A\cup B)=1$, find the value of $P(A/B), P(A/B^c)$ and $P(A^c\cap B^c)$. State whether the events are mutually exclusive. [C.U. B.Com.(H) 2003]

4. A construction company is bidding for two contracts, A and B. The probability that the company will get contract A is $\frac{3}{5}$, the probability that the company will get contract B is $\frac{1}{3}$ and the probability that the company will get both the contracts is $\frac{1}{8}$. What is the probability that the company will get at least one contract?

5. The probability that a contractor will get an electric contract is $\frac{3}{5}$, and the probability that he will not get a plumbing contract is $\frac{7}{10}$. If the probability of getting at least one contract is $\frac{4}{7}$, what is the probability that he will get both the contracts?

6. The probability that a Management Accountant's job applicant has a post-graduate degree is 0.30, that he has had some work experience as a Chief Accountant is 0.70, and that he has both is 0.20. Out of 400 applicants, what number would have either a post-graduate degree or some professional work experience or both?

7. (a) A bag contains 10 red and 6 green balls. Two successive drawings of 3 balls are made without replacement. Find the probability that the first drawing will give 3 red balls and the second will give 3 green balls.

   (b) A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white balls and the second to give 4 black balls when the balls are not replaced before the second draw.

8. A bag contains 5 red and 3 black balls, and the second bag contains 4 red and 5 black balls. One of these is selected at random and a draw of two balls is made from it. What is the probability that one of them is red and the other is black?

9. There are two identical boxes containing 5 white and 3 red balls and 4 white and 6 red balls. A box is chosen at random and a ball is drawn from it. If the ball is white, what is the probability that it is from the second box?

[Hints: Use Bayes' Theorem.]

10. In a bolt factory, machines X, Y, Z manufacture respectively 20, 35 and 45 per cent of the total output. Of their output, 8, 6 and 5 per cent respectively are defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured in the machine Z?

[**Hints**: Use Bayes' Theorem.]

11. A box contains 4 defective and 6 good electronic calculators. Two calculators are drawn out one by one without replacement. (a) What is the probability that the two calculators so drawn are good? (b) One of the two calculators so drawn is tested and found to be good. What is the probability that the other one is also good?

12. There are two identical urns containing respectively 4 white and 3 red balls, and 3 white and 7 red balls. An urn is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball drawn is white, what is the probability that it is from the first urn? [C.U. B.Com. 2002]

[**Hints**: See worked-out Ex. 38 in **Section 9.7**, 2nd Part is the same as Ex. 38.

1st Part. Probability of first selecting the first urn and then drawing 1 white ball from it $= \frac{1}{2} \times \frac{4}{7} = \frac{4}{14}$.

Probability of first selecting the 2nd urn and then drawing 1 white ball from it $= \frac{1}{2} \times \frac{3}{10} = \frac{3}{20}$.

$\therefore$ probability that the ball is white $= \dfrac{4}{14} + \dfrac{3}{20} = \dfrac{40+21}{140} = \dfrac{\mathbf{61}}{\mathbf{140}}$.]

D

1. If 2 balls are drawn one after another from a bag containing 3 white and 5 black balls, what is the probability that (a) the first ball is white and the second is black; (b) one ball is white and the other is black?

2. One urn contains 2 white and 2 black balls; a second urn contains 2 white and 4 black balls. (a) If one ball is chosen from each urn, what is the probability that they will be of the same colour? (b) If an urn is selected at random and one ball is drawn from it, what is the probability that it will be a white ball?

3. X and Y stand in a line at random with 8 other persons. What is the probability that there are 3 persons between X and Y?

4. Given that $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find $P(A/B)$ and $P(B/A)$. Are A and B independent?

[ICWAI June 1991; B.U. B.Com.(H) 2008]

5. What is the probability that (a) a leap year selected at random will contain 53 Sundays, (b) a non-leap year selected at random will contain 53 Sundays?

6. If n biscuits are distributed at random among N beggars, show that the probability that a particular beggar receives $r(< n)$ biscuits is $\frac{{}^nC_r(N-1)^{n-r}}{N^n}$.

7. In a group of equal number of men and women, 10% men and 45% women are unemployed. What is the probability that a person selected at random is employed?

8. An urn contains 4 white and 5 black balls. A second urn contains 5 white and 4 black balls. One ball is transferred from the first to the second urn; then a ball is drawn from the second urn. What is the probability that it is white?

9. Three groups of children contain 3 girls and 1 boy, 2 girls and 2 boys and 3 boys and 1 girl. One child is selected at random from each group. Show that the chance that the three selections consist of 1 girl and 2 boys is 13/32.

10. A speaks the truth in 60% and B in 75% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

11. A six-faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even?

12. A, B and C, in order, toss a coin. The first one to throw a head wins. What are their respective chances of winning, assuming that the game may continue indefinitely?

13. From a pack of 52 cards, an even number of cards is drawn. Show that the probability that these consist of half red and half black cards is $\left\{\frac{52!}{(26!)^2} - 1\right\}\Big/(2^{51} - 1)$.

14. A and B throw with a pair of dice. A wins if he throws 6 before B throws 7, and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.

15. There are three men aged 60, 65 and 70 years. The probability to live 5 years more is 0.8 for a 60-year old, 0.6 for a 65-year old and 0.3 for a 70-year old person. Find the probability that at least two of the three persons will remain alive 5 years hence.

16. A bag contains 5 white and 4 black balls. A ball is drawn at random from the bag and put into another bag which contains 3 white and 7 black balls. A ball is drawn randomly from the second bag. What is the probability that it is white? **[C.U. B.Com.(H) 1997]**

[**Hints:** See worked-out Ex. 50 in **Section 9.9.** Here **Case 1.** Probability of transferring a white ball = $\frac{5}{5+4} = \frac{5}{9}$ and after transfer, the probability of drawing 1 white ball from the 2nd bag = $\frac{4}{4+7} = \frac{4}{11}$. Probability of first transferring 1 white ball from 1st to the 2nd bag and then of drawing a white ball from the 2nd bag = $\frac{5}{9} \times \frac{4}{11} = \frac{20}{99}$.

Case 2. Similarly, probability of first transferring a black ball and then of drawing a white ball from the 2nd bag = $\frac{4}{5+4} \times \frac{3}{3+8} = \frac{12}{99}$. Hence, *required probability* = $\frac{20}{99} + \frac{12}{99} = \frac{32}{99}$.]

E

1. (a) Can the following represent measures of probability?

   (i) $P(A) = 0.2, P(B) = 0.7, P(C) = 0.1$;

   (ii) $P(A) = 0.4, P(B) = 0.6, P(C) = 0.2$;

   where $A \cup B \cup C = S$ (the sample space) and A, B, C are mutually exclusive events.

   (b) Three fair coins are tossed once. Construct the sample space of the outcomes of the random experiment. Mention two mutually exclusive events. Find the probability of (i) at least one Head, (ii) exactly one Tail.

2. If A and B are events connected with a random experiment such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, find (i) $P(A \cup B)$; (ii) $P(A/B)$; (iii) $P(A^c \cap B^c)$; (iv) $P(A^c \cup B^c)$ and (v) $P(A^c \cap B)$.

3. (a) Let A and B be events with $P(A \cup B) = \frac{7}{8}, P(A \cap B) = \frac{1}{4}$ and $P(A^c) = \frac{5}{8}$. Find $P(A), P(B)$ and $P(A \cap B^c)$.

(b) If $P(A)=\frac{1}{3}, P(B)=\frac{2}{7}$ and $P(A\cup B)=\frac{1}{2}$, then find $P(A^c\cap B)$. [C.U. B.Com.(H) 2006]

4. Let A and B be two events connected with a random experiment such that $P(A)=\frac{1}{2}$, $P(A\cup B)=\frac{3}{4}$, and $P(B^c)=\frac{5}{8}$. Find $P(A\cap B), P(A^c\cap B^c), P(A^c\cup B^c)$ and $P(B\cap A^c)$.

5. The nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged in random order to form a nine-digit number. Find the probability that 1, 2 and 3 appear as neighbours in the order mentioned.

6. In a family there are 4 children. Find the probability that (a) all of them will have different birthdays, (b) two of them will have the same birthday. (1 year = 365 days)

7. A bag contains 5 red and 4 black balls. A ball is drawn at random from the bag and put into another bag which contains 3 red and 7 black balls. A ball is drawn randomly from the second bag. What is the probability that it is red?

8. One shot is fired from each of three guns. Let A, B, C denote the events that the target is hit by the first, second and the third guns respectively. Assuming that A, B, C are mutually independent events and that $P(A)=0.5, P(B)=0.6, P(C)=0.8$, find the probability that at least one hit is registered.

9. What is the chance of getting at least one defective item, if 3 items are drawn randomly from a lot containing 10 items, of which just 2 are defective?

10. There are 5 persons in a room. Find the probability that at least 2 of them will have the same birthday in the same calendar month of a year.

11. Mr X is called for interview for 3 separate posts. At the first interview there are 5 candidates; at the second, 4 candidates; at the third, 6 candidates. If selection of each candidate is equally likely, find the probability that Mr X will be selected for (a) at least one post; (b) at least two posts.

12. In a bridge game, North and South have 9 spades between them. Find the probability that either East or West has no spades.

    (There are only 13 spades in a pack of 52 cards and each player has 13 cards. The players are designated by the positions they occupy, viz., North, South, East, West.)

13. It was found that a league match in football ends in WIN, LOSS or DRAW. A club is supposed to play 22 matches. Being a supporter of the club, what is the probability that you can predict 18 correct results?

    [**Hints:** Total no. of cases = 3^{22}, and no. of the favourable cases = ${}^{22}C_{18}\times 2^4$.]

14. A bag contains 3 red and 5 white balls, and a second bag contains 4 red and 6 white balls. One ball is drawn at random from the first bag and put into the second bag. If now, a ball is drawn at random from the second bag, find the probability that it is red. [C.U. B.Com.(H) 1996]

    [**Hints:** See worked-out Ex. 11 in **Section 9.9** or hints of Q.11 in Section D.]

ANSWERS

A

1. $\frac{3}{11}$.
2. (a) $\frac{1}{2}$, $\frac{1}{8}$; (b) $\frac{1}{4}$, $\frac{3}{4}$; (c) $\frac{1}{36}$.
3. (a) $\frac{1}{2}$; (b) $\frac{2}{3}$; (c) $\frac{1}{3}$.
4. (a) $\frac{1}{2}$; (b) $\frac{1}{13}$; (c) $\frac{1}{4}$; (d) $\frac{2}{13}$.
5. (a) $\frac{4}{13}$; (b) $\frac{7}{10}$.
6. (a) (i) $\frac{2}{13}$; (ii) $\frac{11}{13}$; (iii) $\frac{1}{2}$; (iv) $\frac{9}{13}$;
   (b) $S = \{HO, HM, HE, OM, OE, ME\}$;
   (i) $\frac{1}{6}$; (ii) $\frac{5}{6}$; (iii) $\frac{1}{2}$.
7. $\frac{7}{99}$.
8. (a) $\frac{1}{6}$; (b) $\frac{3}{8}$.
9. $\frac{8}{15}$.
10. (a) $\frac{1}{99}$; (b) $\frac{14}{33}$; (c) $\frac{35}{99}$.
11. 14 : 129.
12. (a) $\frac{1}{2}$; (b) $\frac{4}{17}$.
13. 2 : 1; 1 : 2.
14. $\frac{3}{5}$.
15. (a) $\frac{2}{3}$; (b) (i) $\frac{1}{114}$; (ii) $\frac{91}{228}$; $\frac{137}{228}$.
16. (a) $\frac{10}{21}$; (b) $\frac{12}{133}$; (c) $\frac{4}{19}$.
17. 0.7.
18. $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$,
    $E_1 = \{HHT, HTH, THH\}$,
    $E_2 = \{HHH, HHT, HTH, HTT\}$; $\frac{3}{8}$, $\frac{1}{2}$.
19. $\frac{1}{6}$.
20. $\frac{16}{25}$.
21. $\frac{6}{13}$.
22. $\frac{5}{18}$.
23. $\frac{1}{9}$; $\frac{3}{4}$.
24. $\frac{70}{323}$.

B

1. (a) $\frac{1}{17}$; (b) $\frac{1}{221}$; (c) $\frac{13}{102}$.
2. (a) $\frac{5}{11}$; (b) $\frac{53}{66}$.
3. (a) 0.8645; (b) 0.765.
4. (a) $\frac{1}{64}$; (b) $\frac{1}{7}$.
5. (a) (i) $\frac{1}{8}$, (ii) $\frac{1}{8}$, (iii) $\frac{3}{8}$; (b) $\frac{1}{4}$.
6. $\frac{80}{429}$; 349 : 80.
7. $\frac{1}{3}$.
8. (a) 31 : 5; (b) $\frac{5}{18}$, $\frac{7}{12}$; (c) $\frac{5}{12}$.
9. (a) $\frac{1}{216}$, 215 : 1; (b) $\frac{1}{36}$, 35 : 1;
   (c) $\frac{1}{72}$, 71 : 1.
10. (a) $\frac{1}{64}$; (b) $\frac{1}{2197}$; (c) $\frac{1}{8}$.
11. (i) (a) $\frac{1}{4}$; (b) $\frac{5}{24}$; (c) $\frac{13}{24}$; (ii) $\frac{32}{63}$.
12. (a) $\frac{3}{14}$; (b) $\frac{1}{2}$.
13. $\frac{5}{12}$.
14. $\frac{2}{15}$.
15. (a) $\frac{2}{3}$, (b) $\frac{2}{77}$.
16. (a) $\frac{3}{5}$; (b) $\frac{87}{112}$; $\frac{25}{112}$.
17. (a) 0.559; (b) 0.427.
18. (a) $\frac{2}{49}$; (b) $\frac{3}{25}$.
19. $\frac{^{20}C_5 \times 6^{15}}{7^{20}}$.
20. $\frac{207}{625}$.
21. $\frac{5120}{19683}$.
22. $\frac{11}{20}$.
23. $\frac{3}{4}$; $\frac{5}{12}$.
24. 0.37.
25. $\frac{16}{35}$.

C

1. $\frac{3}{20}$, $\frac{3}{8}$, $\frac{3}{5}$.
2. (a) $\frac{2}{5}$, $\frac{13}{15}$; (b) $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$.
3. (a) $\frac{23}{30}$, $\frac{7}{30}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{4}{15}$,
   (b) $\frac{1}{3}$; 1; 0; not mutually exclusive.
4. $\frac{97}{120}$.
5. $\frac{23}{70}$.
6. 320.
7. (a) $\frac{15}{1001}$, (b) $\frac{3}{715}$.
8. $\frac{275}{504}$.
9. $\frac{16}{41}$.
10. $\frac{45}{119}$.
11. (a) $\frac{1}{3}$; (b) $\frac{5}{9}$.
12. $\frac{61}{140}$, $\frac{40}{61}$.

D

1. (a) $\frac{15}{56}$; (b) $\frac{15}{28}$.
2. (a) $\frac{1}{2}$; (b) $\frac{5}{12}$.
3. $\frac{2}{15}$.
4. $\frac{2}{5}:\frac{2}{3}$; A and B are not independent $[\because P(A\cap B)\neq P(A)\cdot P(B)]$.
5. (a) $\frac{2}{7}$; (b) $\frac{1}{7}$.
7. $\frac{29}{40}$.
8. $\frac{49}{90}$.
10. 45% of the cases.
11. $\frac{5}{9}$.
12. $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$.
15. 0.612.
16. $\frac{32}{99}$.

E

1. (a) (i) Yes, (ii) No; (b) $S=\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$; HHH, TTT; $\frac{7}{8}$; $\frac{3}{8}$.
2. (i) $\frac{7}{12}$; (ii) $\frac{3}{4}$, (iii) $\frac{5}{12}$, (iv) $\frac{3}{4}$, (v) $\frac{1}{12}$.
3. (a) $\frac{3}{8}, \frac{3}{4}, \frac{1}{8}$; (b) $\frac{1}{6}$.
4. $\frac{1}{8}, \frac{1}{4}, \frac{7}{8}, \frac{1}{4}$.
5. $\frac{1}{72}$.
6. (a) $\frac{364\times363\times362}{(365)^3}$, (b) $\frac{6\times364\times363}{(365)^3}$.
7. $\frac{32}{99}$.
8. 0.96.
9. $\frac{8}{15}$.
10. $1-\frac{364\times363\times362\times361}{(365)^4}$.
11. (a) $\frac{1}{2}$; $\frac{13}{120}$.
12. $\frac{11}{115}$.
13. $\frac{{}^{22}C_{18}\times2^4}{3^{22}}$.
14. $\frac{35}{88}$.

9.10 Random Variable and its Probability Distribution

We know that 'tossing a coin', 'throwing a die', 'drawing a card from a pack of 52 cards', etc., are all experiments. If a coin is tossed, either 'head ' or 'tail' turns up and the chance of a head is $\frac{1}{2}$ and the chance of a tail is also $\frac{1}{2}$. Similarly, when a die is cast, the possible outcomes (i.e. the numbers that will appear on top) are 1, 2, 3, 4, 5, 6, each of which occurs with probability (or chance) $\frac{1}{6}$. The experiments whose results (or outcomes) depend on chance are called *Random Experiments.* The definition of a *random variable* is based on the results of a random experiment.

Definition 1. *A variable whose numerical value is determined by the outcome (or result) of a random experiment is called a random variable or a chance variable.*

A random variable X can also be regarded as a real-valued function defined on the sample space S of a random experiment such that for each point x of the sample space, $f(x)$ is the probability of occurrence of the event represented by x.

Illustration 1. *Consider a random experiment of tossing* 3 *coins. Then the sample space S is given by* $S=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Let X denote the number of heads observed. Then $X=0$ if the outcome is TTT, $X=1$ if the outcome is HTT or THT or TTH, $X=2$ if the outcome is HHT or HTH or THH, and finally $X=3$ if the outcome is HHH.

$\therefore$ X is a random variable whose values are determined by the outcomes of the random experiment of tossing 3 coins, and it is a function with domain S and range $\{0,1,2,3\}$. We can write

$$\begin{cases} X(TTT)=0, & X(THT)=1, \quad X(TTH)=1, \quad X(HTT)=1, \\ X(THH)=2, & X(HTH)=2, \quad X(HHT)=2, \quad X(HHH)=3. \end{cases}$$

A random variable may either be *discrete* or *continuous.*

9.10.1 Discrete and Continuous Random Variables

Definition 2. *If a variable can assume only a discrete set of values, i.e. a finite set of values (as in the case of 'casting a die') or at most an enumerably infinite set of values (as in Poisson distribution, not discussed in the this book), then it is called a* ***discrete random variable.***

Illustration 2. *The number appearing on top of a die when it is cast.*

Illustration 3. *The number of defectives in a sample of electric bulbs.*

Illustration 4. *The number of printing mistakes per page of a book.*

Illustration 5. *The number of telephone calls received per day, etc.*

If a random variable can assume all real values at a given interval, it is called a ***continuous random variable.*** *In this case no enumeration of favourable and total cases is possible. Heights, weights, temperature, time, etc. are examples of continuous random variables.*

In this chapter, we shall discuss topics concerned with discrete random variables. In case of continuous random variable X with probability density function (p.d.f.) $p(x)$, summation is replaced by integration over the domain of X.

9.10.2 Probability Function (or Density Function) of a Random Variable

If a random variable X assumes the discrete set of values $x_1, x_2, \ldots, x_n$, then the function f defined by

$$f(x_i) = P(X = x_i) = \text{Probability that } X \text{ assumes the value } x_i = p_i \text{ (say)}$$

is called Probability Function (or Density Function or Probability Mass Function) of X. This function gives the probabilities corresponding to the different values $x_1, x_2, \ldots, x_n$ of the variable X and satisfies the following two conditions:

(i) $f(x_i) \geq 0$, for $i = 1, 2, \ldots, n$;

(ii) $\sum_{i=1}^{n} f(x_i) = 1.$

Illustration 6. *If x is the number appearing on top of a die when it is cast, then $f(x) = \frac{1}{6}$; $x = 1, 2, 3, 4, 5, 6$. In this case, the probability function is a constant function.*

9.11 Discrete Probability Distribution

If a random variable x can assume a discrete set of values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ where $p_1 + p_2 + \cdots + p_n = 1$, then the occurrence of the values x_i with their probabilities p_i, $(i = 1, 2, \ldots, n)$, is called the ***discrete probability distribution of x.***

Illustration 1. *If x represents the number appearing on top of a die when it is cast, then the values of x are* 1, 2, 3, 4, 5, 6 *which occur with respective probabilities $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$, where $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$. Thus the values of x with their respective probabilities define the discrete probability distribution of x:*

x	1	2	3	4	5	6	Total
p	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Example 52. *X is a discrete random variate having the following probability distribution:*

x	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) *Determine the constant k;* (ii) *find $P(X<6)$;* (iii) *what will be $P(X \geq 6)$ and $P(0<X<5)$?*

Solution: (i) The probability function $f(x)=P(X=x)$ must satisfy the conditions (i) $f(x) \geq 0$ and (ii) $\Sigma f(x)=1$.

By (ii), we have

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$
$$\text{or,}\quad 10k^2+9k-1=0 \quad \text{or,}\quad 10k^2+10k-k-1=0$$
$$\text{or,}\quad 10k(k+1)-1(k+1)=0 \quad \text{or,}\quad (k+1)(10k-1)=0;$$
$$\therefore \quad k=-1, \tfrac{1}{10}.$$

By (i), $k \neq -1$. For if $k=-1$, then $f(1), f(2), f(3), f(4)$, become negative which contradicts the condition (i).

Hence, $\boldsymbol{k=\frac{1}{10}=0.1}$.

(ii) $P(X<6)=f(0)+f(1)+f(2)+f(3)+f(4)+f(5)=0+k+2k+2k+3k+k^2=8k+k^2=8\times0.1+(0.1)^2=0.8+0.01=\mathbf{0.81}$.

(iii) $P(X \geq 6)=1-P(X<6)$ [$\because \Sigma f(x)=1$] $=1-0.81=\mathbf{0.19}$

and $P(0<X<5)=P(1)+P(2)+P(3)+P(4)=k+2k+2k+3k=8k=\frac{8}{10}=\mathbf{0.8}$.

9.12 Expectation of a Random Variable (or Expected Value or Mathematical Expectation)

Definition 1. *If a discrete random variable x assumes the discrete set of values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, where $p_1+p_2+\cdots+p_n=1$, then the **expectation** or **expected value** or **mathematical expectation** of x, written as $\mathbf{E(x)}$, is defined by*

$$E(x)=p_1x_1+p_2x_2+\cdots+p_nx_n=\sum_{i=1}^{n} p_ix_i \quad \textit{or,} \quad \Sigma px,$$

i.e., it is the sum of the product of the different possible values of x and the corresponding probabilities.

If in this expectation the probabilities p_i are replaced by relative frequencies f_i/N, where $N=\Sigma f_i$, then $E(x)=\Sigma fx/N=$ Mean. As N becomes larger and larger, the relative frequencies f_i/N approach the probabilities p_i. [See Empirical Definition of Probability]

We shall take $\boldsymbol{E(x)}$ **as the Mean of the probability distribution of $\boldsymbol{x}$**, which is a characteristic of the random variable x.

The expected value of x^2 is defined by $E\left(x^2\right)=\Sigma p_ix_i^2$, and the expected value of x^r is defined by $E(x^r)=\Sigma p_ix_i^r$.

Example 53. *If x is a random variable and c is a constant, show that* (i) $E(c)=c$; (ii) $E(cx)=cE(x)$; (iii) $E(x-\bar{x})=0$.

Solution: (i) $E(c)=\Sigma p_i c = c\Sigma p_i = c\cdot 1 = c$ [$\because \Sigma p_i = 1$]

(ii) $E(cx)=\Sigma p_i c x_i = c\Sigma p_i x_i = c\cdot E(x)$.

(iii) $E(x-\bar{x}) = \Sigma p_i(x_i-\bar{x}) = \Sigma p_i x_i - \Sigma p_i \bar{x} = E(x) - \bar{x}\Sigma p_i$ [$\because \bar{x}$ is fixed]

$= \bar{x} - \bar{x}\cdot 1 = 0.$

We shall use these results as **formulae.**

Example 54. *If x denotes the number of points on a die, find the expectation of x.*

Solution: Since x represents the number of points on a die, the different values of x are 1, 2, 3, 4, 5, 6, each having the same probability $\frac{1}{6}$.

By definition,

$$E(x)=\sum p_i x_i = \frac{1}{6}\cdot 1+\frac{1}{6}\cdot 2+\frac{1}{6}\cdot 3+\frac{1}{6}\cdot 4+\frac{1}{6}\cdot 5+\frac{1}{6}\cdot 6=\frac{1}{6}(1+2+3+4+5+6)$$
$$=\frac{1}{6}\cdot\frac{6(6+1)}{2}=\frac{7}{2}=3.5.$$

We state below two theorems on expected values without proofs.

Theorem 1. *The expectation of the sum of two discrete random variables is the sum of their expectations.*

Symbolically, $E(x+y)=E(x)+E(y)$.

Corollary 1. *For three discrete random variables x, y, z, $E(x+y+z)=E(x)+E(y)+E(z)$.*

We have similar results for any number of random variables.

Corollary 2. *If $y=a+bx$, where a and b are constants, then $E(y)=a+bE(x)$.*

$$E(y)=E(a+bx)=E(a)+E(bx)=a+bE(x).$$

Theorem 2. *The mathematical expectation of the product xy of two independent random variables x and y, is equal to the product of their expectations.*

Symbolically, $E(xy)=E(x)E(y)$.

Example 55. (i) *Find the mathematical expectation of the sum of points on n dice.*

(ii) *Two dice are thrown at random. Find the expected value of sum of points shown up.*

[**C.U. B.Com.(H) 2001**]

Solution: (i) Let x_i denote the number of points on the ith die and S denote the sum of the points on n dice. Then $S=x_1+x_2+\cdots+x_n$.

$$\therefore E(S)=E(x_1+x_2+\cdots+x_n)=E(x_1)+E(x_2)+\cdots+E(x_n).$$

But for a single die, x_i has the six different values 1, 2, 3, 4, 5, 6, each of which occurs with the same probability $\frac{1}{6}$ and, therefore,

$$E(x_i) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6$$
$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2} \text{ for } i = 1, 2, \ldots, n.$$

Hence, $E(S) = \frac{7}{2} + \frac{7}{2} + \cdots$ to n terms $= \frac{7n}{2}$.

(ii) Let x_1 denote the number of points on the 1st die and x_2 on the 2nd die. If S be the sum of the points on two dice, then $S = x_1 + x_2$. Now proceed as above; $E(S) = \frac{7}{2} + \frac{7}{2} = 7$.

Example 56. *A number is chosen at random from the set* 1, 2, 3, ..., 100, *and another number is chosen at random from the set* 1, 2, ..., 50. *What is the expected value of the product?*

Solution: Let x be the number chosen at random from the set 1, 2, 3, ..., 100. Then x is a discrete random variable whose probability distribution is given by

x	1	2	3	$\cdots$	100	Total
p	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\cdots$	$\frac{1}{100}$	1

$$\therefore E(x) = \sum p_i x_i = \frac{1}{100}\times 1 + \frac{1}{100}\times 2 + \frac{1}{100}\times 3 + \cdots + \frac{1}{100}\times 100$$
$$= \frac{1}{100}(1+2+3+\cdots+100) = \frac{1}{100}\cdot\frac{100(100+1)}{2} = \frac{101}{2}.$$

If y be the number chosen at random from the set 1, 2, ..., 50, then the probability distribution of y is given by

y	1	2	3	...	50	Total
p'	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	$\cdots$	$\frac{1}{50}$	1

$$\therefore E(y) = \sum p'_i y_i = \frac{1}{50}\times 1 + \frac{1}{50}\times 2 + \frac{1}{50}\times 3 + \cdots + \frac{1}{50}\times 50$$
$$= \frac{1}{50}(1+2+3+\cdots+50) = \frac{1}{50}\cdot\frac{50(50+1)}{2} = \frac{51}{2}.$$

Clearly, x and y are two *independent random variables.*

Hence, *expected value of the product* $= E(xy) = E(x)\cdot E(y) = \frac{101}{2}\times\frac{51}{2} = \frac{5151}{4}$.

9.13 Variance and Standard Deviation of a Random Variable

Like *Mean,* **Variance** is a characteristic of a random variable x and it is used to measure dispersion (or variation) of x.

Variance: *The Variance of a random variable x is the expected value of $(x-\bar{x})^2$ where $\bar{x}$ is the mean of x, i.e. **Variance of x or $\mathbf{var}(x) = E(x-\bar{x})^2$**.*

We can show that $\mathbf{var}(x) = E(x^2) - \{E(x)\}^2$. [C.U. B.Com.(H) 1999]

Proof.

$$\begin{aligned} \text{var}(x) &= E(x-\bar{x})^2 = E\left(x^2 - 2x\bar{x} + \bar{x}^2\right) = E\left(x^2\right) - 2\bar{x}E(x) + E\left(\bar{x}^2\right) \\ &= E\left(x^2\right) - 2\bar{x}^2 + \bar{x}^2 \quad [\because E(x) = \bar{x}] \\ &= E\left(x^2\right) - \bar{x}^2 = \boldsymbol{E\left(x^2\right) - \{E(x)\}^2}. \end{aligned}$$

Standard Deviation (σ)

The Stanadrd Deviation (S.D.) *of a random variable x is the positive square root of the variance of x, i.e.* **S.D.** $\boldsymbol{(\sigma) = \sqrt{\text{var}(x)} = \sqrt{E\left(x^2\right) - \{E(x)\}^2}}$.

9.14 Moments of a Discrete Random Variable[1]

If a discrete random variable x assumes the discrete set of values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ where $p_1 + p_2 + \cdots + p_n = 1$, then *the rth moment* (m'_r) *about any arbitrary point* (*or value*) A is defined by

$$\boldsymbol{m'_r = \sum p_i(x_i - A)^r} \quad \text{or} \quad \boldsymbol{\sum p(x-A)^r} \tag{1}$$

and *the rth moment m_r about mean $\bar{x}$* is defined by

$$\boldsymbol{m_r} = \sum p_i(x_i - \bar{x})^r \quad \text{or} \quad \sum p(x-\bar{x})^r. \tag{2}$$

rth moment about the origin = $\Sigma p_i x_i^r$ or, $\Sigma p x^r$ [from (1)].

In particular, 1st moment about the origin $= \Sigma p_i x_i$ or, $\boldsymbol{\Sigma px}$ = **Mean** ($\boldsymbol{\bar{x}}$).

2nd moment about the mean $= m_2 = \Sigma p_i(x_i - \bar{x})^2$ or, $\Sigma p(x-\bar{x})^2$ = variance, i.e. *2nd moment about the mean is the* **variance.**

Example 57. *A random variable x has the following probability distribution:*

x	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the variance and hence, determine the standard deviation of the distribution.

Solution: Variance of $x = E\left(x^2\right) - \{E(x)\}^2$.

Now $$E(x) = \sum p_i x_i = \frac{1}{8}\times 0 + \frac{3}{8}\times 1 + \frac{3}{8}\times 2 + \frac{1}{8}\times 3 = \frac{3+6+3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

and $$E\left(x^2\right) = \sum p_i x_i^2 = \frac{1}{8}\times 0^2 + \frac{3}{8}\times 1^2 + \frac{3}{8}\times 2^2 + \frac{1}{8}\times 3^2 = 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3.$$

$\therefore$ Variance $= 3 - (1.5)^2 = 3 - 2.25 = \mathbf{0.75}$.

Standard deviation $= \sqrt{\text{var}(x)} = \sqrt{0.75} = \mathbf{0.87}$ (approx.).

Example 58. *A random variable has the following probability distribution:*

x	4	5	6	8
Probability	0.1	0.3	0.4	0.2

Find the expectation and the standard deviation of the random variable. **[C.U. B.Com.(H) 2003]**

[1]Not included in B.Com. (Honours) course of Vidyasagar University.

Solution: We have

x_i	4	5	6	8	Total
p_i	0.1	0.3	0.4	0.2	1

By definition,

$$\text{Expectation of } x = E(x) = \Sigma p_i x_i = 0.1\times4+0.3\times5+0.4\times6+0.2\times8$$
$$=0.4+1.5+2.4+1.6=5.9.$$
$$\text{var}(x)=E\left(x^2\right)-\{E(x)\}^2 \text{ and S.D.}=\sqrt{\text{var}(x)}.$$
$$E\left(x^2\right)=\Sigma p_i x_i^2=0.1\times4^2+0.3\times5^2+0.4\times6^2+0.2\times8^2$$
$$=0.1\times16+0.3\times25+0.4\times36+0.2\times64$$
$$=1.6+7.5+14.4+12.8=36.3.$$
$$\therefore \text{var}(x)=36.3-(5.9)^2=36.3-34.81=1.49.$$

Hence, *standard deviation* $=\sqrt{1.49}=1.22$ (approx.).

Example 59. *A bag contains 5 white and 7 black balls. Find the expectation of a man who is allowed to draw two balls from the bag and who is to receive one rupee for each black ball and two rupees for each white ball drawn.*

Solution: 2 balls can be drawn out of 12 in $^{12}C_2$ ways and 2 black balls can be drawn out of 7 in $^7C_2$ ways.
$\therefore p_i$ = Probability of drawing 2 black balls $=\frac{^7C_2}{^{12}C_2}=\frac{7}{22}$.
Similarly, p_2 = Probability of drawing 1 black and 1 white ball $=\frac{^5C_1\times{}^7C_1}{^{12}C_2}=\frac{35}{66}$ and p_3 = Probability of drawing 2 white balls $=\frac{^5C_2}{^{12}C_2}=\frac{5}{33}$.
If x be the amount of money received in a draw, then the *probability distribution of* x is given by

Result	2 black balls	1 black and 1 white ball	2 white balls	Total
x	₹2	₹2 + ₹1 = ₹3	₹2 × 2 = ₹4	
p	7/22	35/66	5/33	1

Hence, *the required expectation of the man*
$$=E(x)=\Sigma p_i x_i=\frac{7}{22}\times ₹2+\frac{35}{66}\times ₹3+\frac{5}{33}\times ₹4=₹\frac{187}{66}=₹2.83.$$

Example 60. *A and B play for a prize of ₹99. The prize is to be won by a player who first throw a '3' with one die. A first throws and if he fails, B throws and if he fails, A again throws and so on. Find their respective expectations.*

Solution: The probability of throwing '3' with a die $=\frac{1}{6}$ and the probability of not throwing '3' with a die $=1-\frac{1}{6}=\frac{5}{6}$.
A will win the prize if he throws '3' in the first casting, or *A* fails in the first, *B* fails in the second and *A* throws '3' in the third casting, or *A* fails, *B* fails and again *A* fails, *B* fails and then *A* throws '3' in the fifth casting, and so on. This may continue indefinitely.

Hence, the probability that A will win the prize

$$=\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\cdots$$
$$=\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\text{to }\infty\right]$$
$$=\frac{1}{6}\times\frac{1}{1-\left(\frac{5}{6}\right)^2}\qquad\left[\because\text{ for an infinite GP, sum}=\frac{a}{1-r},\text{ where }-1<r<1\right]$$
$$=\frac{1}{6}\times\frac{36}{11}=\mathbf{\frac{6}{11}}.$$

B will win the prize if A fails to throw '3' in the first casting, and B throws '3' in the second casting or A fails, B fails and again A fails and then B throws '3' in the 4th casting and so on. This may continue indefinitely.

Hence, the probability that B will win the prize

$$=\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\cdots$$
$$=\frac{5}{6}\times\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\text{to }\infty\right]$$
$$=\frac{5}{36}\times\frac{1}{1-\left(\frac{5}{6}\right)^2}=\frac{5}{36}\times\frac{36}{11}=\mathbf{\frac{5}{11}}.$$

Hence, A's expectation = ₹99 × $\frac{6}{11}$ = **₹54**

and B's expectation = ₹99 × $\frac{5}{11}$ = **₹45.**

Example 61. *An unbiased coin is tossed four times. If x denotes the number of heads, form the distribution of x by writing down all the possible outcomes and hence calculate the expected value and variance of x.*

Solution: If p denotes the probability of getting x heads, then probability distribution of x is given by

x	0	1	2	3	4	Total
p	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$	1

$$\left[\because P(x=0)=\frac{{}^4C_0}{2^4}=\frac{1}{16};P(x=1)=\frac{{}^4C_1}{2^4}=\frac{4}{16}=\frac{1}{4};\text{ etc.}\right]$$

Expected value of x

$$=E(x)=\Sigma p_i x_i=\frac{1}{16}\times0+\frac{1}{4}\times1+\frac{3}{8}\times2+\frac{1}{4}\times3+\frac{1}{16}\times4$$
$$=0+\frac{1}{4}+\frac{3}{4}+\frac{3}{4}+\frac{1}{4}=\frac{8}{4}=\mathbf{2}.$$

Now

$$E\left(x^2\right)=\sum p_i x_i^2=\frac{1}{16}\times0^2+\frac{1}{4}\times1^2+\frac{3}{8}\times2^2+\frac{1}{4}\times3^2+\frac{1}{16}\times4^2$$
$$=0+\frac{1}{4}+\frac{3}{2}+\frac{9}{4}+1=\frac{1+6+9+4}{4}=\frac{20}{4}=5.$$

$\therefore$ Variance of $x = \text{var}(x) = E(x^2) - \{E(x)\}^2 = 5 - (2)^2 = 1$.

Example 62. (i) *If 'a' is a constant, then* $\text{var}(a) = 0$ *and* $\text{var}(ax) = a^2 \text{var}(x)$;
(ii) *If a and b are constants, then* $\text{var}(a + bx) = b^2 \text{var}(x)$. [C.U. B.Com.(H) 2004]

Solution: (i) $\text{var}(a) = E(a^2) - \{E(a)\}^2$. Since '$a$' is a constant, $E(a) = a, E(a^2) = a^2$.

$$\therefore \text{var}(a) = a^2 - a^2 = 0.$$
$$\text{var}(ax) = E\left(a^2x^2\right) - \{E(ax)\}^2 = a^2 E\left(x^2\right) - \{aE(x)\}^2 = a^2 E\left(x^2\right) - a^2\{E(x)\}^2$$
$$= a^2\left[E\left(x^2\right) - \{E(x)\}^2\right] = \boldsymbol{a^2 \text{var}(x)}.$$

(ii) Let $y = a + bx$; then $\bar{y} = E(y) = E(a + bx) = a + bE(x)$, i.e. $\bar{y} = a + b\bar{x}$ [$\because \bar{x} = \text{Mean} = E(x)$].
$\therefore y - \bar{y} = bx - b\bar{x} = b(x - \bar{x})$.
By definition, $\text{var}(y) = E(y - \bar{y})^2 = E[b(x - \bar{x})]^2 = E[b^2(x - \bar{x})^2] = b^2 E(x - \bar{x})^2 = b^2 \text{var}(x)$.
Hence, $\boldsymbol{\text{var}(a + bx) = b^2 \text{var}(x)}$.

Example 63. *If X and Y are two independent random variables, prove that*

$$\text{var}(aX + bY) = a^2 \text{var}(X) + b^2 \text{var}(Y),$$

where a and b are constants. [C.U. B.Com.(H) 2002]

Solution: Let $u = aX + by$; $\therefore$ $\text{var}(u) = E(u - \bar{u})^2$.
Now $\bar{u} = E(u) = E(aX + bY) = E(aX) + E(bY) = aE(X) + bE(Y) = a\bar{X} + b\bar{Y}$.

$$\therefore E(u - \bar{u})^2 = E(aX + bY - a\bar{X} - b\bar{Y})^2 = E\{a(X - \bar{X}) + b(Y - \bar{Y})\}^2$$
$$= E\{a^2(X - \bar{X})^2 + b^2E(Y - \bar{Y})^2 + 2abE[(X - \bar{X})(Y - \bar{Y})]$$
$$= a^2 \text{var}(X) + b^2 \text{var}(Y) + 2ab \cdot 0$$
$$[\because E(X - \bar{X})(Y - \bar{Y}) = E(X - \bar{X}) \cdot E(Y - \bar{Y}) = 0]$$
$$= \boldsymbol{a^2 \text{var}(X) + b^2 \text{var}(Y)}.$$

9.15 Repeated Trials: Bernoulli Trials

By the word 'trial', we mean an attempt to produce an event A which is neither certain nor impossible. 'Tossing a coin' is a trial and 'getting a head' is an event, say A. The outcome (or result) of a trial is called a 'success' if the event A occurs, and a 'failure' if A fails to occur.

If a trial is repeated, we get a series of trials. A series of trials will be said to be independent with respect to an event A if the probability of this event in any trial remains the same, whether the results of any number of other trials are known or not.

Repeated independent trials are called *Bernoulli trials* or Bernoullian series of trials if there are only two possible outcomes (success or failure, yes or no, hit or miss, etc.) for each trial and their probabilities remain the same throughout the trials.

Theorem 1. *Prove that the probability of r successes in a series of n independent trials* ($r \leq n$) *is* $\boldsymbol{{}^nC_r p^r q^{n-r}}$, *where p is the probability of a success and* $q(= 1 - p)$ *is the probability of a failure in a single trial.*

Proof. Let 'S' denote a success and 'F' denote a failure. Let us first find the probability that the first r trials are successes and the remaining $(n-r)$ trials are failures, i.e. the probability of

$$\underbrace{SS\cdots S}_{\text{1st } r \text{ trials}} \quad \underbrace{FF\cdots F}_{(n-r) \text{ trials}}$$

Since n trials are independent, by the theorem of compound probability, the probability of r successes is

$$\underbrace{p\times p\times\cdots\times p}_{r \text{ times}}=p^r$$

and the probability of $(n-r)$ failures is

$$\underbrace{q\times q\times\cdots\times q}_{(n-r) \text{ times}}=q^{n-r}$$

$\therefore$ the probability that the first r trials are successes and the remaining $(n-r)$ trials are failures is $p^r q^{n-r}$, where $q=1-p$. Similarly, the probability of obtaining r successes and $(n-r)$ failures in any other definite specified order is also $p^r q^{n-r}$.

But we are interested in finding the probability that in n independent trials any r trials are successes [and, therefore, remaining $(n-r)$ trials are failures]. r trials can be selected out of n in ${}^nC_r$ mutually exclusive ways in each of which the probability of r successes is $p^r q^{n-r}$.

Hence, by *the theorem of total probability*, the probability $f(r)$ of r successes (regardless of their order) in a series of n independent trials is $f(r)={}^nC_r p^r q^{n-r}$, where $q=1-p$.

Example 64. (a) *Four coins are tossed simultaneously. What is the probability of getting* 2 *heads and* 2 *tails?*

(b) *Six fair coins are thrown simultaneously. Find the probability of getting* (i) *exactly* 4 *heads,* (ii) *at least* 4 *heads.* [C.U. B.Com.(H) 1998]

Solution: (a) We know that tossing 4 coins simultaneously once is the same thing as tossing one coin 4 times, provided all the coins are unbiased.

Let 'getting head' by tossing a coin represent a 'success'. Then 'getting tail' will be a 'failure'. $\therefore p=\frac{1}{2}$ and $q=\frac{1}{2}$.

Thus we have 4 independent trials in which 2 trials are successes and 2 failures, i.e. $n=4$, $r=2$, $p=\frac{1}{2}$, $q=\frac{1}{2}$.

Hence, the required probability of getting 2 heads and 2 tails

$$={}^4C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{4-2}=\tfrac{4\cdot 3}{2\cdot 1}\times\tfrac{1}{4}\times\tfrac{1}{4}=\mathbf{\tfrac{3}{8}}.$$

(b) We have $p=\frac{1}{2}$, $q=1-\frac{1}{2}=\frac{1}{2}$, $n=6$.

(i) Probability of getting exactly 4 heads $={}^6C_4\left(\frac{1}{2}\right)^4\cdot\left(\frac{1}{2}\right)^{6-4}=15\times\left(\frac{1}{2}\right)^6=\mathbf{\frac{15}{64}}$.

(ii) Probability of getting at least 4 heads

= Probability of getting 4 heads + Probability of getting 5 heads + Probability of getting 6 heads

$$=\frac{15}{64}+{}^6C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{6-5}+\left(\frac{1}{2}\right)^6=\frac{15}{64}+\frac{6}{64}+\frac{1}{64}=\frac{22}{64}=\mathbf{\frac{11}{32}}.$$

Example 65. *Four unbiased coins are tossed simultaneously. Let X be the random variable denoting the number of heads obtained. Obtain the probability distribution of X. Find $E(x)$ and* $\operatorname{var}(X)$. *[Symbols have their usual meanings.]*

Solution: The probability distribution of X is given by

x	0	1	2	3	4	Total
$f(x)$	1/16	4/16	6/16	4/16	1/16	1

[$\because$ 1 head can be obtained in ${}^4C_1$ ways = 4 ways, 2 heads can be obtained in ${}^4C_2$ ways = 6 ways, etc.]

$\therefore E(X)=\Sigma xf(x)=0\times\frac{1}{16}+1\times\frac{4}{16}+2\times\frac{6}{16}+3\times\frac{4}{16}+4\times\frac{1}{16}=\frac{0+4+12+12+4}{16}=\frac{32}{16}=2.$

$$\begin{aligned} E(X^2)&=\Sigma x^2 f(x)=0^2\times\frac{1}{16}+1^2\frac{4}{16}+2^2\times\frac{6}{16}+3^2\times\frac{4}{16}+4^2\times\frac{1}{16}\\ &=\frac{0+4+24+36+16}{16}=\frac{80}{16}=5.\end{aligned}$$

$\therefore \operatorname{var}(X)=E(X^2)-\{E(X)^2\}^2=5-2^2=1.$

Example 66. 13 *cards are drawn simultaneously from a pack of* 52 *cards. If aces count* 1, *face cards* 10 *each, and others according to their denominations, find the expectation of the total score on the* 13 *cards.*

Solution: Let S be the total score on the 13 cards and x_i be the number on the ith card drawn. Then

$$S=\sum_{i=1}^{13}x_i \quad\text{and}\quad E(S)=E(x_1+x_2+\cdots+x_{13})=E(x_1)+E(x_2)+\cdots+E(x_{13}).$$

Now

$$\begin{aligned} E(x_i)&=1\cdot\frac{4}{52}+2\cdot\frac{4}{52}+\cdots+10\cdot\frac{4}{52}+10\cdot\left(\frac{4}{52}+\frac{4}{52}+\frac{4}{52}\right)\\ &=\frac{4}{52}(1+2+\cdots+10)+3\cdot10\cdot\frac{4}{52}=\frac{1}{13}\times\frac{10(10+1)}{2}+\frac{30}{13}=\frac{85}{13}.\\ \therefore E(x_i)&=\frac{85}{13}\quad\text{for } i=1,2,\ldots,13.\end{aligned}$$

Hence, $E(S)=\frac{85}{13}+\frac{85}{13}+\cdots$ to 13 terms $=\frac{85}{13}\times 13=85.$

EXERCISES ON CHAPTER 9(II)

(Random Variable, Expectation and Moment)

Theory

1. Define 'Random Variable'. How do you distinguish between 'discrete' and 'continuous' random variables? Illustrate your answer with suitable examples. [C.U. B.Com.(H) 1996; V.U. B.Com. 2008]

2. What do you mean by 'Discrete Probability Distribution' and 'Probability Function'? Give one example for each.

3. (a) What do you understand by 'the expectation of a random variable'? Explain as clearly as you can. [C.U. B.Com.(H) 2004; V.U. B.Com. 2008]

   (b) Define mathematical expectation of a random variable.

4. (a) Define 'Expectation' and 'Variance' of a random variable. Prove that $E(x-\bar{x})^2 = E(x^2) - \{E(x)\}^2$.

   [C.U. B.Com.(H) 1999]

   (b) Show that $\text{var}(ax) = a^2\,\text{var}(x)$. [C.U. B.Com.(H) 2004]

5. Write notes on: (a) Random Experiment, (b) Random Variable, (c) Dependent and Independent Events.

   [V.U. B.Com.(H) 2011]

6. Define 'Bernoulli Trials' and give an example. Prove that the probability of r successes in a series of n independent trials is ${}^nC_r p^r q^{n-r}$, where p is the probability of a success in a single trial and $q = 1 - p$. [C.U. B.Com.(H) 1998]

Problems

A

1. If x is a random variable, show that (a) $E(1) = 1$; (b) $E(3x) = 3E(x)$; (c) $E(2+3x) = 2+3E(x)$.

2. (a) A random variable X has the following distribution:

X	-1	0	1	2	3	4
$P(X)$	0.1	0.2	m	0.3	$2m$	$3m$

   Compute the value of m. [C.U. B.Com.(H) 2006]

   (b) A random variable X has the following probability distribution:

X	-1	0	1	2
Probability	1/3	1/6	1/6	1/3

   Compute the expectation of X. [C.U. B.Com.(H) 2005]

3. (a) 4 unbiased dice are thrown; find the mathematical expectation of the sum of points on the dice.

   [C.U. B.Com.(H) 2001]

   [**Hints**: See worked-out Example 4, put $n = 4$.]

   (b) Find the mathematical expectation of the sum of points on 10 dice.

4. (a) A random variable x has the following probability distribution:

x	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

   Find the variance and hence determine the standard deviation of the distribution.

(b) If in (a) $y = 2x + 3$, find the expectation and variance of the random variable y.

[B.U. B.Com.(H) 2008]

[**Hints**: $E(y) = E(2x+3) = 2E(x) + 3 = 2 \times \frac{7}{2} + 3 = 10$,
since $E(x) = \Sigma p_i x_i = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6} \times \frac{6 \times 7}{2} = \frac{7}{2}$, from Q.4(a).
$E(y^2) = E(4x^2 + 12x + 9) = 4E(x^2) + 12E(x) + 9$, where
$E(x^2) = \Sigma p_i x_i^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{1}{6} \times \frac{6(6+1)(2\times 6+1)}{6} = \frac{91}{6}$, from Q.4(a).
$\therefore E(y^2) = 4 \times \frac{91}{6} + 12 \times \frac{7}{2} + 9 - 10^2 = \frac{182}{3} + 42 + 9 - 100 = \frac{35}{3}$. Now, find $E(y^2) - \{E(y)\}^3$.]

(c) The probability distribution of a random variable x is as follows:

x	1	2	3	4	5
$f(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

If $y = 3x - 1$, find the expectation and variance of the random variable y.

[B.U. B.Com.(H) 2008]

5. (a) For what value of a will the function $f(x) = ax; x = 1, 2, \ldots, n$, be the probability mass function of a discrete random variable x? Find the mean and variance of x.

[**Hints**: $\Sigma f(x) = 1$ gives $a\Sigma x = 1$ or, $a \cdot \frac{n(n+1)}{2} = 1$, etc. $E(x) = \Sigma x f(x) = a\Sigma x^2 =$ etc. and $E(x^2) = \Sigma x^2 f(x) =$ etc.]

(b) A random variable X has the following probability function:

Values of X	0	1	2	3	4	5	6	7
$P(X)$	0	$2k$	$3k$	k	$2k$	k^2	$7k^2$	$2k^2 + k$

Find the value of k and then evaluate $P(X < 6)$, $P(X \geq 6)$ and $P(0 < X < 5)$.

[CA May 1991]

[**Hints**: See worked-out Example 1.]

(c) A discrete random variable x has the following probability distribution

x	0	1	2	3	4	5	6	7
$p(x)$	a	$4a$	$3a$	$7a$	$8a$	$10a$	$6a$	$9a$

Determine the value of a and $P(X < 3)$. [B.U. B.Com.(H) 2007]

[**Hints**: $a + 4a + 3a + 7a + 8a + 10a + 6a + 9a = 1$ or, $48a = 1$ or, $a = \frac{1}{48}$.
$P(X < 3) = P(0) + P(1) + P(2) = a + 4a + 3a = 8a = \frac{8}{48} = \frac{1}{6}$.]

6. Evaluate k if

$$f(x) = \begin{cases} k, & x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability mass function. Also find its mean and standard deviation. [ICWAI Dec. 1991]

[**Hints**: $\Sigma f(x) = 1$ gives $6k = 1$ or, $k = 1/6$. The probability distribution of x is given by

x	1	2	3	4	5	6	Total
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6	1

Now find $E(x) = \Sigma x f(x)$ and $E(x^2) = \Sigma x^2 f(x)$. See worked-out Example 10.]

7. (a) A random variable has the following probability distribution:

x	4	6	7	10
Probability	0.2	0.4	0.3	0.1

Find the expectation and the standard deviation of the random variable. [C.U. B.Com.(H) 1997]

(b) A random variable has the following probability distribution:

x	4	6	7	8
Probability	0.2	0.3	0.2	0.4

Find the expectation and variance of the random variable. [C.U. B.Com.(H) 2003]

[Note the given probability distribution is not correct, since $\Sigma p_i = 1.1 > 1$.]

8. Prove that: (a) $\text{var}(2) = 0$; (b) $\text{var}(3x) = 9\,\text{var}(x)$; (c) $\text{var}(3+4x) = 16\,\text{var}(x)$.

9. Find the expected value and variance of the following probability distribution:

x	-10	-20	30	75	80
$p(x)$	1/5	3/20	1/2	1/10	1/20

10. (a) In a business venture a man can make a profit of ₹2000 with a probability of 0.4 or have a loss of ₹1000 with a probability of 0.6. What is his expected profit?

(b) A box contains 15 electric bulbs of which 5 are defectives. A man selects 3 bulbs at random. Find the expected number of defective bulbs in his selection.

(c) A player tosses 3 coins. He wins ₹16 if 3 heads appear, ₹8 if 2 heads appear, ₹4 if 1 head appears and ₹2 if no head appears. Find his expected amount of winning.

(d) Three coins, whose faces are marked 1 and 2, are tossed. What is the expectation of the total value of number on their faces?

[**Hints:** If X denote the total value of numbers on the faces of the three coins, then the probability distribution of X is given by

x_i	3	4	5	6	Total
$P(x_i)$	1/8	3/8	3/8	1/8	1

[$\because$ Total value 4 occurs when the outcomes are (1, 1, 2), (1, 2, 1), (2, 1, 1).]
Now find $E(X) = \Sigma x_i P(x_i)$.]

11. A coin is tossed until a head appears. What is the expected number of tosses?

[**Hints:** If x denotes the no. of tosses until a head appears, then

x_i	1	2	3	4	$\cdots$	*Total*
p_i	1/2	1/4	1/8	1/16	$\cdots$	1

$$\therefore E(x) = \Sigma p_i x_i = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \cdots = \frac{1}{2}\left\{1 + 2\times\frac{1}{2} + 3\times\left(\frac{1}{2}\right)^2 + 4\times\left(\frac{1}{2}\right)^3 + \cdots\right\}$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right)^{-2}.]$$

12. The monthly demand for radios is known to have the following probability distribution:

Demand	1	2	3	4	5	6
Probability	0.10	0.15	0.20	0.25	0.20	0.10

Determine the expected demand for radios. Also find the variance. If the cost (₹C) of producing (n) radios is given by $C = 1000 + 200n$, determine the expected cost.

13. A number is chosen at random from the set $1, 2, 3, \ldots, 100$ and another number is chosen at random from the set $1, 2, 3, \ldots, 50$. What is the expected value of their product?

[**Hints:** Consider two random variables which can assume values from the first and the second sets respectively are independent of each other.]

14. Five unbiased coins are tossed simultaneously. What is the probability of getting (a) exactly 2 tails, (b) at most 2 tails, (c) at least 2 tails? [C.U. B.Com.(H) 1999]

15. Eight coins are thrown simultaneously. Find the probability of obtaining (a) exactly 6 heads; (b) at least 6 heads.

16. If a sample of 5 items is drawn randomly from a lot containing 10% defective items, what is the probability of getting not more than one defective item?

17. Let x denote the profit that a man makes in a business. He may earn ₹2,800 with probability 0.5; he may lose ₹5,500 with probability 0.3 and he may neither earn nor lose with probability 0.2. Calculate the mathematical expectation of x.

18. (a) A person tosses a coin once and is to receive ₹4 for head and is to pay ₹2 for tail. Find expectation and variance of his gain.

    (b) A person tosses two coins simultaneously and is to receive ₹8 for two heads, ₹2 for one head and he is to pay ₹6 for no head. Find his expectation.

19. A man draws 2 balls from a bag containing 3 white and 5 black balls. If he is to receive ₹14 for every white ball and ₹7 for every black ball drawn, what is his expectation?

20. (a) The probability that there is at least one error in an accounts statement prepared by A is 0.2, for B and C they are 0.25 and 0.4 respectively. A, B and C prepared 10, 16 and 20 statements respectively. Find the expected number of correct statements in all.

    [**Hints:** If A, B, C be the events that there is at least one error in an accounts statement prepared by A, B, C respectively, then $P(A) = 0.2$, $P(B) = 0.25$, $P(C) = 0.4$ so that $P(A^c) = 0.8$, $P(B^c) = 0.75$, $P(C^c) = 0.6$. Hence, the expected no. of correct statements $= 0.80 \times 10 + 0.75 \times 16 + 0.6 \times 20 = 32$.]

    (b) The probability that there is at least one error in an accounts statement prepared by A is 0.3 and for B and C, they are 0.4 and 0.45 respectively. A, B and C prepared 20, 10, 40 statements respectively. Find the expected number of correct statements in all. [CA Nov. 1991]

21. If a person gains or loses an amount equal to the number appearing when a balanced die is rolled once, according to whether the number is even or odd, how much money can he expect per game in the long run?

22. A box contains 8 tickets. 3 of the tickets carry a prize of ₹5 each and the other 5 carry a prize of ₹2 each.

    (a) If one ticket is drawn, what is the expected value of the prize?

    (b) If two tickets are drawn, what is the expected value of the game?

[**Hints**: (a) If x be the value of the prize in rupees and p the corresponding probability, then

x	2	5	Total
p	5/8	3/8	1

Hence, etc.

(b)

x	4	7	10	Total
p	${}^5C_1/{}^8C_2$	${}^5C_1 \times {}^3C_1/{}^8C_2$	${}^3C_2/{}^8C_2$	1

Hence, etc.]

23. An unbiased coin is tossed four times. If x denotes the number of heads, form the distribution of x by writing down all the possible outcomes and hence calculate the expected value and variance of x.

24. 10,000 tickets are sold in a lottery in which there is a first prize of ₹5,000, two second prizes of ₹1,000 each and ten consolation prizes of ₹100 each. One ticket costs ₹1. Find the expected net gain or loss if you buy a ticket.

[**Hints**: If ₹x be the net gain or loss and p the corresponding probability, then the prob. distribution of x is

x	₹4,999	₹999	₹99	- ₹1	Total
p	1/10,000	2/10,000	10/10,000	9987/10,000	1

, since $\Sigma p_i = 1$.

Hence, etc.]

25. A and B toss in turn an ordinary die for a prize of ₹55. The first to toss a six wins. If A has the first throw, what is his expectation?

[**Hints**: See worked-out Example 9.]

B

1. If it rains, a taxi driver can earn ₹100 per day. If it is fair, he can lose ₹10 per day. If the probability of rain is 0.4, what is his expectation?

2. Throwing 2 unbiased coins simultaneously, Mr. X bets with Mrs. X that he will receive ₹4 from her if he gets 2 heads and he will give ₹4 to her otherwise. Find Mr. X's expectation. [ICWAI June 1991]

3. A player tosses 3 fair coins. He wins ₹10 if 3 heads appear, ₹6 if 2 heads appear and ₹2 if 1 head appears. On the other hand he loses ₹25 if 3 tails appear. Find the expected gain of the player.

[ICWAI Dec.1990]

4. A man runs an ice cream parlour in a holiday resort. If the summer is mild he can sell 2500 cups of ice cream; if it is hot, he can sell 4000 cups; if it is very hot, he can sell 5000 cups. It is known that the probability in any year for the summer to be mild is 1/7 and hot is 4/7. A cup of ice cream costs ₹2 and sold for ₹3.50. What is his expectation? **[ICWAI June 1990]**

[**Hints:** If x denote amount of profit and p the corresponding probability, then the prob. distribution of x is given by

No. of cups sold x(Rs.)	2500 $1.5\times 2500=3750$	4000 $1.5\times 4000=6000$	5000 $1.5\times 5000=7500$	Total
p	1/7	4/7	2/7	1

His expectation $=\Sigma p_i x_i=\frac{1}{7}\times 3750+\frac{4}{7}\times 6000+\frac{2}{7}\times 7500=$ etc.]

ANSWERS

A

2. (a) 1/15; (b) 1/2.
3. (a) 14; (b) 35.
4. (a) 2.92, 1.71; (b) 35/3; (c) 8, 18.
5. (a) $\frac{2}{n(n+1)}$; $\frac{2n+1}{3}$; $\frac{n^2+n-2}{18}$;
   (b) 1/10, 0.81, 0.19, 0.8.
   (c) $\frac{1}{48}$, $\frac{1}{6}$.
6. 1/6; 3.5, $\frac{\sqrt{35}}{2\sqrt{3}}$ or 1.71.
7. 6.3; 1.62.
9. 21.5; 950.25.
10. (a) ₹200; (b) 1; (c) ₹6.75, (d) 4.5.
11. 2.
12. 3.60; 2.14; ₹1,720.
13. 5151/4.
14. (a) 5/16; (b) 1/2; 13/16.
15. 7/64, 37/256.
16. 45, 927/50,000.
17. ₹250 (Loss).
18. (a) ₹1, ₹9; (b) ₹1.50.
19. ₹19.25.
20. (a) 32; (b) 42.
21. 50 paise.
22. ₹3.125 or ₹3.13; and ₹6.25.
23. 2; 1.
24. 20 paise (loss).
25. ₹30.

B

1. ₹34.
2. –₹2, i.e. Mr. X will give ₹2.
3. ₹1.13.
4. ₹6, 107.14.

Chapter 10

Index Numbers

10.1 Introduction

A very useful statistical tool for measuring changes in a variable or a group of related variables w.r.t. time, geographic location or other characteristic is Index Number. Index Number, in its simplest form, represents a special case of an average, generally weighted average, compiled from a sample of items judged to be representative of the whole. In this chapter, we shall study various types of Index Numbers, their uses and the principles involved in the construction of Index Numbers.

10.2 Index Number

An Index Number is a number which is used as a device for comparison between the price, quantity or value of a group of articles (related variables) in different situations, e.g., at a certain place or a period of time and that at another place or period of time. When the comparison is in respect of prices, it is called an Index Number of Prices; when in respect of physical quantities, it is termed as Index Number of Quantities. Other Index Numbers are defined in the same manner.

The period of time, usually a year and sometimes a week or a month, with whose prices the prices of other periods are compared, is called the *Base Period.* Thus, index numbers are meant for comparison of variations arising out of the difference in situations, e.g., change of time or change of place.

The statement that 'the Index Number of wholesale prices in Kolkata for the year 1993 was 150 compared to 1999' means that there was a net increase in the price of wholesale commodities in the Kolkata market to the extent of 50 per cent. In comparing the price in a given year with the price of a particular year called the *base year* (or base period), the price at the base year is taken as 100 and the price at the given period of time (called *current* year), for which the Index Number is required, is expressed as a percentage of the base year price. Thus, if for any item, p_1 be the current price (i.e., price in the current year) and p_0, the base price, then corresponding Index Number is defined by

$$\text{Index Number} = \frac{p_1}{p_0} \times 100 = \frac{\text{Current price}}{\text{Base price}} \times 100.$$

10.3 Price Relative

A *Price Relative* or simply a *Relative* is the ratio of the price of a certain commodity at the current year to the price at the base year, expressed as a percentage. So,

$$\text{Price Relative} = \frac{\text{Current year's price}}{\text{Base year's price}} \times 100 \quad \text{or} \quad \frac{p_1}{p_0} \times 100.$$

Price Relatives are thus the same as *Index Numbers for prices.*

10.4 Selection of Base

Every index number is associated with a base period. The base selected in constructing index numbers should suffer from no abnormalities, as in the case of price index the base period (year, month or day) should be so selected that there is no war, famine, widespread earthquake or flood, etc., causing abnormalities (boom or depression) in the prices in that period.

Usually, bases are selected in four different ways, and according to these four types of base periods, the following are the four methods of constructing Index Numbers:

(a) Fixed-Base Method, (b) Chain-Base Method, (c) Average-Base Method and (d) Laspeyres' Method.

(a) Fixed-Base Method: Simple and Composite Index Numbers

In Fixed-Base Method, a year is fixed as base period and the prices during the base year are represented by 100. The Price Relatives of the other years are then the required Index Numbers.

Simple Index Numbers

Example 1.

TABLE 10.1: PRICE INDEX OF A SINGLE COMMODITY, VIZ., RICE

Year	Price of Rice per Ql (₹)	Index No. with 2000 as Base Year	Index No. with 1997 as Base Year
1997	400	80	100
1998	360	72	90
1999	480	96	120
2000	500	100	125
2001	440	88	110
2002	520	104	130
2003	460	92	115

These Index Numbers are Simple Index Numbers.

Composite Index Numbers

In the case of more than one item, their price relatives w.r.t. a selected base are determined separately. The statistical average of these relatives is called a *Composite Index Number* or simply an *Index Number* in respect of the items. In averaging the relatives, any types of the averages, viz., arithmetic, geometric and harmonic means, median and mode can be used. Of these averages, the geometric and the harmonic means presenting difficulty in calculations and median and mode being unsuitable from some points of view, the *arithmetic mean is the most popularly used average* for the purpose.

Example 2.

TABLE 10.2: PRICE INDEX OF RICE, WHEAT AND JOWAR (Base Year: 1997)

Year	Price of Rice per Ql (₹)	Price of Wheat per Ql (₹)	Price of Jowar per Ql (₹)	Index Numbers			Composite Index No.
				Rice	Wheat	Jowar	
1997	400	250	200	100	100	100	100
1998	360	210	240	90	84	120	98
1999	480	270	210	120	108	105	111
2000	500	260	220	125	104	110	113
2001	440	230	190	110	92	95	99
2002	520	280	230	130	112	115	119
2003	460	240	170	115	96	85	98.67

Here the simple (unweighted) arithmetic mean of the three price relatives of the three commodities, in each year, has been taken as the Composite Index Number.

When weights are attached to different commodities, the weighted arithmetic mean of the price relatives is to be determined for the Composite Index Number.

(b) Chain-Base Method

When year to year comparison is desired, Index Numbers are determined by a method known as *Chain-Base Method.* In this method, each year is taken as the base for the immediately next year, the first year itself being its own base. *For example,* if Index Numbers for 1997, 1998, 1999, 2000 and 2001 are to be constructed, 2000 will be the base for 2001, 1999 for 2000, 1998 for 1999, 1997 for 1998 and 1997 itself for 1997.

Such indices are particularly found suitable for use in business, where changes with reference to immediate past are considered more important than a relative change with remoter period. These indices are comparatively free from seasonal variations.

Some of the advantages of Chain-Base Index Numbers are: new items may be readily included or old ones dropped in their calculations; also such indices are more free from seasonal variation than the Fixed-Base Indices.

Example 3.

TABLE 10.3: PRICE INDEX OF RICE (Chain-Base Method)

Year	Price of Rice per Ql (₹)	Calculations	Chain-Base Index Numbers
1997	400	$\frac{400}{400} \times 100 =$	100
1998	360	$\frac{360}{400} \times 100 =$	90
1999	480	$\frac{480}{360} \times 100 =$	133.3
2000	500	$\frac{500}{480} \times 100 =$	104.2
2001	440	$\frac{440}{500} \times 100 =$	88
2002	520	$\frac{520}{440} \times 100 =$	118.2

The *Chain-Base Index Numbers are also known as Link Index Numbers.*

Example 4.

TABLE 10.4: CONVERSION OF FIXED-BASE INDEX NUMBERS OF RICE TO CHAIN-BASE INDEX NUMBERS

Year	Fixed-Base Index No.	Conversion	Chain-Base Index Numbers
1998	90	$\frac{90}{90} \times 100 =$	100
1999	120	$\frac{120}{90} \times 100 =$	133.3
2000	125	$\frac{125}{120} \times 100 =$	104.2
2001	110	$\frac{110}{125} \times 100 =$	88
2002	130	$\frac{130}{110} \times 100 =$	118.2

Example 5.

Year	Chain-Base Index No.	Conversion	Fixed-Base Index No.
1997	120		120
1998	115	$\frac{120}{100} \times 115 =$	138
1999	75	$\frac{138}{100} \times 75 = \frac{207}{2} =$	103.5
2000	80	$\frac{207}{2\times100} \times 80 = \frac{414}{5} =$	82.8
2001	87.5	$\frac{414}{5\times100} \times \frac{175}{2} = \frac{1449}{20} =$	72.5

TABLE 10.5: CONVERSION OF CHAIN-BASE INDEX NUMBERS TO FIXED-BASE INDEX NUMBERS

(c) Average-Base Method

The average of a number of years' prices may be used as base-price in determining index numbers. This has the effect of minimizing the abnormalities of any particular year.

Example 6. *Prepare index numbers of prices of rice for five years with the average price as base:*

Year	1957	1958	1959	1960	1961
Rate per Rupee	2.500 kg	4 kg	3.125 kg	3.500 kg	2.536 kg

After converting the prices into ₹ per Ql (1 Ql = 100 kg) the whole working is as under:

TABLE 10.6: PRICE INDEX OF RICE WITH THE AVERAGE OF THE FIVE YEARS' PRICE AS BASE-PRICE

Year	Price per Ql (₹)	Average Price (Base-Price) (₹)	Calculations	Index No.
1957	40		$\frac{40}{33} \times 100 =$	121.2
1958	25		$\frac{25}{33} \times 100 =$	75.8
1959	32	33	$\frac{32}{33} \times 100 =$	97.0
1960	28.57		$\frac{28.57}{33} \times 100 =$	86.6
1961	39.43		$\frac{39.43}{33} \times 100 =$	119.5

10.5 Methods of Construction of Index Numbers

The following are the different methods of construction of Index Numbers of Prices of different commodities:

- Fixed-Base Method
  - Method of Aggregates
    1. Simple Aggregate of Prices
    2. Weighted Aggregate of Prices.
  - Method of Relatives
    1. Simple Arithmetic Mean of Price Relatives
    2. Simple Geometric Mean of Price Relatives
    3. Weighted Arithmetic Mean of Price Relatives
    4. Geometric Mean of two weighted Aggregates of Prices (special case).
- Chain-Base Method

  Simple Arithmetic or Geometric Mean of Link Relatives (Chain Index).

Notations

For the purpose of showing the above modes of construction by mathematical formulae, the following symbols will be used:

$$
\begin{aligned}
p_0', p_0'', p_0''', \ldots &= \text{prices of items } 1, 2, 3, \cdots \text{ respectively in the base year, say } Y_0.\\
q_0', q_0'', q_0''', \ldots &= \text{quantities of items } 1, 2, 3, \cdots \text{ respectively in the base year, } Y_0.\\
p_1', p_1'', p_1''', \ldots &= \text{prices of items } 1, 2, 3, \cdots \text{ respectively in the current year, } Y_1.\\
q_1', q_1'', q_1''', \ldots &= \text{quantities of items } 1, 2, 3, \cdots \text{ respectively in the current year, } Y_1.
\end{aligned}
$$

Similar notations for prices and quantities of items $1, 2, 3, \cdots$ respectively in the year Y_2, Y_3, etc., will be used. However, p_0, q_0, and p_1, q_1 will generally refer to the price and quantity at the base year (Y_0) and at the current year (Y_1) respectively, without any specific mention about the different items.

I_{01} = Index Number for the year Y_1 with the year Y_0 as base.

I_{12} = Index Number for the year Y_2 with the year Y_1 as base, and so on.

A. Method of Aggregates

1. Simple Aggregate of Prices

The price per standard unit for each of the items included in the group is aggregated or totalled for the base year as well as for the current year separately. The total price for the current year expressed as a percentage of the total for the base year, gives the Index Number. Thus, symbolically,

Simple Aggregative Price Index,

$$I_{01} = \frac{p_1' + p_1'' + p_1''' + \cdots}{p_0' + p_0'' + p_0''' + \cdots} \times 100 = \frac{\Sigma p_1}{\Sigma p_0} \times 100, \quad \text{i.e.,} \quad \boldsymbol{I_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100.}$$

Example 7.

TABLE 10.7: DETERMINATION OF SIMPLE AGGREGATIVE INDEX NUMBER		
Commodity	**Base (2001) Price (₹ per Ql) p_0**	**Current (2005) Price (₹ per Ql) p_1**
Rice	320	500
Wheat	250	250
Oil (edible)	900	1000
Fish	1200	1400
Potato	350	400
Total	$3020 = \Sigma p_0$	$3550 = \Sigma p_1$

$$\therefore \text{ Simple Aggregative Index Number} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{3550}{3020} \times 100 = 117.5.$$

2. Weighted Aggregate of Prices

In this type, the prices in the current year and the base year are both weighted by some quantities (or weights). The weighted prices are then aggregated and the ratio of the aggregate for the current year to the aggregate for the base year, expressed as a percentage, gives the desired Index.

If $w', w'', \ldots, w^{(n)}$, are the respective weights, then the Weighted Aggregative Price Index I_{01} is given by

$$I_{01} = \frac{p_1' w' + p_1'' w'' + \cdots + p_1^{(n)} w^{(n)}}{p_0' w' + p_0'' w'' + \cdots + p_0^{(n)} w^{(n)}} \times 100 = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100, \quad \text{i.e.,} \quad \boldsymbol{I_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100.}$$

Note: Students should note that the same set of weights is to be used both for the base year and the current year.

Example 8. *Find, by the Weighted Aggregative Method, the Index Number from the following data:*

TABLE 10.8: DETERMINATION OF WEIGHTED AGGREGATIVE INDEX NUMBER					
Commodity	**Base (2001) Price (₹ per Ql) (p_0)**	**Current (2005) Price (₹ per Ql) (p_1)**	**Weight (w)**	$p_1 w$	$p_0 w$
Rice	320	500	8	4000	2560
Wheat	250	250	6	1500	1500
Oil (edible)	900	1000	7	7000	6300
Fish	1200	1400	3	4200	3600
Potato	350	400	5	2000	1750
Total	...	...	...	$18700 = \Sigma p_1 w$	$15710 = \Sigma p_0 w$

$$\therefore \text{ Weighted Aggregative Index Number} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100 = \frac{18700}{15710} \times 100 = 119.03.$$

There are various ways of weighting, depending on the nature of weights used.

1. If base year quantities are used as weights, we have an Index Number known as **Laspeyres' Index.** Thus,

$$\textbf{Laspeyres' Index } (I_{01}) = \frac{p_1' q_0' + p_1'' q_0'' + p_1''' q_0''' + \cdots}{p_0' q_0' + p_0'' q_0'' + p_0''' q_0''' + \cdots} \times 100 = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{\text{Sum of prices of base period quantities at current prices}}{\text{Sum of prices of base period quantities at base prices}} \times 100.$$

2. If the current year quantities are used as weights, the Index Number arrived at is known as **Paasche's Index.**

$$\textbf{Paasche's Index } (I_{01}) = \frac{p_1'q_1' + p_1''q_1'' + p_1'''q_1''' + \cdots}{p_0'q_1' + p_0''q_1'' + p_0'''q_1''' + \cdots} \times 100 = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{\text{Sum of prices of current period quantities at current prices}}{\text{Sum of prices of current period quantities at base prices}} \times 100.$$

Note: 1. In Laspeyres' Index, the weights (base year quantities q_0) do not change from year to year. Thus only information on latest prices is to be collected. But in Paasche's Index, the weights being current year quantities q_1, the latest information on weights (i.e., quantities) as well as prices must be obtained. Thus computation of a Paasche's Index involves more labour in collection of data.

2. According to the law of supply and demand people tend to purchase more when prices are low and to purchase less when prices are high. This law is valid if the need for commodities is not absolutely essential. In Laspeyres' Index $\Sigma p_1 q_0$ will be somewhat higher than it should be by the law of supply and demand, so that total cost would be less than predicted by $\Sigma p_1 q_0$. Thus Laspeyres' Index $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}$ tends to be higher than it should be.

In Laspeyres' Index, if q_0 is replaced by q_1, we get Paasche's Index. This interchange of q_0 and q_1 tends to make the Paasche's Index lower than it should be.

Example 9. (a) *Calculate the Index Number from the following data, by using Laspeyres' formula:*

TABLE 10.9: CALCULATION OF INDEX NUMBER (LASPEYRES' FORMULA)

Commodity	Base Year (1997)	Base Year (1997)	Current Year (1999)	(p_1q_0)	(p_0q_0)
	Quantities (q_0)	Prices (₹) (p_0)	Prices (₹) (p_1)		
Rice	200	400	480	96,000	80,000
Wheat	160	250	270	43,200	40,000
Oil (edible)	80	950	1,050	84,000	76,000
Fish	100	1,100	1,200	1,20,000	1,10,000
Milk	60	800	1,000	60,000	48,000
Total	—	—	—	4,03,200	3,54,000

By **Laspeyres' formula,**

$$\text{required Index Number} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{403200}{354000} \times 100 = 113.9.$$

(b) *Calculate the price index number by using Paasche's method for the year* 2000 *with* 1990 *as base year:*

Commodity	1990		2000	
	Price	Quantity	Price	Quantity
A	65	40	81	46
B	72	35	90	54
C	57	92	77	72

[V.U. B.Com.(H) 2008]

Solution: Paasche's Index Number $(I_{01}) = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100.$

Commodity	1990		2000		$p_0 q_1$	$p_1 q_1$
	Price (p_0)	Quantities (q_0)	Price (p_1)	Quantities (q_1)		
A	65	40	81	46	2990	3726
B	72	35	90	54	3888	4860
C	57	92	77	72	4104	5544
Total	—	—	—	—	10982 $=\Sigma p_0 q_1$	14130 $=\Sigma p_1 q_1$

Hence I_{01} = Paasche's Index $= \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100 = \dfrac{14130}{10982} \times 100 = 128.67.$

3. Another system of weighting makes use of both the base year quantities and the current year quantities. In fact, the sum (or average) of both these quantities is used as weight and we get an Index Number known as **Marshall-Edgeworth's Index.**

$$\textbf{Marshall-Edgeworth's Index } (I_{01}) = \frac{p_1'(q_0'+q_1') + p_1''(q_0''+q_1'') + p_1'''(q_0'''+q_1''') + \cdots}{p_0'(q_0'+q_1') + p_0''(q_0''+q_1'') + p_0'''(q_0'''+q_1''') + \cdots} \times 100$$
$$= \frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100.$$

4. **Bowley's Index (I_{01}) :** The Arithmetic Mean of Laspeyres' Index and Paasche's Index is known as *Bowley's Index*. Thus,

$$\textbf{Bowley's Index } (I_{01}) = \frac{1}{2}(\text{Laspeyres' Index + Paasche's Index})$$
$$= \frac{1}{2}\left(\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}\right) \times 100.$$

B. Method of Relatives

1. Simple Arithmetic Mean of Price Relatives

Expressed in symbols, $100\dfrac{p_1'}{p_0'},\ 100\dfrac{p_1''}{p_0''},\ 100\dfrac{p_1'''}{p_0'''},\ \cdots$

are the price relatives of items 1, 2, 3, $\cdots$ respectively. The *Index Number*, calculated by this method is given by

$$\boldsymbol{I_{01}} = \frac{1}{n}\left(100\frac{p_1'}{p_0'} + 100\frac{p_1''}{p_0''} + 100\frac{p_1'''}{p_0'''} + \cdots\right), \quad \text{(where } n \text{ is the number of items included in the group.)}$$

$$= \frac{100}{n}\left(\frac{p_1'}{p_0'} + \frac{p_1''}{p_0''} + \frac{p_1'''}{p_0'''} + \cdots\right) = \frac{\mathbf{100}}{\boldsymbol{n}}\sum\left(\frac{\boldsymbol{p_1}}{\boldsymbol{p_0}}\right).$$

2. Simple Geometric Mean of Price Relatives

The Index Number computed by this method is given by

$$I_{01} = \sqrt[n]{100\frac{p_1'}{p_0'} \times 100\frac{p_1''}{p_0''} \times 100\frac{p_1'''}{p_0'''} \times \cdots} = 100\sqrt[n]{\frac{p_1'}{p_0'} \times \frac{p_1''}{p_0''} \times \frac{p_1'''}{p_0'''} \times \cdots};$$

$$\therefore\ \log I_{01} = \log 100 + \frac{1}{n}\Sigma \log\frac{p_1}{p_0}.$$

3. Weighted Arithmetic Mean of Price Relatives

When the weights $w_1, w_2, \ldots, w_n$ are attached to the respective commodities, the *Index Number*, calculated as the *weighted arithmetic mean of price relatives*, is given by

$$\boldsymbol{I_{01}} = \frac{100}{\Sigma w}\left(\frac{p_1' w_1}{p_0'} + \frac{p_1'' w_2}{p_0''} + \cdots + \frac{p_1^{(n)} w_n}{p_0^{(n)}}\right), \quad \text{i.e.,} \quad \boldsymbol{I_{01}} = \frac{\mathbf{100}}{\boldsymbol{\Sigma w}}\sum\frac{\boldsymbol{p_1 w}}{\boldsymbol{p_0}}.$$

But, for all practical purposes, the weights adopted in this method are the values (= price × quantity) of items. There are several Index Numbers calculated as the weighted AM of price relatives, depending on the nature of weights used.

- The Index Number calculated as the arithmetic mean of price relatives weighted by the base year values p_0q_0 is given by

$$I_{01} = \frac{\sum\left(100\frac{p_1}{p_0} \times p_0q_0\right)}{\Sigma p_0q_0} = \frac{\Sigma(100p_1q_0)}{\Sigma p_0q_0} \quad \text{or,} \quad \boldsymbol{I_{01}} = \frac{\boldsymbol{\Sigma p_1 q_0}}{\boldsymbol{\Sigma p_1 q_0}} \times \mathbf{100},$$

which is **Laspeyres' Index.**

- The AM of price relatives weighted by values of current year quantities at base year prices p_0q_1 is *Paasche's Index* given by

$$I_{01} = \frac{\sum\left(100\frac{p_1}{p_0} \times p_0q_1\right)}{\Sigma p_0q_1} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100.$$

Weighted Geometric Mean of Price Relatives

The Index Number, calculated by this method, is not very much in use. It is given by

$$I_{01} = 100\left\{\left(\frac{p_1'}{p_0'}\right)^{w_1} \times \left(\frac{p_1''}{p_0''}\right)^{w_2} \times \cdots \times \left(\frac{p_1^{(n)}}{p_0^{(n)}}\right)^{w_n}\right\}^{1/\Sigma w}$$

where $w_1, w_2, \ldots, w_n$ are the weights attached to the respective commodities;

$$\therefore\ \log I_{01} = \log 100 + \frac{1}{\Sigma w}\sum w \log\frac{p_1}{p_0}.$$

Weighted Harmonic Mean of Price Relatives

The Index Number calculated as the Harmonic Mean of price relatives weighted by current year values p_1q_1 is given by

$$I_{01} = \frac{\Sigma p_1q_1}{\sum\left\{(p_1q_1)/\left(\frac{p_1}{p_0}\times 100\right)\right\}} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1}\times 100, \text{ i.e., } \boldsymbol{I_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}\times 100,$$

which is **Paasche's Index Number.**

4. Special Index — Geometric Mean of Two Aggregative Price Index Numbers

The Geometric Mean of Laspeyres' Index and Paasche's Index is known as **Fisher's Ideal Index,** after the name of Irving Fisher.

Fisher's Ideal Index (I_{01})

$$= \sqrt{\textbf{Laspeyres' Index} \times \textbf{Paasche's Index}} = \mathbf{100}\sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times\frac{\Sigma p_1q_1}{\Sigma p_0q_1}}.$$

Note: Fisher's Ideal Index is of special importance, because geometric mean is useful in averaging ratios and percentages, and Index Number shows percentage changes. Also Fisher's Ideal Index Number obeys Time Reversal Test and Factor Reversal Test which are the tests for good Index Numbers.

Example 10. *If Laspeyres' and Paasche's price index numbers are* 125.6 *and* 154.3 *respectively, find Fisher's Ideal Price Index Number.* [C.U. B.Com. 1995]

Solution: We have, Fisher's Ideal Price Index Number

$$= \sqrt{\text{Laspeyres' Price Index Number} \times \text{Paasche's Price Index Number}}$$

$$= \sqrt{125.6\times 154.3} = \sqrt{19380.08} = \mathbf{139.2\ (approx.)}$$

General Index from Group Indices

The items (or commodities) included in constructing an Index Number are divided into some groups — similar or related items falling in the same group. Index Number for each group (called *Group Index*) is then determined. The General Index is the Arithmetic Mean of the group indices and it is given by the formula,

General Index = $\boldsymbol{\frac{\Sigma wI}{\Sigma w}}$, **where I = Group Index and w = Group Weight.**

Example 11. *Find the general cost of living index of* 1993 *from the following table:*

Class	Food	Dress	House Rent	Fuel	Miscellaneous
Class Index	620	575	325	255	280
Weight	30	20	25	15	10

[C.U. B.Com. 1994]

Solution:

TABLE 10.10: CALCULATIONS BEST FITTED LANE COST OF LIVING INDEX OF 1993

Class	Class Index (I)	Weight (w)	Iw
Food	620	30	18600
Dress	575	20	11500
House Rent	325	25	8125
Fuel	255	15	3825
Miscellaneous	280	10	2800
Total	—	$100 = \Sigma w$	$44850 = \Sigma Iw$

Hence, *the required general cost of living Index* $= \dfrac{\Sigma Iw}{\Sigma w} = \dfrac{44850}{100} = 448.5.$

10.6 Illustrative Examples

Example 12. *Find the Price Index Number by the method of relatives using arithmetic mean from the following:*

(a)

Commodity	Base Price	Current Price
Wheat	5	7
Milk	8	10
Fish	25	32
Sugar	6	12

[C.U. B.Com. 1996]

(b)

Commodity	Base Price	Current Price	Weight
Wheat	5	7	4
Milk	8	10	2
Fish	25	32	3
Sugar	6	12	1

[V.U. B.Com.(H) 2007]

[Use weighted AM of Price relatives.]

Solution:

TABLE 10.11: CALCULATIONS FOR INDEX NUMBER BY THE METHOD OF RELATIVES

Commodity	Base Price, p_0 (₹)	Current Price, p_1 (₹)	Price Relative $\left(\frac{p_1}{p_0} \times 100\right)$
Wheat	5	7	$\frac{7}{5} \times 100 = 140$
Milk	8	10	$\frac{10}{8} \times 100 = 125$
Fish	25	32	$\frac{32}{25} \times 100 = 128$
Sugar	6	12	$\frac{12}{6} \times 100 = 200$
Total	—	—	$\Sigma \frac{p_1}{p_0} \times 100 = 593$

The required Price Index Number $(I_{01}) = \dfrac{100}{n}\sum \dfrac{p_1}{p_0} = \dfrac{1}{n}\sum \dfrac{p_1}{p_0} \times 100 = \dfrac{1}{4} \times 593 = \mathbf{148.25.}$

(b)

Commodity	Base price (p_0)	Current price (p_1)	Weight (w)	$p_1 w/p_0$
Wheat	5	7	4	28/5
Milk	8	10	2	20/8
Fish	25	32	3	96/25
Sugar	6	12	1	12/6
Total	—	—	$10 = \Sigma w$	

$$I_{01} = \text{Price Index} = \frac{100}{\Sigma w}\sum \frac{p_1 w}{p_0} = \frac{100}{10}\left(\frac{28}{5} + \frac{20}{8} + \frac{96}{25} + \frac{12}{6}\right)$$
$$= 28 \times 2 + \frac{200}{8} + 2 \times \frac{96}{5} + 10 \times 2 = 56 + 25 + 38.4 + 20 = \mathbf{139.4.}$$

Example 13. *Find, by the Arithmetic Mean Method, the Index Number from the following data:*

Commodity	Base Price (₹)	Current Price (₹)
Rice	30	35
Wheat	22	25
Fish	54	64
Potato	20	25
Coal	15	18

Solution: The required Index Number I_{01}, by the Arithmetic Mean Method, is given by

$$I_{01} = \frac{100}{n}\Sigma\frac{p_1}{p_0} = \frac{100}{5}\left(\frac{35}{30} + \frac{25}{22} + \frac{64}{54} + \frac{25}{20} + \frac{18}{15}\right) = 20\left(\frac{7}{6} + \frac{25}{22} + \frac{32}{27} + \frac{5}{4} + \frac{6}{5}\right)$$
$$= \frac{70}{3} + \frac{250}{11} + \frac{640}{27} + 25 + 24 = 23.333 + 22.727 + 23.704 + 49$$
$$= 118.764 = \mathbf{118.76.}$$

Example 14. *Find, by Arithmetic Mean Method, the Index Number from the following data:*

Commodity	Base Price	Current Price	Weight
Rice	30	52	8
Wheat	25	30	6
Fish	130	150	3
Potato	35	49	5
Oil	70	105	7

Solution:

TABLE 10.12: CALCULATIONS OF WEIGHTED AM OF PRICE RELATIVES

Commodity	Base Price (p_0)	Current Price (p_1)	Weight (w)	$\frac{p_1}{p_0}$	$\frac{p_1}{p_0} \times w$
Rice	30	52	8	1.73	13.84
Wheat	25	30	6	1.20	7.20
Fish	130	150	3	1.15	3.45
Potato	35	49	5	1.40	7.00
Oil	70	105	7	1.50	10.50
Total	—	—	$29 = \Sigma w$	—	$41.99 = \Sigma\left(\frac{p_1}{p_0} \times w\right)$

∴ *the required Index Number* I_{01} *is given by*

$$I_{01} = \frac{100}{\Sigma w}\Sigma\left(\frac{p_1 w}{p_0}\right) = \frac{100}{29} \times 41.99 = \frac{4199}{29} = 144.793 = 144.79.$$

Example 15. *The following table gives weights (base year price × quantity) and price relatives for* 1999 *and* 2000 *with* 1998 *as base year for four commodities*:

Commodity	Price Relatives		Weights
	1999	2000	
A	125	112.5	6
B	120	110	10
C	$83\frac{1}{3}$	$91\frac{2}{3}$	6
D	75	125	1

Calculate the index number for 2000, *using unweighted Arithmetic Mean of the price relatives.*

Find also the index number for 1999, *using weighted AM of price relatives.*

Solution: Using unweighted AM of the price relatives,

$$\text{the required index number} = \frac{1}{4}\left(112.5 + 110 + 91\frac{2}{3} + 125\right) = \frac{1}{4}\left(112\frac{1}{2} + 110 + 91\frac{2}{3} + 125\right)$$

$$= \frac{1}{4} \times \left(439\frac{1}{6}\right) = \frac{1}{4} \times \frac{2635}{6} = \frac{2635}{24} = 109.79.$$

Using weighted AM of the price relatives, the required index number

$$= \frac{100}{\Sigma w}\sum \frac{p_1 w}{p_0} = \frac{1}{\Sigma w}\sum\left(\frac{p_1}{p_0} \times 100\right) w = \frac{1}{23}(125 \times 6 + 120 \times 10 + 83\frac{1}{3} \times 6 + 75 \times 1)$$

$$= \frac{1}{23}(750 + 1200 + 500 + 75) = \frac{1}{23} \times 2525 = 109.78 \quad \left[\because\ \Sigma w = 23 \text{ and } \frac{p_1}{p_0} \times 100 \text{ is given}\right].$$

Example 16. *Prepare Index Numbers of prices for three years with the average prices as base from the following data:*

	Rate per Rupee		
	Wheat	Cotton	Oil
1st Year	4 kg	2 kg	2 kg
2nd Year	3 kg	1500 kg	1250 kg
3rd Year	2500 kg	2 kg	0.750 kg

Solution: After converting the prices into rupee per Ql (1 Ql = 100 kg) the working is as shown below:

TABLE 10.13: INDEX NUMBERS FOR THREE YEARS (AVERAGE BASE METHOD)

Commodities	Base Price	1st Year		2nd Year		3rd Year	
	Average Price (per Ql) ₹	Price (per Ql) ₹	Price Relatives	Price (per Ql) ₹	Price Relatives	Price (per Ql) ₹	Price Relatives
Wheat	32.78	25	76.27	33.33	101.68	40	122.03
Cotton	72.22	50	69.23	66.67	92.32	100	138.46
Oil	87.78	50	56.96	80	91.14	133.33	151.89
Total	—	—	202.46	—	285.14	—	412.38

∴ applying Simple Arithmetic Mean Method of Relatives, the required Index Numbers are

$$\frac{202.46}{3} = 67.5 \text{ for 1st yr.;}$$

$$\frac{285.14}{3} = 95.0 \text{ for 2nd yr.;}$$

$$\text{and } \frac{412.38}{3} = 137.5 \text{ for 3rd yr.}$$

Note: Base price for wheat = (25 + 33.33 + 40) ÷ 3 = 32.78, and price relative for wheat = (25 ÷ 32.78) × 100 = 76.27.

Example 17. *From the table of group index number and group weights given below, calculate the cost of living index:*

Group	Group Index Numbers	Weight
Food	428	45
Clothing	240	15
Fuel and Light	200	8
House Rent	125	20
Others	170	12

Solution: The cost of living index (or general index from group indices) I_{01} is given by $I_{01} = \frac{\Sigma wI}{\Sigma w}$, where I = group index and w = group weight.

TABLE 10.14: CALCULATIONS FOR THE COST OF LIVING INDEX

Group	Index I	Weight w	wI
Food	428	45	19260
Clothing	240	15	3600
Fuel and Light	200	8	1600
House Rent	125	20	2500
Others	170	12	2040
Total	—	$100 = \Sigma w$	$29000 = \Sigma wI$

$\therefore\ I_{01} = \frac{29000}{100} = 290.$

Example 18. *Construct* (i) *Laspeyres' and* (ii) *Paasche's Price Index Number of* 2000 *with* 1990 *as base year from the following data*:

	Price (₹/kg)		Quantity sold (kg)	
	1990	**2000**	**1990**	**2000**
Commodity A	4	5	95	120
Commodity B	60	70	118	130
Commodity C	35	40	50	70

[C.U. B.Com.(H) 2001]

Solution:

TABLE 10.15: COMPUTATIONS FOR LASPEYRES' AND PAASCHE'S INDEX

Commodity	1990 (Base Year)		2000		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	4	95	5	120	380	480	475	600
B	60	118	70	130	7080	7800	8260	9100
C	35	50	40	70	1750	2450	2000	2800
Total	—	—	—	—	9210 $=\Sigma p_0q_0$	10730 $=\Sigma p_0q_1$	10735 $=\Sigma p_1q_0$	12500 $=\Sigma p_1q_1$

(i) Laspeyres' Price Index Number (I_{01})

$$= \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{10735}{9210} \times 100 = \mathbf{116.56\ (approx.)}$$

(ii) Paasche's Price Index Number (I_{01})

$$= \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100 = \frac{12500}{10730} \times 100 = \mathbf{116.5\ (approx.)}$$

Example 19. *Find x from the following table:*

Commodity	Base year		Current year	
	Price (₹)	Quantity (kg)	Price (₹)	Quantity (kg)
A	1	10	2	5
B	1	5	x	2

Given that the ratio between Laspeyre's and Paasche's Index Number is 28 : 27. [V.U. B.Com.(H) 2011]

Solution:

Commodity	Base year		Current year		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	p_0	V_0	p_1	q_1				
A	1	10	2	5	10	5	20	10
B	1	5	x	2	5	2	$5x$	$2x$
Total	—	—	—	—	$15=\Sigma p_0q_0$	$7=\Sigma p_0q_1$	$20+5x=\Sigma p_1q_0$	$10+2x=\Sigma p_1q_1$

Laspeyre's index $= \dfrac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \dfrac{20+5x}{15}$ and Paasche's index $= \dfrac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100 = \dfrac{10+2x}{7}$.

Given,

$$\frac{\text{Laspeyre's index}}{\text{Paasche's index}} = \frac{28}{27} \text{ or, } \frac{(20+5x)\times 7}{15(10+2x)} = \frac{28}{27} \text{ or, } \frac{20+5x}{5(10+2x)} = \frac{4}{9}$$

or, $180+45x = 200+40x$ or, $5x = 20$ or, $\boldsymbol{x = 4}$.

Example 20. *Calculate the Laspeyres' and the Paasche's Index Numbers for the following data and hence find Fisher's Ideal Index number.*

Commodity	Base Year		Current Year	
	Quantity	Price per lb ₹	Quantity	Price per lb ₹
A	10.0	80	11.0	70
B	8.0	85	9.0	90
C	5.0	130	5.5	80

Solution:

TABLE 10.16: CALCULATIONS FOR LASPEYRES' AND PAASCHE'S INDICES

Commodity	Base Year		Current Year		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	80	10.0	70	11.0	800	880	700	770
B	85	8.0	90	9.0	680	765	720	810
C	130	5.0	80	5.5	650	715	400	440
Total	—	—	—	—	2130 $=\Sigma p_0q_0$	2360 $=\Sigma p_0q_1$	1820 $=\Sigma p_1q_0$	2020 $=\Sigma p_1q_1$

$$\therefore \text{ Laspeyres' Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{1820}{2130} \times 100 = 85.446 = 85.45 \text{ (approx.)}$$

$$\text{Paasche's Index} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100 = \frac{2020}{2360} \times 100 = 85.593 = 85.59 \text{ (approx.)}$$

$$\text{Fisher's Ideal Index} = \sqrt{\text{Laspeyres' Index} \times \text{Paasche's Index}}$$

$$= \sqrt{85.45 \times 85.59} = 85.52.$$

Example 21. *It is stated that the Marshall-Edgeworth's index number is a good approximation to Fisher's ideal index number, verify using the following data:*

Commodity	2008		2012	
	Price	Quantity	Price	Quantity
A	2	74	3	82
B	5	125	4	140
C	7	40	6	33

Solution:

TABLE 10.17: CALCULATIONS FOR MARSHALL-EDGEWORTH'S INDEX

Commodity	2008		2012		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	2	74	3	82	148	164	222	246
B	5	125	4	140	625	700	500	560
C	7	40	6	33	280	231	240	198
Total	—	—	—	—	1053 $=\Sigma p_0q_0$	1095 $=\Sigma p_0q_1$	962 $=\Sigma p_1q_0$	1004 $=\Sigma p_1q_1$

$$\text{Marshall-Edgeworth's Index} = \frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100 = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100$$

$$= \frac{962+1004}{1053+1095} \times 100 = \frac{196600}{2148} = 91.527.$$

$$\text{Fisher's Ideal Index} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100 = \sqrt{\frac{962}{1053} \times \frac{1004}{1095}} \times 100 = 91.523.$$

From above we see that the *Marshall-Edgeworth's Index is a good approximation to Fisher's Ideal Index.*

Example 22. *Prepare Index Number for* 2005, *on the basis of* 1995, *from the following information:*

Year	Article I		Article II		Article III	
	Price	Quantity	Price	Quantity	Price	Quantity
1995	5	10	8	6	6	3
2005	4	12	7	7	5	4

Solution: The quantities for the Base Year and the Current Year being different, either *Marshall-Edgeworth's Index* or *Fisher's Ideal Index* may be constructed with the given data. *Fisher's Ideal Index,* being better is determined here.

TABLE 10.18: COMPUTATION OF FISHER'S IDEAL INDEX

Article	1995		2005		p_0q_1	p_1q_0	p_0q_0	p_1q_1
	p_0	q_0	p_1	q_1				
I	5	10	4	12	60	40	50	48
II	8	6	7	7	56	42	48	49
III	6	3	5	4	24	15	18	20
Total	—	—	—	—	140	97	116	117

$\therefore$ *Fisher's Ideal Index Number for* 2005 *on the basis of* 1995 is

$$I = 100 \times \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} = \sqrt{\frac{97}{116} \times \frac{117}{140}} \times 100;$$

$$\therefore \ \log I = \log 100 + \frac{1}{2}(\log 97 + \log 117 - \log 116 - \log 140) = 1.9222;$$

$\therefore$ I = antilog 1.9222 = 83.6.

Note: Index Numbers obtained in Exs 1–8 are all Price Index Numbers.

[Instead of log-table students may use Calculator.]

Example 23. *The following table gives the change in the price and consumption of three commodities. Compute Fisher's ideal price index number:*

Commodity	1995		2005	
	Price (₹)	Quantity (Units)	Price (₹)	Quantity (Units)
Wheat	100	10	110	6
Rice	150	15	170	18
Cloth	5	50	4	30

[C.U. B.Com.(H) 2000]

Solution:

TABLE 10.19: COMPUTATIONS FOR FISHER'S IDEAL PRICE INDEX NUMBER

Commodity	1995		2005		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)				
Wheat	100	10	110	6	1000	600	1100	660
Rice	150	15	170	18	2250	2700	2550	3060
Cloth	5	50	4	30	250	150	200	120
Total	—	—	—	—	3500 $=\Sigma p_0q_0$	3450 $=\Sigma p_0q_1$	3850 $=\Sigma p_1q_0$	3840 $=\Sigma p_1q_1$

∴ *Fisher's Ideal Price Index Number for 2005 on the basis of 1995* is

$$I_{01} = 100\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} = 100\sqrt{\frac{3850}{3500} \times \frac{3840}{3450}} = 100 \times \sqrt{1.224348}$$
$$= 100 \times 1.1065 = 110.65.$$

10.7 Chain-Base Method

In the Chain-Base Method of computing Index Numbers, the price of each item in any year is compared with its price in the preceding year, and expressed as a percentage. These are known as *Link Relatives.* The link relative is, thus, the ratio of the price of an article at any period to its price in the period just preceding it, expressed as a percentage. The link relative for the first year Y_0 is arbitrarily taken as 100. The link relatives for the different items in other years are then averaged for each year separately (the arithmetic or the geometric mean is generally employed) to get the *Link Indices.*

These link indices for the successive years are then chained together by multiplication to obtain the Chain Index Numbers, relative to a common base.

The link relatives are shown below:

LINK RELATIVES				
Items ↓ \ Year ⟶	Y_0	Y_1	$Y_2 \cdots$	Y_k
1	100	$\frac{p_1'}{p_0'} \times 100$	$\frac{p_2'}{p_1'} \times 100 \cdots$	$\frac{p_k'}{p_{k-1}'} \times 100$
2	100	$\frac{p_1''}{p_0''} \times 100$	$\frac{p_2''}{p_1''} \times 100 \cdots$	$\frac{p_k''}{p_{k-1}''} \times 100$
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
n	100	$\frac{p_1^{(n)}}{p_0^{(n)}} \times 100$	$\frac{p_2^{(n)}}{p_1^{(n)}} \times 100 \cdots$	$\frac{p_k^{(n)}}{p_{k-1}^{(n)}} \times 100$

If the Arithmetic Mean is used, then

Link Index for Y_0 = average of col. $Y_0 = 100$

Link Index for Y_1 = average of col. Y_1, so that

$$I_{01}' = \frac{1}{n}\sum\left(\frac{p_1}{p_0} \times 100\right) = \frac{100}{n}\sum\left(\frac{p_1}{p_0}\right).$$

$$\text{Link Index for } Y_2, I_{12}' = \frac{1}{n}\sum\left(\frac{p_2}{p_1} \times 100\right) = \frac{100}{n}\sum\left(\frac{p_2}{p_1}\right).$$

$$\text{Link Index for } Y_3, I_{23}' = \frac{1}{n}\sum\left(\frac{p_3}{p_2} \times 100\right) = \frac{100}{n}\sum\left(\frac{p_3}{p_2}\right), \text{ and so on.}$$

The Chain Index Numbers are calculated as follows:

Chain Index for $Y_0 = 100$

Chain Index for Y_1 (as compared to Y_0), $I_{01} = I'_{01}$

Chain Index for Y_2 (as compared to Y_0), $I_{02} = \dfrac{I_{01} \times I'_{12}}{100}$

Chain Index for Y_3 (as compared to Y_0), $I_{03} = \dfrac{I_{02} \times I'_{23}}{100} = \dfrac{I'_{01} \times I'_{12} \times I'_{23}}{100^2}$, and so on.

Thus, Chain Index for any year $= \dfrac{\text{Link Index for the year} \times \text{Chain Index of previous year}}{100}$.

Similarly, if Geometric Mean is used as the measure for averaging,

Link Index for $Y_0 = 100$

Link Index for Y_1, $I'_{01} = \sqrt[n]{100\dfrac{p'_1}{p'_0} \times 100\dfrac{p''_1}{p''_0} \times 100\dfrac{p'''_1}{p'''_0} \times \cdots}$

Link Index for Y_2, $I'_{12} = \sqrt[n]{100\dfrac{p'_2}{p'_1} \times 100\dfrac{p''_2}{p''_1} \times 100\dfrac{p'''_2}{p'''_1} \times \cdots}$

Link Index for Y_3, $I'_{23} = \sqrt[n]{100\dfrac{p'_3}{p'_2} \times 100\dfrac{p''_3}{p''_2} \times 100\dfrac{p'''_3}{p'''_2} \times \cdots}$ and so on.

The Chain Index Numbers with base Y_0 are calculated from these indices in the same manner as shown above.

Example 24. *Compute the Chain Index Numbers with prices in* 2001 *as base, from the following table giving the average wholesale prices of the commodities, A, B, and C for the years* 2001–2005:

AVERAGE WHOLESALE PRICES					
Commodities	2001	2002	2003	2004	2005
A	20	16	28	35	21
B	25	30	24	36	45
C	20	25	30	24	30

Solution:

TABLE 10.20: CALCULATION OF CHAIN INDEX NUMBERS WITH 2001 AS BASE YEAR

Commodities	Link Relative				
	2001	2002	2003	2004	2005
A	100	$\frac{16}{20} \times 100 = 80$	$\frac{28}{16} \times 100 = 175$	$\frac{35}{28} \times 100 = 125$	$\frac{21}{35} \times 100 = 60$
B	100	120	80	150	125
C	100	125	120	80	125
Total	300	325	375	355	310
Link Indices	$300 \div 3 = 100$	$325 \div 3 = 108.3$ (I'_{01})	$375 \div 3 = 125$ (I'_{12})	$355 \div 3 = 118.3$ (I'_{23})	$310 \div 3 = 103.3$ (I'_{34})
Chain Indices (with 2001 as base)	100	$(I_{01} = I'_{01})$ $\frac{100 \times 108.3}{100}$ $= 108.3$	$\left(I_{02} = \frac{I_{01} \times I'_{12}}{100}\right)$ $\frac{108.3 \times 125}{100}$ $= 135.4$	$\left(I_{03} = \frac{I_{02} \times I'_{23}}{100}\right)$ $\frac{135.4 \times 118.3}{100}$ $= 160.2$	$\left(I_{04} = \frac{I_{03} \times I'_{34}}{100}\right)$ $\frac{160.2 \times 103.3}{100}$ $= 165.5$

Clearly, Link Relative for any year $= \dfrac{\text{Price Relative for the year}}{\text{Price Relative for the previous year}} \times 100.$

10.8 Problems in the Construction of Index Numbers

The various problems in the construction of Index Numbers are:

- Determination of the exact purpose for which the proposed Index Numbers are constructed.
- Selection of items which are to be included in the group.
- Collection of data regarding prices and quantities and selection of sources from which such data are to be collected.
- Choice of the base period.
- Determination of weights to be employed.
- Selection of the measure of Central Tendency (or Average) to be used.

We shall discuss them one by one.

Determination of the Exact Purpose: Before starting to construct an Index Number, a clear idea as to the purpose and nature of its use is essential. All Index Numbers do not serve the same purpose. Thus, for constructing the Index Number of Wholesale Prices, the prices from retail dealers are not necessary, just as for construction of a Cost of Living Index, the quotations of cloth price ex-mill or prices of cotton yarn are useless. One must be sure of what changes the proposed Index Numbers are going to measure.

Selection of Items: It is not possible to include all the commodities in the construction of an Index Number for reasons of economy and ease of calculation. The items included for the construction of an Index Number are the most important ones in each field, i.e., such articles whose price movements appear to be the most representative of the group. Inclusion of a very large number of items creates difficulty in the accurate collection of data regarding prices and quantities, apart from the increased cost. On the other hand, selection of too few items will make the Index Number unrepresentative of the general price level. The selection of items should be such in which vast majority of the people are generally interested and consequently those which are consumed in large quantities than others. With passage of time, the habits and tastes of people change and thus some new items gather importance while some other lose importance. Hence, the less important items should be replaced by new ones in conformity with their relative importance.

Collection of Data and Selection of Source: For a regular source of Index Numbers, a systematic collection of prices should be made from prominent business firms or standard retail stores located at different important centres, at regular intervals of time. The selected shops should be those which are visited by a large majority of customers. The average of the prices so collected from different centres is taken as the representative price. The average of the weekly price may be used for monthly indices, while yearly indices will be computed on the basis of the average of the monthly prices.

Choice of Base Period: The base period should be chosen carefully and be such a time when no abnormal increase or decrease in price was noticed. Once the base period is selected, all the subsequent indices are computed with reference to this fixed-base period until, of course, after a long time it is found necessary to shift the base to a more recent period. Usually, a year is taken as base, preferably a year of some economic importance of the country.

Index Numbers are calculated in three ways according to the nature of base period: (a) Fixed-Base Method, (b) Chain-Base Method and (c) Average-Base Method.

Determination of Weights: Most of the Index Numbers at present used are weighted averages of price relatives or ratio of two weighted aggregates of prices. The weighting of prices is necessary because a change in the prices of some important articles affects the general price level to a large extent than does a similar change in the prices of some other less important items. *For example,* a middle class family in an urban area has to spend more than 50% of the total income on food while only about 15% of the income is spent as house rent. A small percentage increase in the food prices will affect the family such more severely than a similar percentage increase in the house rent.

While computing Index Numbers, the systems of weighting and particularly the allocation of weights to the individual items are of utmost importance. Different methods of weighting are used to obtain different types of Index Numbers. Price relatives are weighted by values, prices by quantities and quantities by prices.

In Laspeyres' Index $\left(=\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0}\right)$, the base year quantities are used as weights, while in Paasche's Index $\left(=\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}\right)$, the weights were the current year quantities. Thus, Laspeyres' Index measures the relative change in cost in the current year of a set of goods which are consumed in the base year; while Paasche's Index measures the relative change in cost of a set of goods consumed in the current year as compared to the consumption of the same set of goods during base year. Since the pattern of consumption changes with the passage of time, both these methods are subject to some criticism.

In Marshall-Edgeworth's Index $=\left(\dfrac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)}\right)$, the sum or the average of the quantities used in the base and current years is used as weights.

When the weighted arithmetic mean of price relatives is used for computation of Index Numbers

$$I_{01} = \left(\frac{\sum 100 \frac{p_1}{p_0} \times p_0 q_0}{\sum p_0 q_0} \right),$$

the weights used are the values (= price × quantity) or total outlay on each commodity.

Selecting the Average to be used: It is evident from our discussions that Index Numbers are actually averages of percentage changes in price of each of the commodities included in the group. The Arithmetic Means (Simple and Weighted) and Geometric Means are most frequently used for the purpose. In certain cases, median is also employed.

The AM is easy to compute and readily understood; but since it gives more weight to the items with high prices (although less important) and less to the ones whose per unit cost is low, the GM is preferred in many cases since it gives equal importance to equal ratios of change and also is not very much affected by high or low prices. However, the computation of GM involves much labour in calculation, and hence it is employed not so often as the AM.

10.9 Quantity (or Volume) Index Numbers

All the formulae used for calculating the Price Index Numbers, discussed so far, may also be used to compute the Quantity Index Numbers. The only modification necessary is that prices (p) should be replaced by quantities (q) and vice versa. Thus, using the Simple Aggregative formula, we get

Simple Aggregative Index

For Price $= \dfrac{\Sigma p_1}{\Sigma p_0} \times 100$; for **Quantity** $= \dfrac{\Sigma q_1}{\Sigma q_0} \times \mathbf{100}$.

Using base year prices as weights, **Quantity Index** $= \dfrac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \mathbf{100}$.

Again, using Laspeyres' formula,

$$\text{Laspeyres' Price Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$\textbf{Laspeyres' Quantity Index} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times 100 = \frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \times 100$$

$$\text{Paasche's Price Index} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$\textbf{Paasche's Quantity Index} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1} \times 100 = \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0} \times 100$$

$$\textbf{Fisher's Quantity Index} = 100 \times \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}.$$

Similarly, we can write down other formulae for Quantity Index Numbers.

Example 25. *From the following data prepare a Quantity Index Number for the year* 2005 *taking* 2000 *as the base year* (*use Fisher's Ideal Index Formula*):

Year	Commodity I		Commodity II		Commodity III	
	Price	Quantity	Price	Quantity	Price	Quantity
2000	5	10	8	6	6	3
2005	4	12	7	7	5	4

Solution:

TABLE 10.21: CALCULATIONS FOR QUANTITY INDEX NUMBER

Commodity	2000		2005		p_0q_0	q_1p_0	q_0p_1	q_1p_1
	p_0	q_0	p_1	q_1				
I	5	10	4	12	50	60	40	48
II	8	6	7	7	48	56	42	49
III	6	3	5	4	18	24	15	20
Total	—	—	—	—	$116 = \Sigma q_0p_0$	$140 = \Sigma q_1p_0$	$97 = \Sigma q_0p_1$	$117 = \Sigma q_1p_1$

$$\text{Fisher's Quantity Index} = \sqrt{\frac{\Sigma q_1p_0}{\Sigma q_0p_0} \times \frac{\Sigma q_1p_1}{\Sigma q_0p_1}} \times 100 = \sqrt{\frac{140}{116} \times \frac{117}{97}} \times 100 = 120.65.$$

Example 26. *Using Paasche's formula, compute the Quantity Index and the Price Index Numbers for* 2005 *with* 2000 *as base year:*

Commodity	Quantity (in Units)		Value (in ₹)	
	2000	2005	2000	2005
A	100	150	500	900
B	80	100	320	500
C	60	72	150	360
D	30	33	360	297

Solution:

TABLE 10.22: CALCULATION FOR PAASCHE'S INDEX

Commodity	Quantity		Value (in ₹)		$p_0 = \frac{p_0q_0}{q_0}$	$p_1 = \frac{p_1q_1}{q_1}$	p_0q_1	p_1q_0
	q_0	q_1	p_0q_0	p_1q_1				
A	100	150	500	900	5	6	750	600
B	80	100	320	500	4	5	400	400
C	60	72	150	360	2.5	5	180	300
D	30	33	360	297	12	9	396	270
Total	—	—	—	$2057 = \Sigma p_1q_1$	—	—	$1726 = \Sigma p_0q_1$	$1570 = \Sigma p_1q_0$

$$\text{Paasche's Quantity Index} = \frac{\Sigma q_1p_1}{\Sigma q_0p_1} \times 100 = \frac{2057}{1570} \times 100 = 131.019 = 131.02 \text{ (approx.)}$$

$$\text{Paasche's Price Index} = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100 = \frac{2057}{1726} \times 100 = 119.177 = 119.18 \text{ (approx.)}$$

Example 27. *Using the following data, verify that Paasche's formula for Index does not satisfy Factor Reversal Test:*

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
X	4	10	6	15
Y	6	15	4	20
Z	8	5	10	4

[C.U. B.Com. 1995]

Solution:

TABLE 10.23: CALCULATIONS FOR PAASCHE'S PRICE INDEX

Commodity	Base Year		Current Year		p_1q_1	p_0q_1	p_1q_0	p_0q_0
	p_0	q_0	p_1	q_1				
X	4	10	6	15	90	60	60	40
Y	6	15	4	20	80	120	60	90
Z	8	5	10	4	40	32	50	40
Total	—	—	—	—	$210=\Sigma p_1q_1$	$212=\Sigma p_0q_1$	1[illegible] $=\Sigma p_1q_0$	$170=\Sigma p_0q_0$

Paasche's Price Index (P_{01}) is given by, $P_{01}=\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}$ (omitting the factor 100) $=\dfrac{210}{212}$.

Interchanging p and q, Paasche's Quantity Index (Q_{01}) is given by

$$Q_{01}=\frac{\Sigma q_1p_1}{\Sigma q_0p_1}=\frac{\Sigma p_1q_1}{\Sigma p_1q_0}=\frac{210}{170}.$$

$$\therefore\ P_{01}\times Q_{01}=\frac{210}{212}\times\frac{210}{170}\neq\frac{210}{170},\ \text{i.e.,}\neq\frac{\Sigma p_1q_1}{\Sigma p_0q_0}.$$

Hence, **Paasche's Formula does not satisfy Factor Reversal Test.**

10.9.1 Deflating an Index Number

During inflationary trends the prices of various commodities or items increase, i.e., there is a rise in price level. So there is a fall in the real incomes, i.e., reduction in the purchasing power of money. Deflating means adjusting, correcting or reducing an inflated value by making allowances for the effect of changing price levels. The purchasing power of money (or rupee) is the reciprocal of the price index. (See illustration given below.) So the income (or wages) is obtained by the relation:

$$\text{Real Wages}=\frac{\text{Money (or Nominal) Wages}}{\text{Price Index}}\times 100,$$

which is also known as *deflated income.* This process is called *deflating.*

If cost of Living Index or Consumer Price Index or General Price Index is given, then

$$\text{Real Income or Wages}=\frac{\text{Money Income (or Wages)}}{\text{Cost of Living Index}}\times 100.$$

For example, if the price of rice increases from ₹ 6 per kg in 1995 to ₹ 12 per kg in 2000, then we can buy only half of rice by the same amount which was spent in 1995. This indicates that the value (i.e., purchasing power) of a rupee is the reciprocal of an appropriate price index. If the said price increases by 50%, the price index of rupee is 150%, i.e., 1.5 and the quantity that a rupee will buy is only 1/1.5, i.e., 2/3 of what it could buy before. Thus the purchasing power of a rupee is 2/3 of what it was before or 67 paise approximately. Similarly, if the said price rises by 25%, then the price index is 1.25 and the purchasing power of a rupee is 1/1.25, i.e., 80 paise. This means that we have to spend ₹1 for buying rice which cost 80 paise in 1995.

Example 28. *The following data relate to the income of the people and General Index Number of Price of a certain region. Calculate* (i) *Real Income and* (ii) *Index Number of real income with* 1999 *as base*:

Year	1999	2000	2001	2002	2003	2004	2005
Income in ₹	800	819	825	876	920	938	936
General Price Index No.	100	105	110	120	125	140	144

Solution:

TABLE 10.24: CALCULATIONS FOR REAL INCOME AND INDEX NUMBER OF REAL INCOME

Year	Income in ₹ x	General Index Number (I)	Real Income in ₹ $y = \frac{x}{I} \times 100$	Index Number of Real Income $= \frac{y}{800} \times 100$ (Base 1999 = 100)
1999	800	100	$\frac{800}{100} \times 100 = 800$	$\frac{800}{800} \times 100 = 100.00$
2000	819	105	$\frac{819}{105} \times 100 = 780$	$\frac{780}{800} \times 100 = 97.50$
2001	825	110	$\frac{825}{110} \times 100 = 750$	$\frac{750}{800} \times 100 = 93.75$
2002	876	120	$\frac{876}{120} \times 100 = 730$	$\frac{730}{800} \times 100 = 91.25$
2003	920	125	$\frac{920}{125} \times 100 = 736$	$\frac{736}{800} \times 100 = 92.00$
2004	938	140	$\frac{938}{140} \times 100 = 670$	$\frac{670}{800} \times 100 = 83.75$
2005	936	144	$\frac{936}{144} \times 100 = 650$	$\frac{650}{800} \times 100 = 81.25$

10.10 Cost of Living Index or Consumer Price Index Number

Cost of Living Index Numbers also known as *Consumer Price Index Numbers* measure the relative (expressed as a percentage) amount of money necessary to derive equal satisfaction during two periods of time, after taking into consideration the fluctuations of the retail prices of consumer goods during these two periods.

Generally, the list of items consumed varies for different classes of people (e.g., the rich, the poor and the middle class) at the same place of residence. Also people of the same class belonging to different geographical regions have different consumer-habits. Thus, the Cost of Living Index always relates to a specified class of people and a specified geographical area, and it helps us in determining the effect of changes in prices on different classes of consumers living in different areas.

Construction of a Cost of Living Index

The first step in the construction of a Cost of Living Index is the decision about the class of people for whom the Index Number is meant, i.e., it should be clearly mentioned whether the Index Number relates to industrial workers or officers or teachers, etc. It is also necessary to decide on the geographical area to be covered by the Index Number.

The second step is to conduct a family budget enquiry covering the class of people for whom the Index Number is intended. The enquiry should be conducted in the base year by the process of random sampling. This would give us information regarding the nature, quality and quantities of commodities consumed by an average family of the class and also the amount spent on different items of consumption. The items on which the money is spent are classified under five major groups: (a) Food, (b) Clothing, (c) Fuel and Light, (d) Housing, and (e) Miscellaneous.

Each of these groups includes a large number of important items. *For example,* the group Food includes Rice, Wheat, Fish, Milk, Egg, Sugar, etc. Among these only important items which are used by the majority of the class of people are included in the construction of a Cost of Living Index.

The next step is to collect retail prices of the items from the localities in which the class of people concerned reside or from the markets where they usually make their purchases. Price quotations should be obtained at least once a week.

After the collection of retail prices, we would observe that each item will have a number of price quotations depending on qualities and markets. In this case, an average price for each item is calculated. Such average is first calculated for the base period of the Index, and later for every week, or month, or year according as the Index is maintained on a weekly, monthly or yearly basis.

As the relative importance of various items for different classes of people is not the same, prices or price relatives are always weighted and, therefore, the Cost of Living Index is always a weighted Index. Amount spent on each commodity by an average family gives the weights.

The percentage expenditure on an item constitutes the weight of the item and the percentage expenditure on the five groups constitute the group weight.

Laspeyres' Formula, $I_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ is widely employed for computing the Cost of Living Index.

The above formula may be written as

$$I_{01} = \frac{\sum\left(\frac{p_1}{p_0} \times 100 \times p_0 q_0\right)}{\Sigma p_0 q_0} = \frac{\Sigma I w}{\Sigma w},$$

where $I = \frac{p_1}{p_0} \times 100 =$ price relative of an item consumed, and $w = p_0 q_0 =$ base price of an item consumed, so that $I_{01} =$ weighted Arithmetic Average of the price relatives, weighted with the weights $p_0 q_0$ representing the value, at base prices, of the items consumed in the base year.

If $\Sigma w = 100$, the formula reduces to $I_{01} = \frac{\Sigma I w}{100}$, where $I =$ price relative and $w =$ percentage of family's budget spent on an item in the base year.

Separate index numbers are first of all determined for each of the five major groups, by calculating the weighted average of price relatives of the selected items in the group. The weights in this case are proportional to the expenditure of an average family on the different items in the group. These group indices are weighted by the percentage of total family budget spent on each of these groups and the weighted average gives the final Cost of Living Index Number.

Important Relations

Real Wage = (Actual Wage/Cost of Living Index) × 100.

The process of finding real wage by using Cost of Living Index (CLI) is called *deflation.*
Equivalent wage in a year Y' relative to the year Y to maintain the same standard of living

$$= \frac{\text{CLI for } Y'}{\text{CLI for } Y} \times \text{actual wage in } Y.$$

Purchasing Power of a Rupee in any year, say Y, in terms of a particular year Y' is given by the formula,
Purchasing Power

$$= \frac{\text{Index Number for } Y'}{\text{Index Number for } Y}.$$

As p_0q_0 and p_1q_0 are the aggregate expenditure in current and base periods respectively, the method of computing Cost of Living Index by the application of Laspeyres' formula is known as *Aggregate Expenditure Method* and the method of computation by the application of the formula

$$\frac{\Sigma Iw}{\Sigma w} \text{ or } \frac{\Sigma Iw}{100}$$

is known as *Family Budget Method.*

Example 29. *From the following data, calculate the cost of living index numbers:*

Group	Weight	Index Number (Base: 2004–05 = 100)
Food	50	241
Clothing	2	221
Fuel and Light	3	204
Rent	16	256
Miscellaneous	29	179

[C.U. B.Com.(H) 2000]

Solution: Cost of Living Index = $\frac{\Sigma Iw}{\Sigma w}$, where I = group index and w = group weight.

TABLE 10.25: CALCULATIONS FOR COST OF LIVING INDEX

Group	Weight	Index Number (Base: 2004–05 = 100)	Iw
Food	50	241	12050
Clothing	2	221	442
Fuel and Light	3	204	612
Rent	16	256	4096
Miscellaneous	29	179	5191
Total	$100 = \Sigma w$	—	$22391 = \Sigma Iw$

Hence, *the required Cost of Living Index number* = $\frac{\Sigma Iw}{\Sigma w} = \frac{22391}{100} = 223.91.$

Example 30. *The group indices and the corresponding weights for the working class cost of living index numbers in an industrial city for the years* 2001 *and* 2005 *are given below:*

Group	Weight	Group Index	
		2001	2005
Food	71	370	380
Clothing	3	423	504
Fuel, etc.	9	469	336
House Rent	7	110	116
Miscellaneous	10	279	283

Compute the cost of living indices for the two years 2001 *and* 2005. *If a worker was getting* 300 *p.m. in* 2001, *do you think that he should be given some extra allowance so that he can mai'‘ain his* 2001 *standard of living? If so, what should be the minimum amount of this extra allowance?*

Solution: Cost of Living Index = $\frac{\Sigma Iw}{\Sigma w}$, where I = group index and w = group weight.

TABLE 10.26: CALCULATIONS FOR COST OF LIVING INDEX

Group	Weight w	Group Index		$I_1 w$	$I_2 w$
		2001 (I_1)	2005 (I_2)		
Food	71	370	380	26270	26980
Clothing	3	423	504	1269	1512
Fuel, etc.	9	469	336	4221	3024
House Rent	7	110	116	770	812
Miscellaneous	10	279	283	2790	2830
Total	$100 = \Sigma w$	—	—	$35320 = \Sigma I_1 w$	$35158 = \Sigma I_2 w$

$$\text{Cost of Living Index for 2001} = \frac{\Sigma I_1 w}{\Sigma w} = \frac{35320}{100} = 353.20.$$

$$\text{Cost of Living Index for 2005} = \frac{\Sigma I_2 w}{\Sigma w} = \frac{35158}{100} = 351.58.$$

Since, *Cost of Living Index for 2005 is slightly less than the cost of living index for 2001, the worker need not be given any extra allowance to maintain his 2001 standard of living.*

[The Cost of Living Index for 2005 relative to that for 2001 = $\frac{351.58}{353.20}$, and to maintain the same standard of living as 2001, the worker needs 300 × $\frac{351.58}{353.20}$ = 298.62. But he is already getting 300.]

10.11 Tests for Index Numbers (Tests of Adequacy)

Various formulae have been discussed for constructing Index Numbers, which are not perfect in measuring changes in prices or in quantities. The problem arises in selecting the most appropriate formula in a given situation. The following tests are used for choosing an appropriate Index Number: (1) Unit Test, (2) Time Reversal Test, (3) Factor Reversal Test, and (4) Circular Test.

- **Unit Test**: This test requires that the formula for an Index Number should be independent of the units in which prices and quantities of commodities are quoted. All Index Number formulae, except the simple aggregate Index Number formula, satisfy this test.

- **Time Reversal Test:** An index number formula satisfies this test if it works both ways, forward and backward, w.r.t. time. In other words, an index number I_{01} for the year Y_1 with base year Y_0, calculated by this formula, should be the reciprocal of the index number I_{10} for the year Y_0 with base year Y_1. Symbolically, $I_{01} \times I_{10} = 1$, omitting the factor 100 from both the indices.

  This test is satisfied by (i) Simple Aggregative Index, (ii) Marshall-Edgeworth's Index, (iii) Fisher's Ideal Index, and (iv) Simple GM of Price Relatives.
- **Factor Reversal Test:** An index number formula satisfies this test if the product of the Price Index and the Quantity Index gives the true value ratio, omitting the factor 100 from each index. Symbolically,

  $$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \text{the true value ratio} = \text{TVR},$$

  where P_{01} = Price Index for Y_1 with base year Y_0 and Q_{01} = Quantity Index for Y_1 with base year Y_0. Fisher's Ideal Index is the only formula which satisfies this test.
- **Circular Test:** This test is based on the shifting of base in a circular fashion. It may be considered as an Extension of Time Reversal Test. An index number formula is said to satisfy the Circular Test if it satisfies $I_{01} \times I_{12} \times I_{23} \times \cdots \times I_{(n-1)n} \times I_{no} = 1$.

  This test is concerned with the measurement of price changes over a period of years when the shiftability of the base is desirable.

  This test is satisfied by (i) Simple Aggregative Index, (ii) Simple Geometric Mean of Price Relatives and (iii) Weighted Aggregative Formula with Fixed Weights.

Example 31. *Show that the simple aggregative formula and weighted aggregative formula (fixed weights) satisfy Circular Test.*

Solution: For three years Y_0, Y_1, Y_2, using simple aggregative formula (omitting the factor 100), we get

$$I_{01} = \frac{\Sigma p_1}{\Sigma p_0}, \; I_{12} = \frac{\Sigma p_2}{\Sigma p_1} \text{ and } I_{20} = \frac{\Sigma p_0}{\Sigma p_2}.$$

$$\therefore \quad I_{01} \times I_{12} \times I_{20} = \frac{\Sigma p_1}{\Sigma p_0} \times \frac{\Sigma p_2}{\Sigma p_1} \times \frac{\Sigma p_0}{\Sigma p_2} = 1.$$

Hence, the *simple aggregative formula satisfies Circular Test.*

Again, by weighted aggregative formula (fixed weights),

$$I_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w}, \; I_{12} = \frac{\Sigma p_2 w}{\Sigma p_1 w} \text{ and } I_{20} = \frac{\Sigma p_0 w}{\Sigma p_2 w}.$$

$$\therefore \quad I_{01} \times I_{12} \times I_{20} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times \frac{\Sigma p_2 w}{\Sigma p_1 w} \times \frac{\Sigma p_0 w}{\Sigma p_2 w} = 1.$$

Hence, *the weighted aggregative formula (fixed weights) also satisfies Circular Test.*

Similarly, we can prove the result for four and more than four years.

Example 32. *Show that neither Laspeyres' formula nor Paasche's formula obeys Time Reversal and Factor Reversal Tests of index numbers.* [C.U. B.Com.(H) 2006]

Solution: Laspeyres' formula (omitting the factor 100) is

$$I_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}. \tag{1}$$

Interchanging the suffixes 0 and 1, we get

$$I_{10} = \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1}. \tag{2}$$

Multiplying (1) and (2), we have

$$I_{01} \times I_{10} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \neq 1.$$

Therefore, *Laspeyres' formula does not obey Time Reversal Test which gives* $I_{01} \times I_{10} = 1$.

Again, Paasche's formula (omitting the factor 100) is

$$I_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}. \tag{3}$$

Interchanging the suffixes 0 and 1, we get

$$I_{10} = \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}. \tag{4}$$

Multiplying (3) and (4), we have

$$I_{01} \times I_{10} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \neq 1.$$

Thus, *Paasche's formula also does not obey Time Reversal Test.*

Factor Reversal Test is

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0},$$

where P_{01} = Price Index and Q_{01} = Quantity Index.

Laspeyres' Price Index P_{01} (omitting the factor 100) is

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}. \tag{5}$$

Interchanging p and q in (5), we get Laspeyres' Quantity Index

$$Q_{01} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} = \frac{\Sigma p_0 q_1}{\Sigma p_0 q_0}. \tag{6}$$

Multiplying (5) and (6), we have

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \neq \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}.$$

This shows that *Laspeyres' formula does not obey Factor Reversal Test.*

Again, Paasche's Price Index P_{01} is given by

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}. \tag{7}$$

Interchanging p and q, Paasche's Quantity Index is

$$Q_{01} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1} = \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0}. \tag{8}$$

Multiplying (7) and (8), we get

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0} \neq \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0},$$

which shows that *Paasche's formula does not obey Factor Reversal Test.*

Example 33. *Examine whether Fisher's Ideal Index Number satisfies the Time Reversal and Factor Reversal Tests.* [C.U. B.Com.(H) 2002]

Solution: Fisher's Ideal Index Number (omitting the factor 100) is

$$I_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}. \tag{1}$$

Interchanging the suffixes 0 and 1, we get

$$I_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}}. \tag{2}$$

Multiplying (1) and (2), we get

$$I_{01} \times I_{10} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}} = \sqrt{1} = 1,$$

i.e., $I_{01} \times I_{10} = 1$, which shows that *Fisher's Ideal Index Number satisfies the Time Reversal Test.*

Again, Fisher's Price Index P_{01} is given by

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}. \tag{3}$$

Interchanging p and q, we get Fisher's Quantity Index

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0}}. \tag{4}$$

Multiplying (3) and (4), we have

$$P_{01} \times Q_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0}} = \sqrt{\frac{(\Sigma p_1 q_1)^2}{(\Sigma p_0 q_0)^2}} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \text{the true value ratio.}$$

This shows that *Fisher's Ideal Index Number satisfies Factor Reversal Test.*

Example 34. *Calculate Fisher's Ideal Index using the following data and check whether it satisfies the Time Reversal Test:*

Commodities	2004		2005	
	Quantity	Price (₹)	Quantity	Price (₹)
X	50	32	50	30
Y	35	30	40	25
Z	55	16	50	18

Solution:

TABLE 10.27: CALCULATIONS FOR FISHER'S IDEAL INDEX

Commodity	2004		2005		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	q_0	p_0	q_1	p_1				
X	50	32	50	30	1600	1600	1500	1500
Y	35	30	40	25	1050	1200	875	1000
Z	55	16	50	18	880	800	990	900
Total	—	—	—	—	3530 $=\Sigma p_0q_0$	3600 $=\Sigma p_0q_1$	3365 $=\Sigma p_1q_0$	3400 $=\Sigma p_1q_1$

Fisher's Price Index (P_{01}) is given by

$$P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100 = \sqrt{\frac{3365}{3530} \times \frac{3400}{3600}} \times 100 = \sqrt{\frac{114410}{127080}} \times 100$$

$$= \sqrt{0.90029} \times 100 = 94.88 \text{ (approx.)}$$

2nd Part $P_{01} = \sqrt{\frac{3365}{3530} \times \frac{3400}{3600}}$, omitting the factor 100.

Again, $P_{10} = \sqrt{\frac{\Sigma p_0q_1}{\Sigma p_1q_1} \times \frac{\Sigma p_0q_0}{\Sigma p_1q_0}} = \sqrt{\frac{3600}{3400} \times \frac{3530}{3365}}$.

$$\therefore\ P_{01} \times P_{10} = \sqrt{\frac{3365}{3530} \times \frac{3400}{3600} \times \frac{3600}{3400} \times \frac{3530}{3365}} = \sqrt{1} = 1.$$

This shows that *Fisher's Index satisfies Time Reversal Test.*

Example 35. *Using the following data, show that Fisher's Ideal Formula satisfies the Factor Reversal Test:*

Commodity	Price (in ₹) per Unit		Number of Units	
	Base Period	Current Period	Base Period	Current Period
A	6	10	50	56
B	2	2	100	120
C	4	6	60	60
D	10	12	30	24
E	8	12	40	36

Solution:

TABLE 10.28: CALCULATIONS FOR FISHER'S IDEAL INDEX

Commodity	Price		Quantity		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	p_0	p_1	q_0	q_1				
A	6	10	50	56	300	336	500	560
B	2	2	100	120	200	240	200	240
C	4	6	60	60	240	240	360	360
D	10	12	30	24	300	240	360	288
E	8	12	40	36	320	288	480	432
Total	—	—	—	—	1360 $=\Sigma p_0q_0$	1344 $=\Sigma p_0q_1$	1900 $=\Sigma p_1q_0$	1880 $=\Sigma p_1q_1$

Omitting the factor 100, *Fisher's Price Index* P_{01} is given by

$$P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} = \sqrt{\frac{1900}{1360} \times \frac{1880}{1344}}.$$

Interchanging p and q, *Fisher's Quantity Index* Q_{01} is

$$Q_{01} = \sqrt{\frac{\Sigma q_1p_0}{\Sigma q_0p_0} \times \frac{\Sigma q_1p_1}{\Sigma q_0p_1}} = \sqrt{\frac{1344}{1360} \times \frac{1880}{1900}}.$$

$$\therefore\ P_{01} \times Q_{01} = \sqrt{\frac{1900}{1360} \times \frac{1880}{1344} \times \frac{1344}{1360} \times \frac{1880}{1900}} = \sqrt{\frac{(1880)^2}{(1360)^2}} = \frac{1880}{1360} = \frac{\Sigma p_1q_1}{\Sigma p_0q_0}.$$

This shows that **Fisher's Ideal Index satisfies Factor Reversal Test.**

Example 36. *When the cost of cigarette was increased by* 50%, *a smoker who maintained his former scale of consumption said that the rise of cigarette price had increased his cost of living by* 5%. *What percentage of his cost of living was due to buying cigarette before the change of price?*

Solution: Let his expenditure on cigarette before the change of price be ₹ x. Then his expenditure on cigarette after increase in price of cigarette = $x + 50\%$ of x = ₹ $1.5x$.

$\therefore$ increase in expenditure = $1.5x - x$ = ₹ $0.5x$.

If ₹ y be his former cost of living, then his cost of living after the increase in price of cigarette (other prices remaining fixed) = y + 5% of y = ₹ $1.05y$.

$\therefore$ increase in expenditure = $1.05y - y$ = ₹ $0.05y$.

$\therefore$ we have $0.5x = 0.05y$, or, $x = \frac{0.05}{0.5}y = \frac{1}{10}y$.

Hence, *the required expenditure on cigarette before the change of price expressed as percentage of his cost of living* = $\frac{x}{y} \times 100 = \frac{1}{10} \times 100 = 10\%$.

[*Otherwise:* Let the required percentage be x. Then we have

Group	Weights (w)	% increase in price (i)	iw
(i) Cigarette	x	50	$50x$
(ii) Other items	$100 - x$	0	0
Total	$100 = \Sigma w$	—	$50x = \Sigma iw$

Average percentage increase for all items = $\frac{\Sigma iw}{\Sigma w}$ or, $5 = \frac{50x}{100}$ or, $50x = 500$; $\therefore$ $\boldsymbol{x = 10}$.]

10.12 Base Shifting

Sometimes it becomes necessary to shift the base period of a series of Index Numbers from one period to another.

If the base is too old, we shift the base to a more recent period to make the data more useful.

Base-shifting is also required if we are given two (or more) series of Index Numbers with different base periods. *For example,* if a series of Index Numbers with the base year 1976 is to be compared with another series whose base year is 1985, then it is necessary to convert either the former series into a new one with 1985 as the base year or the two series into new ones with any other year as the common base. This is done by using the formula:

Index Number for any year with a new base year

$$= \frac{\text{Old Index Number for the year}}{\text{Old Index Number for the new base year}} \times 100.$$

In this case, we take the index number of the new base year as 100.

Example 37. *Shift the base period of the following series of index numbers from* 1995 *to* 2002:

Year	1999	2000	2001	2002	2003	2004	2005
Index Number (Base 1995 = 100)	120	125	132	140	150	158	175

Solution:

TABLE 10.29: SHIFTING OF BASE PERIOD FROM 1995 TO 2002

Year	Index Number with base 1995	Index Number with new base 2002
1999	120	$\frac{120}{140} \times 100 = 85.71$
2000	125	$\frac{125}{140} \times 100 = 89.29$
2001	132	$\frac{132}{140} \times 100 = 94.29$
2002	140	$\frac{140}{140} \times 100 = 100.00$
2003	150	$\frac{150}{140} \times 100 = 107.14$
2004	158	$\frac{158}{140} \times 100 = 112.86$
2005	175	$\frac{175}{140} \times 100 = 125.00$

Example 38. *In* 2001, *an index number is* 100; *it rises* 5% *in* 2002, *falls* 2% *in* 2003, *and again rises* 4% *in* 2004 *and* 6% *in* 2005. *If rise and fall begin w.r.t. previous year, find the index numbers for the given years shifting the base to* 2004.

Solution:

TABLE 10.30: Shifting of base period from 2001 to 2004

Year	Index Number (Base 2001)	Index Number (Base 2004)
2001	100.0	$\frac{100}{107} \times 100 = 93.46$
2002	$\frac{105}{100} \times 100 = 105.0$	$\frac{105}{107} \times 100 = 98.13$
2003	$\frac{98}{100} \times 105 = 102.9$	$\frac{102.9}{107} \times 100 = 96.17$
2004	$\frac{104}{100} \times 102.9 = 107.0$	$\frac{107}{107} \times 100 = 100.00$
2005	$\frac{106}{100} \times 107 = 113.4$	$\frac{113.4}{107} \times 100 = 105.98$

10.12.1 Splicing of Index Numbers

Sometimes, an old series of index numbers is discontinued and a new series of index numbers is constructed with a recent period as the base. As these two series have different base periods, they cannot be compared. To make them comparable, the two series of index numbers are spliced together, i.e., one continuous series of index numbers with a common base period is prepared. Thus splicing is the technique (i.e., statistical procedure) of combining two (or more) series of index numbers with different base periods to obtain a continuous series of index numbers with a common base period. We illustrate the whole procedure with the help of a worked-out example.

Example 39. *Splice the following two series of index members with* (i) *base* 1995 *and* (ii) *base* 2000:

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Series A	100	110	125	150	180	200	...	...	...	...	...
Series B	...	...	...	...	...	100	105	90	120	140	160

Solution: From the given data, we see that series A with base year 1995 is discontinued after 2000 and a new series is constructed with 2000 as base year. We have to splice (i) the series B to A shifting the base of series B to 1995 and (ii) the series A to B shifting the base of A to 2000. The procedure is shown in the following table:

TABLE 10.31: SPLICING OF TWO SERIES OF INDEX NUMBERS

Year	Series A Index No. (Base 1995 = 100)	Series B Index No. (Base 2000 = 100)	New Continuous Series (Spliced Index Nos.)	
			Base 1995 = 100	Base 2000 = 100
1995	100	...	100	$\frac{100}{200} \times 100 = 50$
1996	110	...	110	$\frac{100}{200} \times 110 = 55$
1997	125	...	125	$\frac{100}{200} \times 125 = 62.5$
1998	150	...	150	$\frac{100}{200} \times 150 = 75$
1999	180	...	180	$\frac{100}{200} \times 180 = 90$
2000	200	100	200	100
2001	...	105	$\frac{200}{100} \times 105 = 210$	105
2002	...	90	$\frac{200}{100} \times 90 = 180$	90
2003	...	120	$\frac{200}{100} \times 120 = 240$	120
2004	...	140	$\frac{200}{100} \times 140 = 280$	140
2005	...	160	$\frac{200}{100} \times 160 = 320$	160

Example 40. *Due to change in prices, the cost of living index for the working class in a city rose to* 225 *in April from* 180 *in March,* 2005. *The index of food became* 252 *from* 198, *that of clothing from* 185 *to* 205, *that*

of fuel and lighting from 175 to 195 and that of miscellaneous from 138 to 212. The index of rent, however, remained unchanged at 150. Find the weights of all the groups if the weights of clothing, fuel and lighting and rent were the same.

Solution: Let the total weights $(\Sigma w) = 100$, where w_1 and w_2 are the weights of food and clothing respectively.

TABLE 10.32: CALCULATIONS FOR THE WEIGHTS OF THE DIFFERENT GROUPS

Group	Group Index		Weight	For March	For April
	March (I_1)	April (I_2)	(w)	$I_1 w$	$I_2 w$
Food	198	252	w_1	$198w_1$	$252w_1$
Clothing	185	205	w_2	$185w_2$	$205w_2$
Fuel and Light	175	195	w_2	$175w_2$	$195w_2$
Rent	150	150	w_2	$150w_2$	$150w_2$
Miscellaneous	138	212	$100-w_1-3w_2$	$13800-138w_1-414w_2$	$21200-212w_1-636w_2$
Total	—	—	$100 = \Sigma w$	$60w_1+96w_2+13800 = \Sigma I_1 w$	$40w_1-86w_2+21200 = \Sigma I_2 w$

Since the Cost of Living Indices (CLI) for March and April are respectively 180 and 225;

$\therefore$ CLI for March $= \dfrac{\Sigma I_1 w}{\Sigma w} = 180,$

$$\text{or, } \frac{60w_1+96w_2+13800}{100} = 180 \text{ or, } 5w_1+8w_2=350, \quad (1)$$

and CLI for April $= \dfrac{\Sigma I_2 w}{\Sigma w} = 225,$

$$\text{or, } \frac{40w_1-86w_2+21200}{100} = 225 \text{ or, } 20w_1-43w_2=650. \quad (2)$$

Solving (1) and (2), $w_1 = 54$ and $w_2 = 10$.

Hence, **the required weights of the five groups are** *Food:* **54**, *Clothing:* **10**, *Fuel and Light:* **10**, *Rent:* **10** and *Miscellaneous*: 16.

10.13 Description of a Few Important Index Numbers

1. Index Number of Wholesale Prices in India

This index is compiled by the office of the Economic Adviser to the Government of India under the Ministry of Commerce and Industry, and is a general purpose Index with Base Year ended August 1987 = 100. It covers 78 commodities divided into 5 major groups.

The index is calculated as a weighted geometric mean of price relatives, with weight proportional to the value of quantities marketed during 1988–89.

A Revised Index of Wholesale Price with Base: 2002–2003 = 100 is also compiled by the Economic Adviser to the Government of India, based on 112 commodities and 555 individual quotations divided into

6 main groups. It is a weighted arithmetic mean of price relatives, the weights being proportional to the marketed values of domestic product and value of imports inclusive of duty. Semi-manufactured and manufactured articles have been weighted according to the gross value of products obtained from Census of Manufacturers, 1998. The weight for electricity is based on energy sold by electricity supplier undertakings valued at average All-India rates. Petroleum-weights are based on consumption figures. These weights generally refer to the period 1998–99.

2. Working Class Consumer Price Index

Indices for various important centres of India are published. These indices relate only to working class families, the heads of which are in full-time regular employment and represent the changes in the cost of a given schedule of goods and services consumed by an average working class family in each centre, as determined by periodical enquiries. The items included are grouped under 5 heads, viz., (i) Food; (ii) Clothing; (iii) Fuel and Lighting; (iv) House Rent and (v) Miscellaneous.

Separate indices are worked-out for each of these groups. The index for any group is the weighted arithmetic mean of price relatives of the several items in the group, the weights being proportional to the expenditure of an average working class family on each item. The group indices, thus obtained, are weighted by the percentage of total expenditure devoted to each of these groups to obtain the final Index Number. The arithmetic mean of the monthly Index Numbers gives the annual Index.

The Index for Kolkata is based on the price of 487 items included in 80 subgroups under the 5 major groups. This Index is computed by the State Statistical Bureau, West Bengal.

Example 41. *The following are the group index numbers and the group weights of an average working class family's budget. Construct the cost of living index number by assigning the given weights:*

Group	Index Number	Weight
Food	352	48
Fuel	220	10
Clothing	230	8
Rent	160	12
Miscellaneous	190	15

Solution:

TABLE 10.33: CONSTRUCTION OF COST OF LIVING INDEX NUMBER

Group	Index Number (I)	Weight (w)	Iw
Food	352	48	16,896
Fuel	220	10	2200
Clothing	230	8	1840
Rent	160	12	1920
Miscellaneous	190	15	2850
Total	—	$93 = \Sigma w$	$25{,}706 = \Sigma Iw$

$\therefore$ *the Cost of Living Index No.* $= \frac{\Sigma Iw}{\Sigma w} = \frac{25706}{93} = 276.4.$

Example 42. *Given below the average wages in ₹ per hour of unskilled workers of a factory during the years* 2000–2005. *Also shown is Consumer Price Index for these years (taking* 2000 *as Base Year with Price Index* 100*). Determine the real wages of the workers during* 2000–2005 *compared with their wages in* 2000:

Year	2000	2001	2002	2003	2004	2005
Consumer Price Index	100	120.2	121.7	125.9	129.3	140
Average Wage (rupees/hour)	1.19	1.94	2.13	2.28	2.45	3.10

How much is the worth of one rupee of 2000 in subsequent years?

Solution: We have

$$\text{Real Wage} = \frac{\text{Actual Wage for a period}}{\text{Cost of Living Index for the period (or Consumer Price Index)}} \times 100.$$

TABLE 10.34: CALCULATION OF REAL WAGES

Year	Consumer Price Index (Base: 2000)	Average Wage (₹/hour)	Real Wages of Workers (₹/hour)	Worth of ₹1 of 2000 (₹) 100 ÷ Col. 2
2000	100.0	1.19	$(1.19 \div 100) \times 100 = 1.19$	1.00
2001	120.2	1.94	$(1.94 \div 120.2) \times 100 = 1.61$	0.83
2002	121.7	2.13	$(2.13 \div 121.7) \times 100 = 1.75$	0.82
2003	125.9	2.28	$(2.28 \div 125.9) \times 100 = 1.81$	0.79
2004	129.3	2.45	$(2.45 \div 129.3) \times 100 = 1.89$	0.77
2005	140.0	3.10	$(3.10 \div 140) \times 100 = 2.21$	0.71

Note: Purchasing power in a year of one rupee of 2000 = $\frac{\text{Index Number for 2000}}{\text{Index Number of the year}}$.

3. Index Number of Security Prices

This is a general purpose index compiled by the Reserve Bank of India with the financial year 1950–51 as base and is computed by the Chain Index Formula. In the compilation of the index, 468 scrips are included. These scrips were selected from among those quoted in the Mumbai, Kolkata, Chennai and Delhi Stock Exchanges on the basis of the importance of the concerns to which they are related and their activity in the market. These scrips were classified into 4 groups and 38 sub-groups. The weights allocated to each of them were proportional to the total amount of loans outstanding, in the case of Government and Semi-Government securities and to the market value of shares during the base period in the case of fixed-dividend and variable-dividend industrials. The indices are calculated from weekly average prices for the selected scrips. The average quotations of the scrips during the base year represent the arithmetic means of the daily quotations for the whole year. Monthly and annual indices are similar averages of weekly figures.

4. Index of Industrial Production

The index of Industrial Production is a special purpose Index Number. It is designed to measure the relative change (increase or decrease) in the level of industrial production in a given period compared to some base period. To construct such an index, data about the level of industrial output are collected under the following 6 major heads:

(1) Textile Industries; (2) Mining Industries; (3) Mechanical Industries; (4) Metallurgical Industries; (5) Industries subject to excise duties; and (6) Miscellaneous.

Each Head includes a number of important items. *For example,* Textile Industries include cotton, silk, wool, etc.

Data on industrial output are collected on a monthly, quarterly or yearly basis. Weights are used to various industries on the basis of the values of their output. The Index of the Industrial Production is the AM (or GM) of the quantity relatives and it is given by

Index of Industrial Production

$$= \frac{\sum\left(\frac{q_1}{q_0} \times w\right)}{\Sigma w} \times 100,$$

where q_1 = quantity produced in the given period, q_0 = quantity produced in the base period and w = relative importance of different outputs.

5. Index Numbers of Quantum of Price Level of Imports and Exports

- **Base:** 1952–53 = 100.
- **Coverage:** It covers trade by all routes, i.e., by Sea, Air and Land, and the items included account for 92% of total trade of India in respect of export and 75% of the import.
- **Computation:**

  Price Index: The formula employed is that of Paasche's Index $I_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$. The method consists in revaluing the trade of the current period at the prices prevailing in the base period and then in comparing the recorded value of the current period with the corresponding value recalculated on the above basis.

  For Index of Quantum of Trade: The formula employed is that of Laspeyres' Index (with quantities and prices interchanged), the quantities in the current and the base years being weighted by the prices in the base year.

$$Q_{01} = 100 \times \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0}.$$

The Index Numbers of Quantum of Trade are designed to show the movements in the aggregate value of imports and exports after elimination of the effect of the price changes.

These Index Numbers are constructed by the Director General of Commercial Intelligence and Statistics under the Ministry of Commerce and Industry, Government of India.

6. Index Number of Agricultural Production in India

- **Base:** Year ending June 1950 = 100.
- **Coverage:** The Index covers 28 crops divided into 2 main groups and 6 sub-groups.
- **Weights:** Weights have been assigned to different crops in proportion to the total value of production of each item during the base period. The output has been evaluated at annual harvest prices prevailing during the year.
- **Construction:** The Index Numbers are constructed by the Chain-Base Method in order to provide for changes in the estimates of production due to extension in the geographical coverage or to variation in the method of estimation over time. The sub-groups, the groups and all commodity Index Numbers are the weighted Arithmetic Mean of gross Production Indices of crops covered.

  This Index is constructed by the Directorate of Economics and Statistics, Ministry of Food and Agriculture, Government of India.

Errors in Index Numbers: Limitations

Index Numbers play a very important role in studying the business and economic activity of a country. But they have limitations and persons using them must know all these before their uses to avoid errors of interpretation. The main limitations of Index Numbers are as follows:

As Index Numbers are based on a sample, they do not depend on all the items in their construction. So they are only approximate indicators and may not properly represent the changes in prices (or quantities).

There may be error in selection of commodities, base period, collection of data relating to prices or quantities, choice of formula and choice of average, at each stage of the construction of the Index numbers.

As selection of a sample at random from a large population of millions of commodities is neither practical nor representative, Index Numbers are constructed from deliberately selected samples. So error is likely to occur in the construction of Indices and every effort should be made to minimize these errors.

As it is often not possible to take into account the changes in the quality of products, differences in qualities of commodities and, therefore, difference in prices are not considered over a period of time. This makes comparisons by Index Number over a long time less reliable.

As there are large number of methods for constructing Index Numbers and different methods of computation give different results, the selection of an appropriate formula often creates problems in comparisons by Index Numbers.

Index Numbers can also be misused in drawing the desired results. A dishonest person can consider the boom period (year) of profits as base to show the subsequent years' profits to be very low.

10.14 Uses of Index Numbers

Index Numbers are widely used to study the relative positions of business and economic conditions. At the Government level, index numbers are used for wage policy, price policy, taxation and general economic policies.

Index Numbers were originally developed to measure the changes in prices of commodities. But they are now increasingly used to measure changes in physical quantities or total value of production, the volume of trade, the intensity of business activity. We have, thus, the Index Number of the Wholesale Prices, the Index Numbers of the Retail Prices, the Index Number of the Industrial Production, the Cost of Living Index Number, the Index Number of the Business Activity and of many others. They are all useful in their own fields.

Index Numbers help in framing suitable economic and business policies. *For example,* the Cost of Living Index helps the employer to adjust wages and salaries, specially, dearness allowance of an employee in accordance with the cost of living. If the wages, salaries and dearness allowance are not adjusted in accordance with the cost of living, it leads very often to strikes and lock-outs causing considerable loss of production and the consequent loss of National Income.

The Index Number of the Industrial Production is used to study the comparative position in production. The Wholesale Price Index and the Retail Price Index indicate the changes that take place in the value of money. Index Numbers of Stock Prices help the speculators to forecast the future trend in the Stock Market.

Index Numbers are also useful in many other fields, e.g., Sociologists speak of population indices; Psychologists measure Intelligence Quotients; Health Authorities construct indices to display changes in the adequacy of the Hospital Facilities, etc.

Index Numbers are very useful in deflation. The process of obtaining the real wages from the nominal wages on adjustment by the Cost of Living Index is known as *deflation.*

LIST OF FORMULAE

Price Index Numbers

1. Simple Aggregative Price Index = $\dfrac{\Sigma p_1}{\Sigma p_0} \times 100.$

2. Weighted Aggregative Price Index = $\dfrac{\Sigma p_1 w}{\Sigma p_0 w} \times 100.$

3. Laspeyres' Index = $\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100.$

4. Paasche's Index = $\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100.$

5. Marshall-Edgeworth's Index = $\dfrac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100.$

6. Simple AM of Price Relatives Index = $\dfrac{100}{n}\sum\left(\dfrac{p_1}{p_0}\right).$

7. Simple GM of Price Relatives Index = $100\sqrt[n]{\dfrac{p_1'}{p_0'} \times \dfrac{p_1''}{p_0''} \times \dfrac{p_1'''}{p_0'''} \times \cdots \times \dfrac{p_1^{(n)}}{p_0^{(n)}}}.$

8. Weighted AM of Price Relatives Index = $\dfrac{100}{\Sigma w}\sum\dfrac{p_1 w}{p_0}.$

9. (a) Fisher's Ideal Index = $100\sqrt{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}.$

   (b) Bowley's Index = $\dfrac{1}{2}\left(\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}\right) \times 100.$

10. General Index from Group Indices = $\dfrac{\Sigma I w}{\Sigma w}.$

11. (a) Chain Index for any year = $\dfrac{\text{Link Index for the year} \times \text{Chain Index of previous year}}{100}.$

    (b) Link Relative for any year = $\dfrac{\text{Price Relative for the year}}{\text{Price Relative for the previous year}} \times 100.$

Quantity Index Numbers

Quantity Index Formulae may be obtained from Price Index Formulae replacing p by q, and q by p.

12. Time Reversal Test: $I_{01} \times I_{10} = 1.$

13. Factor Reversal Test: $P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$ = True Value Ratio.

14. Circular Test: $I_{01} \times I_{12} \times I_{23} \times \cdots \times I_{(n-1)n} \times I_{no} = 1.$

Cost of Living Index

15. (a) Cost of Living Index $= \dfrac{\Sigma I w}{\Sigma w}$, where I = group index and w = group weight.

(b) Cost of Living Index $= \dfrac{\Sigma I W}{100}$, where I = group index $= \dfrac{\Sigma\,(\text{Price Relative} \times w)}{\Sigma w}$

and W = percentage expenditure of an average family on the group (w = percentage expenditure of an average family or an item).

EXERCISES ON CHAPTER 10

(Theory)

1. (a) What is meant by an Index Number? State the uses of Index Numbers. Define Price Relative. State Fisher's Ideal Index Formula. [C.U. B.Com.(H) 2003]

   (b) Write down Laspeyres' and Paasche's Price Index Formulae. [B.U. B.Com.(H) 2008]

2. (a) What is Index Number? How is it prepared? How is it weighted? Mention its importance in business and commerce. [V.U. B.Com.(H) 2007]

   (b) What is Index Number? State the importance of 'base year' in calculating index number. [C.U. B.Com.(H) 2000; V.U. B.Com.(H) 2010]

3. (a) Discuss the various problems in the construction of Index Numbers. [Utkal U. B.Com. 2000]

   (b) What are the problems faced in the construction of Index Numbers?

   (c) Mention any two methods of calculating Index Numbers. [B.U. B.Com.(H) 2008]

4. Discuss the importance and use of weights in the construction of Index Numbers.

5. Explain the different methods of construction of Index Numbers. [V.U. B.Com.(H) 2009]

6. What is the Chain-Base Method of construction of Index Numbers and how does it differ from the Fixed-Base Method?

7. Explain with formula, the construction of the following Index Numbers of Price: (a) Laspeyres' Index; (b) Paasche's Index; (c) Marshall-Edgeworth's Index; (d) Fisher's Ideal Index. [C.U. B.Com.(H) 1999]

8. Define Laspeyres' and Paasche's Index Numbers. It is sometimes said that Laspeyres' Price Index tends to overestimate price changes while Paasche's Price Index tends to underestimate them. Put forward a possible explanation to substantiate this statement.

9. (a) What weights are used in Laspeyres', Paasche's and Marshall-Edgeworth's Price Index Numbers? Prove that Marshall-Edgeworth's Price Index Number lies between the Laspeyres' and Paasche's Price Index Numbers.

   (b) Distinguish between Laspeyres' and Paasche's Price Index Numbers. [B.U. B.Com.(H) 2008]

10. What are the tests prescribed for a good Index Number? Describe the Index Number which satisfies these tests.

11. What are the tests to be satisfied by a good Index Number? Examine how far they are met by Fisher's Ideal Index Number.

12. Explain the Time Reversal Test and Factor Reversal Test of Index Numbers, and examine whether the following Index Numbers satisfy the above tests: Laspeyres' Index Number, Paasche's Index Number, Marshall-Edgeworth's Index Number and Fisher's Ideal Index Number.

13. (a) What is a Cost of Living Index Number? What does it measure? How is it prepared?
[C.U. B.Com. 2004; V.U. B.Com.(H) 2011]

(b) Is the Cost of Living Index Number the same as the Consumer Price Index Number?

(c) Write short notes on Link Relatives and Chain-Base Index Numbers. [C.U. B.Com.(H) 1999]

14. What data are essential for the construction of an Index Number of Industrial Production? What are the weights used and how are they calculated? Explain the method employed for construction of such an Index Number.

15. Write a short note on each of the following: (a) Time Reversal Test; (b) Factor Reversal Test.

16. Prove that Laspeyres' and Paasche's index formulae do not satisfy time reversal and factor reversal tests of index numbers. Prove that if the geometric mean of Laspeyres' price index and Paasche's price index be taken as a price index, then that satisfies time reversal and factor reversal tests.
[C.U. B.Com.(H) 2002]

[**Hints:** See worked-out Exs 30, 31 in Section 11.11.

Fisher's Ideal Index Number is the geometric mean of Laspeyres' Price Index and Paasche's Price Index, and it satisfies Time Reversal and Factor Reversal Tests. Hence, the results follow.]

Problems (A)

1. (a) The average prices of mustard oil per quintal in the years 2001 to 2005 are given below:

PRICE OF MUSTARD OIL PER QUINTAL IN RS.	
Year	Price
2001	585
2002	500
2003	635
2004	780
2005	895

Find the corresponding index numbers taking 2002 as the base year.

(b) The average prices of Rice, Wheat and Jowar per quintal in the years 2001 to 2005 are given below:

AVERAGE PRICE PER QUINTAL IN RS.			
Year	Rice	Wheat	Jowar
2001	150	82	80
2002	175	72	72
2003	200	90	80
2004	220	99	96
2005	198	81	88

Find the Price Relatives of Rice, Wheat and Jowar, and hence calculate the Composite Index Numbers. (Take 2003 as the base year.)

(c) In 2004, the average price of a commodity was 20% more than that in 2003 but 20% less than that in 2002; and moreover, it was 50% more than that in 2005. Reduce the data to price relatives using 2003 as the base year (2003 Price Relative = 100).

[**Hints:** Price relative of 2004 = 80% of price relative of 2002 or, 120 = (80/100) $\times x$ or, $x = 150$. Again, price relative of 2004 = 150% of price relative of 2005 or, 120 = (150/100) $\times y$ or, $y = 80$. Hence, etc.]

2. From the arranged price of Rice given below, find the Chain-Base Index Numbers:

Year	2000	2001	2002	2003	2004	2005
Price in ₹ (per quintal)	108	96	120	144	132	180

3. From the Fixed-Base Index Numbers given below, prepare Chain-Base Index Numbers:

Year	2000	2001	2002	2003	2004	2005
Index	94	98	102	95	98	100

4. From the Chain-Base Index Numbers given below, prepare Fixed-Base Index Numbers:

Year	2001	2002	2003	2004	2005
Index	110	160	140	200	150

5. Find, (a) by the Method of Aggregates and (b) by the Arithmetic Mean Method, Index Number from the following data:

Commodity	Base Price	Current Price
Rice	30	36
Wheat	25	28
Oil	90	108
Potato	15	21
Fish	96	120

6. Find, by the Arithmetic-Mean method, the Index Number from the following data:

Commodity	Base Price	Current Price
Rice	30	35
Wheat	22	25
Fish	54	64
Potato	20	25
Coal	15	18

7. Find the Index Number by (a) the Method of Relatives (using arithmetic mean) and (b) the Method of Aggregates from the following data:

Commodity	Base Price	Current Price
Rice	35	42
Wheat	30	35
Pulse	40	38
Fish	107	120

8. Construct Laspeyres' Price Index number from the following data:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

9. (a) Construct (i) Laspeyres' and (ii) Paasche's Price Index Number of 2005 with 1995 as base from the following data:

Commodity	Price (₹/kg)		Quantity sold (kg)	
	1995	2005	1995	2005
A	4	5	35	120
B	60	70	118	130
C	35	40	50	70

[**Hints**: See worked-out Ex. 19 in Section 10.6. Here $\Sigma p_0q_0 = 9210$, $\Sigma p_1q_0 = 10735$, $\Sigma p_0q_1 = 10730$ and $\Sigma p_1q_1 = 12500$.]

(b) Find x from the following data:

Commodity	Base year		Current year	
	Price (₹)	Quantity	Price (₹)	Quantity
A	1	10	2	5
B	1	5	x	2

Given that the ratio between Laspeyres' and Paasche's Index Numbers is 28 : 27.

[C.U. B.Com. 2003]

(c) If Laspeyres' and Paasche's Price Index Numbers are 125.6 and 154.3 respectively, find Fisher's Ideal Price Index Number. [C.U. B.Com. 2008]

[**Hints**: Fisher's Ideal Price Index Number $= \sqrt{125.6 \times 154.3} = \sqrt{19380} = 139.2$.]

10. Using the following data (given in suitable units) calculate Fisher's Ideal Index Number for 2005 with 2004 as base year and examine whether it satisfies the Time Reversal Test:

Commodity	2004		2005	
	Quantity	Price	Quantity	Price
A	50	32	50	40
B	35	30	40	35
C	55	16	50	18

11. Calculate the price index numbers by (a) Paasche's method, (b) Laspeyres' method, (c) Bowley's method, (d) Fisher's Ideal formula:

Commodity	2004		2005	
	Price in ₹	Quantity in kg	Price in ₹	Quantity in kg
A	20	8	40	6
B	50	10	60	5
C	40	15	50	10
D	20	20	20	15

12. The following table gives the change in prices and consumption of three commodities. Compute Fisher's Ideal Price Index Number:

Commodity	1995		2005	
	Price (₹)	Quantity (Units)	Price (₹)	Quantity (Units)
Wheat	100	10	110	6
Rice	150	15	170	18
Cloth	5	50	4	30

[**Hints:** See worked-out Ex. 22 in **Section 10.6.**]

13. The data below show the percentage increases in price of a few selected food items and the weights attached to each of them. Calculate the index number for the food group:

Food items	Rice	Wheat	Dal	Oil	Ghee	Spices	Milk	Meat	Veg.	Ref.
Weights	40	8	10	7	2	3	7	8	12	3
% increase in Price	200	150	105	225	300	450	200	400	250	175

(Here Ref. = Refreshment.)

[**Hints:** Current Index of an item (I) = 100 + percentage increase in price. Calculate the current indices of all the food items and then find the index number by using the formula $I_{01} = \frac{\Sigma Iw}{\Sigma w}$.]

14. In calculating a certain cost of living index number the following weights were used: Food 15, Clothing 3, Rent 4, Fuel and light 2, Miscellaneous 1. Calculate the index for a date when the average percentage increases in prices of items in the various groups over the base period were 32, 54, 47, 78 and 58 respectively.

15. Obtain Laspeyres' Price Index Number and Paasche's Quantity Index Number from the following data:

Item	Price (₹ per unit)		Quantity	
	Base year	Current year	Base year	Current year
1	2	5	20	15
2	4	8	4	5
3	1	2	10	12
4	5	10	5	6

[Mangalore U. B.Com. 1997]

16. (a) From the data given below construct Fisher's Quantity Index Number:

Commodity	2004		2005	
	Quantity	Price	Quantity	Price
A	2	4	6	18
B	5	3	2	2
C	7	8	4	24

[Hints: Fisher's Quantity Index Number $(Q_{01}) = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0}} \times 100.$

Commodity	q_0	p_0	q_1	p_1	p_0q_1	p_0q_0	p_1q_1	p_1q_0
A	2	4	6	18	24	8	108	36
B	5	3	2	2	6	15	4	10
C	7	8	4	24	32	56	96	168
Total	—	—	—	—	$62 = \Sigma p_0 q_1$	$79 = \Sigma p_0 q_0$	$208 = \Sigma p_1 q_1$	$214 = \Sigma p_1 q_0$

$\therefore$ Fisher's Quantity Index Number $= \sqrt{\frac{62}{79} \times \frac{208}{214}} \times 100 = 0.87338 \times 100 = 87.34.$]

(b) Calculate the quantity index number for the current year using Fisher's formula for the following data and show that it satisfies the Time Reversal Test:

Commodity	2000		2003	
	Price	Quantity	Price	Quantity
X	6	70	8	120
Y	8	90	10	100
Z	12	140	16	280

[C.U. B.Com. 2004]

Problems (B)

1. Find, by Weighted Aggregative Method, the index number of the following data:

Commodity	Base Price	Current Price	Weight
Rice	140	180	10
Oil	400	550	7
Sugar	100	250	6
Wheat	125	150	8
Fish	200	300	4

2. Construct, by the Method of Weighted Aggregate of Prices, the index number for 2005 on the basis of 1997 from the following information:

Commodity	Average Price for 1997	Average Price for 2005	Weights
Rice	30	30	20
Pulse	35.50	32.50	6
Potato	12.50	10	4
Oil	90	100	3.5
Salt	8	10	1.5

3. (a) Find Price Index Number by the method of relatives using Arithmetic Mean from the following:

Commodity	Basic Price	Current Price	Weight
Wheat	5	7	4
Milk	8	10	2
Fish	25	32	3
Sugar	6	12	1

[C.U. B.Com. 1996; V.U. B.Com.(H) 2007]

(b) Find, by Arithmetic Mean Method, the index number from the following data:

Commodity	Base Price	Current Price	Weight
Rice	25	28	20
Wheat	20	22	12
Oil	30	27	10
Fish	90	99	8

4. Construct an index number for the following data using Weighted Average of Price Relatives method:

Commodity	Current Year (2005)	Base Year (1995)	Weights
	Price in ₹	Price in ₹	
A	5.20	4.25	30
B	3.75	2.95	40
C	1.95	2.15	15
D	8.10	8.85	15

5. The following are the prices of commodities in 1998 and 2005. Calculate a Price Index Based on Price Relatives, using the Geometric Mean:

Year	Commodity					
	A	B	C	D	E	F
1998	45	60	20	50	85	120
2005	55	70	30	75	90	130

6. Calculate, by the method of Weighted Arithmetic Mean of Price Relatives, the index number of prices for 2005 on the basis of 2004 from the data given below:

Commodities	Weights	Price per unit 2004	Price per unit 2005
A	40	16.00	20.00
B	25	40.00	60.00
C	5	0.50	0.50
D	20	5.12	6.25
E	10	2.00	1.50

7. Prepare Index Number for 1995 on the basis of 2005, by applying Laspeyres' Method, where the following information is given:

Commodity	1995		2005
	Price	Quantity	Price
Rice	4	50	10
Wheat	3	10	8
Jowar	2	5	4

8. Calculate Paasche's Index Number for prices from the following data taking 2000 as the base year:

Commodity	2000		2005	
	Price	Quantity	Price	Quantity
A	1	6	3	5
B	3	5	8	5
C	4	8	10	6

9. Calculate Marshall-Edgeworth's Index Number for prices from the following data, taking 2004 as the base year:

Commodity	2004		2005	
	Price	Quantity	Price	Quantity
A	2	74	3	82
B	5	125	4	140
C	7	40	6	33

10. (a) Construct Fisher's Ideal Index for prices from the following data:

Commodity	1997 (Base Year)		2005 (Current Year)	
	Price	Quantity	Price	Quantity
A	8	6	12	5
B	10	5	11	6
C	7	8	8	5

[**Hints:** See worked-out Exs 21, 23 in Art. 10.6.]

[V.U. B.Com.(H) 2010]

(b) Compute Fisher's Index Formula (the quantity index) from the data given below:

Commodity	Base Year		Current Year	
	Price	Total Value	Price	Total Value
A	10	100	8	96
B	16	96	14	98
C	12	36	10	40

[Punjab U. B.Com. 1994]

11. "Marshall-Edgeworth's Index Number is a good approximation to the Fisher's Ideal Index Number." Verify the truth of this statement from the following data:

Year	Rice		Wheat		Jowar	
	Price	Quantity	Price	Quantity	Price	Quantity
1998	9.3	100	6.4	11	5.1	5
2005	4.5	90	3.7	10	2.7	3

12. (a) The prices of a commodity in the years 2000 and 2005 were 25 and 30 respectively. Find the price relatives (i) taking 2000 as base year; (ii) taking 2005 as base year. Verify the truth of Time Reversal property.

(b) From the following data find the index numbers for the current year and the base year based on each other and show that the GM makes it reversible but the AM does not:

Commodity	Prices	
	Base Year	Current Year
A	25	55
B	30	45

[Hints: For A, $\frac{p_1}{p_0} \times 100 = \frac{55}{25} \times 100 = 220$; $\frac{p_0}{p_1} \times 100 = \frac{25}{55} \times 100 = \frac{500}{11}$.

For B, $\frac{p_1}{p_0} \times 100 = \frac{45}{30} \times 100 = 150$; $\frac{p_0}{p_1} \times 100 = \frac{30}{45} \times 100 = \frac{200}{3}$.

I_{01} as AM of price relatives $= \frac{220+150}{2} = 185$.

I_{10} as AM of price relatives $= \frac{1}{2}\left(\frac{500}{11} + \frac{200}{3}\right) = \frac{3770}{66} = 56.06$.

I_{01} as GM of price relatives $= \sqrt{220 \times 150} = 181.66$.

$I_{10} = \sqrt{\frac{500}{11} \times \frac{200}{3}} = 55.05$.

For AM, $I_{01} \times I_{10} = \frac{185}{100} \times \frac{56.06}{100} \neq 1$ and for GM, $I_{01} \times I_{10} = \frac{181.66}{100} \times \frac{55.05}{100} = 1$.

Hence, the result follows.]

13. Using Fisher's Ideal Index Number, calculate the index number for the following data:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	4	40	8	34
B	2	18	2	28
C	5	12	5	52
D	1	24	10	46

[C.U. B.Com.(H) 1999]

14. Construct Chain Index Numbers (Base 2000 ≡ 100) for the year 2001–05:

Year	2001	2002	2003	2004	2005
Link Index	103	98	105	112	108

[**Hints**:

Year	2000	2001	2002	2003	2004	2005
Link Index	100	103	98	105	112	108

Chain Index Numbers (C.I.) 100, $\frac{103\times 100}{100} = 103$, $\frac{98\times 103}{100} = 100.94$, $\frac{105\times 100.94}{100} = 106$, $\frac{112\times 106}{100} = 118.72$, $\frac{108\times 118.72}{100} = 128.22$.]

Problems (C)

1. (a) Construct the Cost of Living Index Number from the following group data:

Group	Weights	Group Index Number for a given year
(1) Food	47	247
(2) Fuel and Light	7	293
(3) Clothing	8	289
(4) House Rent	13	100
(5) Miscellaneous	14	236

(b) From the following data, calculate the Cost of Living Index Number:

Group	Weight	Index Number (Base: 2004–05 = 100)
Food	50	241
Clothing	2	221
Fuel and Light	3	204
Rent	16	256
Miscellaneous	29	179

[**Hints**: See worked-out Ex. 29 in Section 10.10.]

2. Annual production (in million tonnes) for four commodities is given below:

Commodity	Production in Year			Weights
	2000	2004	2005	
A	160	200	266	20
B	24	42	45	30
C	50	72	68	13
D	250	168	156	17

Calculate quantity index numbers for the years 2004 and 2005 with 2000 as base year, using (a) Simple Arithmetic Mean, and (b) Weighted Arithmetic Mean of the Relatives.

3. Find index numbers for the years 2003, 2004, 2005 by the Chain-Base Method, with base year 2002, from the following table:

Year	2002	2003	2004	2005
Link Index	100	110	95.5	109.5

4. (a) Calculate quantity index for 2005 from the following data using (a) Laspeyres' formula, (b) Paasche's formula and Fisher's formula with 2000 as the base year:

Commodity	2000		2005	
	Price	Quantity	Price	Quantity
A	1	6	5	4
B	3	5	8	5
C	4	·8	10	6

(b) Using the following data verify that Paasche's formula for Index does not satisfy Factor Reversal Test:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
X	4	10	6	15
Y	6	15	4	20
Z	8	5	10	4

[C.U. B.Com. 1995]

5. Compute, by Fisher's formula, the Quantity Index Number from the data given below:

Article	2003		2005	
	Price (₹)	Total Value (₹)	Price (₹)	Total Value (₹)
A	5	50	4	48
B	8	48	7	49
C	6	18	5	20

6. For the following data, calculate Quantity Index Number for the current year (i.e., Q_{01}) by Fisher's formula:

Commodity	Base Year		Current Year	
	Quantity	Price	Quantity	Price
I	6	5	8	6
II	8	3	10	2
III	12	2	10	2
IV	2	8	2	7
V	5	9	6	9

7. With the data given below, calculate the Price Index and Quantity Index by Fisher's Ideal Formula and then verify that Fisher's Ideal Formula satisfies the Factor Reversal Test:

Commodity	Base Year		Current Year	
	Price (₹)	Quantity ('000 tons)	Price (₹)	Quantity ('000 tons)
A	56	72	50	26
B	32	107	30	83
C	41	62	28	48

8. (a) Using the following data, show that Fisher's Ideal Index satisfies both the Time Reversal Test and Factor Reversal Test:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	10	50	12	60
B	8	30	9	32
C	6	35	7	40

(b) State Fisher's Ideal Index Number and using this, calculate the index number for the following data and hence show that it satisfies Factor Reversal Test.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	4	40	8	34
B	2	18	2	28
C	5	12	5	52
D	1	24	10	46

[**Hints:** See worked-out Ex. 33 and 34 in Section 10.11]

[C.U. B.Com.(H) 1999]

9. The price relatives and weights of a set of commodities are given in the following table:

Commodity	A	B	C	D
Price Relative	120	127	125	119
Weight	$2w_1$	w_2	w_1	w_2+3

If the index for the set is 122 and the sum of the weights is 40, find w_1 and w_2.

[**Hints**: $2w_1 + w_2 + w_1 + (w_2+3) = 40$ or, $3w_1 + 2w_2 = 37$ (1)

122 = Index No. = AM of Price Relatives $= \frac{120\times 2w_1 + 127w_2 + 125w_1 + 119(w_2+3)}{2w_1 + w_2 + w_1 + (w_2+3)}$

$= \frac{365w_1 + 246w_2 + 357}{40}$ or, $365w_1 + 246w_2 = 4523.$ (2)

Solve (1) and (2).]

10. From the following data construct Fisher's Ideal Index Number:

Commodity	1994		1996	
	Price	Value (₹)	Price	Value (₹)
A	5	50	6	72
B	7	84	10	80
C	10	80	12	96
D	4	20	5	30
E	8	56	8	64

[Gulbarga U. B.Com. 1997; Madras U. B.Com. 1998]

[**Hints**: See worked-out Ex. 23 in **Section 10.6.**]

11. From the following data construct a Price Index Number of the group of four commodities by using Fisher's Ideal Formula:

Commodity	Base Year		Current Year	
	Price per unit	Expenditure (₹)	Price per unit	Expenditure (₹)
A	2	40	5	75
B	4	16	8	40
C	1	10	2	24
D	5	25	10	60

[**Hints**: See worked-out Ex. 22 in **Section 10.6**. Here, expenditure = value.]

[H.P.U. B.Com. 1996]

Problems (D)

1. The following shows the Consumer Price Index Numbers (base: November 2002 = 100) for December 2005 for a certain class of families in West Bengal, separately for the five broad groups of items. Compute the overall Consumer Price Index from the given data:

Item Group	Weights of Group	Group Index
Food	58.55	91.4
Clothing	5.37	106.5
Fuel and Light	6.15	102.2
Housing	9.61	100.0
Miscellaneous	20.32	100.3

2. (a) During a certain period the Cost of Living Index Number goes up from 110 to 200 and the salary of a worker is also raised from ₹ 325 to ₹ 500. Does the worker really gain, and if so, by how much in real terms? [C.U. B.Com.(H) 1990]

[**Hints**: Use Real Wage = (Actual Wage/CLI) × 100 and show that Real Wage decreases from ₹ 295.45 to ₹ 250.]

(b) Net monthly income of an employee was ₹ 800 in 1980. The Consumer Price Index Number was 160 in 2000. It rises to 200 in 2004.

Calculate the additional dearness allowance to be paid to the employee if he has to be rightly compensated.

(c) The net monthly salary of an employee was ₹ 3000 in 1990. The consumer price index number in 1995 is 250 with 1990 as base year. Calculate the dearness allowances to be paid to the employee if he has to be rightly compensated. [C.U. B.Com. 2002]

[**Hints**: Consumer Price Index in the base year 1990 is 100. In 1995, the employee should get ₹ 3000 × $\frac{250}{100}$ = ₹ 7500.
∴ extra D.A. to be paid to the employee if he has to be rightly compensated = ₹ 7500– ₹3000 = ₹4500.]

3. Using the following data, show that Laspeyres' Price Index formula does not satisfy the Time Reversal Test:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	6	50	10	56
B	2	100	2	120
C	4	60	2	60
D	10	30	12	24
E	8	40	12	36

4. Calculate the Cost of Living Index Number from the following data:

Items	Price		Weight
	Base Year	Current Year	
Food	30	47	4
Fuel	8	12	1
Clothing	14	18	3
House Rent	22	15	2
Miscellaneous	25	30	1

5. Compute Index Number from the following data by using (a) Laspeyres' Method, (b) Paasche's Method and (c) Fisher's Ideal Method:

Commodity	Base Year		Current Year	
	Quantity	Price	Quantity	Price
A	12	10	15	12
B	15	7	20	5
C	24	5	20	9
D	5	16	5	14

6. Calculate Fisher's Quantity Index from the following data and show that it satisfies Time Reversal Test:

Commodity	Base Year		Current Year	
	Price (₹)	Value	Price (₹)	Value
P	10	200	12	300
Q	8	108	10	220
R	20	160	25	250
S	18	144	20	140
T	35	280	30	300

7. When the cost of tobacco was increased by 50%, a certain hardened smoker, who maintained his former scale of consumption, said that the rise had increased his cost of living by 5%. What percentage of his cost of living was due to buying tobacco before the change in price?

8. Monthly wages average in different years are as follows:

Year	1999	2000	2001	2002	2003	2004	2005
Wages (₹)	200	240	350	360	360	380	400
Price Index	100	150	200	220	230	250	250

Calculate real wages index numbers. How much increase in money wages is required to maintain real wages intact?

9. The cost of living index uses the following weights: Food–40, Rent–15, Clothing–20, Fuel–10, Miscellaneous–15.

During the period 1995–2005, the cost of living index number rose from 100 to 205.65. Over the same period the percentage rise in prices were: Rent–60, Clothing–180, Fuel–75, Miscellaneous–165. What was the percentage change in the price of food?

10. The cost of living index for the working class families for 2001 was 267.22. Using the data, find the weight of the fuel and lighting group:

Groups	Weights	Index Number
Food	46	352
Fuel and Lighting	?	220
Clothing	9	230
House Rent	13	160
Miscellaneous	22	190

11. The following table gives the cost of living index numbers for different commodity groups together with respective weights for 2005 (Base Year = 1982):

Group	Food	Clothing	Fuel and Light	Rent	Miscellaneous
Group Index	425	475	300	400	250
Group Weight	62	4	6	12	16

Obtain the overall cost of living index number. Suppose a person was earning ₹6000 in 1982, what should be his salary in 2005 if his standard of living in that year to be the same as in 1982?

[C.U. B.Com. 2008]

12. The total value of retained imports into India in 1990 was ₹71.5 million p.m. The corresponding total for 1997 was ₹87.6 million p.m. The index of volume of retained imports in 1997 compared with 1990 (= 100) was 62.0. Calculate the price index for retained imports for 1997 on 1990 as base year.

13. Shift the base period of the following series of index numbers from 1990 to 1997:

Year	1994	1995	1996	1997	1998	1999	2000
Index Numbers (Base year 1990 = 100)	125	130	142	150	160	168	180

Assuming that an index number is 100 in 1998, it rises 3% in 1999, falls 1% in 2000 and rises 2% in 2001 and 3% in 2002; rise and fall begin w.r.t. the previous year. Calculate the index for the five years using 2002 as the base year.

14. Splice the following two series of index numbers with (a) base 1995 and (b) base 2000:

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Series A	100	120	135	160	178	200	...	...	...	...	...
Series B	...	...	...	...	...	100	110	98	115	125	150

ANSWERS

[A]

1. (a) 117, 100, 127, 156, 179;

   (b) Composite Index Numbers: 88.33, 85.83, 100, 113.33, 99.67;

   (c) 150, 100, 120, 80.

2. 100, 88.89, 125, 120, 91.67, 136.36.
3. 100, 104.3, 104.1, 93.1, 103.2, 102.
4. 110, 176, 246.4, 492.8, 739.2.
5. (a) 122.3;

(b) 123.4.

6. 118.77.

7. (a) 110.95 or 110.0;

(b) 110.85 or 110.9.

8. 125.

9. (a) (i) 116.56, (ii) 116.5;

(b) $x = 4$;;

(c) 139.2.

10. 119.4.

11. (a) 125.23;

(b) 124.70;

(c) 124.96;

(d) 124.96.

12. 110.65.

13. 319.

14. 141.76.

15. 221.98; 98.52.

16. (a) 87.34.

(b) 126.9.

[B]

1. 144.7.

2. 96.

3. (a) 139.4;

(b) 106.8.

4. 114.9.

5. 124.2.

6. 124.4.

7. 250.

8. 261.36.

9. 91.527 or 91.5.

10. (a) 123.49 or 123.5;

(b) 120.65.

11. M.E. Index = 49.135 and F.I. Index = 49.134; hence, etc.

12. (a) 120; (b) 83.33.

13. 222.

14. 103, 100.94, 106, 118.72, 128.22.

[C]

1. (a) 231.19;

(b) 223.91.

2. (a) 127.8, 138.04;

(b) 134.55, 147.23.

3. 110, 105.1, 115.

4. (a) 81.13;

(b) 80; 80.56.

5. 120.65.

6. 112.88.

7. $P_{01} = 84.95$ and $Q_{01} = 60.79$.

8. (b) 222.04.

9. $w_1 = 7$, $w_2 = 8$.

10. 121.96.

11. 219.1.

[D]

1. 95.51.

2. (a) No. Real wage decreases by ₹45.45;

(b) ₹200;

(c) ₹4500.

4. 128.98.

5. (a) 118.8;

(b) 112.8;

(c) 115.7.

6. 103.88.

7. 10%.

8. 100, 80, 87.5, 81.8, 78.3, 76, 80 (base year 1999); 0, 60, 50, 80, 100, 120, 100 (in Rs).

9. 71.

10. 10.

11. 388.5; ₹23,310.

12. 197.6.

Chapter 11

Time Series Analysis

11.1 Introduction

In Economics, Business and Commerce, it is important to estimate for the future, e.g., an economist is interested in knowing the likely population in the coming year for his future planning or a businessman likes to estimate (or predict) his likely sales in the coming year to adjust his production accordingly. For making such estimates, one has to collect information from the past, i.e., one has to deal with statistical data collected and recorded at successive intervals of time (or points of time). Such statistical data relating to time are referred to as Time Series. Although Time Series usually refers to economic data, it also applies to data arising in natural and other Social Science. In this chapter, we shall discuss various methods for analyzing Time Series and procedures for resolving it into four important components.

11.2 Time Series

A Time Series is a set of observations taken at specified times, usually (but not always) at equal intervals. Thus, a set of data depending on the time (which may be year, quarter, month, week, day, etc.) is called a *Time Series.*

Examples of Time Series are: (i) The Annual Production of Steel in India over the last 10 years; (ii) The Monthly Sales of a Chemical Industry for the last 6 months; (iii) The daily closing price of a share in the Calcutta Stock Exchange; (iv) Hourly Temperature recorded by the Me. orological Office in a city; (v) Yearly Price or Quantity Index Numbers.

Mathematically, a Time Series is defined by the values $Y_1, Y_2, \ldots, Y_n$ of a variable Y at times $t_1, t_2, \ldots, t_n$. Here Y is a function of time t and Y_t denotes the value of the variable Y at time t.

Necessity of Time Series Analysis

Analysis of Time Series is of special importance to Businessmen, Economists, Scientists, Sociologists, Geologists, Research Workers in various disciplines, etc.

Analysis of Time Series helps us to understand the past behaviour of time series data (i.e., one can understand the changes that took place in the past). With the knowledge of the past behaviour, it would be possible, within certain limits, to forecast for the probable future variations (or movements) of such data. Thus, it helps in planning future operations.

With the help of Time Series Analysis, we can compare the actual performance with the expected performance and analyze the cause of variation. Different time series can be compared and therefrom important conclusions can be drawn.

Analysis of Time Series shows that the observed values of the variable are always fluctuating from time to time. The fluctuations are due to various factors (or forces) like increase of population, changes in habits and tastes of people, weather conditions, etc. On the action of these forces, the values of the variable are changing with time. The object of time series analysis is to isolate and ascertain these forces (i.e., the various components).

11.3 Components of Time Series

Fluctuations in a Time Series are mainly due to four basic types of variations (or movements). These four types of movements are called *the four components* or *elements of Time Series*. The four components are:

(a) *Secular Trend or Simply Trend* (T),

(b) *Seasonal Variation* (S),

(c) *Cyclical Variation or Cyclical Fluctuation* (C),

(d) *Irregular or Random Movement* (I).

The changes in Time Series data are the result of the combined effect of these four components.

In traditional or classical time series analysis, a multiplicative relationship between the four components is usually assumed, i.e., any particular observation is considered to be the product of the effects of four components. Symbolically,

$$Y = T \times S \times C \times I, \tag{1}$$

where Y = the result of the four components (or original data).

Instead of the multiplicative model (1), some statisticians may prefer an additive model:

$$Y = T + S + C + I,$$

in which Y is the sum of the four components.

In practice, the decision as to which model should be assumed depends on the degree of success achieved in applying the assumption.

1. Secular Trend or Simply Trend (T)

In Business, Economics and in our daily conversation, the term Secular Trend or simply Trend is popularly used. When we speak of rising trend of population or prices, we mean the gradual increase in population or prices over a period of time. Similarly, by declining trend of production or sales, we mean gradual decrease in production or sales over a period of time. The concept of Trend does not include short range oscillations, but refers to the steady movements over a long period of time.

Secular trend is the smooth, regular and long-term movement of a series showing continuous growth, stagnation or decline over a long period of time. Graphically, it exhibits general direction and shape of Time Series.

The long-term growth or upward trend in economic and business time series is due to the increase in population, advances in technology, etc. The decline or downward trend is because of decreasing demand of the product concerned or availability of a better substitute in its place or change in the Government's economic policy, etc. In our practical life, we may notice upward trends in a series concerning Population, National Income, Bank-deposits, etc., and downward trends in a series concerning Birth and Death rates.

2. Seasonal Variation (S)

Seasonal Variation is a short-term periodic movement which occurs more or less regularly within a specified period of one year or less. It recurs periodically year after year. Due to the presence of seasonal variation, business activities are found to have brisk and slack periods at different parts of the year. The major factors *that cause seasonal variations are climatic and weather conditions, customs and habits of people, religious festivals, etc. For example,* the demand for electric fans rises in summer and falls in winter. The sales and profits of departmental stores show a sudden rise before the Durga Puja, Diwali and Christmas. The prices of grain vary between the harvest and the non-harvest seasons.

Although the period of seasonal variations refers to a year in business and economics, it can also be taken as a month, week, day, hour, etc., depending on the type of the data available. *For example,* seasonal fluctuations can be observed in the sales of a departmental store during 12 months of a year, withdrawals (or deposits) in a bank during the days of a month, number of books issued by a library during the 7 days in a week, temperatures recorded during the 24 hours of a day, etc.

Seasonal variations give a clear idea about the relative position of each season and on this basis, it is possible to plan for the season.

3. Cyclical Variation or Cyclical Fluctuation (C)

Cyclical fluctuation is a long-term periodic movement which occurs over a long period of time — usually two or more years. It is oscillatory in nature, but it is not as regular as seasonal variation. One complete period which is more than a year is called a cycle. Cyclical fluctuations are found to exist in almost all business and economic time series where it is known as business cycle or trade cycle. The ups and downs (or rises and declines) in business recurring at intervals of times are the effects of cyclical variations. A business cycle showing the recurrence of the ups and downs movements of business activity consists of four phases: (1) *prosperity,* (2) *recession,* (3) *depression and* (4) *recovery. Each phase changes gradually into the next phase in the order given above until one business cycle is completed.*

The study of cyclical fluctuations is very useful in framing suitable policies for avoiding periods of booms and depressions in business activity as both are equally bad for an economy. Depression may cause a complete disaster and shatter the economy.

4. Random or Irregular Movement (I) [B.U. B.Com.(H) 2008]

Irregular or Random movements, as the name indicates, are such variations in business activity which are caused by factors of irregular (or erratic) nature. These are purely random and unpredictable. These include all movements not already covered in trend, seasonal variation and cyclical fluctuation. Irregular movements are caused by unforeseen events like floods, wars, earthquake, strikes, elections, etc. Random movements, also known as Residual Variations, do not recur in a definite pattern.

11.4 Some Adjustments of Time Series Data

Before analyzing time series data, it is necessary to make certain adjustments in the original data. Some important adjustments are due to: (a) Calendar Variations, (b) Population Changes, (c) Price Changes and (d) Others.

(a) **Calendar Variations:** We know that the number of days in different calendar months of a year varies from month to month (the number of days in February being the least). If holidays and weekends are taken into account, variations will be much higher. The adjustment for calendar variations is made by dividing each monthly total (or figure) by the number of days (sometimes by the number of working days) in the month and this gives the daily average of each month.

(b) **Population Changes:** Sometimes, it is necessary to adjust the data for population changes; say, when National income (or Production) is increasing but per capita income (or consumption) is decreasing due to changes in the size of population. In such cases, the original data (or figures) are divided by the appropriate population totals to reduce them to per capita value which is appropriate for comparison.

(c) **Price Changes**: Adjustments for price changes is necessary when a sale-value (product of price and quantity sold) series is given and we are interested in knowing quantity changes alone. In this case, the effect of price changes can be eliminated by dividing each item in the sale value series by an appropriate price index.

(d) **Others**: Adjustments are necessary when different units of measurement have been used to find out the real value of money.

11.5 Measurement of Trend

There are four methods for determining trend in time series: 1. *Freehand* (or *Graphic*) *Method*, 2. *Semi-Average Method*, 3. *Moving Average Method*, 4. *Method of Least Squares*.

1. Freehand Method (or Graphic Method)

Freehand (or Graphic) method is the simplest method for studying trend. In this method, the actual figures (given data) are first plotted as points on a graph paper showing the time series data Y_t along the vertical axis OY and time t along the horizontal axis OX. Then a straight line (i.e., a freehand smooth curve of first degree) is drawn to fit as closely as possible the plotted points (To draw the line, leave equal number of points on both sides of it at more or less equal distances). The line so obtained shows the direction of the trend and the vertical distance of this line from OX gives the trend value for each time period.

By this method a quick estimate of the trend can be obtained, but this depends too much on the judgement of the investigator. Different people will locate the line in different positions. This method should be used only when a quick approximate idea of the trend is required.

Example 1. *Fit a trend line to the following data by the freehand method*:

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales of a firm (in million ₹)	62	64	66	63.5	67	64.5	69	67

Solution: Required trend line by the freehand method is drawn in the following diagram:

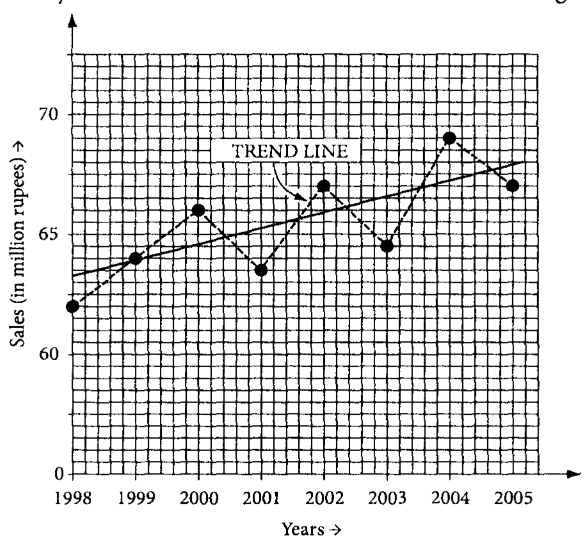


Fig. 11.1: Trend by the Freehand Method.

11.6 Semi-Average Method

In the Semi-Average Method, the given data is first divided into two parts (preferably equal) and an average (i.e., AM) for each part is found. Then these two averages are plotted on a graph paper as points against the midpoints of the time intervals covered by the respective two parts. These two points are joined by a straight line. This straight line is the required trend line and the distances of the line from the horizontal axis *OX* give the trend values.

Although this method is simple to apply, it may lead to poor results when used indiscriminately. It is applicable only where the trend is linear or approximately linear.

Example 2. *Draw a trend line by the Semi-Average Method using the following data:*

Year	2000	2001	2002	2003	2004	2005
Production of Steel (in lac tonnes)	253	260	255	263	259	264

Solution: The average production of Steel for the first three years

$$= \frac{253+260+255}{3} = \frac{768}{3} = 256 \text{ lac tonnes}$$

and the average production for the last three years

$$= \frac{263+259+264}{3} = \frac{786}{3} = 262 \text{ lac tonnes.}$$

Thus, we get two points 256 and 262 which are plotted against the respective middle years (midpoints) 2001 and 2004 of two parts 2000–02 and 2003–05. By joining these two points, the required trend line is obtained (See Fig. 11.2 given below).

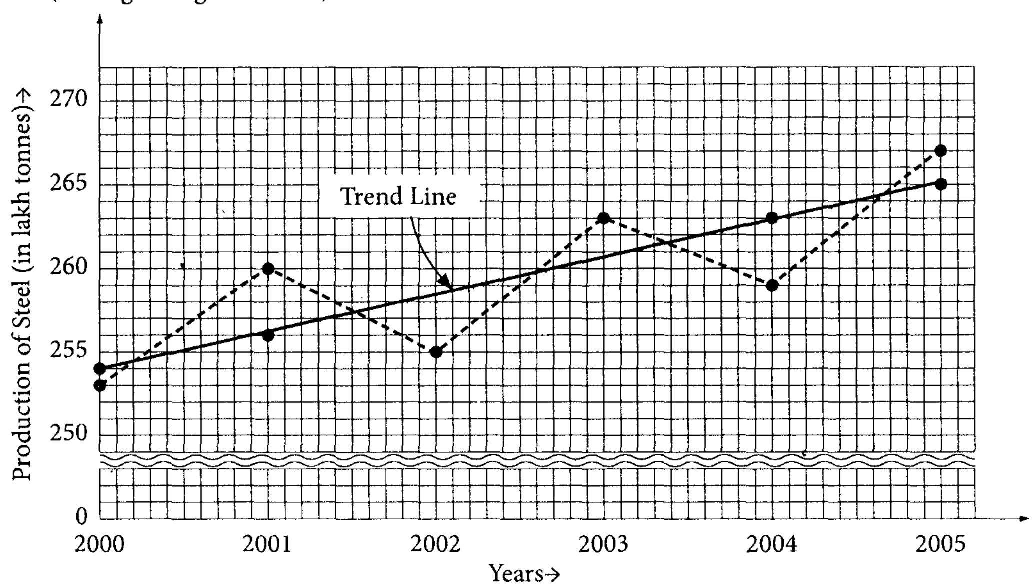


Fig. 11.2: Trend Line by the Semi-Average Method.

Note: In the above problem (Ex. 2), the number of years is six which is even. In case of odd number of years like 7, 9, 11, etc., the middle year is omitted for dividing the given data into two equal parts.

11.7 Moving Average Method

For a given set of numbers $Y_1, Y_2, Y_3, \ldots$, we define moving totals of order N by the sums

$$Y_1 + Y_2 + \cdots + Y_N, Y_2 + Y_3 + \cdots + Y_{N+1}, Y_3 + Y_4 + \cdots + Y_{N+2}, \ldots$$

and moving averages of order N by the sequence of arithmetic means

$$\frac{Y_1 + Y_2 + \cdots + Y_N}{N}, \frac{Y_2 + Y_3 + \cdots + Y_{N+1}}{N}, \frac{Y_3 + Y_4 + \cdots + Y_{N+2}}{N}, \ldots$$

In Moving Average Method, a series of moving averages of specific order is calculated. Starting from the beginning of the given series, an average for a specific number of years for yearly data or a time interval (called period) is calculated and this is placed against the midpoint of the time interval. Keeping the period fixed the process is repeated by dropping the first yearly figure of the given values and adding the figure of the next year we had not added before. We continue with this till the end of the series is reached.

If the period of moving average is odd, the moving totals and moving averages correspond to the given years or time. But if the period is even, a two-point moving average of the moving averages is to be found for centring them, i.e., for synchronizing the moving averages with the original data [See Ex. 2(ii) given below].

This method is commonly used for measuring trend. By using moving averages of appropriate orders, cyclical fluctuation, seasonal and irregular movements may be eliminated, leaving only the trend movement.

If the moving averages are strongly affected by extreme values, a weighted moving average with appropriate weights is sometime used.

Advantages

- This method is used to measure trend, seasonal, cyclical and irregular fluctuations.
- Moving average method is easy to apply as this method does not involve any difficult calculation.
- If an appropriate period is chosen (i.e., if the period of the moving average coincides with the period of cyclical fluctuations), then these fluctuations are automatically eliminated from the data by using this method.
- The choice of the period of moving average is made by observing the oscillatory movements in the data and not by the personal judgement of the Statistician.
- This method is quite flexible in the sense that when a few more observations are added to the given data, the trend values already obtained will not be affected, only some more trend values will be included in the series.

Limitations (or Disadvantages)

- Some trend values at the beginning and at the end of the series cannot be determined.
- It is not easy to determine the period of moving average when the oscillatory movement does not exhibit any regular periodic cycle.
- This method cannot be used to forecast future trend values as the moving averages do not obey any law.
- This method is used to find only linear trend. Non-linear trend values obtained by this method are biased and deviate from the actual trend values.
- This method may generate cycles or other movements which were not present in the original data.

Example 3. *Using three-yearly working averages, find the trend values and short-term fluctuations for the following series:*

Year	1	2	3	4	5	6	7
Value	2	4	5	7	8	10	13

[C.U. B.Com. 1999 Type]

Solution:

TABLE 11.1: CALCULATIONS OF THREE-YEARLY MOVING AVERAGES

Year	Value	Three-yearly moving total	Three-yearly moving averages (Trend Values)	Short-term fluctuations
1	2	...	...	...
2	4 →	11	3.67	$4-3.67=0.33$
3	5 →	16	5.33	$5-5.33=-0.33$
4	7 →	20	6.67	$7-6.67=0.33$
5	8 →	25	8.33	$8-8.33=-0.33$
6	10 →	31	10.33	$10-10.33=-0.33$
7	13 →	...	...	...

Hence, the trend values are **3.67, 5.33, 6.67, 8.33, 10.33.**

Example 4. (i) *Obtain the five-year moving averages for the following series of observations:*

Year	1997	1998	1999	2000	2001	2002	2003	2004
Annual Sales (₹'00000)	3.6	4.3	4.3	3.4	4.4	5.4	3.4	2.4

(ii) *Construct also the 4-year centred moving averages.*

Solution: (i)

TABLE 11.2: CALCULATIONS OF 5-YEAR MOVING AVERAGES

Year, (1)	Annual Sales (₹'00000) (2)	5-year moving total (3)	5-year moving average (₹'00000) (4)
1997	3.6	—	—
1998	4.3	—	—
1999	4.3	20.0	4.00
2000	3.4	21.8	4.36
2001	4.4	20.9	4.18
2002	5.4	19.0	3.80
2003	3.4	—	—
2004	2.4	—	—

Note: The first moving total 20.0 of column 3 is the sum of the first 5 values 3.6, 4.3, 4.3, 3.4, 4.4. The second moving total is 4.3 + 4.3 + 3.4 + 4.4 + 5.4 = 21.8, which can also be easily obtained by adding (5.4 - 3.6), i.e., 1.8 with the first moving total. Similarly, the 3rd moving total is 21.8 + (3.4 - 4.3) = 20.9 and so on.

The five-year moving averages (or trend values) for the years 1999–2002 are shown in column 4 (Note that the moving averages correspond to the given years). For the other years 1997, 1998 and 2003, 2004, moving averages cannot be determined.

(ii) First Method:

TABLE 11.3: CALCULATIONS OF 4-YEAR CENTRED MOVING AVERAGES

Year (1)	Annual Sales (₹'0000) (2)	4-year moving total (3)	2-point moving total of col. 3 (centred) (4)	4-year centred moving average (₹'0000) (Col. 4 ÷ 8)
1997	3.6	...	...	...
1998	4.3	...	...	...
		15.6		
1999	4.3		32.0	4.00
		16.4		
2000	3.4		33.9	4.24
		17.5		
2001	4.4		34.1	4.26
		16.6		
2002	5.4		32.2	4.03
		15.6		
2003	3.4	...	...	...
2004	2.4	...	...	...

In the above table 4-year moving totals are shown against the midpoints of the time intervals in col. 3. As the moving totals do not correspond to the given years, 2-point moving totals of col. 3 are found in col. 4 for centring them (i.e., for synchronizing them with the original data).

Second Method:

TABLE 11.4: CALCULATIONS FOR 4-YEAR CENTRED MOVING AVERAGES

Year (1)	Data (Annual sales in ₹'00000) (2)	4-year moving total (3)	4-year moving average (4)	2-year moving total of col. 4 (centred) (5)	4-year centred moving average (Col. 5 ÷ 2)
1997	3.6	...	...	...	...
1998	4.3	...	...	...	...
		15.6	3.9		
1999	4.3			8.0	4.0
		16.4	4.1		
2000	3.4			8.5	4.2
		17.5	4.4		
2001	4.4			8.6	4.3
		16.6	4.2		
2002	5.4			8.1	4.0
		15.6	3.9		
2003	3.4	...	...	...	...
2004	2.4	...	...	...	...

Example 5. *Construct 5-yearly moving averages of the number of students studying in a college shown below:*

Year	Number	Year	Number
1996	332	2001	405
1997	317	2002	410
1998	357	2003	427
1999	392	2004	405
2000	402	2005	431

Solution:

TABLE 11.5: CALCULATIONS OF 5-YEARLY MOVING AVERAGES

Year	No. of students	5-yearly moving total	5-yearly moving averages
1996	332	...	...
1997	317	...	...
1998	357	1800	360.0
1999	392	1873	374.6
2000	402	1966	393.2
2001	405	2036	407.2
2002	410	2049	409.8
2003	427	2078	415.6
2004	405	...	...
2005	431	...	...

Example 6. *Find the trend for the following series using a three-year weighted moving average with weights* 1, 2, 1:

Year	1	2	3	4	5	6	7
Values	2	4	5	7	8	10	13

Solution:

TABLE 11.6: CALCULATIONS OF 3-YEAR WEIGHTED MOVING AVERAGE

Year (1)	Values (2)	3-year weighted moving total (3)	3-year weighted moving average (Col. 3 ÷ 4) (4)
1	2	...	...
2	4	$2\times1+4\times2+5\times1=15$	3.75
3	5	$4\times1+5\times2+7\times1=21$	5.25
4	7	$5\times1+7\times2+8\times1=27$	6.75
5	8	$7\times1+8\times2+10\times1=33$	8.25
6	10	$8\times1+10\times2+13\times1=41$	10.25
7	13	...	...

Col. 4 = Col. 3 ÷ total weight, where total weight = 1 + 2 + 1 = 4.

Example 7. *For the following series of observations, verify that the* 4*-year centred moving average is equivalent to a* 5*-year weighted moving average with weights* 1, 2, 2, 2, 1 *respectively.*

Year	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Sales (₹'00000)	2	6	1	5	3	7	2	6	4	8	3

[C.U. B.Com. 2008]

Solution:

TABLE 11.7: CALCULATIONS OF 4-YEAR CENTRED MOVING AVERAGE

Year (1)	Annual Sales (₹'00000) (2)	4-year moving total (3)	2-year moving total of col. 3 (4)	4-year centred moving average (Col. 4 ÷ 8)
1994	2	...	...	...
1995	6	...	...	...
		14		
1996	1		29	3.625
		15		
1997	5		31	3.875
		16		
1998	3		33	4.125
		17		
1999	7		35	4.375
		18		
2000	2		37	4.625
		19		
2001	6		39	4.875
		20		
2002	4		41	5.125
		21		
2003	8	...	...	...
2004	3	...	...	...

TABLE 11.8: CALCULATIONS OF 5-YEAR WEIGHTED MOVING AVERAGE

Year (1)	Annual Sale (₹'00000) (2)	5-year weighted moving total (3)	5-year weighted moving average (Col. 3 ÷ total wt. 8)
1994	2	...	...
1995	6	...	...
1996	1	$2\times1+6\times2+1\times2+5\times2+3\times1=29$	3.625
1997	5	$6\times1+1\times2+5\times2+3\times2+7\times1=31$	3.875
1998	3	$1\times1+5\times2+3\times2+7\times2+2\times1=33$	4.125
1999	7	$5\times1+3\times2+7\times2+2\times2+6\times1=35$	4.375
2000	2	$3\times1+7\times2+2\times2+6\times2+4\times1=37$	4.625
2001	6	$7\times1+2\times2+6\times2+4\times2+8\times1=39$	4.875
2002	4	$2\times1+6\times2+4\times2+8\times2+3\times1=41$	5.125
2003	8	...	...
2004	3	...	...

From the last columns of the two tables 11.7 and 11.8, we see that the 4-year centred moving average is equivalent to a 5-year weighted moving average with weights 1, 2, 2, 2, 1 respectively.

Example 8. *Using 3-year moving average method determine the trend and short-term fluctuations for the following data:*

Year	1991	1992	1993	1994	1995	1996	1997
Values	21	34	45	28	40	57	73

[C.U. B.Com.(H) 1999]

Solution:

TABLE 11.9: CALCULATION OF 3-YEAR MOVING AVERAGES AND SHORT-TERM FLUCTUATIONS

Year (1)	Values (2)	3-year moving totals (3)	3-year moving averages (4)	Short-term fluctuations (5) = (2) – (4)
1991	21	...	...	...
1992	34	100	33.33	$34-33.33=0.67$
1993	45	107	35.67	$45-35.67=9.33$
1994	28	113	37.67	$28-37.67=-9.67$
1995	40	125	41.67	$40-41.67=-1.67$
1996	57	170	56.67	$57-56.67=0.33$
1997	73	...	...	...

11.8 Method of Least Squares

This method is widely used for the measurement of trend.

Linear Trend: Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$ be N pairs of observations, where Y_i represents time series and X_i represents time. Suppose the equation of the straight line to be fitted to the time series data by the Method of Least Squares is

$$Y = a + bX. \tag{1}$$

For a given value of X, say X_1, the corresponding value of Y obtained from (1) is $a+bX_1$. The difference $E_1 = Y_1 - (a+bX_1)$ or, $Y_1 - a - bX_1$, which may be positive, negative or zero, is called an error or residual.

Similarly, we obtain $E_2 = Y_2 - a - bX_2, \ldots, E_N = Y_N - a - bX_N$.

By the Principle of Least Squares, the line best fitted is obtained when the sum of the squares of the differences E_i between the observed values Y_i and the corresponding calculated values $a+bX_i$ is minimum, i.e., when $\sum_{i=1}^{N} E_i^2 = \sum_{i=1}^{N} (Y_i - a - bX_i)^2$ is minimum.

When $\sum_{i=1}^{N} E_i^2$ is minimum, we obtain the normal equations

$$\Sigma Y = Na + b\Sigma X \tag{2}$$

and

$$\Sigma XY = a\Sigma X + b\Sigma X^2. \tag{3}$$

Solving these two equations, a and b can be determined and substituting these values of a and b in (1), the required equation of the straight line trend is obtained. From this equation, we can compute the trend values.

If we take the midpoint in time as the origin, the negative values in the first half of the series balance out the positive values in the second half so that $\Sigma X = 0$. The normal equations (2) and (3) would reduce to $\Sigma Y = Na$ and $\Sigma XY = b\Sigma X^2$.

$$\therefore\ a = \frac{\Sigma Y}{N} \quad \text{and} \quad b = \frac{\Sigma XY}{\Sigma X^2}.$$

Example 9. *Determine the equation of a straight line which best fits the following data:*

Year	2001	2002	2003	2004	2005
Sales (in ₹ '00000)	35	56	79	80	40

Compute the trend values for all the years from 2001 *to* 2005.

Solution: Let the equation of the straight line best fitted with the origin at the middle year 2003 and unit of X as 1 year be

$$Y = a + bX. \tag{1}$$

By the Method of Least Squares, the values of a and b are given by

$$a = \Sigma Y/N \quad \text{and} \quad b = \Sigma XY/\Sigma X^2. \tag{2}$$

Here N = number of years = 5.

TABLE 11.10: CALCULATIONS FOR THE LINE OF BEST FIT

Year	Sales (₹'00000) Y	X	X^2	XY
2001	35	− 2	4	− 70
2002	56	− 1	1	− 56
⟶2003	79	0	0	0
2004	80	1	1	80
2005	40	2	4	80
Total	$290 = \Sigma Y$	0	$10 = \Sigma X^2$	$34 = \Sigma XY$

Using (2),

$$a = \frac{\Sigma Y}{N} = \frac{290}{5} = 58 \text{ and } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{34}{10} = 3.4.$$

From (1), the required equation of the best fitted straight line is $Y = 58 + 3.4X$.

Year	X	Trend Values ($Y = 58 + 3.4X$)
2001	−2	$58 + 3.4 \times (-2) = 51.2$
2002	−1	$58 + 3.4 \times (-1) = 54.6$
2003	0	$58 + 3.4 \times 0 = 58.0$
2004	1	$58 + 3.4 \times 1 = 61.4$
2005	2	$58 + 3.4 \times 2 = 64.8$

Note: Unless otherwise specified, we shall assume that the values of Y refer to mid-year values, i.e., as of July 1. Thus, in Ex. 9 in Section 11.8, $X = 0$ corresponds to July 1, 2003, $X = -1$ to July 1, 2002, $X = 1$ to July 1, 2004, etc.

Example 10. *Fit a suitable straight line to the following data by the method of least squares and estimate the percentage of insured people in* 1997:

Year	1989	1990	1991	1992	1993
Percentage of Insured People	11.3	13.0	9.7	10.6	10.7

[C.U. B.Com.(H) 2001]

Solution: Let the equation of the straight line to be fitted to the given data by the method of least squares with the origin at the middle year 1991 and unit of X as 1 year be

$$Y = a + bX. \tag{1}$$

By the method of least squares, the values of a and b are given by

$$a = \frac{\Sigma Y}{N} \text{ and } b = \frac{\Sigma XY}{\Sigma X^2}. \tag{2}$$

Here N = Number of years = 5.

TABLE 11.11: CALCULATIONS FOR FITTING THE LINE (1) TO THE GIVEN DATA

Year	Percentage of Insured People (Y)	X	X^2	XY
1989	11.3	−2	4	−22.6
1990	13.0	−1	1	−13.0
→1991	9.7	0	0	0
1992	10.6	1	1	10.6
1993	10.7	2	4	21.4
Total	$55.3 = \Sigma Y$	0	$10 = \Sigma X^2$	$-3.6 = \Sigma XY$

Using (2), we get

$$a = \frac{\Sigma Y}{N} = \frac{55.3}{5} = 11.06 \text{ and } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{-3.6}{10} = -0.36.$$

Hence, from (1), the required equation of the best fitted straight line to the given data is $Y = 11.06 - 0.36X$, with origin at 1991 and unit of $X = 1$ year.

In 1997, $X = 6$ and $Y = 11.06 - 0.36 \times 6 = 11.06 - 2.16 = 8.9$.

Hence, estimated percentage of people in 1997 = 8.9.

Example 11. *The following table gives the annual profits (in thousand ₹) in a factory:*

Year	1991	1992	1993	1994	1995	1996	1997
Profit ('000 ₹)	60	72	75	65	80	85	90

(i) *Fit a straight line trend by the method of least squares*; (ii) *Find the gradient of the fitted trend line*; (iii) *Calculate the projected profit for* 1998. [C.U. B.Com.(H) 1999]

Solution: Let the equation of the straight line to be fitted to the given data by the method of least squares with origin at the middle year 1994 and unit of x as 1 year be

$$y = a + bx. \tag{1}$$

Then the normal equations giving the values of a and b are

$$a = \frac{\Sigma y}{n} \text{ and } b = \frac{\Sigma xy}{\Sigma x^2}. \tag{2}$$

TABLE 11.12: CALCULATIONS FOR THE STRAIGHT LINE TREND

Year	Profit ('000 ₹) y	x	x^2	xy
1991	60	−3	9	−180
1992	72	−2	4	−144
1993	75	−1	1	−75
→1994	65	0	0	0
1995	80	1	1	80
1996	85	2	4	170
1997	90	3	9	270
Total	$527 = \Sigma y$	0	$28 = \Sigma x^2$	$520 - 399 = 121 = \Sigma xy$

∴ from (2),

$$a = \frac{527}{7} = 75.29 \text{ and } b = \frac{121}{28} = 4.32.$$

Hence, from (1), the required fitted straight line trend is $y = 75.29 + 4.32x$.

(ii) The gradient of the fitted trend line $= b = 4.32$.

(iii) In the year 1998, $x = 4$ and projected profit for 1998 is

$$y = 75.29 + 4.32 \times 4 = ₹92.57 \text{ thousand} = ₹92{,}570.$$

Example 12. *Fit a straight line trend equation by the method of least squares and estimate the trend values:*

Year	1991	1992	1993	1994	1995	1996	1997	1998
Values	80	90	92	83	94	99	92	104

Solution: Here N = number of years = 8, which is even.

Let the straight line trend equation by the method of least squares with the origin at the midpoint of 1994 and 1995 and unit of X as $\frac{1}{2}$ year be

$$Y = a + bX. \tag{1}$$

Then a and b are given by

$$a = \frac{\Sigma Y}{N} \text{ and } b = \frac{\Sigma XY}{\Sigma X^2}. \tag{2}$$

TABLE 11.13: CALCULATIONS FOR FITTING THE STRAIGHT LINE TREND

Year	Values Y	X	X^2	XY
1991	80	−7	49	−560
1992	90	−5	25	−450
1993	92	−3	9	−276
1994	83	−1	1	−83
⟶				
1995	94	1	1	94
1996	99	3	9	297
1997	92	5	25	460
1998	104	7	49	728
Total	$734 = \Sigma Y$	0	$168 = \Sigma X^2$	$210 = \Sigma XY$

Using (2),

$$a = \frac{\Sigma Y}{N} = \frac{734}{8} = 91.75 \text{ and } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{210}{168} = 1.25.$$

$\therefore$ from (1), *the required equation of the straight line trend is* $Y = 91.75 + 1.25X$.

TABLE 11.14: CALCULATIONS FOR TREND VALUES

Year	X	Trend Values ($Y = 91.75 + 1.25X$)
1991	−7	$91.75 + 1.25 \times (-7) = 83.0$
1992	−5	$91.75 + 1.25 \times (-5) = 85.5$
1993	−3	$91.75 + 1.25 \times (-3) = 88.0$
1994	−1	$91.75 + 1.25 \times (-1) = 90.5$
1995	1	$91.75 + 1.25 \times 1 = 93.0$
1996	3	$91.75 + 1.25 \times 3 = 95.5$
1997	5	$91.75 + 1.25 \times 5 = 98.0$
1998	7	$91.75 + 1.25 \times 7 = 100.5$

Note: If the number of years is even, there is no middle year and in this case the midpoint, which is taken as the origin, lies midway between the two middle years. In example 12 in **Section 11.8**, the midpoint (i.e., the origin) lies midway between July 1, 1994 and July 1, 1995, which is January 1, 1995 (or December 31, 1994). To avoid fractions, the units of X are taken as $\frac{1}{2}$ year (or 6 months).

Example 13. *The weights of a calf taken at weekly intervals are given below:*

Age in Weeks (x)	1	2	3	4	5	6	7	8	9	10
Weight (y) in kg Unit	52.5	58.7	65	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Fit a straight line using the method of least squares and calculate the average rate of growth per week.

[C.U. B.Com.(H) 2002]

Solution: Let $y = a + bx$ be the straight line to be fitted to the given data by the method of least squares with unit of x as 1 week without shifting the origin. Then the normal equations giving the values of a and b are

$$\Sigma y = na + b\Sigma x \tag{1}$$

and

$$\Sigma xy = a\Sigma x + b\Sigma x^2. \tag{2}$$

TABLE 11.15: CALCULATIONS FOR FITTING A STRAIGHT LINE TO THE GIVEN DATA

Age in weeks x	Wt. in kg y	x^2	xy
1	52.5	1	52.5
2	58.7	4	117.4
3	65.0	9	195.0
4	70.2	16	280.8
5	75.4	25	377.0
6	81.1	36	486.6
7	87.2	49	610.4
8	95.5	64	764.0
9	102.2	81	919.8
10	108.4	100	1084.0
$55 = \Sigma x$	$796.2 = \Sigma y$	$385 = \Sigma x^2$	$4887.5 = \Sigma xy$

Here $n = 10$. From (1) and (2), we get

$$10a + 55b = 796.2 \tag{3}$$

and

$$55a + 385b = 4887.5. \tag{4}$$

Now,

$$(4) \times 2 - (3) \times 11 \Rightarrow 165b = 1016.8 \Rightarrow b = 6.162 \text{ (approx.)}$$

Substituting $b = 6.162$ in (3), we get

$$10a + 55 \times 6.162 = 796.2,$$

$$\text{or,} \quad 10a = 796.2 - 338.91 = 457.29,$$

$$\text{or,} \quad a = 45.729 = 45.73 \text{ (approx.)}$$

The equation of the best fitted straight line is $\boldsymbol{y = 45.73 + 6.162x}$.

Slope of this line is **6.162** which is the average rate of growth per week.

11.8.1 Non-linear Trend: Second Degree or Quadratic Trend

Method of Fitting Parabolic Curves

Let

$$Y = a + bX + cX^2 \tag{1}$$

be the equation of the parabola to be fitted to the given time series data $(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$. Using Method of Least Squares, the normal equations for determining the constants a, b, c are

$$\begin{aligned} \Sigma Y &= Na + b\Sigma X + c\Sigma X^2, \\ \Sigma XY &= a\Sigma X + b\Sigma X^2 + c\Sigma X^3 \\ \text{and } \Sigma X^2 Y &= a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4. \end{aligned}$$

If the midpoint of time is taken as the origin, then $\Sigma X = 0$, $\Sigma X^3 = 0$, and the above normal equations reduce to

$$\begin{aligned} \Sigma Y &= Na + c\Sigma X^2, \\ \Sigma XY &= b\Sigma X^2 \\ \text{and } \Sigma X^2 Y &= a\Sigma X^2 + c\Sigma X^4. \end{aligned}$$

Solving these equations, we get the values of a, b and c, and substituting these values in (1), we obtain the trend equation.

Example 14. *Fit a parabolic curve of second degree to the data given below and estimate the value for 1999 and comment on it:*

Years	1993	1994	1995	1996	1997
Sales (in ₹ '00000)	10	12	13	10	8

Solution: Let

$$Y = a + bX + cX^2 \tag{1}$$

be the equation of the parabolic curve to be fitted to the given data. By the method of least squares, the normal equations referred to midpoint of time as the origin are $\Sigma Y = Na + c\Sigma X^2$, $\Sigma XY = b\Sigma X^2$ and $\Sigma X^2 Y = a\Sigma X^2 + c\Sigma X^4$.

Here $N = 5$.

TABLE 11.16: CALCULATIONS FOR FITTING A PARABOLIC CURVE

Years	Sales (in ₹'00000) Y	X	X^2	X^4	XY	X^2Y
1993	10	−2	4	16	−20	40
1994	12	−1	1	1	−12	12
→ 1995	13	0	0	0	0	0
1996	10	1	1	1	10	10
1997	8	2	4	16	16	32
Total	$53 = \Sigma Y$	0	$10 = \Sigma X^2$	$34 = \Sigma X^4$	$-6 = \Sigma XY$	$94 = \Sigma X^2 Y$

Using the normal equations, we get

$$\begin{aligned} 53 &= 5a + c \times 10 \qquad (2) \\ -6 &= b \times 10 \qquad (3) \\ 94 &= a \times 10 + c \times 34. \qquad (4) \end{aligned}$$

From (3), $b = -\frac{6}{10} = -0.6$.

Multiplying (2) by 2 and then subtracting from (4), we get

$$-12 = 14c \text{ or, } c = -\frac{6}{7} = -0.857.$$

From (2),

$$5a = 53 - 10c = 53 - 10 \times (-0.857) = 61.57; \ \therefore \ a = 12.314.$$

Hence, the required equation of the fitted parabolic curve is $Y = 12.314 - 0.6X - 0.857X^2$ with origin at the middle year 1995 and unit of X as 1 year.

In 1999, $X = 4$,

$$Y = 12.314 - 0.6 \times 4 - 0.857 \times 16 = 12.314 - 16.112 = -3.798.$$

Negative trend value for the year 1999 indicates that the model (i.e., fitting parabolic curve) is not suitable for the given data.

11.8.2 Exponential Trend

The equation of the exponential curve is

$$Y = ab^X, \qquad (1)$$

where X represents time, Y represents time series data and a, b are constants.

In the curve (1), the given Y values form a GP while the corresponding X values form an AP. Data from the fields of biology, banking and economics often exhibit exponential trend. (The growth of bacteria is exponential). In business, sales or earnings may grow exponentially over a short period.

Taking logarithms of both sides of (1), we get

$$\log Y = \log a + X \log b.$$

When plotted on a semi-logarithmic graph, the curve will be a straight line.

Taking $y = \log Y$, $A = \log a$ and $B = \log b$, the above equation reduces to

$$y = A + BX.$$

Using the method of least squares, the normal equations are

$$\begin{aligned} \Sigma y &= NA + B\Sigma X \\ \text{and } \Sigma Xy &= A\Sigma X + B\Sigma X^2. \end{aligned}$$

When deviations are taken from the middle year, these equations become

$$\Sigma y = NA \text{ and } \Sigma Xy = B\Sigma X^2.$$

$$\therefore \ A = \Sigma y / N \text{ and } B = \Sigma Xy / \Sigma X^2.$$

Example 15. *The sales of a company in thousand of rupee for the year* 1995 *through* 2001 *are given below:*

Year	1995	1996	1997	1998	1999	2000	2001
Sales	32	47	65	92	132	190	275

Estimate sales figures for the year 2002 *using an equation of the form* $Y = ab^X$, *where* X = *years and* Y = *sales.*

Solution: Taking origin at the middle year 1998, the normal equations for the exponential trend $Y = ab^X$ are

$$\log a = \Sigma y/N \text{ and } \log b = \Sigma Xy/\Sigma X^2; \text{ where } y = \log Y.$$

TABLE 11.17: CALCULATIONS FOR FITTING EXPONENTIAL TREND

Year	Sales (Y) (₹'000)	X	$y = \log Y$	X^2	Xy
1995	32	−3	1.5051	9	−4.5153
1996	47	−2	1.6721	4	−3.3442
1997	65	−1	1.8129	1	−1.8129
1998	92	0	1.9638	0	0
1999	132	1	2.1206	1	2.1206
2000	190	2	2.2788	4	4.5576
2001	275	3	2.4393	9	7.3179
Total	...	...	$13.7926 = \Sigma y$	$28 = \Sigma X^2$	$4.3237 = \Sigma Xy$

Substituting the values of Σy, ΣX^2, ΣXy in the normal equations, we get

$$\log a = \frac{13.7926}{7} = 1.9704 \text{ and } \log b = \frac{4.3237}{28} = 0.1544.$$

[a = antilog (1.9704) = 93.42 and b = antilog (0.1544) = 1.427]

From the given equation, we have

$$\begin{aligned} \log Y &= \log a + X \log b \\ &= 1.9704 + 0.1544X. \end{aligned} \tag{1}$$

For the year 2002, $X = 4$; therefore, from (1), we get

$$\log Y = 1.9704 + 0.1544 \times 4 = 2.5880.$$

$$\therefore\ Y = \text{antilog}\,(2.5880) = 387.3.$$

Hence, *the required sale figures for the year* 2002 *is* ₹3,87,300.

11.8.3 Computation of Monthly Trend from Annual Trend

Computation of Monthly Trend depends on the nature of the given annual data.

Case I. *Let annual data for odd number of years be given.*

If $Y = a + bX$ be the equation of the straight line trend fitted to the annual data with the origin at the middle year (on 30th June or 1st July) and unit of $X = 1$ year, then the equation of the straight line trend fitted to monthly averages is

$$Y = \frac{a}{12} + \frac{b}{12}X,$$

where a and b are divided by 12.

Now, slope $\frac{b}{12}$ of this trend line is the rate of change in monthly average w.r.t. 'x' for 12 months. So, monthly change will be $\frac{1}{12} \cdot \frac{b}{12}$, i.e., $\frac{b}{144}$ and hence the monthly trend equation will be

$$Y = \frac{a}{12} + \frac{b}{144}X,$$

with origin at 30th June or 1st July and unit of $X = 1$ month.

Case II. *Let annual data for even number of years be given.*

If $Y = a+bX$ be the equation of the straight line trend fitted to the annual data with origin at the midpoint of two middle most years (i.e., at 31st December or 1st January) and unit of X as $\frac{1}{2}$ year, then the equation of the straight line trend fitted to the monthly averages is

$$Y = \frac{a}{12} + \frac{b}{12}X.$$

Now, slope $\frac{b}{12}$ of this line is the rate of change in monthly average for 6 months. So monthly change will be $\frac{1}{6} \cdot \frac{b}{12}$, i.e., $\frac{b}{72}$ and hence the monthly trend equation will be

$$Y = \frac{a}{12} + \frac{b}{72}X,$$

with origin at 31st December or 1st January and unit of $X = \frac{1}{2}$ year.

Example 16. *Below are given the annual productions (in '000 tons) of a fertilizer factory:*

Year	1999	2000	2001	2002	2003	2004	2005
Production	70	75	90	91	95	98	100

(i) *Fit a straight line trend by the method of least squares and tabulate the trend values.* (ii) *Convert your annual trend equation into a monthly trend equation.*

Solution:

TABLE 11.18: CALCULATIONS FOR FITTING A STRAIGHT LINE TREND

Year	Production Y	X	X^2	XY	Trend Values ($Y = 88.43 + 5.04X$)
1999	70	−3	9	−210	$88.43 + 5.04 \times (-3) = 73.31$
2000	75	−2	4	−150	$88.43 + 5.04 \times (-2) = 78.35$
2001	90	−1	1	−90	$88.43 + 5.04 \times (-1) = 83.39$
→2002	91	0	0	0	$88.43 + 5.04 \times 0 = 88.43$
2003	95	1	1	95	$88.43 + 5.04 \times 1 = 93.47$
2004	98	2	4	196	$88.43 + 5.04 \times 2 \doteq 98.51$
2005	100	3	9	300	$88.43 + 5.04 \times 3 = 103.55$
Total	$619 = \Sigma Y$	—	$28 = \Sigma X^2$	$141 = \Sigma XY$	

(i) If $Y = a + bX$ be the straight line trend by the method of least squares with origin at 2002 and unit of $X = 1$ year, then the normal equations are

$$a = \frac{\Sigma Y}{N} = \frac{619}{7} = 88.43 \text{ and } b = \frac{\Sigma XY}{\Sigma X^2} = \frac{141}{28} = 5.04.$$

Hence, the required equation of the best fitted trend line is

$$Y = 88.43 + 5.04X.$$

(ii) Annual trend equation is $Y = 88.43 + 5.04X$.

Here number of years = 7, which is odd and unit of X is 1 year.

The required monthly trend equation is

$$Y = \frac{88.43}{12} + \frac{5.04}{144}X, \text{ i.e., } Y = 7.37 + 0.035X$$

with the origin at 30th June or 1st July of 2002 and unit of $X = 1$ month.

11.9 Measurement of Seasonal Variation [C.U. B.Com. 2008]

To measure seasonal variation, we first estimate how the time series data vary from quarter to quarter or month to month (or week to week, etc.) throughout a year. A series of numbers showing relative values of a variable during the quarters or months (or weeks, etc.) of the year is called *seasonal index* for the variable. Generally, Seasonal Index of a quarter or a month or a week (or etc.) is obtained by taking the average Seasonal Index of the whole year as 100.

For example, if sales of a firm during the four quarters of the year 1998 are 80, 110, 120, 90 per cent of the average quarterly sales for the whole year, then the numbers 80, 110, 120, 90 are called the *Seasonal Indices* for the four quarters of the year 1998. Clearly, the sum of the four quarterly Seasonal Indices is 400 and the average quarterly Seasonal Index of the whole year is 100. Similarly, for the monthly sales in a year, the sum of the 12 monthly Seasonal Indices is 1200 and the average Seasonal Index for the whole year is 100. If ₹x be the average quarterly (or monthly) sales in a year and I be the Seasonal Index of that quarter (or month), then sales for the quarter (or month) = I% of $x = \frac{I}{100} \times x$.

Seasonal indices are the measures of seasonal variation.

The following methods are popularly used for measuring seasonal variations: **1.** Method of Averages (Quarterly, Monthly or Weekly), **2.** Moving Average Method, **3.** Ratio to Trend Method, **4.** Link Relative Method.

11.9.1 Method of Averages (Quarterly, Monthly or Weekly)

This is the simplest method of computing seasonal variation and this method is used when trend and cyclical fluctuation, if any, have little effect on the time series.

If quarterly data are given, we first find quarterly totals for each quarter and the averages $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4$ for the 4 quarters of the years. To find these averages, we divide the quarterly totals by the number of years for which the data are given. Then we find an average (called *grand average*) of the 4 quarterly averages. Thus, grand average (G)

$$= (\bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \bar{X}_4) \div 4.$$

If we use additive model, then seasonal variations for the 4 quarters are $\bar{X}_1 - G, \bar{X}_2 - G, \bar{X}_3 - G, \bar{X}_4 - G$.

If the multiplicative model is used, then the seasonal indices are the 4 quarterly averages expressed as percentages of the grand average G, i.e., they are

$$\frac{\bar{X}_1}{G} \times 100, \ \frac{\bar{X}_2}{G} \times 100, \ \frac{\bar{X}_3}{G} \times 100, \ \frac{\bar{X}_4}{G} \times 100.$$

When monthly (or weekly) data are given, we find monthly (or weekly) averages for the 12 months (or 52 weeks) and proceed as above to obtain the seasonal index for each month (or week).

Example 17. *Compute the seasonal index for the following data:*

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
2000	75	60	54	59
2001	86	65	63	80
2002	90	72	66	85
2003	100	78	72	93

Solution: Assuming that trend and cyclical fluctuation are absent in the given data, the calculations are shown below using an additive model.

TABLE 11.19: CALCULATIONS FOR SEASONAL INDEX

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
2000	75	60	54	59
2001	86	65	63	80
2002	90	72	66	85
2003	100	78	72	93
Total	351	275	255	317
Average $\bar{X}_t$ [Total ÷4]	87.75	68.75	63.75	79.25

Grand Average

$$(G) = \frac{87.75 + 68.75 + 63.75 + 79.25}{4} = \frac{299.50}{4} = 74.875.$$

Using additive model, Seasonal Index = Average $(\bar{X}_i)$ – Grand Average (G).

∴ seasonal Indices for the first, second, third and fourth quarters are respectively

$$\bar{X}_1 - G, \quad \bar{X}_2 - G, \quad \bar{X}_3 - G, \quad \bar{X}_4 - G,$$

i.e., 87.75–74.875, 68.75–74.875, 63.75–74.875, 79.25–74.875,

or, **12.875**, **–6.125**, **–11.125**, 4.375.

Note: The sum of seasonal indices for the four quarters is 0.

Using multiplicative model, Seasonal Index = $\frac{\text{Average } (\bar{X}_i)}{\text{Grand Average } (G)} \times 100.$

∴ seasonal Indices for the 1st, 2nd, 3rd and 4th quarters are respectively

$$\frac{87.75}{74.875} \times 100, \ \frac{68.75}{74.875} \times 100, \ \frac{63.75}{74.875} \times 100, \ \frac{79.25}{74.875} \times 100,$$

i.e., **117.20, 91.82, 85.14, 105.84.**

The sum of seasonal indices is 400.

11.9.2 Moving Average Methods (Ratio to Moving Average Method)

This method is widely used for measuring seasonal variation. In this method, if quarterly data are given, we find 4-quarter centred moving averages (for monthly figures, 12-month centred moving averages are found) which represent trend and then eliminate the effect of trend by using additive or multiplicative model.

If the additive model is used, to eliminate trend we subtract the moving averages from the original data and obtain deviations from trend. Now applying the method of averages (quarterly or monthly as the case may be) to these deviations, the required seasonal variations will be obtained (See Ex. 11).

If the multiplicative model is used, then instead of obtaining deviations from trend, we express the original data as percentage of the corresponding moving averages (i.e., ratios to moving averages expressed as percentages). These percentages for corresponding quarters (or months) are then averaged by the method of averages, giving the required seasonal indices. This method is known as *Ratio to Moving Average Method* (See Ex. 12).

[Assuming multiplicative model $Y = T \times S \times C \times I$ and taking ratios to moving averages, trend (T) is eliminated from the original data leaving $\frac{Y}{T} = S \times C \times I$. When these ratios are averaged by the method of averages, cyclical fluctuation (C) and irregular movement (I) are removed giving the required Seasonal Index (S).]

Example 18. *Obtain seasonal fluctuation from the following time series using moving average method:*

Year	Quarterly output ('000 tons) of coal for 4 years			
	I	II	III	IV
2002	65	58	56	61
2003	58	63	63	67
2004	70	59	56	52
2005	60	55	51	58

Solution: Let us first find 4-quarter moving averages, deviations from trend and then seasonal fluctuation using the additive model.

TABLE 11.20: CALCULATIONS OF MOVING AVERAGES AND DEVIATIONS FROM TREND

Year/ Quarter (1)		Output (2)	4-quarter total (3)	2-point moving total of Col. 3 (4)	4-quarter moving average [Col. 4 ÷ 8] (5)	Deviations from trend [Col. 2–Col. 5] (6)
2002	I	65	...	...	...	...
	II	58	...	...	...	...
			240			
	III	56		473	59.13	– 3.13
			233			
	IV	61		471	58.88	2.12
			238			
2003	I	58		483	60.38	– 2.38
			245			
	II	63		496	62.00	1.00
			251			
	III	63		514	64.25	– 1.25
			263			
	IV	67		522	65.25	1.75
			259			
2004	I	70		511	63.88	6.12
			252			
	II	59		489	61.13	– 2.13
			237			
	III	56		464	58.00	– 2.00
			227			
	IV	52		450	56.25	– 4.25
			223			
2005	I	60		441	55.13	4.87
			218			
	II	55		442	55.25	– 0.25
			224			
	III	51	...	...	...	...
	IV	58	...	...	...	...

TABLE 11.21: CALCULATIONS FOR SEASONAL FLUCTUATIONS

Year \ Quarter	Deviations from Trend			
	I	II	III	IV
2002	...	...	− 3.13	2.12
2003	− 2.38	1.00	− 1.25	1.75
2004	6.12	− 2.13	− 2.00	− 4.25
2005	4.87	− 0.25	...	...
Total	8.61	− 1.38	− 6.38	− 0.38
Average ($\bar{X}_i$) [Total ÷ 3]	2.87	− 0.46	− 2.13	− 0.13

Grand Average

$$(G)=\frac{2.87-0.46-2.13-0.13}{4}=\frac{0.15}{4}=0.04.$$

The seasonal fluctuations $(\bar{X}_i-G)$ are $2.87-0.04, -0.46-0.04, -2.13-0.04, -013-0.04$, i.e., $\mathbf{2.83, -0.50, -2.17, -0.17}$.

Note: The sum of seasonal fluctuations is $0.01=0$ (approx.).

Example 19. *Calculate seasonal indices by the ratio to moving average method from the following data:*

Wheat Prices (in ₹ per Quintal)

Quarter \ Year	2002	2003	2004	2005
Q_1	75	86	90	100
Q_2	60	65	72	78
Q_3	54	63	66	72
Q_4	59	80	85	93

Solution:

TABLE 11.22: CALCULATIONS OF MOVING AVERAGE AND RATIO TO MOVING AVERAGE

Year Quarter (1)	Output (2)	4-quarter total (3)	2-point moving total of Col. 3 (4)	4-quarter moving average [Col. 4 ÷ 8] (5)	Deviations from trend [Col. 2 ÷ Col. 5] ×100 (6)
2002 Q_1	75	…	…	…	…
Q_2	60	…	…	…	…
		248			
Q_3	54		507	63.375	85.21
		259			
Q_4	59		523	65.375	90.25
		264			
2003 Q_1	86		537	67.125	128.12
		273			
Q_2	65		567	70.875	91.71
		294			
Q_3	63		592	74.000	85.14
		298			
Q_4	80		603	75.375	106.14
		305			
2004 Q_1	90		613	76.625	117.46
		308			
Q_2	72		621	77.625	92.75
		313			
Q_3	66		636	79.500	83.02
		323			
Q_4	85		652	81.500	104.29
		329			
2005 Q_1	100		664	83.000	120.48
		335			
Q_2	78		678	84.750	92.04
		343			
Q_3	72	…	…	…	…
Q_4	93	…	…	…	…

TABLE 11.23: CALCULATIONS FOR SEASONAL INDICES

Year \ Quarter	Ratio to Moving Average (in %)			
	Q_1	Q_2	Q_3	Q_4
2002	...	...	85.21	90.25
2003	128.12	91.71	85.14	106.14
2004	117.46	92.75	83.02	104.29
2005	120.48	92.04	...	...
Total	366.06	276.50	253.37	300.68
Average ($\bar{X}_i$)	122.02	92.17	84.46	100.23

Grand Average (G) is given by

$$G = \frac{122.02 + 92.17 + 84.46 + 100.23}{4} = \frac{398.88}{4} = 99.72.$$

Seasonal Index = (Average $\div$ Grand Average) $\times 100 = (\bar{X}_i \div G) \times 100$.

$\therefore$ seasonal Indices for the 4 quarters Q_1, Q_2, Q_3, Q_4 are respectively

$$\frac{122.02}{99.72} \times 100, \ \frac{92.17}{99.72} \times 100, \ \frac{84.46}{99.72} \times 100, \ \frac{100.23}{99.72} \times 100,$$

i.e., **122.36, 92.43, 84.70, 100.51.**

Note: The sum of the seasonal indices is 400.

11.9.3 Ratio to Trend Method

In this method trend values are first determined by the method of least squares fitting a mathematical curve (i.e., a straight line, parabola, etc.) and the given data are expressed as percentages of the corresponding trend values. Using the multiplicative model these percentages are then averaged by the method of averages described earlier.

Link Relative Method

This is the most difficult method of measuring seasonal variation. In this method, the given data for each quarter (or month) are expressed as percentages of data for the preceding quarter (or month). These percentages are called *link relatives*. The link relative for the 1st quarter (or 1st month) of the first year cannot be determined. An appropriate averages (Arithmetic Mean or Median) of the link relatives for each quarter (or month) is then found. From these average link relatives, we find the chain relatives with respect to 1st quarter (or 1st month January) for which the chain relative is taken as 100. If Q_1, Q_2, Q_3, Q_4 represent the 1st, 2nd, 3rd, 4th quarters respectively and CR represents chain relative, LR link relative, then

$$\text{CR for } Q_2 = (\text{Average LR for } Q_2 \times \text{CR for } Q_1) \div 100, \quad (1)$$

$$\text{CR for } Q_3 = (\text{Average LR for } Q_3 \times \text{CR for } Q_2) \div 100, \quad (2)$$

$$\text{CR for } Q_4 = (\text{Average LR for } Q_4 \times \text{CR for } Q_3) \div 100, \quad (3)$$

and $$\text{2nd CR for } Q_1 = (\text{Average LR for } Q_1 \times \text{CR for } Q_4) \div 100. \quad (4)$$

Usually, the 2nd CR for Q_1 will be either higher or lower than the 1st CR 100 for Q_1 depending on the presence of an increase or decrease in trend. We can now adjust the chain relatives for this trend.

If d = 2nd CR for Q_1 - 100, i.e., the difference between the 1st and 2nd CR for Q_1, then we subtract $\frac{1}{4}d$, $\frac{2}{4}d$, $\frac{3}{4}d$, $\frac{4}{4}d$ from the chain relatives for Q_2, Q_3, Q_4 and the 2nd CR for Q_1 respectively to obtain the adjusted chain relatives. These adjusted chain relatives expressed as percentages of their AM give the required seasonal indices (See Ex. 4).

Example 20. *Calculate Seasonal Indices by the method of link relatives from the data given in Ex.* 18. (*See data given for the four years* 2002–05.)

Solution:

TABLE 11.24: CALCULATIONS OF AVERAGE LINK RELATIVES

Quarter / Year	Link Relatives			
	Q_1	Q_2	Q_3	Q_4
2002	...	89.23	96.55	108.93
2003	111.48	92.65	100.00	106.35
2004	104.48	84.29	94.92	92.86
2005	115.38	91.67	92.73	113.73
Total	331.34	357.84	384.20	421.87
Average (AM)	110.45	89.46	96.05	105.47

The link relative (LR) for the 1st quarter Q_1 of the 1st year 2002 cannot be determined. For the other three quarters of 2002,

$$\text{LR for } Q_2 = (58 \div 65) \times 100,$$
$$\text{LR for } Q_3 = (56 \div 58) \times 100,$$
$$\text{LR for } Q_4 = (61 \div 56) \times 100.$$

Similarly, we determine the other link relatives.

From the average link relatives obtained in the last row of the above **Table 11.24**, we now find chain relatives, taking 100 as the chain relative (CR) for Q_1.

CR for Q_1 $= 100,$

CR for Q_2 $= (89.46 \times 100) \div 100 = 89.46,$ [using definition (1) in **Section 11.9.3**]

CR for Q_3 $= (96.05 \times 89.46) \div 100 = 85.93,$ [using definition (2)]

CR for Q_4 $= (105.47 \times 85.93) \div 100 = 90.63,$ [using definition (3)]

2nd CR for $Q_1 = (110.45 \times 90.63) \div 100 = 100.10.$ [using definition (4)]

$\therefore\ d = 100.10 - 100 = 0.10; \therefore\ \frac{1}{4}d = 0.025,\ \frac{2}{4}d = 0.050,\ \frac{3}{4}d = 0.075,\ \frac{4}{4}d = 0.10.$

The adjusted chain relatives are respectively 100, 89.46 – 0.025, 85.93 – 0.05, 90.63 – 0.075, i.e., 100, 89.435, 85.88, 90.555, or, 100, 89.44, 85.88, 90.56.

AM of the adjusted chain relatives $= (100 + 89.44 + 85.88 + 90.56) \div 4 = 91.47.$

The *required seasonal indices are*

$$\frac{100}{91.47}\times 100,\ \frac{89.44}{91.47}\times 100,\ \frac{85.88}{91.47}\times 100,\ \frac{90.56}{91.47}\times 100,$$

i.e., **109.33, 97.78, 93.89, 99.00.**

Note: The total of the seasonal indices is 400.

11.10 Deseasonalization of Data

Deseasonalization of data means elimination of seasonal variation from the original data.

If the additive model is used, we first find trend (T) by the moving average method and it is removed from the original data leaving deviations from trend

$$Y-T=S+C+I.$$

These deviations are then averaged to eliminate the cyclical fluctuation (C) and irregular movement (I). Thus, we obtain seasonal variations (S) which are suitably adjusted so that the total of the seasonal variations is zero. The adjusted seasonal variations are then subtracted from the corresponding data to get deseasonalized data.

If the multiplicative model is used, seasonal indices are first determined by the ratio to moving average method and then seasonal effects (= seasonal index ÷ 100) are found. The original data are now divided by the corresponding seasonal effects to give the deseasonalized data. Such data may include trend, cyclical and irregular movements.

In short, original data expressed as percentages of seasonal indices give the deseasonalized data.

Example 21. *A large company estimates its average monthly sales in a particular year to be* ₹2,00,000. *The seasonal indices* (S.I.) *of the sales data are as follows:*

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
S.I.	76	77	98	128	137	122	101	104	100	102	82	73

Using this information, draw up a monthly sales budget for the company (Assume that there is no trend).

Solution: If ₹x be the average monthly sales in a year and I, the seasonal index of a month, then estimated sales for the month = I% of x = ₹$x \times \frac{I}{100}$.

TABLE 11.25: CALCULATIONS OF MONTHLY SALES BUDGET

Month	Seasonal Index	Estimated Sales (₹)
January	76	$2,00,000 \times \frac{76}{100} = 1,52,000$
February	77	$2,00,000 \times \frac{77}{100} = 1,54,000$
March	98	$2,00,000 \times \frac{98}{100} = 1,96,000$
April	128	$2,00,000 \times \frac{128}{100} = 2,56,000$
May	137	$2,00,000 \times \frac{137}{100} = 2,74,000$
June	122	$2,00,000 \times \frac{122}{100} = 2,44,000$
July	101	$2,00,000 \times \frac{101}{100} = 2,02,000$
August	104	$2,00,000 \times \frac{104}{100} = 2,08,000$
September	100	$2,00,000 \times \frac{100}{100} = 2,00,000$
October	102	$2,00,000 \times \frac{102}{100} = 2,04,000$
November	82	$2,00,000 \times \frac{82}{100} = 1,64,000$
December	73	$2,00,000 \times \frac{73}{100} = 1,46,000$
Total	1200	24,00,000

Example 22. *Deseasonalize the data given in Ex.* 18 *by the method of moving average using the additive model.*

Solution: Seasonal variations are first determined as shown in Ex. 18 by the method of moving average. Seasonals are 2.83, −0.50, −2.17, −0.17 for the 4 quarters, I, II, III, IV respectively. To deseasonalize the given data, the seasonal variations are subtracted from the corresponding quarterly data for each year.

TABLE 11.26: CALCULATIONS OF DESEASONALIZED DATA			
Year/Quarter	Output ('000 tons) Y	Seasonal Variations S	Deseasonalized Data $Y-S$
2002 I	65	2.83	62.17
II	58	−0.50	58.50
III	56	−2.17	58.17
IV	61	−0.17	61.17
2003 I	58	2.83	55.17
II	63	−0.50	63.50
III	63	−2.17	65.17
IV	67	−0.17	67.17
2004 I	70	2.83	67.17
II	59	−0.50	59.5
III	56	−2.17	58.17
IV	52	−0.17	52.17
2005 I	60	2.83	57.17
II	55	−0.50	55.50
III	51	−2.17	53.17
IV	58	−0.17	58.17

Example 23. *Deseasonalize the following sales data using a multiplicative model:*

Quarter	Sales (₹'00000)	Seasonal Index
I	23.7	78
II	25.2	124
III	21.4	50
IV	65.4	148

Solution: Deseasonalized Data $= \dfrac{\text{Original Data}}{\text{Seasonal Effect}}$ or $\dfrac{\text{Original Data}}{\text{Seasonal Index}} \times 100,$

where seasonal effect = (Seasonal Index) ÷ 100.

TABLE 11.27: COMPUTATIONS OF DESEASONALIZED DATA				
Quarter	Sale (₹'00000)	Seasonal Index	Seasonal Effect	Deseasonalized Data
I	23.7	78	0.78	$\frac{23.7}{0.78}=30.38$
II	25.2	124	1.24	$\frac{25.2}{1.24}=20.32$
III	21.4	50	0.50	$\frac{21.4}{0.50}=42.80$
IV	65.4	148	1.48	$\frac{65.4}{1.48}=44.19$
Total		400	4.00	

Note: If the original data were not affected by seasonal variation, then sales in the four quarters I, II, III, IV would have been 30.38, 20.32, 42.80, 44.19 lac rupee respectively.

11.11 Measurement of Cyclical Fluctuation

To measure cyclical fluctuation, we first find trend (T) and seasonal variation (S) by any suitable method, say, by the method of moving average, and then eliminate them from the original data by using additive or multiplicative model. We are now left with only cyclical fluctuation (C) and irregular movement (I). Irregular movement (I) is then removed by using moving average of appropriate period depending on the average duration of I leaving only the cyclical fluctuation.

11.11.1 Estimation of Irregular Movement

Irregular Movement can be obtained by eliminating trend (T), seasonal variation (S) and cyclical fluctuation (C) from the original data. In practice, irregular movements are found to be of small magnitude.

11.11.2 Business Forecasting

Successful business activity demands a reasonably accurate forecasting of future business conditions upon which decisions regarding production, inventories, price-fixation, etc., depend. To eliminate guesswork, modern statistical methods are employed as a very useful tool of forecasting. The methods essentially consist of a knowledge of the past and present conditions and a proper analysis of time series data, isolating them into the four standard components, viz., Trend, Seasonal, Cyclical and Irregular movements which we have discussed in this chapter. Another useful aid in business-prediction is the construction of Index Numbers (discussed in detail in Ch. 10) associated with various business activities.

11.11.3 Business Barometers

A useful aid in practical forecasting is the construction of Index Numbers associated with different business activities, e.g., Wholesale prices, Consumer prices, Stock prices, Industrial production, etc. These terms are loosely referred to as Barometers in Business Statistics. Sometimes, the term Barometer is used to mean an indicator of the present economic situations or to designate an indicator of future conditions. Index numbers of different business activities may be combined to yield a Composite Business Activity Index. However, the study of general business conditions as revealed by this Index should be supplemented by special studies of individual businesses based on their respective Index Numbers.

EXERCISES ON CHAPTER 10

Theory

1. What do you mean by time series? Mention the chief components of time series.

   [C.U. B.Com.(H) 2003;B.U. B.Com.(H) 2008; V.U. B.Com.(H) 2010, 11]

2. (a) Define trend. Enumerate the methods of determining trend in time series.

   (b) Explain what is meant by Secular Trend in time series analysis. Briefly mention the important types of forces which influence an economic time series.

   (c) Explain the object and utility of Time Series Analysis. [C.U. B.Com.(H) 2000]

3. (a) What is the time series? What is the need to analyze a time series? Enumerate the different methods of finding the secular trend. [V.U. B.Com.(H) 2010]

   (b) Explain clearly what is meant by Time Series Analysis.

(c) Enumerate the factors that explain the utility of analyzing time series.

4. (a) What is a 'moving average'? What are its uses in Time Series? [C.U. B.Com. 2003]

   (b) What are the merits and limitations of moving average technique for computing the trend values? [C.U. B.Com.(H) 2001]

5. What are the components of Time Series? With which component of a time series would you mainly associate each of the following:

   (a) A fire in a factory delaying production by four weeks;

   (b) An after-Puja sale in a department store;

   (c) The increased food production due to a constant increase in population;

   (d) A recession;

   (e) General increase in the sale of TV sets?

   [*Ans.* (a) Irregular Movement; (b) Seasonal Variation; (c) Secular Trend; (d) Cyclical Fluctuation; (e) Secular Trend.]

6. (a) What do you mean by Seasonal Variation? Explain with a few examples the utility of such a study. [C.U. B.Com. 2008]

   (b) State the various methods of finding seasonal variation. [V.U. B.Com.(H) 2008]

   (c) What is a seasonal index? Briefly explain the various methods of constructing such an index.

7. Give an account of the moving average method of trend determination. How does one choose the length of the moving average?

8. Briefly describe Link Relative Method for finding the seasonal variations.

9. (a) What are the objectives of analyzing a time series?

   (b) Describe briefly the different components of a time series. [C.U. B.Com. 1997]

10. Define Time Series and describe what is meant by analysis of time series. Briefly state the causes that bring about changes and the utility of analysis of time series.

11. What do you mean by Seasonal Fluctuations in time series? Give an example. What are the major uses of Seasonal Indices in time series analysis?

12. Explain the meaning of deseasonalizing data. What purpose does it serve?

13. Define a Time Series and state its main components. What are the different methods of determining the trend component of a time series?

14. Distinguish between Seasonal, Cyclical and Random fluctuations. Describe any method of eliminating their influence.

15. (a) Give reasons for selecting a particular model for analyzing time series data. Explain how seasonal indices will be determined for a time series data.

    (b) Explain briefly, how the seasonal element in a Time Series data is isolated and eliminated.

16. Explain the nature of cyclical variations in a time series. How do seasonal variations differ from them? Give an outline of the moving average method of measuring seasonal variations.

17. Discuss some of the adjustments for population changes, calendar variation and price changes, which are necessary to make the time series data homogeneous and comparable.

18. Describe the ratio to moving average and the ratio to trend methods of estimating seasonal indices. Compare the two methods.

19. Write short notes on:

    (a) Moving Average;
    (b) Trend;
    (c) Seasonal Variations; [C.U. B.Com. 2004]
    (d) Seasonal Indices;
    (e) Business Forecasting;
    (f) Business Barometers;
    (g) Components of a Time Series;
    (h) Usefulness of Seasonal Index. [C.U. B.Com.(H) 1991]

Problems

A

1. Fit a trend line to the following data by the free-hand method:

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales of a Firm (in million ₹)	63	65	67	64	68	65	70	68

2. (a) Draw a trend line by the semi-average method using the following data:

Year	2000	2001	2002	2003	2004	2005
Production of Steel (in '000000 tonnes)	350	360	355	365	358	363

   (b) Plot the following data on a graph paper and ascertain trend by the method of semi-averages:

Year	1999	2000	2001	2002	2003	2004	2005
Production (in million tonnes)	100	120	95	105	108	102	112

3. Given the numbers 1, 0, – 1, 0, 1, 0, – 1, 0, 1; find the moving average of order four.

4. (a) Obtain the 5-year moving averages for the following data:

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005
Annual Sales (in crore ₹)	36	43	43	34	44	54	34	24	14

   Construct also the 4-year centred moving averages.

   (b) Construct 5-yearly moving averages of the number of students studying in a college shown below:

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Number	332	317	357	392	402	405	410	427	405	431

5. (a) Given the numbers 2, 6, 1, 5, 3, 7, 2. Obtain moving averages of order 3.

(b) Find the trend for the following series using a three-year moving average:

Year	1	2	3	4	5	6	7
Values	2	4	5	7	8	10	13

[B.U. B.Com.(H) 2008]

(c) Using 3-year moving average method determine the trend and short-term fluctuations for the following data:

Year	1991	1992	1993	1994	1995	1996	1997
Values	21	34	45	28	40	57	73

[C.U. B.Com.(H) 1999]

(d) Find the 3-year weighted moving average with weights 1, 4, 1 for the following series:

Year	1	2	3	4	5	6	7
Values	2	6	1	5	3	7	2

6. (a) The table below gives the figures of production of a commodity during the years 2001–05 in the State of Punjab:

Year (X)	2001	2002	2003	2004	2005
Production (Y) (in '000 tons)	10	12	8	10	14

Use the method of least squares to fit a straight line to the data. From this result find the trend values for different years (Take 2003 as origin of X-series).

(b) Fit a least square trend line to the following data:

Year	1994	1995	1996	1997	1998	1999
Volume of Sale (in suitable units)	12	15	17	22	24	30

Estimate the volume of sale for 2000. [C.U. B.Com.(H) 2000; V.U. B.Com.(H) 2008]

(c) Fit a suitable straight line to the following data by the method of least squares and estimate the percentage of insured people in 1997:

Year	1989	1990	1991	1992	1993
Percentage of Insured People	11.3	13.0	9.7	10.6	10.7

[C.U. B.Com.(H) 2001]

[**Hints:** See worked-out Ex. 10.
Here $n = 5$; $a = \frac{\Sigma Y}{n} = \frac{55.3}{5} = 11.06$, $b = \frac{\Sigma XY}{\Sigma X^2} = -\frac{3.6}{10} = -0.36$.]

(d) Fit a linear trend equation by the method of least squares from the following data and estimate the trend value for the year 2004:

Year	1997	1998	1999	2000	2001	2002
Price (₹)	250	207	228	240	281	392

[C.U. B.Com.(H) 2004]

7. (a) Find the values of the trend ordinates by the method of least squares from the data given below:

Year	1991	1992	1993	1994	1995	1996	1997
Sales (₹ '000)	125	128	133	135	140	141	143

[C.U. B.Com.(H) 1999; V.U. B.Com.(H) 2008]

[**Hints**: Fit the line $Y = a + bX$ to the given data, taking origin at the middle year 1994 and unit of X as 1 year.]

(b) Fit a straight line trend by the method of least squares (taking 1995 as the year of origin) to the following data concerning sales of a certain firm:

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999
Sales ('000 ₹)	48	55	63	65	72	84	90	87	82

[**Hints**: Fit the line $Y = a + bX$ to the given data, taking origin at the middle year 1995 and unit of X as 1 year.]

(c) Fit a straight line trend equation by the method of least squares from the following data and estimate the profit for the year 2005:

Year	1975	1980	1985	1990	1995	2000
Profit (₹ lac)	10	13	15	20	22	28

[C.U. B.Com. 2005]

8. Calculate the seasonal indices in the cases of the following quarterly data in certain units using the method of averages:

(a) Total Production of Paper ('000 tons)

Year	Quarters			
	I	II	III	IV
2003	37	38	37	40
2004	41	34	25	31
2005	35	37	35	41

(b)

Year	Quarters			
	Q_1	Q_2	Q_3	Q_4
2002	39	21	52	81
2003	45	23	63	76
2004	44	26	69	75
2005	53	23	64	84

[**Hints**: S.I. = (average for each quarter × 100) ÷ grand average)]

9. Calculate trend values from the following data relating to the production of tea in India by the moving average method, on the assumption of a four-yearly cycle:

Year	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Production (million lb)	464	515	518	467	502	540	557	571	586	612

10. Fit a second degree parabola ($Y = a + bX + cX^2$) to the following data:

Year	2001	2002	2003	2004	2005
Purchases [in crore (₹)]	1	5	10	22	38

11. Find the seasonal indices by the method of moving averages from the following series of observations:

Quarter	Sales of Woollen Yarn ('00000 ₹)			
	2002	2003	2004	2005
I	97	100	106	110
II	88	93	96	101
III	76	79	83	88
IV	94	98	103	106

12. Find the trend by the method of moving averages from the following table:

Total Production of cement (1,00,000 tons)				
Year	Quarter			
	I	II	III	IV
1990	34	32	31	36
1991	37	34	33	41
1992	43	40	38	48

[C.U. B.Com. 1994]

B

1. Fit a trend line to the following data by the Least Squares Method:

Year	1995	1997	1999	2001	2003
Production (in '000 tonnes)	18	21	23	27	16

Estimate the productions in 2000 and 2005.

[**Hints**: Take 1999 as the origin and unit of X as 2 years.]

2. Fit a straight line trend to the data and estimate the profit for the year 1987:

Year	1980	1981	1982	1983	1984	1985	1986
Profits (₹ lac)	60	72	75	65	80	85	95

3. Obtain the straight line trend equation and tabulate against each year after estimation of the trend and short-term fluctuation:

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005
Value	380	400	650	720	690	620	670	950	1040

4. Fit a straight line trend equation by the method of least squares and estimate the value for 2005.

Year	1996	1997	1998	1999	2000	2001	2002	2003
Value	380	400	650	720	690	600	870	930

[C.U. B.Com. 2008 Type]

5. The following series of observations are known to have a business cycle with a period of 4 years. Find the trend values by the moving average method:

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Production ('000 tonnes)	506	620	1036	673	588	696	1116	738	663	773	1189

6. Calculate the Seasonal Index from the following data using the average method:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
2001	72	68	80	70
2002	76	70	82	74
2003	74	66	84	80
2004	76	74	84	78
2005	78	74	86	82

7. Using 4-quarterly moving average in respect of the following data, find: (a) the trend, (b) short-term fluctuations, (c) seasonal variations:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
2001	35	86	67	124
2002	38	109	91	176
2003	47	158	104	226
2004	61	177	134	240
2005	72	206	141	307

8. Obtain the seasonal indices by the ratio to moving average method from the following data:

Year	Quarters			
	I	II	III	IV
2002	101	93	79	98
2003	106	96	83	103
2004	110	101	88	106

9. Construct indices of seasonal variations from the following time series data on consumption of cold drinks which contain only seasonal and irregular variations, using multiplicative model:

Year	Quarters			
	I	II	III	IV
2002	90	75	87	70
2003	75	80	78	75
2004	80	75	75	72
2005	85	82	80	81

(in '000 bottles).

10. Fit a straight line trend by the method of least squares to the following data and find by which year the production will reach 63 million tons:

Year	1998	1999	2000	2001	2002	2003	2004	2005
Production (million tons)	50.3	52.7	49.3	57.3	56.8	60.7	62.1	58.6

11. Obtain seasonal indices for the following data:

Year / Season	Output in Thousand Units				
	2001	2002	2003	2004	2005
Summer	31	42	49	47	51
Rain	39	44	53	51	54
Winter	45	57	65	62	66

12. Bank debits giving a good measure of volume of transactions are given below for 2002–05. Find the seasonal indices by the method of moving averages using additive model:

Year	Bank Debits in million ₹			
	1st	2nd	3rd	4th
2002	16.00	13.50	14.70	17.00
2003	15.90	12.20	15.60	18.10
2004	16.30	11.90	16.90	19.20
2005	17.10	13.20	15.00	18.70

13. Agricultural outputs, in million tonnes, for 5 years are given below:

Years	1999	2000	2001	2002	2003
Outputs	80	85	87	93	100

Obtain a least square linear estimate of the output for the year 2005.

14. Fit a linear trend equation to the following data. Hence, estimate the value of sales for the year 2005:

Years	1999	2000	2001	2002	2003
Sales (in lac of ₹)	100	120	140	160	180

15. (a) Fit an equation of the form $Y = a + bX + cX^2$ to the data given below:

X	1	2	3	4	5
Y	25	28	33	39	46

(b) Fit a quadratic trend to the following series of production data:

Years	1999	2000	2001	2002	2003	2004	2005
Production (Y)	37	38	37	40	41	45	50

Y-values being the average production in thousand tons.

16. (a) Fit an exponential trend of the form $Y = ab^X$ to the data giving the population figures below:

Census Year	1945	1955	1965	1975	1985	1995	2005
Population (in crore)	25.0	25.1	27.9	31.9	36.1	43.9	54.7

Estimate the population figures for the year 2015 using the fitted exponential trend equation.

(b) The following table gives the profits of a concern for 5 years ending 2005:

Years	2001	2002	2003	2004	2005
Profit (in ₹ thousand)	1.6	4.5	13.6	40.2	125.0

Fit an equation of the type $Y = ab^X$.

C

1. Fit by the method of least squares a linear equation $y = mx + c$ to the following data:

Years	1998	1999	2000	2001	2002	2003	2004	2005
Production ('00000 tons)	9.7	10.1	10.2	10.7	11.9	12.9	12.4	14.8

Obtain the trend values for the years 2002 and 2003.

2. Construct a 4-year centred moving averages from the following data:

Year	1945	1955	1965	1975	1985	1995	2005
Imported Cotton consumption in India (in '000 bales)	129	131	106	91	95	84	93

3. Calculate the trend values by the method of moving averages, assuming a four-yearly cycle, from the following data relating to sugar production in India:

Year	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Sugar Production (lac tonnes)	37.4	31.1	38.7	39.5	47.9	42.6	48.4	64.6	58.4	38.6	51.4	84.4

4. Fit a parabolic curve of second degree $y = a + bx + cx^2$ to the data given below by the method of least squares:

Year (x)	2001	2002	2003	2004	2005
Import (y) (in '000 bales)	10	12	13	10	8

[Take 2003 as origin and unit of x as 1 year.]

5. Deseasonalize the following data with the help of the seasonal data given against:

Month	Jan.	Feb.	Mar.	Apr.	May	June
Cash Balance ('00000 ₹)	360	400	350	360	360	550
Seasonal Index	120	80	110	90	70	100

[Hints: Deseasonalized value = $\frac{Y}{S} \times 100$.]

6. Obtain the seasonal indices by the method of moving averages (using additive model) from the following data:

Quarter	Quarterly Output ('00000 tons)			
	2002	2003	2004	2005
I	31	42	49	47
II	39	44	53	51
III	45	57	65	62
IV	36	45	55	50

7. Using additive model, estimate the seasonal indices by the method of moving averages from the table given below. Deseasonalize the given production figures with the help of the seasonal indices obtained, and explain the significance of the deseasonalized data:

Year	Total Production ('00000 tons) Quarters			
	I	II	III	IV
2003	37	38	37	40
2004	41	34	25	31
2005	35	37	35	41

8. Deseasonalize the following production data by the method of moving averages:

Quarters	Quarterly Output ('000 tons)			
	2002	2003	2004	2005
I	30	49	35	75
II	49	50	62	79
III	50	61	60	65
IV	35	20	25	70

9. A company estimates its average monthly sales in a particular year to be ₹2,00,000. The seasonal indices (S.I.) of the sales data are as follows:

Month	Jan.	Feb.	Mar.	April	May	June
S.I.	76	77	98	128	137	122

Assuming that there is no trend, draw up a monthly sales budget for the months January to June.

10. A large manufacturing company estimates its average monthly sales in a particular year to be ₹20,00,000. The seasonal indices of the sales data are given below:

Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
S.I.	78	75	100	126	138	121	101	104	99	103	80	75

Ignoring the possible existence of a trend, use the above information to draw up a monthly sales budget for the company. [V.U. B.Com.(H) 2011]

[Hints: See worked-out Ex. 21 in Art. 11.10.]

11. The following data gives the value of sales of a company for the years 1995–2005.

Year (X)	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Sales (Y) (₹'000)	50.0	36.5	43.0	44.5	38.9	38.1	32.6	38.7	41.7	41.1	33.8

Use the method of least squares to fit a straight line trend to the data given above. Compute the trend values for 1998 and 2003 (Take $X = 0$ for the year 1999 and the unit of X as 1 year).

12. Calculate the quarterly seasonal indices in respect of the following data by using the simple average method:

Year	Quarter I	Quarter II	Quarter III	Quarter IV
2001	71	68	79	71
2002	76	69	82	74
2003	74	66	84	80
2004	76	73	84	78
2005	78	74	86	82

13. Use the method of monthly averages to find the monthly indices for the following data of production of a commodity for the years 2003, 2004, 2005:

PRODUCTION IN LACS OF TONNES			
Months	2003	2004	2005
January	12	15	16
February	11	14	15
March	10	13	14
April	14	16	16
May	15	16	15
June	15	15	17
July	16	17	16
August	13	12	13
September	11	13	10
October	10	12	10
November	12	13	11
December	15	14	15

[**Hints**: First find the total for each of the 12 months and divide each of these totals by 3 (no. of years) to find the monthly averages $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_{12}$ for 12 months. Then find the grand average G of these 12 monthly averages. The required monthly indices are $\bar{X}_1 - G, \bar{X}_2 - G, \ldots, \bar{X}_{12} - G$.]

14. Calculate the seasonal indices by the method of link relatives from the following data:

Quarter	Year			
	2002	2003	2004	2005
I	75	86	90	100
II	60	65	72	78
III	54	63	66	72
IV	59	80	82	93

[**Hints**: See worked-out Ex. 20.]

ANSWERS

A

3. 0, 0, 0, 0, 0.

4. (a) –, –, 40.0, 43.6, 41.8, 38.0, 34.0, –, –;
–, –, 40.0, 42.38, 42.63, 40.25, 35.25; –, –,

(b) 360, 374.6, 393.2, 407.2, 409.8, 415.6 for 1993–98.

5. (a) 3, 4, 3, 5, 4;

(b) –, 3.67, 5.33, 6.67, 8.33, 10.33, –;

(c) –, 33.33, 35.67, 37.67, 41.67, 56.67, –;
–, 0.67, 9.33, −9.67, −1.67, 0.33, –;

(d) 4.5, 3.5, 4.0, 4.0, 5.5.

6. (a) $Y = 10.8 + 0.6X$; 9.6, 10.2, 10.8, 11.4, 12.0;

(b) $Y = 20 + 1.74X$; 32.2 units;

(c) $y = 11.06 - 0.4x$; 8.18;

(d) $y = 266.33 + 13.49x$; ₹387.74.

7. (a) $Y = 135+3.107X$, 125.679, 128.786, 131.893, 135, 138.107, 141.214, 144.321,

(b) $Y = 71.778 + 5.0833X$; ₹97.1943 thousand;

(c) $y = 18 + 1.743x$; unit of $x = 2.5$ years; ₹30.20 lac.

8. (a) 1.75, 0.42, −3.59, 1.42;

(b) 86.4, 44.4, 118.4, 150.8.

9. 495.8, 503.6, 511.6, 529.5, 553.0, 572.5.

10. $Y = 10.78 + 9.1X + 2.21X^2$ with origin at 1996.

11. 10.12, 0.13, −14.08, 3.83 using additive model.

12. 33.625, 34.25, 34.75, 35.625, 37, 38.5, 39.875, 41.375 in lac tons.

B

1. $Y = 21 + 0.1X$; 21,100 tons and 21,600 tons.
2. $Y = 76 + 4.857X$; ₹95.428 lac.
3. $Y = 680 + 70.5X$, −18.0, −68.5, 111.0, 110.5, 10.0, −130.5, −151.0, 58.5, 78.0.
4. $Y = 655 + 35.83X$ referred to midpoint of 1983–84 as origin and unit of X as 6 months; 1049.13.
5. 719.00, 738.75, 758.25, 776.38, 793.88, 812.88, 831.63. The trend values for the 1st two and the last two years cannot be determined.
6. 96.4, 92.1, 106.9, 100.5.
7. (a) Trend: 78.38, 81.63, 87.50, 97.00, 104.63, 111.88, 119.33, 127.50, 135.50, 139.63, 145.75, 151.25, 154.38, 159.38, 163.88, 173.13;

   (b) Short-term fluctuations: −11.38, 42.37, −49.50, 12.00, −13.63, 64.12, −72.63, 30.50, −31.50, 86.37, −84.75, 25.75, −20.38, 80.62, −91.88, 32.87;

   (c) Seasonal indices: −74.625, 25.345, −19.155, 68.435.
8. 110.9, 99.9. 84.9, 104.3.
9. 104.8, 99.0, 101.6, 94.6.
10. $Y = 55.975 + 0.825X$; Year 1994.
11. −4.49, −2.27, 6.76, in '00 units.
12. 0.77, −3.31, 0.11, 2.43 in million ₹
13. 108.2.
14. $Y = 140 + 20X$; 220 lac ₹.
15. (a) $Y = 22.78 + 1.46X + 0.64X^2$;

    (b) $Y = 39.181 + 2.036X + 0.488X^2$.
16. (a) $Y = 33.56(1.142)^X$; 157.15 crore.

    (b) $Y = 13.79(2.982)^X$.

C

1. $y = 0.336x + 11.588$ referred to midpoint of 1995–96 as origin and unit of x as 6 months; 11.92, 12.60 (in '00000 tons).
2. 110.000, 99.875, 92.375 (in '000 bales) for 1965, 1985, 1995.
3. 151.95, 162.95, 173.55, 190.95, 208.75, 212, 211.50, 222.90.
4. $y = 12.314 - 0.6X - 0.857X^2$ with origin at 1975 and unit of x as 1 year.
5. 300, 500, 500, 400, 500, 550 (in '00000 ₹) (using Multiplicative Model).
6. −3.655, −1.615, 8.385, −3.115 (in '00000 tons).
7. Seasonal indices ('00000 tons): 3.12, 0.69, −4.25, 0.44. Deseasonalized data: 33.88, 37.31, 41.25, 39.56, 37.88, 33.31, 29.25, 30.56, 31.88, 36.31, 39.25, 40.56. Significance: If the climatic conditions due to change of season or the festivals and customs in a year had not affected the production, the expected production would be the above deseasonalized data.
8. Deseasonalized data ('00000 tons): 28.28, 38.70, 39.28, 57.75, 47.28, 39.70, 50.28, 42.74, 33.28, 51.70, 49.28, 47.74, 73.28, 68.70, 54.28, 92.74.
9. ₹1,52,000, 1,54,000; 1,96,000; 2,56,000, 2,74,000, 2,44,000;
10. Estimated monthly sales (₹lac): 15.6, 15.0, 20.0, 25.2, 27.6, 24.2, 20.2, 20.8, 19.8, 20.6, 16.0, 15.0.
11. $Y = 39.9 - 0.767x$, where $x = X - 1973$; 41.434, 37.599 (in ₹'000).
12. −1.25, −6.25, 6.75, 0.75.
13. 104.9, 97.5, 90.3, 112.3, 112.3, 114.6, 119.5, 92.5, 82.9, 78.0, 87.8, 107.3.
14. 124.2, 93.5, 82.5, 99.8.

APPENDIX

FOUR-FIGURE LOGARITHMIC TABLES

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	6	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4165	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7733	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9196	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

FOUR-FIGURE ANTILOGARITHMIC TABLES

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	8	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	12
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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